HOMOCLINIC TANGENCY AND VARIATION OF ENTROPY

M. BRONZI AND A. TAHZIBI

ABSTRACT. In this paper we study the effect of a homoclinic tangency in the variation of the topological entropy. We prove that a diffeomorphism with a homoclinic tangency associated to a basic hyperbolic set with maximal entropy is a point of entropy variation in the C^∞-topology. We also prove results about variation of entropy in other topologies and when the tangency does not correspond to a basic set with maximal entropy. We also show an example of discontinuity of the entropy among C^∞ diffeomorphisms of three dimensional manifolds.

1. INTRODUCTION

Topological entropy is one of the most important invariants of topological conjugacy in dynamical systems. By the $\Omega-$stability of Axiom A diffeomorphisms with no cycle condition, it comes out that the entropy is a C^1-locally constant function among such dynamics. We say that a diffeomorphism f is a point of constancy of topological entropy in C^k topology if there exists a C^k-neighborhood $\mathcal{U}$ of f such that for any diffeomorphism $g \in \mathcal{U}, h(g) = h(f)$. We also call a diffeomorphism as a point of entropy variation if it is not a point of constancy.

In [11], Pujals and Sambarino proved that surface diffeomorphisms far from homoclinic tangency are the constancy points of topological entropy in C^∞ topology. In this paper we address the reciprocal problem. We are interested in the effect of a homoclinic tangency to the variation of the topological entropy for a surface diffeomorphism. Of course after unfolding a homoclinic tangency, new periodic points will emerge, but it is not clear whether they contribute to the variation of the topological entropy. We mention that Díaz-Rios [3] studied unfolding of critical saddle-node horseshoes and when the saddle-node horseshoe is not an attractor they proved that the entropy may decrease after the bifurcation. In our context, the tangency occurs outside a basic hyperbolic set.

For Axiom A diffeomorphisms, by the spectral decomposition theorem of Smale (see *e.g.* [14])), we have $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_k$, where

each Λ_i is a basic set, i.e, an isolated f-invariant hyperbolic set with a dense orbit. By the definition of topological entropy we have

$$h(f) = \max_{0 \leq i \leq k} h(f|_{\Lambda_i}).$$

So, we conclude that there exists a set (at least one) which is "*responsible*" for the topological entropy of an Axiom A. By this we mean, there exists some $k_0 \in \{1, \ldots, k\}$ such that $h(f) = h(f|_{\Lambda_{k_0}})$. Such Λ_{k_0} is called responsible for entropy.

We consider a class of diffeomorphisms on the frontier of Axiom A systems which exhibit a homoclinic tangency corresponding to a periodic point of Λ_{i_0}. We show that the topological entropy increases after small C^∞ perturbations.

More precisely, consider a parametrized family $f_\mu : M \to M$ of diffeomorphisms of a closed surface M unfolding generically (like in [10]) a homoclinic tangency at $\mu = 0$ where $\Omega(f_0) = \Lambda_1 \cup \cdots \cup \Lambda_k \cup \mathcal{O}(q)$ where each Λ_i is an isolated hyperbolic set and $\mathcal{O}(q)$ is the unique homoclinic tangency orbit associated to a saddle fixed point p of some Λ_{i_0}.

Theorem 1. *Let f_μ be a one parameter family of C^2 surface diffeomorphism as above, then if Λ_{i_0} is responsible for the entropy, f_0 is a variation point of the topological entropy in C^r topology, for $2 \leq r \leq \infty$.*

A natural question is what happens if the tangency corresponds to a piece which is not responsible for the entropy. We prove that;

Theorem 2. *Let f_μ be a one parameter family of C^2 surface diffeomorphism as above, then*

- *If Λ_{i_0} is not responsible for the entropy, then f_0 is a constancy point of the topological entropy in the C^∞ topology.*
- *f_0 is a point of constancy in C^k−topology ($1 \leq k < \infty$) if $h_{top}(f) - h(f|\Lambda_{i_0}) > \alpha_k$ where $\alpha_k > 0$ is a constant depending on f_0.*

So the above theorems assert that, in the C^∞ topology, if the tangency is at the "correct" place (the responsible basic set), then the entropy varies and, if the tangency is at the "wrong" place, then the entropy remains constant. It is interesting to note that if we consider C^1-topology then even a tangency "in a wrong place" may cause entropy variation:

Theorem 3. *There exists a diffeomorphism f of S^2 fixing a saddle with homoclinic transversal intersection and another one with homoclinic tangency such that f **is** a point of entropy variation in C^1-topology.*

By the existence of transverse homoclinic point in the example of above theorem the entropy of system is positive. Moreover the entropy is localized on the part of non wandering set far from homoclinic tangency. However, C^1-perturbations make the total entropy of the system increase.

We recall a method for perturbation of surface dynamics with homoclinic tangency, due to Newhouse, which is so called the "Snake like" perturbation. Although after such perturbation the non wandering set becomes richer, the topological entropy does not necessarily increase. See theorem 1.1 for the relation between an estimate of entropy after the perturbation and the eigenvalues of the periodic point corresponding to the homoclinic tangency.

Theorem 1.1 ([9])**.** *Let p be a (conservative) saddle point of a C^1-diffeomorphism f, such that $W^u(\mathcal{O}(p))$ is tangent to $W^s(\mathcal{O}(p))$ at some point. Given $\varepsilon > 0$, for any neighborhood $\mathcal{N}$ of f there exists $g \in \mathcal{N}$ such that*

$$h(g) > \frac{1}{\tau(p)} \log |\lambda(p)| - \varepsilon,$$

where $\tau(p)$ is the period of p.

