Another proof of Ricci flow on incomplete surfaces with bounded above Gauss curvature

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Abstract

We give a simple proof of an extension of the existence results of Ricci flow of G. Giesen and P.M. Topping [GiT1],[GiT2], on incomplete surfaces with bounded above Gauss curvature without using the difficult Shi's existence theorem of Ricci flow on complete non-compact surfaces and the pseudolocality theorem of G. Perelman [P1] on Ricci flow. We will also give a simple proof of a special case of the existence theorem of P.M. Topping [T] without using the existence theorem of W.X. Shi [S1].

Key words: Ricci flow, incomplete surfaces, negative Gauss curvature
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Recently there is a lot of study on the Ricci flow on manifold by R. Hamilton [H1], [H2], S.Y. Hsu [Hs1–4], G. Perelman [P1], [P2], W.X. Shi [S1], [S2], L.F. Wu [W1], [W2], and others because Ricci flow is a powerful tool in the study of geometric problems. We refer the readers to the book [CLN] by B. Chow, P. Lu and L. Ni, for the basics of Ricci flow and the papers [P1], [P2], of G. Perelman and the book [Z] by Qi S. Zhang for the most recent results on Ricci flow.

In 1982 R. Hamilton [H1] proved that if M is a compact manifold and $g_{ij}(x)$ is a metric of strictly positive Ricci curvature, then there exists a unique metric g that evolves by the Ricci flow

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij} \tag{1}$$

on $M \times (0,T)$ for some $T > 0$ with $g_{ij}(x,0) = g_{ij}(x)$ where $R_{ij}(\cdot,t)$ is the Ricci curvature of $g(\cdot,t)$.

Short time existence of solutions of the Ricci flow on complete non-compact Riemannian manifold with bounded curvature was proved by W.X Shi [S1]. Global existence and uniqueness of solutions of the Ricci flow on non-compact manifold $\mathbb{R}^2$ was obtained by S.Y. Hsu in [Hs1]. Existence and uniqueness of the Ricci flow on incomplete surfaces with negative Gauss curvature was obtained by G. Giesen and P.M. Topping in [GiT1]. In [GiT1] G. Giesen and P.M. Topping proved the following theorem.

Theorem 1. *(Theorem 1.1 of [GiT1]) Suppose M is a surface (i.e. a 2-dimensional manifold without boundary) equipped with a smooth Riemannian metric g_0 whose Gauss curvature satisfies $K[g_0] \le -\eta < 0$, but which need not be complete. Then exists a unique smooth Ricci flow $g(t)$ for $t \in [0,\infty)$ with the following properties:*

(i) $g(0) = g_0$;

(ii) $g(t)$ *is complete for all* $t > 0$;

(iii) the curvature of $g(t)$ *is bounded above for any compact time interval within* $[0,\infty)$;

(iv) the curvature of $g(t)$ *is bounded below for any compact time interval within* $(0,\infty)$.

Moreover this solution satisfies $K[g(t)] \le -\frac{\eta}{1+t\eta}$ *for* $t \ge 0$ *and* $-\frac{1}{2t} \le K[g(t)]$ *for* $t > 0$.

By abuse of notation we will write $K[u] = K[g]$ for the Gauss curvature of a metric of the form $g = e^{2u}\delta_{ij}$. As observed by G. Giesen and P.M. Topping [GiT1] in order to prove Theorem 1 it suffices to assume that $M = \mathcal{D}$ is a unit disk in $\mathbb{R}^2$ and $g_0 = e^{2u_0}\delta_{ij}$ is a conformal metric on $\mathcal{D}$. Then by scaling Theorem 1 is equivalent to the following two theorems.

Theorem 2. *(cf. Theorem 3.1 and Lemma 2.2 of [GiT1]) Let $g_0 = e^{2u_0}\delta_{ij}$ be a smooth conformal metric on the unit disk $\mathcal{D}$ with*

$$K[u_0] \le -1. \tag{2}$$

Then there exists a smooth solution $g(t) = e^{2u}\delta_{ij}$ of (1) in $\mathcal{D} \times [0,\infty)$ with $g(0) = g_0$ such that $g(t)$ is complete for every $t > 0$ with the Gauss curvature $K[u(t)]$ satisfying

$$K[u(t)] \ge -\frac{1}{2t} \quad \forall t > 0, \tag{3}$$

$$K[u(t)] \le -\frac{1}{2t+1} \quad \forall t \ge 0, \tag{4}$$

$$u(x,t) \ge \log\frac{2}{1-|x|^2} + \frac{1}{2}\log(2t) \quad \forall x \in \mathcal{D}, t > 0, \tag{5}$$

$$u(x,t) \le \log\frac{2}{1-|x|^2} + \frac{1}{2}\log(2t+1) \quad \forall x \in \mathcal{D}, t \ge 0, \tag{6}$$

and

$$u(x,t) \geq u_0(x) + \frac{1}{2}\log(2t+1) \qquad \text{in } \mathcal{D} \times [0,\infty). \tag{7}$$

Moreover $g(t)$ is maximal in the sense that if $\widetilde{g}(t)$ for $t \in [0,\varepsilon]$ is another Ricci flow with $\widetilde{g}(0) = g_0$ for some $\varepsilon > 0$, then

$$\widetilde{g}(t) \leq g(t) \quad \forall 0 \leq t \leq \varepsilon. \tag{8}$$

Theorem 3. *(cf. Theorem 4.1 of [GiT1]) Let $e^{2u_0}\delta_{ij}$ be a smooth metric on the unit disk $\mathcal{D}$ satisfying the upper curvature bound* (2)*. Let $e^{2u}\delta_{ij}$ be a solution of* (1) *in $\mathcal{D} \times (0,\infty)$ with $u(\cdot,0) = u_0$ which satisfies* (3) *and* (4)*. Then u is unique among solutions that satisfy* (3) *and* (4)*.*

