Singular continuous spectrum of one-dimensional Schrödinger operator with point interactions on a sparse set

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Abstract

We say that a discrete set $X = \{x_n\}_{n\in\mathbb{N}_0}$ on the half-line

$$0 = x_0 < x_1 < x_2 < x_3 < \cdots < x_n < \cdots < +\infty$$

is sparse in the case the distances $\Delta x_n = x_{n+1} - x_n$ between neighboring points satisfy the condition $\frac{\Delta x_n}{\Delta x_{n-1}} \to +\infty$. Half-line Schrödinger operators with point δ- and δ'-interactions on a sparse discrete set are considered. Assuming that strengths of point interactions tend to ∞ we give simple sufficient conditions for such Schrödinger operators to have non-empty singular continuous spectrum and to have purely singular continuous spectrum coinciding with $\mathbb{R}_+$.

Keywords: Schrödinger operator, sparse set, point interactions, singular continuous spectrum.
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1 Introduction

One-dimensional Schrödinger operator with δ-interactions on a discrete set describes the motion of a non-relativistic charged particle in a one-dimensional lattice. Periodic models of such a type were considered first by Kronig and Penney in [KrPe]. Classical results and sufficiently complete list of references on the theory of one-dimensional Schrödinger operators with δ- and δ'-interactions on a discrete set are given in the monograph [AGHHE]. Schrödinger operators with point interactions were considered for example in [AKM, B, BuStW, ChrSt, GeKi, GoO, Ko, KM, Mi1, Mi2, N, SaSh] and in many other works. Our list of references is far from being complete, but many of recent significant works are mentioned.

In the present paper we are interested in the effect first discovered by Pearson in [P] for one-dimensional Schrödinger operators with regular sparse potentials. Sparse potentials were also discussed by Gordon, Molchanov and Zagany in [GMoZ], where some results were given without proofs. Under some assumptions on the degree of sparseness of the potential one gets purely singular continuous spectrum. An example of such a potential was constructed by Simon and Stolz in [SiSt]. According to the results of [SiSt] half-line Schrödinger operator

$$-\frac{d^2}{dx^2}+V$$

with the potential

$$V(x)=\begin{cases} n, & \left|x-e^{2n^{3/2}}\right|<\frac{1}{2},\\ 0, & \text{otherwise},\end{cases} \tag{1.1}$$

and arbitrary self-adjoint boundary condition at the origin has the following structure of the spectrum

$$\sigma_{\rm p}=\sigma_{\rm ac}=\varnothing,\quad \sigma_{\rm sc}=\big[0,+\infty\big).$$

The main achievement of this construction is the stability of the singular continuous spectrum under "small" variations of the potential V in (1.1) and arbitrary self-adjoint variations of the boundary condition at the origin. This situation is non-typical for other known examples with singular continuous spectrum.

Recently sparse potentials attracted the attention again. In works of Breuer and Frank [Br, BrF] sufficient conditions for the spectrum of the Laplace operator on discrete and metric sparse radial trees to be purely singular continuous are given.

In the present paper we establish the existence of such an effect for Schrödinger operators with point δ- and δ'-interactions on a sparse discrete set. As in the classical case the obtained singular continuous spectrum is stable under "small" variations of interactions strengths and the discrete set itself.

Let $\alpha=\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Let $X=\{x_n\}_{n\in\mathbb{N}_0}$ be a discrete set on the half-line

$$0=x_0<x_1<x_2<x_3<\cdots<x_n<\cdots<+\infty$$

such that the sequence $\Delta x_n=x_{n+1}-x_n$ satisfies the condition

$$\inf_{n\in\mathbb{N}_0}\Delta x_n>0. \tag{1.2}$$

We are interested in the half-line Schrödinger operators $H_{\delta,X,\alpha}$, and $H_{\delta',X,\alpha}$ formally given by expressions

$$H_{\delta,X,\alpha}=-\frac{d^2}{dx^2}+\sum_{n\in\mathbb{N}}\alpha_n\delta_{x_n}\quad\text{and}\quad H_{\delta',X,\alpha}=-\frac{d^2}{dx^2}+\sum_{n\in\mathbb{N}}\alpha_n\langle\delta'_{x_n},\cdot\rangle\delta'_{x_n}, \tag{1.3}$$

where δ_x and δ'_x are delta distribution and its derivative supported by the point $x\in\mathbb{R}_+$. We fix for simplicity Dirichlet boundary condition at the origin. Strictly defined operators $H_{\delta,X,\alpha}$ and $H_{\delta',X,\alpha}$ are self-adjoint in $L^2(\mathbb{R}_+)$, see Section 2.

We say that a discrete set X is sparse if the following condition holds

$$\frac{\Delta x_n}{\Delta x_{n-1}} \to +\infty. \tag{1.4}$$

Our main results are listed in the following theorem.