In the above theorem $\det(Df(p)) = 1$. However, if this is not the case, $|\lambda(p)|$ in the above theorem can be substitutes by $\min\{\lambda(p), \mu(p)^{-1}\}$ where $\lambda(p) > 1 > \mu(p)$ are the eigenvalues of $Df(p)$. As a corollary of continuity of topological entropy for the surface C^∞ diffeomorphisms we conclude that:

Theorem 4. *It is not possible to substitute C^∞ instead of C^1 in the above theorem.*

The continuity property of entropy is a challenging problem in smooth ergodic theory. Newhouse [8] and Yomdin [15] results give semi-continuity of entropy in C^∞ topology and using a Katok's result [7] the entropy is continuous for C^∞ surface diffeomorphisms.

Let us also mention a result of Hua, Saghin and Xia [5] where they prove that the topological entropy is locally constant for some partially hyperbolic diffeomorphisms with one dimensional central bundle. They have shown that for a large class of partially hyperbolic diffeomorphisms with bi-dimensional central foliations, the entropy varies continuously. The authors also claim that without the homological conditions the result is not true, and they exhibit examples in higher dimension (≥ 4) where the result fails without such hypothesis. So it

arises a natural question about the continuity of the topological entropy in the $C^\infty-$ topology for partially hyperbolic systems defined on 3-dimensional manifolds.

Here we give an example of $3-$dimensional manifold diffeomorphism which is not the point of continuity of entropy in C^∞-topology. However, this example is far from being partially hyperbolic.

Theorem 5. *There exists a diffeomorphism on 3-dimensional ball which is a discontinuity point of the topological entropy in the C^∞-topology.*

2. Main Ingredients

Topological Entropy. Consider $\Sigma_N = \{1, \ldots, N\}^{\mathbb{Z}}$ and the *shift* σ : $\Sigma_N \to \Sigma_N$ given by $\sigma(\mathbf{x}) = \mathbf{y}$ where $y_i = x_{i+1}$, $i \in \mathbb{Z}$. For $A = (a_{ij})_{i,j=1}^N$ a square $0 - 1$-matrix of order N, the correspondent *subshift of finite type* is the restriction of σ to $\Sigma_A = \{\mathbf{x} \in \Sigma_N \mid a_{x_i x_{i+1}} = 1 \text{ for } i \in \mathbb{Z}\}$.

It is well known (see [12]) that if $\sigma|_A : \Sigma_A \to \Sigma_A$ is a subshift of finite type, then

$$h(\sigma_A) = \log(\lambda_{max}),$$

where λ_{max} is the biggest eigenvalue of A in modulus.

Ω-Homoclinic Explosions and Markov Partitions. One of the important features of hyperbolic dynamics is the existence of Markov partitions. with rectangles of arbitrarily small diameter.

In particular, for basic set from the spectral decomposition of a Axiom A diffeomorphism there are Markov partitions with arbitrarily small rectangles and this implies conjugacy between $f|_\Lambda$ and a subshift of finite type $\sigma|_{\Sigma_A}$.

To prove our first theorem, we focus on $\Omega-$explosion like in the model in Palis-Takens result [10], of course, without any hypothesis on the fractal dimensions.

That is, we are considering a one parameter family f_μ where for the parameter $\mu = 0$ the nonwandering set $\Omega(f_0) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \tilde{\Lambda}_{i_0} \cup \cdots \cup \Lambda_k$ such that $\Lambda_i, i \neq i_0$ is a hyperbolic basic and $\tilde{\Lambda}_{i_0} = \Lambda_{i_0} \cup \mathcal{O}(q)$ where Λ_k is a basic set and $\mathcal{O}(q)$ is the orbit of a homoclinic tangency associated with a saddle fixed point $p \in \Lambda_k$.

For $\mu > 0$ we can consider the basic sets $\Lambda_i(\mu)$ as the continuation of Λ_i. Thereby, we have that $\Lambda_i(\mu)$ is hyperbolic and $f_\mu|_{\Lambda_i(\mu)}$ is conjugated to $f_0|_{\Lambda_i}$. Then, we have

$$h(f_\mu|_{\Lambda_i(\mu)}) = h(f_0|_{\Lambda_i})$$

for all $i = 1, \ldots, k$ and all μ positive or negative.

However, when we unfold the family f_μ new periodic points are created and the entropy of the nonwandering sets may increase for positive parameters μ. We will see that, in fact, the entropy increases for small positive parameters. This can be shown by constructing a subsystem of f_μ with a dynamics richer than $f_0|_{\Lambda_{i_0}}$.

To construct such a subsystem, we find a subset of $\Omega(f_\mu)$ containing $\Lambda_{i_0}(\mu)$ using Markov partitions. Take a parameter μ very close to $\mu = 0$. Since f_μ unfolds generically, the map f_μ has transversal homoclinic intersections close to $\mathcal{O}(q_0)$, the tangency orbit of f_0. We have the situation represented below in the figure 1.
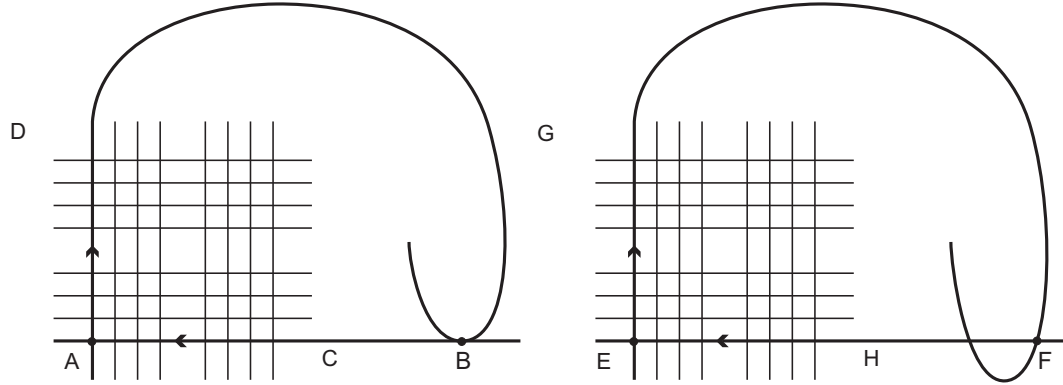

FIGURE 1. Unfolding of a homoclinic tangency close to $\mu = 0$.