The proof of Theorem 3.1 and Lemma 2.2 of [GiT1] uses the results of [T], the Schwartz Lemma of S.T. Yau [Y], and the difficult existence theorem of W.X. Shi [S1] for Ricci flow on complete non-compact manifolds. In this paper we will give a simple proof of Theorem 2 using the results of K.M. Hui in [Hu3] and [Hu4]. We will assume that $M = \mathcal{D} \subset \mathbb{R}^2$ is a unit disk for the rest of the paper. Note that for a metric g on a 2-dimensional manifold, $\text{Ric}[g] = K[g]g$. We will also give simple proofs of the following extension of the existence results of G. Giesen and P.M. Topping [GiT2] and a special case of Theorem 1.1 of [T] without using the existence theorem of W.X. Shi [S1] and the pseudolocality Theorem of G. Perelman [P1] on Ricci flow.

Theorem 4. *(cf. Theorem 3.1 of [GiT2] and Theorem 1.1 of [T]) Let $g_0 = e^{2u_0}\delta_{ij}$ be a smooth (possible incomplete) Riemannian metric on $\mathcal{D}$. Then there exists a maximal instantaneous smooth complete Ricci flow $g(t) = e^{2u}\delta_{ij}$ on $\mathcal{D}$ for all time $t \in [0,\infty)$ with $g(0) = g_0$ which satisfies* (3) *and* (5)*. Suppose in addition the Gauss curvature satisfies*

$$K[g_0] \leq K_0 \tag{9}$$

for some constant $K_0 \geq 0$. Then the following holds.

(i) If $K_0 > 0$, then

$$K[u(t)] \leq \frac{1}{K_0^{-1} - 2t} \quad \forall 0 \leq t < (2K_0)^{-1} \tag{10}$$

and

$$u(x,t) \geq u_0(x) + \frac{1}{2}\log(1 - 2K_0 t) \quad \forall 0 \leq t < (2K_0)^{-1}. \tag{11}$$

(ii) If $K_0 = 0$, then

$$K[u(t)] \leq 0 \quad \forall t \geq 0 \tag{12}$$

and

$$u(x,t) \geq u_0(x) \quad \forall t \geq 0. \tag{13}$$

Theorem 5. *(cf. [DP1], [Hu2], [Hs1], Theorem 1.1 of [T] and Theorem 3.2 of [GiT2]) Let* $g_0 = e^{2u_0}\delta_{ij}$ *be a smoooth metric on* $\mathbb{R}^2$ *which need not be complete. Then there exists a maximal instantaneous smooth complete Ricci flow* $g(t) = e^{2u}\delta_{ij}$ *on* $\mathbb{R}^2$ *for* $t \in [0, T)$ *with* $g(0) = g_0$ *which satisfies* (3) *and for any* $0 < T_1 < T$ *and* $r_0 > 1$ *there exists a constant* $C > 0$ *such that*

$$u(x,t) \geq -C - \log(|x| \log |x|) + \frac{1}{2}\log(2t) \quad \forall |x| \geq r_0, 0 \leq t \leq T_1 \tag{14}$$

where

$$T = \begin{cases} \dfrac{Vol_{g_0}(\mathbb{R}^2)}{4\pi} & \text{if } Vol_{g_0}(\mathbb{R}^2) < \infty \\ \infty & \text{if } Vol_{g_0}(\mathbb{R}^2) = \infty \end{cases} \tag{15}$$

and

$$Vol_{g(t)}(\mathbb{R}^2) = \begin{cases} 4\pi(T-t) & \forall 0 \leq t < T \quad \text{if } Vol_{g_0}(\mathbb{R}^2) < \infty \\ \infty & \forall t > 0 \quad\quad\;\; \text{if } Vol_{g_0}(\mathbb{R}^2) = \infty. \end{cases} \tag{16}$$

If in addition the Gauss curvature satisfies (9) *for some constant* $K_0 \geq 0$*, then the following holds.*

(i) If $K_0 > 0$*, then* (10) *and* (11) *holds on* $\mathbb{R}^2$ *for any* $0 \leq t < (2K_0)^{-1}$ *and* $T \geq (2K_0)^{-1}$*.*

(ii) if $K_0 = 0$*, then* (12) *and* (13) *holds on* $\mathbb{R}^2$ *for any* $t \geq 0$ *and* $T = \infty$*.*

We start with some definitions. For any $r_1 > 0$, $T_1 > 0$, let $B_{r_1} = \{x \in \mathbb{R}^2 : |x| < r_1\}$, $Q_{r_1} = B_{r_1} \times (0, \infty)$, $Q_{r_1}^{T_1} = B_{r_1} \times (0, T_1)$, and $\partial_p Q_{r_1} = (\partial B_{r_1} \times [0, \infty)) \cup (\overline{B}_{r_1} \times \{0\})$. For any set $A \subset \mathbb{R}^2$, let χ_A be the characteristic function of the set A. Note that for any metric of the form $e^{2u(x,t)}\delta_{ij}$ in $Q_{r_1}^T$, we have

$$K[u] = -\frac{\Delta u}{e^{2u}}$$

and $e^{2u(x,t)}\delta_{ij}$ is a solution of (1) in $Q_{r_1}^T$ if any only if

$$\frac{\partial u}{\partial t} = e^{-2u}\Delta u \quad \text{in } Q_{r_1}^T. \tag{17}$$

where Δ is the Euclidean Laplacian on $\mathbb{R}^2$. Let

$$v = e^{2u}. \tag{18}$$

Then (17) is equivalent to

$$\frac{\partial v}{\partial t} = \Delta \log v \quad \text{in } Q_{r_1}^T. \tag{19}$$

