MANIN'S CONJECTURE FOR A QUARTIC DEL PEZZO SURFACE WITH A_3 SINGULARITY AND FOUR LINES

by

Pierre Le Boudec

Abstract. — We give a proof of Manin's conjecture for a quartic del Pezzo surface split over $\mathbb{Q}$ and having a singularity of type $\mathbf{A}_3$ and containing exactly four lines.

Résumé. — Nous donnons une preuve de la conjecture de Manin pour une surface de del Pezzo de degré quatre déployée sur $\mathbb{Q}$, ayant une singularité de type $\mathbf{A}_3$ et contenant exactement quatre droites.

Contents

1. Introduction

Manin's conjecture (see [**FMT89**]) gives a precise description of the distribution of rational points of bounded height on singular del Pezzo surfaces. More precisely, let $V \subset \mathbb{P}^n$ be such a surface, U be the open subset formed by deleting the lines from V and

$$N_{U,H}(B) = \#\{x \in U(\mathbb{Q}), H(x) \leq B\},$$

where $H : \mathbb{P}^n(\mathbb{Q}) \to \mathbb{R}_+$ is the exponential height defined by

$$H(x_0 : \ldots : x_n) = \max\{|x_i|, 0 \leq i \leq n\},$$

for $(x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$ satisfying the condition $\gcd(x_0, \ldots, x_n) = 1$. If $\widetilde{V}$ denotes the minimal desingularization of V and $\rho_{\widetilde{V}}$ the rank of the Picard group of $\widetilde{V}$, then it is expected that

$$N_{U,H}(B) = c_{V,H} B \log(B)^{\rho_{\widetilde{V}}-1}(1 + o(1)), \tag{1.1}$$

where $c_{V,H}$ is a constant which is expected to follow Peyre's prediction [**Pey95**].

2000 ***Mathematics Subject Classification.*** — 11D45, 14G05.

Key words and phrases. — Rational points, Manin's conjecture, del Pezzo surfaces, universal torsors.

Singular del Pezzo surfaces are classified by their degrees, their singularity types and the number of lines they contain. Two surfaces are said to have the same singularity type if they have the same number of singularities and if their Dynkin diagrams match up. We are interested here in singular del Pezzo surfaces of degree four, their classification can be found in the work of Coray and Tsfasman [**CT88**]. Up to isomorphism over $\overline{\mathbb{Q}}$, there are fifteen types of such surfaces (see the table in [**Bro07**, section1.2]). Here is a quick overview of the results already obtained towards a proof of Manin's conjecture for all singular quartic del Pezzo surfaces split over $\mathbb{Q}$. The conjecture is already known to hold for nine of these fifteen types. Using techniques coming from harmonic analysis on adelic groups and studying the height Zeta function

$$Z_{U,H}(s) \quad = \sum_{x \in U(\mathbb{Q})} H(x)^{-s},$$

Batyrev and Tschinkel have proved it for toric varieties [**BT98**] (which covers the three types $4\mathbf{A}_1$, $2\mathbf{A}_1 + \mathbf{A}_2$ and $2\mathbf{A}_1 + \mathbf{A}_3$) and Chambert-Loir and Tschinkel have proved it for equivariant compactifications of vector groups [**CLT02**] (which covers the type $\mathbf{D}_5$). Note that for the $\mathbf{D}_5$ surface, la Bretèche and Browning have obtained this result independently [**BB07**]. Finally, the conjecture has been obtained for five other singularity types, the type $\mathbf{D}_4$ by Derenthal and Tschinkel [**DT07**], the type $\mathbf{A}_1 + \mathbf{A}_3$ by Derenthal [**Der09**], the type $\mathbf{A}_4$ by Browning and Derenthal [**BD09**] and the types $3\mathbf{A}_1$ and $\mathbf{A}_1 + \mathbf{A}_2$ by the author [**LB10**]. These proofs are very different from those using the fact that the varieties considered are equivariant compactifications of algebraic groups. They all use a lift to *universal torsors*. This consists in defining a bijection between the set of the points to be counted on U and some integral points on an affine variety of higher dimension (which is equal to eight for quartic surfaces). Note that Derenthal has calculated the equations of the universal torsors for all singular quartic del Pezzo surfaces in his thesis [**Der**]. This can also be achieved using only elementary techniques, see section 3 for an example.