Consider q_μ a transversal homoclinic intersection point between $W^s(p_\mu)$ and $W^u(p_\mu)$ close to q_0 (the tangency for f_0). Since $\Lambda_{i_0}(\mu)$ is hyperbolic and maximal invariant set for f_μ, there exist a isolating neighborhood of $\Lambda_{i_0}(\mu)$, say V_{i_0}. Suppose that $q_\mu \notin V_{i_0}$. Moreover, we can use the Bowen's construction of Markov partition [2]. Consider $\{R_1, \ldots, R_s\}$ a Markov partition for $\Lambda_{i_0}(\mu)$ such that

$$\Lambda_{i_0}(\mu) = \bigcup_{j=1}^{s} R_j \subset V_{i_0}.$$

Furthermore, as $q_\mu \notin V_{i_0}$ we have that a part of $\mathcal{O}(q_\mu)$ remains out of V_{i_0}. Take $N_1, N_2 \in \mathbb{N}$ such that $f_\mu^{N_1}(q_\mu) \in R_s$, $f_\mu^{-N_2}(q_\mu) \in R_1$ and $f_\mu^j(q_\mu) \notin \bigcup_{j=1}^{s} R_j$ for $j = -N_2+1, \ldots, 0, \ldots, N_1-1$. In other words, R_s is the rectangle containing the first forward iterated of q_μ that belongs to V_{i_0}, and R_1 is the rectangle containing the first backward iterated of q_μ that belongs to V_{i_0}.
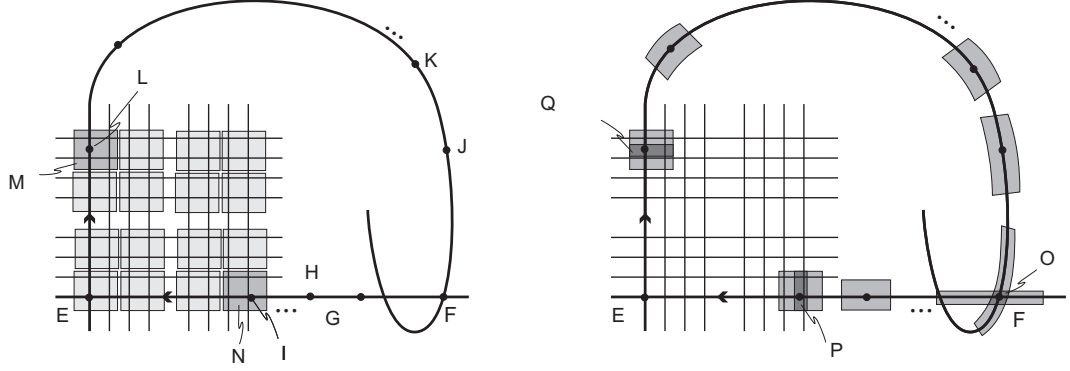
FIGURE 2. Construction of the Markov partition.

Given the Markov partition for $\Lambda_{i_0}(\mu)$, we extend it for a larger set which contains $\Lambda_{i_0}(\mu) \cup \mathcal{O}(q_\mu)$ by constructing other rectangles containing $\{f_\mu^{-N_2+1}(q_\mu), \ldots, q_\mu, \ldots, f_\mu^{N_1+1}(q_\mu)\}$ in the following way: if we iterate R_1 under f^{N_2}, we get a narrow strip around $W^u(p_\mu)$ containing q_μ. And if we iterate R_s under $f_\mu^{-N_1}$, we get a narrow strip around $W^s(p_\mu)$ containing q_μ. We know that $W^s(p_\mu)$ and $W^u(p_\mu)$ have transversal intersection on q_μ. As we could take the diameter of the partition small enough, it comes out that $f_\mu^{-N_1}(R_s)$ and $f_\mu^{N_2}(R_1)$ are transversal. Let $C := f_\mu^{-N_1}(R_s) \cap f_\mu^{N_2}(R_1)$. It is clear that C is disjoint from $\bigcup_{i=1}^s R_i$ and contains q_μ.

Note that $f_\mu^{N_1}(C)$ is a vertical strip of full height contained in R_s and $f_\mu^{-N_2}(C)$ is a horizontal strip of full weight contained in R_1. Consider the disjoint sets S_i defined as

$$S_j = f_\mu^{-N_2+j}(C)$$

for $j = 1, 2, \ldots, N_2, N_2+1, \ldots, N_1 + N_2 - 1$. Note that $S_{N_2} = C$. Now denote $\ell = N_1 + N_2 - 1$ and consider $\mathcal{P} = \{R_1, \ldots, R_s, S_1, \ldots, S_\ell\}$ and

$$R = \bigcup_{i=1}^s R_i \cup \bigcup_{j=1}^\ell S_j.$$

So $\Lambda_R = \bigcap_{n \in \mathbb{Z}} f_\mu^n(R)$ is an isolated hyperbolic set such that $\Lambda_{i_0}(\mu) \subset \Lambda_R \subset \Omega(f_\mu)$. The desired subsystem is the restriction $f_\mu : \Lambda_R \to \Lambda_R$.

Lemma 2.1. $\mathcal{P}$ *is Markov partition for* Λ_R.