Theorem 1.1. *Let $X = \{x_n\}_{n\in\mathbb{N}_0}$ be a sparse discrete set on the half-line. Let $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\alpha_n \to \infty$. Let $H_{\delta,X,\alpha}$ and $H_{\delta',X,\alpha}$ be self-adjoint half-line Schrödinger operators as in* (1.3)*, see Section 2 for strict definitions. Let us define a value $a \in \mathbb{R}_+ \cup \{+\infty\}$ as the following limit*

$$a := \liminf_{n\to\infty} \frac{\Delta x_n}{\Delta x_{n-1}\alpha_n^2}. \tag{1.5}$$

Then the following assertions hold:

(i) if $0 < a < +\infty$, then the spectrum of the operator $H_{\delta,X,\alpha}$ has the following structure:

(pp) $\sigma_{\mathrm{pp}} \cap \mathbb{R}_+ \subseteq [0, 1/a]$,

(sc) $[1/a, +\infty) \subseteq \sigma_{\mathrm{sc}} \subseteq [0, +\infty)$,

(ac) $\sigma_{\mathrm{ac}} = \varnothing$,

and the spectrum of the operator $H_{\delta',X,\alpha}$ has the structure:

(pp) $\sigma_{\mathrm{pp}} \cap \mathbb{R}_+ \subseteq [a, +\infty)$,

(sc) $[0, a] \subseteq \sigma_{\mathrm{sc}} \subseteq [0, +\infty)$,

(ac) $\sigma_{\mathrm{ac}} = \varnothing$,

it is worth noting that in this case the singular continuous spectrum of both operators is non-empty;

(ii) if $a = +\infty$ and the sequence α contains only positive real numbers, then the spectrum of both operators $H_{\delta,X,\alpha}$ and $H_{\delta',X,\alpha}$ is purely singular continuous and coincides with $\mathbb{R}_+$.

As an example the spectrum of Schrödinger operator formally given by the expression

$$-\frac{d^2}{dx^2} + \sum_{n\in\mathbb{N}} n^{1/4}\delta_{n!}$$

is purely singular continuous and coincides with $\mathbb{R}_+$, in this case $a = +\infty$. The singular continuous spectrum of Schrödinger operator formally given by the expression

$$-\frac{d^2}{dx^2} + \sum_{n\in\mathbb{N}} n^{1/2}\delta_{n!}$$

is non-empty and contains the interval $[1, +\infty)$, in this case $a = 1$.

Notations: By $\mathbb{N}_0$ we denote $\mathbb{N} \cup \{0\}$. We write $T \in \mathfrak{S}_\infty$ in the case the operator T is compact. By $\sigma_{\mathrm{p}}, \sigma_{\mathrm{pp}}, \sigma_{\mathrm{ac}}$ and σ_{sc} we denote point, pure point, absolutely continuous and singular continuous spectra. By σ_{ess} we denote the essential spectrum. We write $\psi \in \mathrm{AC}_{\mathrm{loc}}(I)$ if the function ψ is locally absolutely continuous on the set I.

2 Definitions of operators with point interactions

In this section we give strict definitions of operators $H_{\delta,X,\alpha}$ and $H_{\delta',X,\alpha}$ on the language of boundary conditions.

Let $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Let $X = \{x_n\}_{n\in\mathbb{N}_0}$ be a discrete set of points on the half-line arranged from the origin to $+\infty$ such that the sequence $\Delta x_n = x_{n+1} - x_n$ satisfies the condition (1.2).

Let us introduce two classes of functions ψ on the half-line

$$\mathcal{S}_{\delta,X,\alpha} = \left\{\psi, \psi' \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+ \setminus X) \colon \psi(0) = 0, \begin{smallmatrix} \psi(x_n+)=\psi(x_n-)=\psi(x_n) \\ \psi'(x_n+)-\psi'(x_n-)=\alpha_n\psi(x_n) \end{smallmatrix}\right\} \tag{2.1}$$

and

$$\mathcal{S}_{\delta',X,\alpha} = \left\{\psi, \psi' \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+ \setminus X) \colon \psi(0) = 0, \begin{smallmatrix} \psi'(x_n+)=\psi'(x_n-)=\psi'(x_n) \\ \psi(x_n+)-\psi(x_n-)=\alpha_n\psi'(x_n) \end{smallmatrix}\right\}. \tag{2.2}$$

The operator $H_{\delta,X,\alpha}$ is defined in the following way

$$H_{\delta,X,\alpha}\psi = -\psi'', \quad \mathrm{dom}\, H_{\delta,X,\alpha} = \left\{\psi \in L^2(\mathbb{R}_+) \cap \mathcal{S}_{\delta,X,\alpha} \colon -\psi'' \in L^2(\mathbb{R}_+)\right\}. \tag{2.3}$$

Analogously we define the operator $H_{\delta',X,\alpha}$

$$H_{\delta',X,\alpha}\psi = -\psi'', \quad \mathrm{dom}\, H_{\delta',X,\alpha} = \left\{\psi \in L^2(\mathbb{R}_+) \cap \mathcal{S}_{\delta',X,\alpha} \colon -\psi'' \in L^2(\mathbb{R}_+)\right\}. \tag{2.4}$$

Operators $H_{\delta,X,\alpha}$ and $H_{\delta',X,\alpha}$ are both self-adjoint in $L^2(\mathbb{R}_+)$ according to [GeKi, Ko].