Our aim is to prove Manin's conjecture for another surface split over $\mathbb{Q}$, having singularity type $\mathbf{A}_3$ and containing exactly four lines. Such a surface $V \subset \mathbb{P}^4$ can be defined as the intersection of the two following quadrics

$$\begin{aligned} &x_0 x_1 - x_2^2 = 0, \\ &(x_0 + x_1 + x_3) x_3 - x_2 x_4 = 0. \end{aligned}$$

The lines on V are given by $x_i = x_2 = x_3 = 0$ and $x_i = x_2 = x_0 + x_1 + x_3 = 0$ for $i \in \{0, 1\}$ and the unique singularity is $(0:0:0:0:1)$. We see that V is actually split over $\mathbb{Q}$ and thus, if $\widetilde{V}$ denotes the minimal desingularization of V, the Picard group of $\widetilde{V}$ has rank $\rho_{\widetilde{V}} = 6$. Define the open subset U and the quantity $N_{U,H}(B)$ as explained above. In section 3, we define a bijection between the set of the points to be counted on U and certain integral points of an open subset of the affine variety embedded in $\mathbb{A}^{10} \simeq \operatorname{Spec} \mathbb{Q}[\eta_1, \dots, \eta_7, \alpha_1, \alpha_2, \alpha_4]$ defined by

$$\begin{aligned} \eta_1^2 \eta_2 \eta_4^2 \eta_7 + \eta_5 \alpha_1 - \eta_6 \alpha_2 &= 0, \\ \eta_2 \eta_3^2 \eta_5^2 \eta_6 + \eta_7 \alpha_2 - \eta_4 \alpha_4 &= 0. \end{aligned}$$

The universal torsor corresponding to our present problem actually has five equations and can be embedded in $\mathbb{A}^{11} \simeq \operatorname{Spec} \mathbb{Q}[\eta_1, \dots, \eta_7, \alpha_1, \alpha_2, \alpha_3, \alpha_4]$ but we will neither use these three other equations nor the variable α_3. Let us insist on the fact that it is the first time that Manin's conjecture is proved for a split singular quartic del Pezzo surface whose universal torsor has several equations. Our result is the following.

Theorem 1. — *As B tends to $+\infty$, we have the estimate*

$$N_{U,H}(B) \quad = \quad c_{V,H} B \log(B)^5 \left(1 + O\left(\frac{1}{\log(B)}\right)\right),$$

where $c_{V,H}$ agrees with Peyre's prediction.

Since $\rho_{\widetilde{V}} = 6$, this estimate proves that V satisfies Manin's conjecture. Let us note here that Derenthal has proved that V is not toric [**Der06**, Proposition 12] and Derenthal and Loughran have proved that it is not an equivariant compactification of $\mathbb{G}_a^2$ [**DL10**], so theorem 1 does not follow from the general results [**BT98**] and [**CLT02**]. In view of this result, there are only five types of surfaces in the list of fifteen for which we still do not know if Manin's conjecture holds.

It is worth pointing out that the summations over the variables $\eta_1, \dots, \eta_7$ could have been carried out studying a certain Dirichlet series in two variables linked to the height Zeta function of the surface and using a tauberian theorem. The error term coming from these summations might have certainly been improved to $B^{1-\delta}$ for some $\delta > 0$. However, being unable to get a better error term than $B \log(B)^2$ for the summations over α_1, α_2 and α_4, the author has chosen not to take this path. This latter error term therefore seems to be the only reason why a proof of a meromorphic continuation of the height Zeta function of the surface on the left of $\Re(s) = 1$ is hard to attain.

In the following section, we prove several lemmas about summations of arithmetic functions. The next two sections are respectively devoted to the calculations of the universal torsor and of Peyre's constant. Finally, the last section is dedicated to the proof of theorem 1.

It is a great pleasure for the author to thank his supervisor Professor de la Bretèche both for his encouragement and his advice during this work.

This work has received the financial support of the ANR PEPR (Points Entiers Points Rationnels).

2. Arithmetic functions

We need to introduce the following collection of arithmetic functions

$$\varphi^*(n) = \frac{\varphi(n)}{n} = \prod_{p|n}\left(1 - \frac{1}{p}\right), \qquad \varphi^\circ(n) = \prod_{\substack{p|n \\ p\neq 2}}\left(1 - \frac{1}{p-1}\right),$$

$$\varphi^\dagger(n) = \prod_{p|n}\left(1 - \frac{1}{p^2}\right), \qquad \varphi^\flat(n) = \prod_{\substack{p|n \\ p\neq 2}}\left(1 + \frac{1}{p(p-2)}\right).$$

We can note here that if n is odd then $\varphi^\circ(n)\varphi^\flat(n) = \varphi^*(n)$ and if n is even then $\varphi^\circ(n)\varphi^\flat(n) = 2\varphi^*(n)$. Moreover, for $a, b \geq 1$, we define

$$\psi_{a,b}(n) = \begin{cases} \varphi^\circ(\gcd(a,n))^{-1} & \text{if } \gcd(n,b) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\psi'_{a,b}(n) = \begin{cases} \varphi^\circ(\gcd(a,n))^{-1}\varphi^*(n)\varphi^*(\gcd(a,n))^{-1} & \text{if } \gcd(n,b) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, for $\delta > 0$, we set $\sigma_{-\delta}(n) = \sum_{k|n} k^{-\delta}$.