Proof. We already know that all R_i's satisfy the Markov property. It remains to verify the Markov property for S_j's. By construction we have all R_i's and S_j's pairwise disjoint. Furthermore $f_\mu(S_j) = S_{j+1}$ for

$j = 1, \ldots, \ell - 1$. In particular, we have $f_\mu(S_\ell) \subset R_s$ is a vertical strip of full height. So

$$f_\mu(S_\ell) \cap R_s \neq \varnothing,$$
$$f_\mu(S_\ell) \cap R_i = \varnothing, \text{ for } i = 1, \ldots, s \text{ and}$$
$$f_\mu(S_\ell) \cap S_j = \varnothing, \text{ for } j = 1, \ldots, \ell.$$

On the other hand, only R_1 has image by f_μ that intersects some S_j. In fact,

$$f_\mu(R_1) \cap S_1 \neq \varnothing \text{ and } f_\mu(R_1) \cap S_j = \varnothing, \text{ for } j = 2, \ldots, \ell.$$

Note that since $f_\mu^{-1}(S_1) \subset R_1$ is a horizontal strip of full weight in R_1, then $S_1 = f_\mu(f_\mu^{-1}(S_1)) \subset f_\mu(R_1)$. So, $f_\mu(R_1) \cap S_1 \neq \varnothing$ and, by the construction of S_1, this intersection satisfies the transversality condition of Markov partitions. Thus $\mathcal{P}$ is a Markov partition for Λ_R.
□

We can associate to $f_\mu : \Lambda_R \to \Lambda_R$ a subshift of finite type as follows. We consider the Markov partition $\mathcal{P} = \{P_1, \ldots, P_{s+\ell}\}$ as above and we define a transition matrix $A_\mu = (a_{ij})_{(s+\ell) \times (s+\ell)}$ for f_μ taking

$$a_{ij} = \begin{cases} 1, & \text{if } f_\mu(P_i) \cap P_j \neq \varnothing; \\ 0, & \text{if } f_\mu(P_i) \cap P_j = \varnothing \end{cases}$$

for $i, j \in \{1, \ldots, s + \ell\}$. In this way we obtain a topological conjugacy between the systems $f_\mu : \Lambda_R \to \Lambda_R$ and the subshift of the finite type $\sigma_{A_\mu} : \Sigma_{A_\mu} \to \Sigma_{A_\mu}$, where $\Sigma_{A_\mu} \subset \Sigma_{s+\ell}$. The transition matrix A_μ has the following form

$$a_{ij} = \begin{cases} H_{ij}, & \text{if } 1 \leq i, j \leq s; \\ 1, & \text{if } i = 1, j = s + 1 \text{ or } i = s + \ell, j = s; \\ 1, & \text{if } j = i + 1 \text{ for } s + 1 \leq i \leq s + \ell - 1; \\ 0, & \text{in other cases.} \end{cases}$$

where $H_\mu = (H_{ij})_{s \times s}$ is the transition matrix of $f_\mu : \Lambda_{i_0}(\mu) \to \Lambda_{i_0}(\mu)$ which is irreducible, because $f_\mu|_{\Lambda_{i_0}(\mu)}$ is topologically transitive (see the next section).

3. Proof of Theorem 1

Let f_μ be the one parameter family as in the theorem 1. In the previous section we constructed a Markov partition for the subsystem $f_\mu|_{\Lambda_R}$, for $\mu \geq 0$. By means of this Markov partition, one may give a conjugacy between such invariant subsystem $f_\mu|_{\Lambda_{i_0}(\mu)}$ and the dynamics of a subshift of finite type.

Let A_μ be the transition matrix of $f_\mu|_{\Lambda_R}$, for $\mu > 0$ small enough. Recall that $h(f_\mu) = \log \lambda_\mu$ where λ_μ is the largest eigenvalue of A_μ. By construction of Markov partition in the previous section, we conclude that

$$
A_\mu = \left(
\begin{array}{cc}
\left[\begin{array}{c} \\ \quad H_\mu \quad \\ \\ \end{array}\right] &
\left[\begin{array}{ccccc} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{array}\right] \\[20pt]
\left[\begin{array}{cccc} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{array}\right] &
\left[\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{array}\right]
\end{array}
\right). \qquad (3.1)
$$

The following proposition asserts that the largest eigenvalue of the matrix A_μ is strictly bigger than the largest eigenvalue of the matrix $A_0 = H_0$. From this proposition we can conclude that the entropy of the system $f_\mu|_{\Lambda_{i_0}(\mu)}$ is bigger than the entropy of $f_0|_{\Lambda_{i_0}}$.

Proposition 3.1. *Let A_μ as defined above. If λ_μ is the largest eingenvalue of A_μ in modulus, then for any $\mu > 0$ near to zero, $\lambda_\mu > \lambda_0$.*

Proof. To proof the proposition we use Perron and Frobenius theorem.

Theorem 3.2 (Perron-Frobenius, [4])**.** *Every non-negative $s \times s$ matrix A has a non-negative eigenvector, $AU = \lambda U$, with the property that the associated λ is equal to the spectral radius $|\lambda|_{max}$. If the matrix A is irreducible, then there is just one non-negative eigenvector up to multiplication by positive constant, and this eigenvector is strictly positive. Furthermore, the maximal eigenvalue λ' of every principal minor (of order less than s) of A satisfies $\lambda' \leq \lambda$. If A is irreducible, then $\lambda' \lneqq \lambda$.*

To use this theorem we need the matrix A_μ be irreducible, that is, for any pair i, j there is some power $n(i,j)$ of A_μ such that $A_\mu^{n(i,j)} > 0$.