3 Sufficient conditions for absence of point spectrum on a subinterval of positive semi-axis

In this section we establish sufficient conditions on X and α, which give operators $H_{\delta,X,\alpha}$ and $H_{\delta',X,\alpha}$ with absence of point spectra on a subinterval of $\mathbb{R}_+$. We adapt the approach suggested by Simon and Stolz in [SiSt] for regular potentials to the case of point interactions. Similar ideas were used recently by Breuer and Frank in [BrF] in order to prove absence of point spectrum for certain classes of sparse trees.

Let us consider a function ψ such that for some $\lambda > 0$

$$-\psi''(x) = \lambda\psi(x), \quad \text{for all} \quad x \in \mathbb{R}_+ \setminus X. \tag{3.1}$$

If we show that any non-trivial function $\psi \in \mathcal{S}_{\delta,X,\alpha}$ satisfying (3.1) for some $\lambda > 0$ does not belong to $L^2(\mathbb{R}_+)$, then $\lambda \notin \sigma_{\mathrm{p}}(H_{\delta,X,\alpha})$. Analogously if we show, that any non-trivial function $\psi \in \mathcal{S}_{\delta',X,\alpha}$ satisfying (3.1) for some $\lambda > 0$ does not belong to $L^2(\mathbb{R}_+)$, then $\lambda \notin \sigma_{\mathrm{p}}(H_{\delta',X,\alpha})$.

We need the following subsidiary lemma.

Lemma 3.1. *Let $X = \{x_n\}_{n\in\mathbb{N}_0}$ be a discrete set on the half-line satisfying the condition* (1.2)*. Let a function $\psi \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+ \setminus X)$ be such that $\psi' \in$*

$\mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+ \setminus X)$. *Assume* ψ *satisfies* (3.1) *for some* $\lambda > 0$. *If the sequence of vectors* $\xi_n := \begin{pmatrix} \psi(x_n+) \\ \psi'(x_n+) \end{pmatrix}$ *satisfies the condition*

$$\sum_{n=0}^{\infty} \Delta x_n \|\xi_n\|^2_{\mathbb{C}^2} = \infty, \tag{3.2}$$

then

$$\int_0^{\infty} |\psi(x)|^2 dx = \infty.$$

Proof. For a point $x \in (x_n, x_{n+1})$ one has the following identity

$$\xi_n = M_\lambda(x_n - x) \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix}, \tag{3.3}$$

where the matrix $M_\lambda(d)$ is the fundamental matrix of the differential equation $-\psi'' = \lambda\psi$ having the following explicit form

$$M_\lambda(d) = \begin{pmatrix} \cos(d\sqrt{\lambda}) & \frac{\sin(d\sqrt{\lambda})}{\sqrt{\lambda}} \\ -\sqrt{\lambda}\sin(d\sqrt{\lambda}) & \cos(d\sqrt{\lambda}) \end{pmatrix}. \tag{3.4}$$

It follows from the identity (3.3) that

$$\left\| \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix} \right\|_{\mathbb{C}^2} \geq \frac{\|\xi_n\|_{\mathbb{C}^2}}{\|M_\lambda(x - x_n)\|}.$$

The following estimate takes place

$$\sup_{d \in \mathbb{R}} \|M_\lambda(d)\| \leq C_\lambda < +\infty. \tag{3.5}$$

According to (3.2) we have

$$\int_0^{\infty} \|\left(\begin{smallmatrix} \psi(x) \\ \psi'(x) \end{smallmatrix} \right)\|^2_{\mathbb{C}^2} dx \geq \frac{1}{(C_\lambda)^2} \sum_{n=0}^{\infty} \int_{x_n}^{x_{n+1}} \|\xi_n\|^2_{\mathbb{C}^2} dx = \frac{1}{(C_\lambda)^2} \sum_{n=0}^{\infty} \Delta x_n \|\xi_n\|^2_{\mathbb{C}^2} = \infty. \tag{3.6}$$

If $\psi \in L^2(\mathbb{R}_+)$, then $\psi'' = -\lambda\psi \in L^2(\mathbb{R}_+)$. Taking into account the inequality, see, e.g., [EE, §III.10],

$$\|\psi'\|^2_{L^2(\mathbb{R}_+)} \leq a\|\psi\|^2_{L^2(\mathbb{R}_+)} + b\|\psi''\|^2_{L^2(\mathbb{R}_+)}, \tag{3.7}$$

which holds for some constants $a, b > 0$, we get $\psi' \in L^2(\mathbb{R}_+)$. Finally we come to the conclusion that for the divergence of the integral on the left hand side in (3.6) one needs $\psi \notin L^2(\mathbb{R}_+)$. □

Let $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers, then we introduce two sequences of real-valued functions $\{A_n(\lambda)\}_{n\in\mathbb{N}_0}$ and $\{B_n(\lambda)\}_{n\in\mathbb{N}_0}$ of the argument $\lambda \in (0, +\infty)$ in the following way

$$A_n(\lambda) := \prod_{i=1}^{n} \Big(1 + \frac{|\alpha_i|}{\sqrt{\lambda}}\Big) \quad \text{and} \quad B_n(\lambda) := \prod_{i=1}^{n} \Big(1 + |\alpha_i|\sqrt{\lambda}\Big). \tag{3.8}$$

Further we need two lemmas, which give asymptotic estimates of the behavior of functions in the classes $\mathcal{S}_{\delta,X,\alpha}$ and $\mathcal{S}_{\delta',X,\alpha}$ satisfying (3.1) for some $\lambda > 0$.