Existence and various properties of (19) were studied by P. Daskalopoulos and M.A. Del Pino [DP1], S.H. Davis, E. Dibenedetto, and D.J. Diller [DDD], J.R. Esteban, A. Rodriguez, and J.L. Vazquez [ERV1], [ERV2], S.Y. Hsu [Hs1], K.M. Hui [Hu2], etc. We refer the readers to the book [DK] by P. Daskalopoulos and C.E. Kenig and the book [V] by J.L. Vazquez for the recent results on the equation (19).

For any $0 \le v_0 \in L^1_{loc}(B_{r_1})$, we say that v is a solution of

$$\begin{cases} v_t = \Delta \log v & \text{in } Q_{r_1} \\ v > 0 & \text{in } Q_{r_1} \\ v(x,t) = \infty & \text{on } \partial B_{r_1} \times (0,\infty) \\ v(x,0) = v_0(x) & \text{in } B_{r_1} \end{cases} \tag{20}$$

if v is a classical solution of (19) in Q_{r_1},

$$\lim_{y \to x} v(y,t) = \infty \quad \forall x \in \partial B_{r_1}, t > 0,$$

$$\inf_{B_{r_1} \times (t_1,t_2)} v(x,t) > 0 \quad \forall t_2 > t_1 > 0,$$

and

$$\lim_{t \to 0} \|v(\cdot,t) - v_0\|_{L^1(K)} = 0 \tag{21}$$

for any compact set $K \subset B_{r_1}$. We say that v is a solution of (19) in $\mathbb{R}^2 \times (0,T)$ with initial value v_0 if v is a classical solution of (19) in $\mathbb{R}^2 \times (0,T)$, $v > 0$ in $\mathbb{R}^2 \times (0,T)$, and (21) holds for any compact set $K \subset \mathbb{R}^2$.

Note that for any solution v of (19) the equation (19) is uniformly parabolic on K for any compact set $K \subset Q^T_{r_1}$. Hence by the Schauder estimates [LSU] and a bootrap argument $v \in C^\infty(Q^T_{r_1})$ for any solution v of (19). We first observe that by the same argument as the proof of Theorem 1.6 of [Hu3] we have the following result.

Theorem 6. *(cf. Theorem 1.6 of [Hu3]) Let $r_1 > 0$. Suppose $v_0 \ge 0$ satisfies*

$$v_0 \in L^p_{loc}(B_{r_1}) \quad \text{for some constants } p > 1.$$

Then exists a solution v of (20) which satisfies

$$v_t \le \frac{v}{t} \tag{22}$$

in Q_{r_1} and

$$v(x,t) \ge C \frac{t}{\phi(x)} \quad \text{in } Q_{r_1}$$

for some constant $C > 0$ depending on ϕ and λ where ϕ and λ are the first positive eigenfunction and the first positive eigenvalue of the Laplace operator $-\Delta$ on B_{r_1} with $\|\phi\|_{L^2(B_{r_1})} = 1$.

Lemma 7. *Let $r_1 > 0$. Suppose v_0 is a positive smooth function on B_{r_1} and v is a solution of (20). Then $v \in C^\infty(B_{r_1} \times [0,\infty))$.*

Proof: We will first use a modification of the technique of Lemma 1.6 of [Hu1] to show that v is uniformly bounded below by a positive constant on $\overline{B}_{r_3} \times [0,1]$ for any $0 < r_3 < r_1$. Let $0 < r_3 < r_2 < r_1$, $\delta_1 = r_3 - r_2$, and

$$\varepsilon = \frac{1}{2} \min_{\overline{B}_{r_2}} v_0.$$

Since v_0 is a smooth positive function on B_{r_1}, $\varepsilon > 0$. Let w be the maximal solution of the equation (19) in $\mathbb{R}^2\times(0,T)$ constructed in [DP1] and [Hu2] with initial value $w(x,0) = \varepsilon\chi_{\overline{B}_{\delta_1}}$ and $T = \varepsilon|\overline{B}_{\delta_1}|/4\pi$ which satisfies

$$\int_{\mathbb{R}^2} w(x,t)\,dx = \int_{\mathbb{R}^2} w(x,0)\,dx - 4\pi t \quad \forall 0 \le t < T.$$

For any $\alpha > 0$, let

$$w_\alpha(x,t) = w(\alpha x, \alpha^2 t) \quad \forall x \in \mathbb{R}^2, 0 \le t < T/\alpha^2.$$

and

$$v_{\alpha,p}(x,t) = v(\alpha x + p, \alpha^2 t) \quad \forall p \in \overline{B}_{r_3}, |x| < \delta_1/\alpha, t > 0.$$

Since $v_0(x+p) \ge w(x,0)$ for any $p \in \overline{B}_{r_3}$, $x+p \in B_{r_1}$, and $v = \infty$ on $\partial B_{r_1} \times (0,\infty)$, by Corollary 1.8 and Lemma 2.9 of [Hu4],

$$\begin{aligned} & v(x+p,t) \ge w(x,t) \quad \forall p \in \overline{B}_{r_3}, x+p \in B_{r_1}, 0 \le t < T \\ \Rightarrow\quad & v_{\alpha,p}(x,t) \ge w_\alpha(x,t) \quad \forall p \in \overline{B}_{r_3}, |x| < \delta_1/\alpha, 0 \le t < T/\alpha^2. \end{aligned} \tag{23}$$