It is easy to see that A_μ is irreducible. Indeed, irreducibility is satisfied by the rectangles R_i's because the system $f_\mu|_{\Lambda_{i_0}(\mu)}$ is transitive. Since for each S_j, the iterated $f^{\ell-j}(S_j)$ intersects R_s and the iterated $f^j(R_1)$ intersects S_j, we obtain the desired property for all elements of $\mathcal{P}$.

Now we can apply the Perron-Frobenius Theorem to the sub-matrix $A_{\mu,1}$ of the irreducible transition matrix A_μ, obtained by excluding the

last line and the last column of A_μ. So we obtain that the largest eigenvalue λ_μ of A_μ is strictly bigger than the largest eigenvalue $\lambda_{\mu,1}$ of $A_{\mu,1}$. Even though $A_{\mu,1}$ is not necessarily an irreducible matrix, we can use the Perron-Frobenius theorem again to the sub-matrix $A_{\mu,2}$, with largest eigenvalue $\lambda_{\mu,2}$ and obtain that $\lambda_{\mu,2} \leq \lambda_\mu$. We repeat this process to obtaining the sub-matrix H_μ, whose largest eigenvalue $\lambda_{\mu,\ell}$ is equal to λ_0, because the systems $f_\mu|_{\Lambda_{i_0}(\mu)}$ and $f_0|_{\Lambda_{i_0}}$ are topologically conjugated. Thus we have $\lambda_\mu \gneq \lambda_0$. $\qquad\square$

To conclude the proof of Theorem 1 observe that for all C^2-neighborhood $\mathcal{V}$ of $f = f_0$ we can take f_μ with μ very close to 0 such that $f_\mu \in \mathcal{V}$ and the Proposition 3.1 holds. So, since Λ_{i_0} is responsible for the entropy of f_0, $h(f_\mu) \geq h(f_\mu|_{\Lambda_{i_0}(\mu)}) > h(f_0|_{\Lambda_{i_0}}) = h(f_0)$. Then $h(f_\mu) \neq h(f_0)$ and thus f_0 is a point of entropy variation.

$\qquad\square$

4. Proof of Theorem 2

Here we recall some results of Yomdin [15] and Newhouse [8] for the calculation of the defect of continuity of the entropy function in the space of C^k-diffeomorphisms. We use their result for maximal invariant subsets of dynamics.

Let $\Lambda_f = \bigcap_{n \in \mathbb{Z}} f^n(U)$ be an isolated maximal invariant subset and $r(n, \epsilon, f)$ denote the maximal cardinality of an $(n, \epsilon)-$separated subset of Λ_f. Denote $r(\epsilon, f) = \limsup_{n \to \infty} \frac{1}{n} r(n, \epsilon, f)$. By definition $h(f|\Lambda_f) = \lim_{\epsilon \to 0} r(\epsilon, f)$. We need to find an upper bound for $h(f|\Lambda_f) - r(\epsilon, f)$. Local entropy turns out to be such upper bound. The local entropy can be defined for any invariant measure as follows: Take Λ any compact subset of M and for any $x \in \Lambda, \epsilon > 0$ let

$$W^s(x, n, \epsilon) := \{y \in M, d(f^i(x), f^i(y)) \leq \epsilon \quad \text{for} \quad i \in [0, n)\}.$$

Let $r(n, \delta, \epsilon, \Lambda, x) = max\, Card(F)$ where

$$F \subset \Lambda \cap W^s(x, n, \epsilon)$$

is $(n, \delta)-$separated. Taking $r(n, \delta, \epsilon, \Lambda) = \sup_{x \in \Lambda} r(n, \delta, \epsilon, \Lambda, x)$ we define

$$r(\epsilon, \Lambda) = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \delta, \epsilon, \Lambda)$$

and for any invariant measure μ we define $h_{loc}^\mu(\epsilon, f) = \lim_{\sigma \to 1} \inf_{\mu(\Lambda) > \sigma} r(\epsilon, \Lambda)$. Finally define $h_{loc}(\epsilon, f) = \sup_\mu h_{loc}^\mu(\epsilon, f)$. Newhouse proved (Theorem 1 of [8]) that:

$$h(f) \leq r(\epsilon, f) + h_{loc}(\epsilon, f).$$

Take any g close to f define $\Lambda_g := \bigcap_{n \in \mathbb{Z}} g^n(U)$.

Proposition 4.1. *The map $g \to h(g|\Lambda_g)$ is upper semicontinuous in $C^\infty-$topology.*

Proof. The proof is an adaptation of Newhouse proof. We use that for any $g, h(g|\Lambda_g) \le r(\epsilon, g|\Lambda_g) + h_{loc}(\epsilon, g|\Lambda_g)$ repeating the arguments of Newhouse. For any C^∞ diffeomorphism f and g close enough in $C^\infty-$topology, the local entropy of g is small (This comes from Yomdin result [15]) and $r(\epsilon, g|\Lambda(g))$ is also upper semicontinuous. More precisely, Let $\mathcal{A}$ be any cover of U with $diam(U) \le \epsilon$ then $r(\epsilon, g|\Lambda_g) \le h(\mathcal{A}, g)$ where $h(\mathcal{A}, g)$ is the entropy of the covering. It is easy to see that $g \to h(\mathcal{A}, g)$ is upper semicontinuous, because it is infimum of upper semicontinuous functions.

A similar proof to Yomdin Theorem [15] implies

Proposition 4.2. *Let $f : M \to M$ be C^k e $g_n \to f$ in C^k topology then,*

$$\limsup_{n \to \infty} h(g_n|\Lambda(g_n)) \le h(f|\Lambda(f)) + \frac{2m}{k} R(f), \qquad (4.1)$$

where $k \ge 1$, $m = \dim M$ e $R(f) = \lim_{n \to \infty} \frac{1}{n} \log \max_{x \in M} \|Df^n(x)\|$.