Lemma 3.2. *Let $X = \{x_n\}_{n\in\mathbb{N}_0}$ be a discrete set on the half-line satisfying the condition* (1.2)*. Let $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Let a function $\psi \in \mathcal{S}_{\delta,X,\alpha}$ satisfy* (3.1) *for some $\lambda > 0$. Let the sequence $\{A_n(\lambda)\}_{n\in\mathbb{N}_0}$ be defined as in* (3.8)*. Then norms of vectors $\xi_n := \begin{pmatrix}\psi(x_n+)\\ \psi'(x_n+)\end{pmatrix}$ satisfy the estimate*

$$\|\xi_n\|_{\mathbb{C}^2} \geq c_\lambda \frac{\|\xi_0\|_{\mathbb{C}_2}}{A_n(\lambda)}, \quad n \in \mathbb{N}, \tag{3.9}$$

with some positive constant $c_\lambda > 0$.

Proof. The sequence of vectors $\{\xi_n\}_{n\in\mathbb{N}_0}$ is a solution of the discrete linear system

$$\xi_n = \Lambda_n \xi_{n-1}, \quad n \in \mathbb{N}, \tag{3.10}$$

with the sequence of matrices $\{\Lambda_n\}_{n\in\mathbb{N}}$ having the explicit form

$$\Lambda_n = \underbrace{\begin{pmatrix}1 & 0\\ \alpha_n & 1\end{pmatrix}}_{J_\delta(\alpha_n)} M_\lambda(\Delta x_{n-1}), \tag{3.11}$$

where $M_\lambda(d)$ is the fundamental matrix defined in (3.4) and $J_\delta(\alpha)$ is the δ-jump matrix.

One can do the substitution in the discrete linear system (3.10) of the type

$$\widetilde{\xi}_n = \underbrace{\begin{pmatrix}\frac{1}{2} & -\frac{i}{2\sqrt{\lambda}}\\ \frac{1}{2} & \frac{i}{2\sqrt{\lambda}}\end{pmatrix}}_{U_\lambda^{-1}} \xi_n. \tag{3.12}$$

The sequence $\{\widetilde{\xi}_n\}_{n\in\mathbb{N}_0}$ is a solution of a new discrete linear system

$$\widetilde{\xi}_n = \widetilde{\Lambda}_n \widetilde{\xi}_{n-1}, \quad n \in \mathbb{N}, \tag{3.13}$$

where matrices $\widetilde{\Lambda}_n$ can be expressed in the following way

$$\widetilde{\Lambda}_n = U_\lambda^{-1} J_\delta(\alpha_n) M_\lambda(\Delta x_{n-1}) U_\lambda.$$

Using that $(M_\lambda(d))^{-1} = M_\lambda(-d)$ and $(J_\delta(\alpha))^{-1} = J_\delta(-\alpha)$ we get

$$\widetilde{\Lambda}_n^{-1} = U_\lambda^{-1} M_\lambda(-\Delta x_{n-1}) J_\delta(-\alpha_n) U_\lambda. \tag{3.14}$$

Substituting in (3.14) the matrix $M_\lambda(d)$ by its explicit form given in (3.4), the matrix $J_\delta(\alpha)$ by its explicit form given in (3.11) and the matrix U_λ by its explicit form in (3.12) we get after simple calculations

$$\widetilde{\Lambda}_n^{-1} = \begin{pmatrix} e^{-i\sqrt{\lambda}\Delta x_{n-1}} & 0\\ 0 & e^{i\sqrt{\lambda}\Delta x_{n-1}}\end{pmatrix} \left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{i\alpha_n}{2\sqrt{\lambda}} \begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix} \right). \tag{3.15}$$

Now it is clear that

$$\|\widetilde{\Lambda}_n^{-1}\| \leq 1 + \frac{|\alpha_n|}{\sqrt{\lambda}}. \tag{3.16}$$

Taking into account the discrete linear system (3.13) and the estimate (3.16) we get

$$\|\widetilde{\xi}_n\|_{\mathbb{C}^2} \geq \frac{\|\widetilde{\xi}_{n-1}\|_{\mathbb{C}^2}}{\|\widetilde{\Lambda}_n^{-1}\|} \geq \frac{\|\widetilde{\xi}_{n-1}\|_{\mathbb{C}^2}}{1+\frac{|\alpha_n|}{\sqrt{\lambda}}}. \tag{3.17}$$

Expanding the estimate (3.17) we get

$$\|\widetilde{\xi}_n\|_{\mathbb{C}^2} \geq \frac{\|\widetilde{\xi}_{n-1}\|_{\mathbb{C}^2}}{1+\frac{|\alpha_n|}{\sqrt{\lambda}}} \geq \frac{\|\widetilde{\xi}_{n-2}\|_{\mathbb{C}^2}}{\left(1+\frac{|\alpha_{n-1}|}{\sqrt{\lambda}}\right)\left(1+\frac{|\alpha_n|}{\sqrt{\lambda}}\right)} \geq \cdots \geq \frac{\|\widetilde{\xi}_0\|_{\mathbb{C}^2}}{A_n(\lambda)}. \tag{3.18}$$