Since $w_\alpha(x,0) = w(\alpha x, 0) = \varepsilon\chi_{\overline{B}_{\delta_1/\alpha}}$,

$$0 \le w_{\alpha_1}(x,0) \le w_{\alpha_2}(x,0) \le \varepsilon \quad \forall x \in \mathbb{R}^2, \alpha_1 \ge \alpha_2 > 0. \tag{24}$$

By (24) and the comparison theorems in [DP1] and [Hu2],

$$0 < w_{\alpha_1}(x,t) \le w_{\alpha_2}(x,t) \le \varepsilon \quad \forall x \in \mathbb{R}^2, 0 < t < \varepsilon|\overline{B}_{\delta_1}|/(4\pi\alpha_1^2), \alpha_1 \ge \alpha_2 > 0. \tag{25}$$

We choose $0 < \alpha_0 < \delta_1$ such that $\varepsilon|\overline{B}_{\delta_1}|/(4\pi\alpha_0^2) > 1$. By (25) the equation (19) for w_α, $0 < \alpha \le \alpha_0$, is uniformly parabolic on every compact set $K \subset \mathbb{R}^2 \times (0,1]$. Then by the Schauder estimates [LSU] $\{w_\alpha\}_{\{0<\alpha\le\alpha_0\}}$ are equi-Holder continuous in $C^2(K)$ for any compact set $K \subset \mathbb{R}^2 \times (0,1]$. Hence by (25) w_α increases and converges uniformly on every compact subset K of $\mathbb{R}^2 \times (0,1]$ to the constant function ε in $\mathbb{R}^2 \times (0,1]$ as $\alpha \to 0$. Then there exists $\alpha_1 \in (0,\alpha_0)$ such that

$$w_\alpha(x,1) \ge \frac{\varepsilon}{2} \quad \forall |x| \le 1, 0 < \alpha \le \alpha_1. \tag{26}$$

By (23) and (26),

$$\begin{aligned} & v(\alpha x + p, \alpha^2) = v_{\alpha,p}(x,1) \ge \frac{\varepsilon}{2} \quad \forall p \in \overline{B}_{r_3}, |x| \le 1, 0 < \alpha \le \alpha_1 \\ \Rightarrow\quad & v(p+y,t) \ge \frac{\varepsilon}{2} \quad \forall p \in \overline{B}_{r_3}, |y| \le \sqrt{t}, 0 < t \le \alpha_1^2. \end{aligned} \tag{27}$$

Let λ and ϕ be the first positive eigenvalue and the first positive eigenfunction of $-\Delta$ on B_{r_2} with $\|\phi\|_{L^2(B_{r_2})} = 1$. Let $C_1 = \max(2\lambda\|\phi\|_{L^\infty}, 12\|\nabla\phi\|_{L^\infty}, \|v_0\|_{L^\infty(\overline{B}_{r_1})} + 1)$ and

$$\psi(x,t) = C_1(t+1)e^{1/\phi(x)}.$$

By the computation on P.784-785 of [Hu3], ψ is a supersolution of (19). Then an argument similar to the proof Theorem 2.9 of [Hu4],

$$\int_{B_{r_2}} (v-\psi)_+(x,t)\,dx \le \int_{B_{r_2}} (v-\psi)_+(x,t_1)\,dx \quad \forall t \ge t_1 > 0. \tag{28}$$

Since

$$\int_{B_{r_2}} (v-\psi)_+(x,t_1)\,dx \le \int_{B_{r_2}} |v(x,t_1)-v_0(x)|\,dx + \int_{B_{r_2}} (v_0(x)-\psi(x,t_1))_+\,dx$$
$$\to 0 \qquad \text{as } t_1 \to 0,$$

letting $t_1 \to 0$ in (28),

$$\int_{B_{r_2}} (v-\psi)_+(x,t)\,dx = 0 \quad \forall t > 0.$$

Hence

$$v(x,t) \le \psi(x,t) = C_1(t+1)e^{1/\phi(x)} \quad \forall |x| < r_2, t \ge 0.$$

Thus

$$v(x,t) \le C'(t+1) \quad \forall |x| \le r_3, t \ge 0 \tag{29}$$

for some constant $C' > 0$. By (27) and (29), the equation (19) is uniformly parabolic on $\overline{B}_{r_3} \times [0,1]$ for any $0 < r_3 < r_1$. By standard parabolic estimates [LSU] and a bootstrap argument, $v \in C^\infty(B_{r_1} \times [0,\infty))$ and the lemma follows. □

We will now prove Theorem 2.