The above semi continuity results are key points of the proof of Theorem 2. So we state our setting again: $f : M \to M$ is a diffeomorphism (Here f stands for f_0 in Theorem 2.) such that the homoclinic tangency $\mathcal{O}(q)$ corresponds to the **non** responsible basic set. The nonwandering set $\Omega(f) = \bigcup_{i=1}^k \Lambda_i \cup \mathcal{O}(q)$ where q is a tangency corresponding to Λ_{i_0}.

Shub in [13] defined a filtration adapted to a homeomorphism $f : M \to M$ as a sequence $\varnothing = M_0 \subset M_1 \subset \cdots \subset M_k = M$, where each M_i is a compact $C^\infty-$submanifold with boundary of M such that

- $\dim M_i = \dim M$;
- $f(M_i) \subset int(M_i)$.

Given a filtration $\mathcal{M}$ adapted to f, $K_i^f(\mathcal{M}) = \bigcap_{n \in \mathbb{Z}} f^n(M_i \setminus M_{i-1})$ is the maximal f-invariant subset of $M_i \setminus M_{i-1}$, which is compact. Furthermore, we denote $K^f(\mathcal{M}) = \bigcup_{i=1}^k K_i^f(\mathcal{M})$ and for the nonwandering set $\Omega(f)$ we have

$$\Lambda_i = \Omega(f) \cap (M_i \setminus M_{i-1}).$$

Proposition 4.3 ([13])**.** *Let $\mathcal{M}$ be a filtration adapted to f and U a neighborhood of $K^f(\mathcal{M})$. Then there exists a C^0-neighborhood $\mathcal{U}$ of f in the space of homeomorphisms on M such that, for each $g \in \mathcal{U}$, $\mathcal{M}$ is a filtration adapted to g and $K^g(\mathcal{M})$ is contained in U. Moreover, taking $U_i = (M_i \setminus M_{i-1}) \cap U$, we can choose a neighborhood of $\mathcal{U}$ such that $K_i^g(\mathcal{M}) \subset U_i$.*

Suppose that Λ_{i_0} is the piece with external homoclinic tangency. By Palis-Takens [10], for each diffeomorphism with a homoclinic tangency as above, there exists a filtration $\mathcal{M}$ for the decomposition $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_k \cup \mathcal{O}(q)$ such that

(i) $\Lambda_i = \bigcap_{n \in \mathbb{Z}} f^n(M_i \setminus M_{i-1})$, for $i \neq i_0$;
(ii) $\Lambda_{i_0} \cup \mathcal{O}(q) = \bigcap_{n \in \mathbb{Z}} f^n(M_{i_0} \setminus M_{i_0-1})$.

Consider g a $C^\infty-$perturbation of f. As $\mathcal{O}(q)$ is the unique orbit of tangency and all pieces $\Lambda_i, i \neq i_0$ are hyperbolic sets then $h(f|\Lambda_i) = h(g|\Lambda_i), i \neq i_0$ and using Proposition 4.1 we obtain that $h(g|\Lambda_{i_0}(g)) \leq h(f|\Lambda_{i_0}(f)) + \epsilon$ where $\epsilon \to 0$ as g converges to f and as $h(f|\Lambda_{i_0}(f)) < h(f)$ we conclude that $h(f) = h(g)$.

To prove the second item of Theorem (in C^k-topology), we use Proposition 4.2. It is enough to consider $\alpha_k = \frac{2m}{k} R(f)$ and using filtrations as the proof of the first item of Theorem.

$\square$

5. Proof of theorem 3

To prove Theorem 3 we construct a system with a horseshoe and a homoclinic tangency corresponding to a hyperbolic fixed point outside the horseshoe. Then we perturb the system in a small neighborhood of the tangency to create a transversal intersections (using C^1-perturbations "Snake like" as in Newhouse [9]) to obtain a new system with larger topological entropy.

Consider the system f on the sphere $\mathbb{S}^2$ whose orbits follow the meridians from a (the North Pole) to b (the South Pole). Suppose that the system has a transverse homoclinic point and a homoclinic loop in two disjoint regions. These regions are sorrounded by meridians. See the Figure 3. Indeed, fistly consider a system with two homoclinic loop. By making an small perturbation on one of the homoclinic loops one obtain a transverse homoclinic orbit and consequently horseshoes.

We suppose this homoclinic loop is associated to a fixed hyperbolic point p which has derivative with eigenvalues $\lambda(p) = 3$ and $\lambda(p)^{-1} = 3^{-1}$. The horseshoe Γ in the first region is a two legs horseshoe, p_∞ is the source which send the orbits to a topological disc Q whose interior is a trapping neighborhood for Γ. The sink of this horseshoe coincides with p_0. See the figure 4.

Thus the nonwandering set $\Omega(f)$ consists of three sinks, a source, a (isolated) horseshoe and a hyperbolic point on the homoclinic loop. Then, the topological entropy of f is $h(f|_\Gamma) = \log 2$. Observe that this horseshoe is responsible
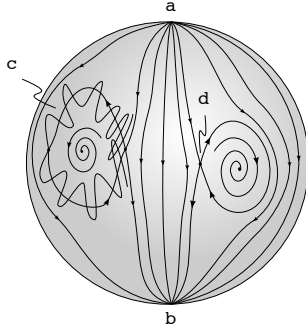
FIGURE 3. System
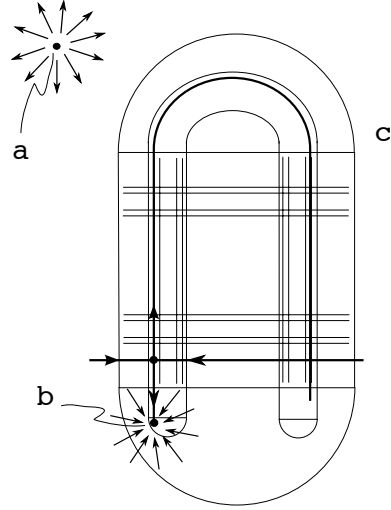with a horseshoe and
a homoclinic loop.