Returning from $\widetilde{\xi}$ to ξ we get estimates

$$\|\widetilde{\xi}_n\|_{\mathbb{C}^2} \leq \|\xi_n\|_{\mathbb{C}^2}\|U_\lambda^{-1}\|, \quad \|\widetilde{\xi}_0\|_{\mathbb{C}^2} \geq \frac{\|\xi_0\|_{\mathbb{C}^2}}{\|U_\lambda\|}. \tag{3.19}$$

Hence from (3.18) and (3.19) we get the claim

$$\|\xi_n\|_{\mathbb{C}^2} \geq c_\lambda \frac{\|\xi_0\|_{\mathbb{C}^2}}{A_n(\lambda)}, \tag{3.20}$$

where $c_\lambda = \left(\|U_\lambda\|\|U_\lambda^{-1}\|\right)^{-1}$. □

Lemma 3.3. *Let $X = \{x_n\}_{n\in\mathbb{N}_0}$ be a discrete set on the half-line satisfying the condition* (1.2)*. Let $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Let a function $\psi \in \mathcal{S}_{\delta',X,\alpha}$ satisfy* (3.1) *for some $\lambda > 0$. Let the sequence $\{B_n(\lambda)\}_{n\in\mathbb{N}_0}$ be defined as in* (3.8)*. Then norms of vectors $\xi_n := \begin{pmatrix}\psi(x_n+)\\ \psi'(x_n+)\end{pmatrix}$ satisfy the estimate*

$$\|\xi_n\|_{\mathbb{C}^2} \geq c_\lambda \frac{\|\xi_0\|_{\mathbb{C}_2}}{B_n(\lambda)}, \quad n \in \mathbb{N}, \tag{3.21}$$

with some constant $c_\lambda > 0$.

Proof. The proof of this lemma is almost the same as the proof of the previous lemma. One should substitute the δ-jump matrix $J_\delta(\alpha)$ in (3.11) by the δ'-jump matrix $J_{\delta'}(\alpha) = \begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}$. Repeating the calculations of the previous lemma we get

$$\widetilde{\Lambda}_n^{-1} = \begin{pmatrix} e^{-i\sqrt{\lambda}\Delta x_{n-1}} & 0\\ 0 & e^{i\sqrt{\lambda}\Delta x_{n-1}}\end{pmatrix}\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{i\alpha_n\sqrt{\lambda}}{2}\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\right).$$

Now it is clear that

$$\|\widetilde{\Lambda}_n^{-1}\| \leq 1 + |\alpha_n|\sqrt{\lambda}.$$

Analogously to the previous lemma we get

$$\|\xi_n\|_{\mathbb{C}^2} \geq c_\lambda \frac{\|\xi_0\|_{\mathbb{C}^2}}{B_n(\lambda)}, \tag{3.22}$$

where $c_\lambda = \left(\|U_\lambda\|\|U_\lambda^{-1}\|\right)^{-1}$. □

Further we prove two theorems, which contain sufficient conditions on X and α for some subinterval of $\mathbb{R}_+$ to be free of point spectra of operators $H_{\delta,X,\alpha}$ and $H_{\delta',X,\alpha}$.

Theorem 3.1. *Let $X = \{x_n\}_{n\in\mathbb{N}_0}$ be a discrete set satisfying the condition* (1.2)*. Let $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Let the self-adjoint operator $H_{\delta,X,\alpha}$ be defined as in* (2.3)*. Let the sequence $\{A_n(\lambda)\}_{n\in\mathbb{N}_0}$ be defined as in* (3.8)*. If for some $\lambda_0 > 0$*

$$\sum_{n=0}^{\infty} \frac{\Delta x_n}{A_n(\lambda_0)^2} = \infty, \tag{3.23}$$

then the point spectrum of $H_{\delta,X,\alpha}$ satisfies

$$\sigma_{\mathrm{p}} \cap \mathbb{R}_+ \subset [0, \lambda_0). \tag{3.24}$$

Proof. Let us fix $\lambda \in [\lambda_0, +\infty)$. Let ψ be a non-trivial function from the class $\mathcal{S}_{\delta,X,\alpha}$ satisfying (3.1) for this λ. Let us introduce a sequence $\xi_n = \begin{pmatrix} \psi(x_n+) \\ \psi'(x_n+) \end{pmatrix}$. According to Lemma 3.2

$$\|\xi_n\|_{\mathbb{C}^2} \geq c_\lambda \frac{\|\xi_0\|_{\mathbb{C}^2}}{A_n(\lambda)}. \tag{3.25}$$

Functions $A_n(\lambda)$ are monotonously decreasing by λ for all $n \in \mathbb{N}$ and hence the divergence of the series in (3.23) implies

$$\sum_{n=0}^{\infty} \|\xi_n\|^2_{\mathbb{C}^2} \Delta x_n \geq c_\lambda \|\xi_0\|^2_{\mathbb{C}^2} \sum_{n=0}^{\infty} \frac{\Delta x_n}{A_n(\lambda)^2} \geq c_\lambda \|\xi_0\|^2_{\mathbb{C}^2} \sum_{n=0}^{\infty} \frac{\Delta x_n}{A_n(\lambda_0)^2} = \infty. \tag{3.26}$$