Proof of Theorem 2: Let $v_0(x) = e^{2u_0(x)}$. For any $k = 2, 3, \dots$, by Theorem 6 there exists a solution v_k of (20) with $r_1 = 1-(1/k)$ and initial value v_0 which satisfies (22) in $B_{1-(1/k)} \times (0,\infty)$ and

$$v_k(x,t) \ge C_k \frac{t}{\phi_k(x)} \quad \text{ in } B_{1-(1/k)} \times (0,\infty) \tag{30}$$

for some constant $C_k > 0$ depending on ϕ_k and λ_k where ϕ_k and λ_k are the first positive eigenfunction and the first positive eigenvalue of the Laplace operator $-\Delta$ on $B_{1-(1/k)}$ with $\|\phi\|_{L^2(B_{1-(1/k)})} = 1$. By Lemma 7, $v_k \in C^\infty(B_{1-(1/k)} \times [0,\infty))$. By Corollary 1.8 and Theorem 2.9 of [Hu4],

$$v_k \ge v_{k+1} \quad \text{ in } B_{1-(1/k)} \times (0,\infty) \quad \forall k \ge 2. \tag{31}$$

Since ϕ_k and λ_k converges to ϕ and λ as $k \to \infty$ where ϕ and λ are the first positive eigenfunction and the first positive eigenvalue of the Laplace operator $-\Delta$ on B_1 with $\|\phi\|_{L^2(B_1)} = 1$, by the proof of Theorem 1.2 and Theorem 1.6 of [Hu3] there exists a constant $C > 0$ such that

$$C_k \ge C \quad \forall k \ge 2. \tag{32}$$

Let $k_0 \ge 2$. Then by (30) and (32) there exists a constant $C_0 > 0$ such that

$$v_k(x,t) \ge C_0 t \quad \text{ in } B_{1-(1/k_0)} \times (0,\infty) \quad \forall k > k_0. \tag{33}$$

By Corollary 1.8 of [Hu4] for any $0 < t_1 < t_2$ there exists a constant $C' > 0$ such that

$$v_k(x,t) \le C' \quad \forall |x| \le 1-(1/k_0), t_1 \le t \le t_2, k > k_0. \tag{34}$$

By (33) and (34) the equation (19) for v_k are uniformly parabolic on $\overline{Q}_{1-(1/k_0)}$ for all $k > k_0$. By the parabolic Schauder estimates [LSU] v_k are equi-Holder continuous in $C^2(K)$ for every compact set $K \subset \overline{Q}_{1-(1/k_0)}$ and all $k > k_0$. Hence by (31) v_k decreases and converges uniformly on K to a solution v of (19) in $\mathcal{D} \times (0,\infty)$ as $k \to \infty$ for any compact set $K \subset \mathcal{D} \times (0,\infty)$. Since v_k satisfies (22) in $Q_{1-(1/k)}$ for any $k \in \mathbb{Z}^+$, by an argument similar to the proof of Theorem 2.4 of [Hu3] v has initial value v_0 and satisfies (22) in $\mathcal{D} \times (0,\infty)$. Letting $k \to \infty$ in (30), by (32),

$$\begin{aligned} & v(x,t) \ge C\frac{t}{\phi(x)} \quad \text{in } \mathcal{D} \times (0,\infty) \\ \Rightarrow \quad & \lim_{x\to x_0} v(x,t) = \infty \quad \forall |x_0| = 1, t > 0. \end{aligned}$$

Hence v is a solution of (20) with $B_{r_1} = \mathcal{D}$. By Lemma 7, $v \in C^\infty(\mathcal{D} \times [0,\infty))$.

Let u be given by (18) and $g(t) = e^{2u}\delta_{ij}$. Then $g(t)$ is a smooth solution of (1) in $\mathcal{D} \times [0,\infty)$ with initial value $e^{2u_0}\delta_{ij}$. By (19) and (22), we get (3). By (3),

$$\Delta u(x,t) \le \frac{1}{2t} e^{2u(x,t)} \quad \text{in } \mathcal{D} \quad \forall t > 0. \tag{35}$$

For any $0 < \delta < 1$, let

$$\psi_\delta(x,t) = \log\frac{2(1+\delta)}{(1+\delta)^2 - |x|^2} + \frac{1}{2}\log(2t) \quad \forall x \in \mathcal{D}, t > 0.$$

Then ψ_δ satisfies

$$\Delta \psi_\delta(x,t) = \frac{1}{2t} e^{2\psi_\delta(x,t)} \quad \text{in } \mathcal{D} \quad \forall t > 0. \tag{36}$$

Let $\Omega(t) = \{x \in \mathcal{D} : u(x,t) < \psi_\delta(x,t)\}$. By (35) and (36),

$$\Delta(u(x,t) - \psi_\delta(x,t)) \le \frac{1}{2t}(e^{2u(x,t)} - e^{2\psi_\delta(x,t)}) < 0 \quad \text{in } \Omega(t) \quad \forall t > 0. \tag{37}$$

Since $u(x,t) - \psi_\delta(x,t) \to \infty$ as $|x| \to 1$, $\overline{\Omega(t)} \subset \mathcal{D}$. Hence by (37) and the maximum principle [GT],

$$\begin{aligned} & u(x,t) - \psi_\delta(x,t) \ge \min_{x \in \partial\Omega(t)} (u(x,t) - \psi_\delta(x,t)) = 0 \quad \text{in } \Omega(t) \quad \forall t > 0 \\ \Rightarrow \quad & u(x,t) \ge \psi_\delta(x,t) \quad \text{in } \mathcal{D} \quad \forall t > 0. \end{aligned} \tag{38}$$