FIGURE 4. Region
of the horseshoe Γ.

Now we perturb f in the C^1-topology to obtain a new system g,
breaking the homoclinic loop. Here we have an interval of tangency
corresponding to the saddle p and can apply Theorem 1.1.

By theorem 1.1 we obtain a diffeomorphism g such that for small ϵ,
$h(g|_\Lambda) \geq \log|\lambda(p)| - \varepsilon$. As the tangency for f is associated to a fixed
point and $h(g) > h(f)$ the proof is complete.

6. PROOF OF THEOREM 4

It is well known that topological entropy function $f \to h_{top}(f)$ is
a continuous function in C^∞ topology for systems defined on a bi-
dimensional compact manifold. Using this observation, our example in
the proof of the second statement of theorem 1 shows that the New-
house perturbation Theorem 1.1 can not be applied in C^∞ topology.

7. LACK OF LOWER SEMI CONTINUITY

We exhibit an example of discontinuity (lower semicontinuity) of the
topological entropy in dimension three in C^∞ topology . Observe that
in C^∞ toplogy the topological entropy if upper semi continuous.

Theorem 7.1. *There exists a diffeomorphism on the closed 3-dimensional
ball which is a discontinuity point of the topological entropy in the
$C^\infty-$topology.*

Proof. Consider a horseshoe map $f : D \to D$ of class C^∞, where D is the unitary disc of $\mathbb{R}^2$ as the Picture 5. We divide it in three parts cut by the lines $y = \frac{1}{3}$ and $y = -\frac{1}{3}$.
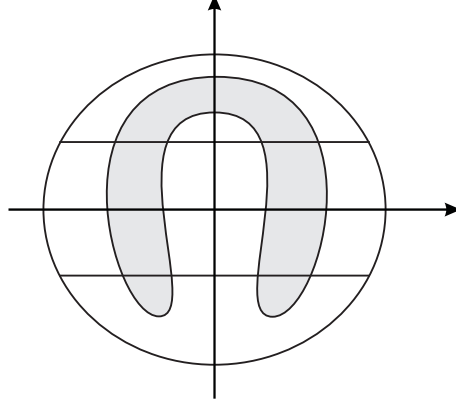

FIGURE 5. Horseshoe map on the disc.

We will now construct an isotopy which deforms the horseshoe map to a contraction. Let us produce a family f_t as follows. First of all consider the maps $\alpha_t : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\alpha_t(x, y) = \left(\frac{2}{3}x, (1-t)y - \frac{5}{3}t \right)$, with $\varepsilon_0 \leq t \leq 1$, where $\varepsilon_0 > 0$ is small enough such that the range $\alpha_{\varepsilon_0}(D)$ is under the x−axis. Now define $f_t = f \circ \alpha_t$ and note that for $t_0 = \varepsilon_0$, the map f_{t_0} is a contraction with a unique fixed point, which is an attractor. We can assume, unless reparametrization, that the family f_t is defined for t in $t \in [0, 1]$ and that $f_0 = f$. Extend this family for $t \in [-1, 1]$ making $f_t = f_{-t}$ when $t \in [-1, 0)$. Moreover, for each $t \in (-1, 1)$ contract the domain of each f_t is defined by decreasing the radius of D to $\sqrt{1 - t^2}$, i.e., define a map $C_t : D \to D_t$, where $D_t = \{(x, y) : x^2 + y^2 \leq 1 - t^2\}$, given by $C_t(x, y) = \left((\sqrt{1 - t^2})x, (\sqrt{1 - t^2})y \right)$. Thus for each $t \in (-1, 1)$ define $\tilde{f}_t = C_t^{-1} \circ f_t \circ C_t$.

Now construct a new map G defined on the 3-dimensional disc $\mathbb{D}^3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ in the following way: far each 2-dimensional disc $\mathbb{D}^2_t = \{(x, y, z) \in \mathbb{D}^3 : z = t\}$, for $t \in (-1, 1)$, define $F(x, y, z = t) = (\tilde{f}_t(x, y), t)$.