Then according to Lemma 3.1 we get $\psi \notin L^2(\mathbb{R}_+)$ and hence $\lambda \notin \sigma_{\mathrm{p}}(H_{\delta,X,\alpha})$. □

Corollary 3.1. *If we are in the conditions of Theorem 3.1 and the sequence α contains only positive real numbers, then the point spectrum of $H_{\delta,X,\alpha}$ satisfies*

$$\sigma_{\mathrm{p}} \subset \big(0, \lambda_0\big). \tag{3.27}$$

Proof. We need only to show that there are no eigenvalues on $\mathbb{R}_-$. Let ψ be an arbitrary function from $\operatorname{dom}(H_{\delta,X,\alpha})$. The scalar product $\big(H_{\delta,X,\alpha}\psi, \psi\big)_{L^2(\mathbb{R}_+)}$ can be rewritten according to boundary conditions (2.1) in the form

$$\|\psi'\|^2_{L^2(\mathbb{R}_+)} + \sum_{n\in\mathbb{N}} \alpha_n |\psi(x_n)|^2. \tag{3.28}$$

Hence $H_{\delta,X,\alpha} \geq 0$ and therefore $\sigma(H_{\delta,X,\alpha}) \cap \big(-\infty, 0\big) = \varnothing$. If ψ is such that $H_{\delta,X,\alpha}\psi = 0$, then according to (3.28), $\psi'(x) = 0$ on $\mathbb{R}_+ \setminus X$ and $\psi(x_n) = 0$ for all $n \in \mathbb{N}$, i.e. the function ψ is a constant on each interval (x_n, x_{n+1}), $n \in \mathbb{N}_0$, and it takes the value zero at the points x_n, $n \in \mathbb{N}$, therefore $\psi(x) = 0$ for all $x \in \mathbb{R}_+$ and hence $0 \notin \sigma_{\mathrm{p}}(H_{\delta,X,\alpha})$. □

Theorem 3.2. *Let $X = \{x_n\}_{n\in\mathbb{N}_0}$ be a discrete set satisfying the condition* (1.2)*. Let $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Let the self-adjoint operator*

$H_{\delta',X,\alpha}$ *be defined as in* (2.4). *Let the sequence* $\{B_n(\lambda)\}_{n\in\mathbb{N}_0}$ *be defined as in* (3.8). *If for some* $\lambda_0 > 0$

$$\sum_{n=0}^{\infty} \frac{\Delta x_n}{B_n(\lambda_0)^2} = \infty, \tag{3.29}$$

then the point spectrum of the operator $H_{\delta',X,\alpha}$ *satisfies*

$$\sigma_{\mathrm{p}} \cap \mathbb{R}_+ \subset \big(\lambda_0, +\infty\big).$$

Proof. The proof of this theorem repeats the proof of Theorem 3.1 with the only one difference: functions $B_n(\lambda)$ are monotonously increasing by λ for all $n \in \mathbb{N}$, unlike monotonously decreasing functions $A_n(\lambda)$, $n \in \mathbb{N}$. □

Corollary 3.2. *If we are in the conditions of Theorem 3.2 and the sequence* α *contains only positive real numbers, then the point spectrum of the operator* $H_{\delta',X,\alpha}$ *satisfies*

$$\sigma_{\mathrm{p}} \subset \big(\lambda_0, +\infty\big).$$

Proof. The proof is analogous to the proof of Corollary 3.1. □

4 Non-empty singular continuous spectrum and purely singular continuous spectrum

In this section we give sufficient conditions on X and α for the operators $H_{\delta,X,\alpha}$ and $H_{\delta',X,\alpha}$ to have non-empty singular continuous spectra and to have even purely singular continuous spectra. Finally we give the proof of Theorem 1.1 formulated in the introduction. We use results of Section 3, the compact perturbation argument and some results of Christ, Stolz [ChrSt] and Mikhailets [Mi1, Mi2].

Lemma 4.1. *Let* $X = \{x_n\}_{n\in\mathbb{N}_0}$ *be a discrete set on the half-line such that* $\Delta x_n \to +\infty$. *Let* $H_{l,\mathrm{D}}$ *be one-dimensional Laplacian on the interval of a length* $l > 0$ *with Dirichlet boundary conditions on each end. Let* $H_{l,\mathrm{N}}$ *be one-dimensional Laplacian on the interval of a length* $l > 0$ *with Neumann boundary conditions on each end. Let self-adjoint operators* $H_{X,\mathrm{D}}$ *and* $H_{X,\mathrm{N}}$ *be defined as direct sums:*

$$H_{X,\mathrm{D}} = \bigoplus_{n=0}^{\infty} H_{\Delta x_n,\mathrm{D}} \quad \text{and} \quad H_{X,\mathrm{N}} = \bigoplus_{n=0}^{\infty} H_{\Delta x_n,\mathrm{N}}. \tag{4.1}$$

Then the essential spectra of operators $H_{X,\mathrm{D}}$ *and* $H_{X,\mathrm{N}}$ *coincide with* $\mathbb{R}_+$.