Letting $\delta \to 0$ in (38), (5) follows. We now let $v_{k,m}$ be the solution of

$$\begin{cases} v_t = \Delta \log v & \text{in } Q_{1-(1/k)} \\ v > 0 & \text{in } Q_{1-(1/k)} \\ v(x,t) = v_0(x) e^{mt^2+2t} & \text{on } \partial B_{1-(1/k)} \times (0,\infty) \\ v(x,0) = v_0(x) & \text{in } B_{1-(1/k)} \end{cases} \tag{39}$$

for any $k \geq 2$, $m \in \mathbb{Z}^+$, which can be constructed by similar technique as that of [Hu3]. By an argument similar to the proof of [Hu3] $v_{k,m}$ increases and converges uniformly to v_k on every compact subset K of $Q_{1-(1/k)}$ as $m \to \infty$ and $v_{k,m}$ satisfies

$$0 < \min_{|x| \leq 1-k^{-1}} v_0(x) \leq v_{k,m}(x,t) \leq e^{mT_1^2+2T_1} \max_{|x|\leq 1-k^{-1}} v_0(x) \quad \forall |x| \leq 1-k^{-1}, 0 \leq t \leq T_1, k \geq 2, \tag{40}$$

for any $T_1 > 0$. By (40) the equation (19) for $v_{k,m}$ is uniformly parabolic on $\overline{Q_{1-(1/k)}^{T_1}}$ for any $T_1 > 0$. Hence by (39), (40), and parabolic estimates [LSU], $v_{k,m} \in C^\infty(\overline{Q}_{1-(1/k)})$. Let $p_{k,m} = \partial_t v_{k,m}/v_{k,m}$. Then $p_{k,m} \in C^\infty(\overline{Q}_{1-(1/k)})$. By (2) and (39) $p_{k,m}$ satisfies

$$p_t = e^{-v_{k,m}} \Delta p - p^2 \quad \text{ in } Q_{1-(1/k)} \tag{41}$$

and

$$\begin{cases} p = 2mt + 2 & \text{on } \partial B_{1-(1/k)} \times [0,\infty) \\ p(x,0) \geq 2 & \forall |x| \leq 1-k^{-1}. \end{cases} \tag{42}$$

Note that the function $2/(2t+1)$ satisfies (41) and by (42),

$$p_{k,m} \geq 2/(2t+1) \quad \text{ on } \partial_p Q_{1-(1/k)}.$$

Hence by the maximum principle (cf. [A]),

$$\begin{aligned} &\frac{\partial_t v_{k,m}}{v_{k,m}} = p_{k,m}(x,t) \geq \frac{2}{2t+1} \quad && \text{in } Q_{1-(1/k)} \quad \forall k \geq 2, m \geq 1 \\ \Rightarrow \quad &\frac{v_t}{v} \geq \frac{2}{2t+1} && \text{in } \mathcal{D} \times [0,\infty) \quad \text{ as } m \to \infty, k \to \infty, \end{aligned}$$

and (4) follows. By (4) and an argument similar to the proof of (5) we get (6). By (5) $g(t)$ is complete for any $t > 0$. Now by (4),

$$\begin{aligned} &K[u(t)] \leq -\frac{1}{2t+1} && \text{in } \mathcal{D} \times [0,\infty) \\ \Rightarrow \quad &\frac{\partial u}{\partial t} \geq \frac{1}{2t+1} && \text{in } \mathcal{D} \times [0,\infty) \\ \Rightarrow \quad &u(x,t) \geq u_0(x) + \frac{1}{2}\log(2t+1) \quad && \text{in } \mathcal{D} \times [0,\infty) \end{aligned}$$

and (7) follows. Suppose $\widetilde{g}(t)$, $0 \leq t \leq \varepsilon$, is a solution of (1) in $\mathcal{D} \times (0,\varepsilon)$ with $g(0) = g_0$. As in [GiT1] we can write $\widetilde{g} = e^{2\widetilde{u}}\delta_{ij}$. Let $\widetilde{v} = e^{2\widetilde{u}}$. Then $\widetilde{v}$ is a solution of (20) with $r_1 = 1$. Hence by Corollary 1.8 and Theorem 2.9 of [Hu4],

$$\begin{aligned} &\widetilde{v}(x,t) \leq v_k(x,t) \quad \forall |x| \leq 1-(1/k), 0 \leq t \leq \varepsilon \\ \Rightarrow \quad &\widetilde{v}(x,t) \leq v(x,t) \quad \forall |x| < 1, 0 \leq t \leq \varepsilon \quad \text{ as } k \to \infty \end{aligned}$$

and (8) follows.

□

Proof of Theorem 4: Let $v_0 = e^{2u_0}, v_k, v_{k,m}, v, u, g(t), p_{k,m}$ be as in the proof of Theorem 3. By the proof of Theorem 3 $v \in C^\infty(\mathcal{D} \times [0,\infty))$ satisfies (3) and (5) and $g(t)$ is a smooth maximal instanteous complete solution of (1) for all time $t \geq 0$.

Suppose now (9) holds for some constant $K_0 \geq 0$. We will show that the curvature $K[u(t)]$ satisfies (10). Let $T_0 = (2K_0)^{-1}$. By (9) and (39), $p_{k,m}$ satisfies

$$\begin{cases} p(x,t) = 2mt + 2 \geq -\dfrac{1}{(2K_0)^{-1} - t} & \forall |x| = 1 - k^{-1}, t \geq 0 \\ p(x,0) \geq -2K_0 & \forall |x| \leq 1 - k^{-1}. \end{cases} \tag{43}$$

Since both $p_{k,m}$ and $-1/((2K_0)^{-1} - t)$ satisfy (41), by (43) and the maximum principle,

$$\begin{aligned} &\frac{\partial_t v_{k,m}}{v_{k,m}} = p_{k,m}(x,t) \geq -\frac{1}{(2K_0)^{-1} - t} \quad \forall |x| < 1 - (1/k), 0 \leq t < (2K_0)^{-1} \\ \Rightarrow \quad &\frac{v_t}{v} \geq -\frac{1}{(2K_0)^{-1} - t} \quad \forall |x| < 1 - (1/k), 0 \leq t < (2K_0)^{-1} \quad \text{as } m \to \infty, k \to \infty \end{aligned}$$

and (10) follows. By (10) and (17),

$$u_t(x,t) \geq -\frac{1}{K_0^{-1} - 2t} \quad \forall |x| < 1, 0 < t < (2K_0)^{-1}.$$

Integrating the above equation with respect to t and (11) follows.