Consider now a C^∞−flow $\phi(t, (x, y, z))$ on the 3-dimensional disc $\mathbb{D}^3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ with just two singularities: the north pole $N = (0, 0, 1)$ and the south pole $S = (0, 0, -1)$, and such that the orbit of each point in $\mathbb{D}^3 \setminus \{N, S\}$ has N as α-limit and S as ω-limit. Moreover, suppose that for each orbit the time t is parametrized such that each 2-dimensional disc $\mathbb{D}^2_{z_0} = \{(x, y, z) \in \mathbb{D}^3 : z = z_0\}$, for
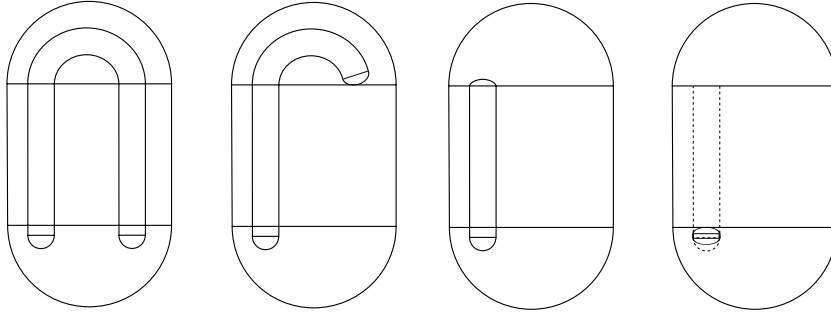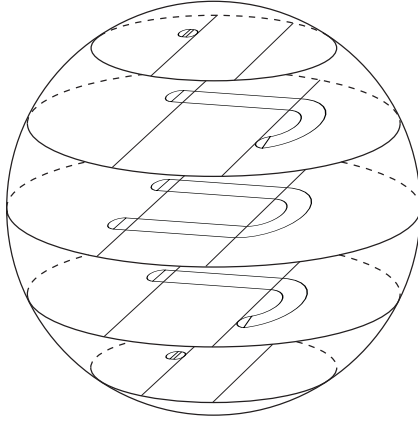
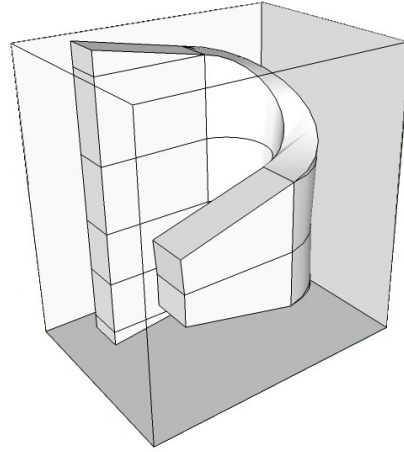FIGURE 6. Family f_t on the disc.

FIGURE 7.   FIGURE 8.

$z_0 \in (-1, 1)$, is send in another disc $\mathbb{D}^2_{z_1}$, with $z_1 < z_0$. Denote by $\phi_\tau : \mathbb{D}^3 \to \mathbb{D}^3$ the time τ diffeomorphism of this flow. It forms an one parameter family of diffeomorphisms such that in $\tau = 0$ we get the identity map on $\mathbb{D}^3$.

Considere the family of diffeomorphisms $G_\tau = \phi_\tau \circ F$. For $\tau = 0$, we have that $G_0 = F$, that the disc $\mathbb{D}^2_0$ is F-invariant and $F|_{\mathbb{D}^2_0} = f$. Furthermore, the entropy of f is $h(f) = \log 2$, which implies that $h(F) \geq h(F|_{\mathbb{D}^2_0}) = \log 2$. Nevertheless, if $\tau > 0$ is small, the entropy of G_τ vanishes, because the set $\Omega(G_\tau)$ is constituted just by S and N.

In short, we got an arc of diffeomorphisms G_τ such that for each $\tau > 0$ the entropy of G_τ is zero and for $\tau = 0$ the entropy jumps to $\log 2$. It proves that the topological entropy is not lower semicontinuous in G_0. $\qquad\square$

References

[1] ABRAHAM, R. & ROBBIN, J., *Transversal mappings and flows*, An appendix by Al Kelley, W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[2] BOWEN, R. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Math. 470. Springer, 1975.

[3] DÍAZ, L.J.; RIOS, I.L. *Critical saddle-node horseshoes: bifurcations and entropy*, Nonlinearity, 16(3):897-928, 2003.

[4] GANTMACHER, F.R. *The theory of matrices. Vols. 1, 2.* (Translated by K. A. Hirsch) Chelsea Publishing Co., New York, Vol. 1, x+374 p. Vol. 2, ix+276 p.,1959.

[5] HUA, Y.; SAGHIN, R.; XIA, Z. *Topological entropy and partially hyperbolic diffeomorphisms*, Ergodic Theory & Dynamical Systems, p.1, 2008.

[6] HASSELBLATT, B., KATOK, A. *Introduction to the theory of dynamical systems*, Cambridge University Press, 1995

[7] KATOK, A. *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Publications Mathematiques de l'IHS, 51:137-173, 1980.

[8] NEWHOUSE, S.E. *Continuity Properties of Entropy.* The Annals of Mathematics, 2a Ser., 129(1):215-235, 1989.

[9] NEWHOUSE, S.E. *Topological entropy and Hausdorff dimension for area preserving diffeomorphisms of surfaces*, Soc. Math. France (Asterisque), Paris, 51:323-334, 1978.

[10] PALIS, J.; TAKENS, F. *Hyperbolicity and sensitive-chaotic dynamics at homoclinic bifurcations.* Cambridge: Cambridge University Press, 53-91, 1993. (Cambridge Studies in Advanced Mathematics)

[11] PUJALS, S.; SAMBARINO, M. *On homoclinic tangencies, hyperbolicity, creation of homoclinic orbits and variation of entropy.* (English summary) Nonlinearity, 13(3):921-926, 2000.

[12] ROBINSON, C. *Introduction to the theory of dynamical systems.* Boca Raton: CRC Press, 1995. (Studies in Advances Mathematics)

[13] SHUB, M. *Stabilité globale des systèmes dynamiques*, Astérisque. Vol. 56. (English preface and summary) Société Mathématique de France, Paris, 1978.

[14] SMALE, S. *Differentiable dynamical systems.* Bull. Amer. Math. Soc., 73:747-817, 1967.

[15] YOMDIN, Y. *Volume growth and entropy.* Israel J. Math., 57(3):285-300, 1987.

Departamento de Matemática, ICMC-USP São Carlos, Caixa Postal 668, 13560-970 São Carlos-SP, Brazil.
E-mail address: bronzi@icmc.sc.usp.br

Departamento de Matemática, ICMC-USP São Carlos, Caixa Postal 668, 13560-970 São Carlos-SP, Brazil.
E-mail address: tahzibi@icmc.sc.usp.br
URL: http://www.icmc.sc.usp.br/∼tahzibi