Proof. Let us prove the claim only for the operator $H_{X,\mathrm{D}}$. The proof for $H_{X,\mathrm{N}}$ is analogous. The operator $H_{X,\mathrm{D}}$ is positive as the direct sum of positive operators. Let $s > 0$ be an arbitrary positive real number. Let us consider the sequence

$$\lambda_{s,n} = \left(\frac{\pi\big\lceil \sqrt{s}\frac{\Delta x_n}{\pi}\big\rceil}{\Delta x_n}\right)^2, \quad n \in \mathbb{N}, \tag{4.2}$$

where $\lceil \cdot \rceil$ is the ceiling function. Since $\lambda_{s,n} \in \sigma_{\rm p}(H_{\Delta x_n,{\rm D}})$, then by the definition of $H_{X,{\rm D}}$ we get $\lambda_{s,n} \in \sigma_{\rm p}(H_{X,{\rm D}})$. The claim for $H_{X,{\rm D}}$ follows from the fact that

$$\lim_{n\to\infty} \lambda_{s,n} = s. \tag{4.3}$$

□

The proof of the following lemma is the main step toward the proof of the main result: Theorem 1.1.

Lemma 4.2. *Let $X = \{x_n\}_{n\in\mathbb{N}_0}$ be a sparse discrete set on the half-line. Let $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\alpha_n \to \infty$. Let the self-adjoint operators $H_{\delta,X,\alpha}$ and $H_{\delta',X,\alpha}$ be defined as in* (2.3) *and as in* (2.4)*, respectively. Then the following assertions hold:*

(i) if for some $\lambda_0 > 0$

$$\sum_{n=0}^{\infty} \frac{\Delta x_n}{\prod_{i=1}^{n}\left(1 + \frac{|\alpha_i|}{\sqrt{\lambda_0}}\right)^2} = \infty, \tag{4.4}$$

then the spectrum of $H_{\delta,X,\alpha}$ has the following structure:

- *(ess)* $\sigma_{\rm ess} = \big[0, +\infty\big)$,
- *(pp)* $\sigma_{\rm pp} \cap \mathbb{R}_+ \subseteq [0, \lambda_0]$,
- *(sc)* $\big[\lambda_0, +\infty\big) \subseteq \sigma_{\rm sc} \subseteq \big[0, +\infty\big)$,
- *(ac)* $\sigma_{\rm ac} = \varnothing$;

(ii) if for some $\lambda_0 > 0$

$$\sum_{n=0}^{\infty} \frac{\Delta x_n}{\prod_{i=1}^{n}\left(1 + |\alpha_i|\sqrt{\lambda_0}\right)^2} = \infty, \tag{4.5}$$

then the spectrum of $H_{\delta',X,\alpha}$ has the structure:

- *(ess)* $\sigma_{\rm ess} = \big[0, +\infty\big)$,
- *(pp)* $\sigma_{\rm pp} \cap \mathbb{R}_+ \subseteq \big[\lambda_0, +\infty\big)$,
- *(sc)* $\big[0, \lambda_0\big] \subseteq \sigma_{\rm sc} \subseteq \big[0, +\infty\big)$,
- *(ac)* $\sigma_{\rm ac} = \varnothing$.

Proof. (i) Since $\alpha_n \to \infty$, then according to [ChrSt, Theorem 3] we have $\sigma_{\rm ac}(H_{\delta,X,\alpha}) = \varnothing$. According to [Mi1, Theorem 1]

$$(H_{\delta,X,\alpha} - \mu) - (H_{X,{\rm D}} - \mu)^{-1} \in \mathfrak{S}_\infty$$

for all $\mu \in \rho(H_{\delta,X,\alpha}) \cap \rho(H_{X,{\rm D}})$. Therefore by the compact perturbation argument and Lemma 4.1

$$\sigma_{\rm ess}(H_{\delta,X,\alpha}) = \sigma_{\rm ess}(H_{X,{\rm D}}) = \mathbb{R}_+. \tag{4.6}$$

By Theorem 3.1

$$\sigma_{\rm pp}(H_{\delta,X,\alpha}) \cap \mathbb{R}_+ \subseteq \big[0, \lambda_0\big]. \tag{4.7}$$

Taking into account the emptiness of the absolutely continuous spectrum we get from (4.6) that

$$\big(\sigma_{\rm sc}\cup\sigma_{\rm pp}\big)(H_{\delta,X,\alpha}) \supseteq \sigma_{\rm ess}(H_{\delta,X,\alpha}) = \mathbb{R}_+.$$

Then according to (4.7) we get

$$\big[\lambda_0,+\infty\big) \subseteq \sigma_{\rm sc}(H_{\delta,X,\alpha}) \subseteq \big[0,+\infty\big).$$

(ii) The idea is almost the same as in the proof of the item (i). In order to prove that $\sigma_{\rm ac}(H_{\delta',X,\alpha}) = \varnothing$ one should make some minor changes in [ChrSt, Theorem 3], see also [Mi2, Theorem 1]. According to [Mi1, Theorem 1] the operator $H_{\delta',X,\alpha}$ is a compact perturbation of the operator $H_{X,{\rm N}}$ in the resolvent sense. Hence by the compact perturbation argument and Lemma 4.1

$$\sigma_{\rm ess}(H_{\delta',X,\alpha}) = \sigma_{\rm ess}(H_{X,{\rm N}}) = \mathbb{R}_+. \tag{4.8}$$