Suppose now (9) holds with $K_0 = 0$. Then by (9) and (39),

$$p_{k,m} \geq 0 \quad \text{on } \partial_p Q_{1-(1/k)}.$$

Hence by the maximum principle,

$$\begin{aligned} &\frac{\partial_t v_{k,m}}{v_{k,m}} = p_{k,m}(x,t) \geq 0 \quad \forall |x| < 1 - k^{-1}, t \geq 0 \\ \Rightarrow \quad &v_t \geq 0 \quad \forall |x| < 1, t \geq 0 \quad \text{as } m \to \infty, k \to \infty \end{aligned}$$

and (12) follows. By (12) and (18),

$$u_t(x,t) \geq 0 \quad \forall |x| < 1, t > 0$$

and (13) follows.

□

Proof of Theorem 5: We will use a modification of the technique of [DP2] and [Hu3] to prove the theorem. Let $v_0 = e^{2u_0}$. by the results of [DP1], [Hu2], and [Hs1], there exists a unique maximal solution v of (19) in $\mathbb{R}^2 \times (0,T)$ which satisfies (16) and (22) in $\mathbb{R}^2 \times (0,T)$ where T is given by (15). By Theorem 3.4 of [ERV2] and the result of [Hs1] for any $0 < T_1 < T$ and $r_0 > 1$ there exists a constant $C > 0$ such that

$$v(x,t) \geq \frac{Ct}{|x|^2(\log|x|)^2} \quad \forall |x| \geq r_0, 0 \leq t < T. \tag{44}$$

Let u be given by (18) and $g(t) = e^{2u}\delta_{ij}$. Then by (22) and (44), (3) and (14) hold.

For any $k \in \mathbb{Z}^+$, by the same argument as the proof of Theorem 4 there exists a solution $\widetilde{v}_k$ of (20) with $r_1 = k$ which satisfies (22) in $B_k \times (0, \infty)$. By Corollary 1.8 and Theorem 2.9 of [Hu4],

$$\widetilde{v}_k \geq \widetilde{v}_{k+1} \geq v \quad \text{in } B_k \times (0, T) \quad \forall k \in \mathbb{Z}^+. \tag{45}$$

Let $k_0 \in \mathbb{Z}^+$ and $0 < t_1 < t_2 < T$. By Corollary 1.8 of [Hu4] there exists a constant $C > 0$ such that

$$\widetilde{v}_k \leq C \quad \text{in } \overline{B}_{k_0} \times [t_1, t_2] \quad \forall k > k_0. \tag{46}$$

Hence by (45) and (46) the equation (19) for $\widetilde{v}_k$ is uniformly parabolic on $\overline{B}_{k_0} \times [t_1, t_2]$ for any $k > k_0$. By the Schauder estimates [LSU] the sequence $\{\widetilde{v}_k\}_{\{k>k_0\}}$ is equi-Holder continuous in $C^2(\overline{B}_{k_0} \times [t_1, t_2])$. Hence by (45) as $k \to \infty$, $\widetilde{v}_k$ decreases and converges unformly on every compact subset K of $\mathbb{R}^2 \times (0, T)$ to a solution $\widetilde{v}$ of (19) in $\mathbb{R}^2 \times (0, T)$. Similar to the proof of Theorem 1.2 of [Hu3] $\widetilde{v}$ has initial value v_0. By (45),

$$\widetilde{v} \geq v \quad \text{in } \mathbb{R}^2 \times (0, T) \quad \forall k \in \mathbb{Z}^+. \tag{47}$$

Since v is the maximal solution of (19) in $\mathbb{R}^2 \times (0, T)$ with initial value v_0, by (47), $\widetilde{v} \equiv v$ in $\mathbb{R}^2 \times [0, T)$.

Since $v_0 > 0$ on $\mathbb{R}^2$, by (45) and an argument similar to the proof of Lemma 7,

$$0 < \inf_{\overline{B}_{r_1} \times [0,1]} v \leq \sup_{\overline{B}_{r_1} \times [0,1]} v < \infty \quad \forall r_1 > 0$$

and $v \in C^\infty(\mathbb{R}^2 \times [0, T))$. Then $g(t)$ is the smooth maximal solution of (1) in $0 \leq t < T$ with initial value g_0. By (14), $g(t)$ is complete for any $0 < t < T$.

Finally if in addition the Gauss curvature satisfies (9) for some constant $K_0 \geq 0$, then by an argument similar to the proof of Theorem (4) we get that (10) and (11) hold on $\mathbb{R}^2$ for any $0 \leq t < \min((2K_0)^{-1}, T)$ if $K_0 > 0$ and (12) and (13) hold on $\mathbb{R}^2$ for any $0 \leq t < T$ if $K_0 = 0$. By (13), $T = \infty$ if $K_0 = 0$.

If $K_0 > 0$, we claim that $T \geq (2K_0)^{-1}$. Suppose not. Then $T < (2K_0)^{-1}$. Hence by (11),

$$\text{Vol}_{g(T)}(\mathbb{R}^2) = \int_{\mathbb{R}^2} v(x, T)\, dx \geq (1 - 2K_0 T) \int_{\mathbb{R}^2} v_0(x)\, dx > 0. \tag{48}$$

This contradicts (16). Hence $T \geq (2K_0)^{-1}$ and the theorem follows.

□

References

[A] J.R. Anderson, *Local existence and uniqueness of solutions of degenerate parabolic equation*, Comm. in P.D.E. 16 (1991), 105–143.

[CLN] B. Chow, P. Lu and L. Ni, *Hamilton's Ricci flow*, Graduate Studies in Mathematics vol. 77, Amer. Math. Soc., Providence, R.I., U.S.A. (2006).

[DP1] P. Daskalopoulos and M.A. Del Pino, *On a singular diffusion equation*, Comm. in Analysis and Geometry 3 (1995), no. 3, 523–542.

[DP2] P. Daskalopoulos and M.A. Del Pino, *On the Cauchy problem for* $u_t = \log u$ *in higher dimensions*, Math. Ann. 313 (1999), 189V206.

[DK] P. Daskalopoulos and C.E. Kenig, *Degenerate diffusion-initial value problems and local regularity theory*, Tracts in Mathematics 1, European Mathematical Society, 2007.

[DDD] S.H. Davis, E. Dibenedetto, and D.J. Diller, *Some a priori estimates for a singular evolution equation arising in thin-film dynamics*, SIAM J. Math. Anal. 27 (1996), no. 3, 638–660.

[ERV1] J.R. Esteban, A. Rodriguez, and J.L. Vazquez, *The fast diffusion equation with logarithmic nonlinearity and the evolution of conformal metrics in the plane*, Advances in Diff. Eqn. 1 (1996), no. 1, 21–50.

[ERV2] J.R. Esteban, A. Rodriguez, and J.L. Vazquez, *The maximal solution of the logarithmic fast diffusion equation in two space dimensions*, Advances in Diff. Eqn. 2 (1997), no. 6, 867–894.

[GiT1] G. Giesen and P.M. Topping, *Ricci flow of negatively curved incomplete surfaces*, Calculus of Variation 38 (2010), 357–367.

[GiT2] G. Giesen and P.M. Topping, *Existence of Ricci flows of incomplete surfaces*, http://arxiv.org/abs/1007.3146.

[GT] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 1983.

[H1] R.S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom.**17**, no. 2, (1982), 255–306.

[H2] R.S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, International Press, Cambridge, MA, 1995.

[Hs1] S.Y. Hsu, *Global existence and uniqueness of solutions of the Ricci flow equation*, Differential and Integral Equations 14 (2001), no. 3, 305–320.

[Hs2] S.Y. Hsu, *Generalized* $\mathcal{L}$*-geodesic and monotonicity of the generalized reduced volume in the Ricci flow*, J. Math. Kyoto Univ. 49 (2009), no. 3, 503 - 571.

[Hs3] S.Y. Hsu, *Maximum principle and convergence of fundamental solutions for the Ricci flow*, Tokyo Journal of Mathematics 32 (2009), 501–516.

[Hs4] S.Y. Hsu, *Uniqueness of solutions of Ricci flow on complete noncompact manifolds*, http://arxiv.org/abs/math/0704.3468.

[Hu1] K.M. Hui, *Fatou Theorem for the solutions of some nonlinear equations*, J. Math. Anal. Appl. 183 (1994), 37–52.

[Hu2] K.M. Hui, *Existence of solutions of the equation* $u_t = \Delta \log u$, Nonlinear Analysis 37 (1999), 875–914.

[Hu3] K.M. Hui, *On Some Dirichlet and Cauchy Problems for a Singular Diffusion Equation,* Differential Integral Equations 15 (2002), no. 7, 769-804.

[Hu4] K.M. Hui, *Singular limit of solutions of the very fast diffusion equation*, Nonlinear Analysis TMA 68 (2008), 1120–1147.

[LSU] O.A. Ladyzenskaya, V.A. Solonnikov and N.N. Uraltceva *Linear and Quasilinear Equations of Parabolic Type,* Transl. Math. Mono. vol. 23, Amer. Math. Soc., Providence, R.I., 1968.

[P1] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, http://arxiv.org/abs/math/0211159.

[P2] G. Perelman, *Ricci flow with surgery on three-manifolds*, http://arxiv.org/abs /math/0303109.

[S1] W.X. Shi, *Deforming the metric on complete Riemannian manifolds*, J. Differential Geom. 30 (1989), 223–301.

[S2] W.X. Shi, *Ricci deformation of the metric on complete non-compact Riemannian manifolds*, J. Differential Geom. 30 (1989), 303–394.

[T] P.M. Topping, *Ricci flow compactness via pseudolocality, and flows with incomplete initial metrics*, J. Eur. Math. Soc. (JEMS) (to appear).

[V] J.L. Vazquez, *Smoothing and decay estimates for nonlinear parabolic equations of porous medium type*, Oxford University Press Inc., New York, 2006.

[W1] L.F. Wu, *The Ricci flow on complete* R^2, Comm. in Analysis and Geometry 1 (1993), 439–472.

[W2] L.F. Wu, *A new result for the porous medium equation*, Bull. Amer. Math. Soc. 28 (1993), 90–94.

[Y] S.T. Yau, *Remarks on conformal transformations*, J. Diff. Geom. 8 (1973), 369–381.

[Z] Qi S. Zhang, *Sobolev inequalities, heat kernels under Ricci flow, and the Poincaré conjecture*, CRC Press, Boca Raton, FL, USA, 2011.