By Theorem 3.2

$$\sigma_{\rm pp}(H_{\delta',X,\alpha})\cap\mathbb{R}_+ \subseteq \big[\lambda_0,+\infty\big). \tag{4.9}$$

Again taking into account that the absolutely continuous spectrum is empty we get from (4.8) that

$$\big(\sigma_{\rm sc}\cup\sigma_{\rm pp}\big)(H_{\delta',X,\alpha}) \supseteq \sigma_{\rm ess}(H_{\delta',X,\alpha}) = \mathbb{R}_+.$$

Then according to (4.9) we get

$$\big[0,\lambda_0\big] \subseteq \sigma_{\rm sc}(H_{\delta',X,\alpha}) \subseteq \big[0,+\infty\big).$$

□

Corollary 4.1. *Assume we are in the conditions of Lemma 4.2 and the sequence α contains only positive real numbers. Then the following assertions hold:*

(i) if the series in (4.4) *diverges for all $\lambda_0 > 0$, then the spectrum of $H_{\delta,X,\alpha}$ is purely singular continuous and coincides with $\mathbb{R}_+$;*

(ii) if the series in (4.5) *diverges for all $\lambda_0 > 0$, then the spectrum of $H_{\delta',X,\alpha}$ is purely singular continuous and coincides with $\mathbb{R}_+$.*

Proof. The item (i) follows from Lemma 4.2 (i) and Corollary 3.1. The item (ii) follows from Lemma 4.2 (ii) and Corollary 3.2. □

Remark 4.1. *Conditions in the items (i) and (ii) of Corollary 4.1 are indeed equivalent.*

Further we give the proof of the main result.

Proof of Theorem 1.1

Recall from the introduction that the value a is defined as the following limit

$$a := \liminf_{n\to\infty} \frac{\Delta x_n}{\Delta x_{n-1}\alpha_n^2}.$$

Let us apply d'Alembert principle to the series (4.4) from Lemma 4.2 (i). For the divergence of this series it is sufficient to satisfy the condition

$$\liminf_{n\to\infty} \frac{\Delta x_n}{\Delta x_{n-1}\Big(1+\frac{2}{\sqrt{\lambda_0}}|\alpha_n|+\frac{1}{\lambda_0}\alpha_n^2\Big)} = \liminf_{n\to\infty} \lambda_0 \frac{\Delta x_n}{\Delta x_{n-1}\alpha_n^2} = \lambda_0 a > 1. \tag{4.10}$$

If $0 < a + \infty$, then for all $\lambda_0 > \frac{1}{a}$ the series (4.4) diverges and we get the first part of the item (i).

Analogously let us apply d'Alembert principle to the series (4.5) from Lemma 4.2 (ii). For the divergence of this series it is sufficient to satisfy the condition

$$\liminf_{n\to\infty} \frac{\Delta x_n}{\Delta x_{n-1}\Big(1+2\sqrt{\lambda_0}|\alpha_n|+\lambda_0\alpha_n^2\Big)} = \liminf_{n\to\infty} \frac{1}{\lambda_0} \frac{\Delta x_n}{\Delta x_{n-1}\alpha_n^2} = \frac{a}{\lambda_0} > 1. \tag{4.11}$$

If $0 < a < +\infty$, then for all $\lambda_0 \in \big(0, a\big)$ the series (4.5) diverges and we get the second part of the item (i).

If $a = +\infty$, then according to d'Alembert principle both series (4.4) and (4.5) diverge for all $\lambda_0 > 0$ and we get from Corollary 4.1 the item (ii).

5 Discussion based on examples

Let us consider Schrödinger operator formally given by the expression

$$-\frac{d^2}{dx^2} + \sum_{n\in\mathbb{N}} n^{1/4}\delta_{n!}. \tag{5.1}$$

Since

$$\liminf_{n\to\infty} \frac{n\cdot n!}{(n-1)\cdot(n-1)!n^{1/2}} = \liminf_{n\to\infty} \frac{n^2}{n^{3/2}-n^{1/2}} = +\infty, \tag{5.2}$$

then by Theorem 1.1 (ii) the spectrum is purely singular continuous and coincides with $\mathbb{R}_+$.

Let us consider Schrödinger operator formally given by the expression

$$-\frac{d^2}{dx^2} + \sum_{n\in\mathbb{N}} n^{1/2}\delta_{n!}. \tag{5.3}$$

Since

$$\liminf_{n\to\infty} \frac{n\cdot n!}{(n-1)\cdot(n-1)!n} = \liminf_{n\to\infty} \frac{n}{n-1} = 1, \tag{5.4}$$

then by Theorem 1.1 (i) the singular continuous spectrum is non-empty and contains the interval $[1, +\infty)$

Our results do not allow to define exactly the structure of the spectrum on the interval $[0, 1]$ in the last example. The spectrum on this interval may be purely singular continuous, only pure point or a mixture of these two kinds of spectra. It is the question of interest for the author to construct an operator $H_{\delta,X,\alpha}$ with δ-interactions on a sparse discrete set X having non-empty positive singular continuous spectrum and non-empty positive pure point spectrum or to establish that this situation can not occur.

Another question of interest is to determine the Hausdorff dimension of the obtained singular continuous spectrum.

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