Lemma 1. — *Let $\delta > 0$. We have the estimate*

$$\sum_{n \leq X} \psi_{a,b}(n) = \Psi(a,b)X + O_\delta\left(X^\delta \sigma_{-\delta}(ab)\right),$$

where

$$\Psi(a,b) = \varphi^*(b)\frac{\varphi^\flat(a)}{\varphi^\flat(\gcd(a,b))}.$$

Proof. — We start by calculating the Dirichlet convolution of $\psi_{a,b}$ with the Möbius function μ.

$$\begin{aligned}(\psi_{a,b} * \mu)(n) &= \sum_{d|n} \psi_{a,b}(n/d)\mu(d) \\ &= \prod_{p^\nu || n} \left(\psi_{a,b}\left(p^\nu\right) - \psi_{a,b}\left(p^{\nu-1}\right)\right).\end{aligned}$$

But $\psi_{a,b}(1) = 1$ and for all $\nu \geq 1$

$$\psi_{a,b}\left(p^\nu\right) = \psi_{a,b}(p) = \begin{cases} (1 - 1/(p-1))^{-1} & \text{if } p|a, p \neq 2 \text{ and } p \nmid b, \\ 1 & \text{if } p \neq 2, p \nmid ab, \\ 1 & \text{if } p = 2, 2 \nmid b, \\ 0 & \text{if } p|b. \end{cases}$$

Thus, we easily obtain

$$(\psi_{a,b} * \mu)(n) = \begin{cases} \mu(n) \prod_{p| \gcd(a,n), p \nmid b} (-1/(p-2)) & \text{if } n|ab \text{ and } (2 \nmid n \text{ or } 2|b), \\ 0 & \text{otherwise.} \end{cases}$$

Let us now write $\psi_{a,b} = (\psi_{a,b} * \mu) * 1$, we get

$$\begin{aligned}\sum_{n \leq X} \psi_{a,b}(n) &= \sum_{n \leq X} \sum_{d|n} (\psi_{a,b} * \mu)(d) \\ &= \sum_{d=1}^{+\infty} (\psi_{a,b} * \mu)(d) \sum_{k \leq X/d} 1.\end{aligned}$$

Let $\delta > 0$. Let us replace the inner sum of the right-hand side by $X/d + O\left(X^\delta/d^\delta\right)$ and use $|(\psi_{a,b} * \mu)(n)| \leq 1$, we get

$$\sum_{d=1}^{+\infty} \frac{|(\psi_{a,b} * \mu)(d)|}{d^\delta} \leq \sigma_{-\delta}(ab).$$

We have proved that

$$\sum_{n \leq X} \psi_{a,b}(n) = X \sum_{d=1}^{+\infty} \frac{(\psi_{a,b} * \mu)(d)}{d} + O\left(X^\delta \sigma_{-\delta}(ab)\right).$$

Finally, a straigthforward calculation gives

$$\sum_{d=1}^{+\infty} \frac{(\psi_{a,b} * \mu)(d)}{d} = \prod_{p|b} \left(1 - \frac{1}{p}\right) \prod_{\substack{p|a, p \nmid b \\ p \neq 2}} \left(1 + \frac{1}{p(p-2)}\right),$$

which concludes the proof. □

Lemma 2. — *Let $\delta > 0$. We have the estimate*

$$\sum_{n \leq X} \psi'_{a,b}(n) = \Psi'(a,b)X + O_\delta\left(X^\delta \sigma_{-\delta}(b)\right),$$

where

$$\Psi'(a,b) = \frac{\varphi^\flat(a)}{\varphi^\flat(\gcd(a,b))}\varphi^*(b)\frac{\zeta(2)^{-1}}{\varphi^\dagger(ab)}.$$

Proof. — We proceed exactly as for the proof of lemma 1. Let

$$f(n) = \mu(n) \prod_{\substack{p|n, p\nmid ab \\ p\neq 2}} \frac{1}{p} \prod_{\substack{p|\gcd(a,n), p\nmid b \\ p\neq 2}} \frac{-1}{p-2}.$$

A calculation provides

$$(\psi'_{a,b} * \mu)(n) = \begin{cases} f(n) & \text{if } 2\nmid n \text{ or } 2|b, \\ f(n)/2 & \text{if } 2|n \text{ and } 2\nmid ab, \\ 0 & \text{otherwise.} \end{cases}$$

Now we see that $|(\psi'_{a,b} * \mu)(n)| \ll \gcd(b,n)/n$, which easily yields

$$\sum_{d=1}^{+\infty} \frac{|(\psi'_{a,b} * \mu)(d)|}{d^\delta} \ll \sigma_{-\delta}(b).$$

Another straightforward calculation gives

$$\sum_{d=1}^{+\infty} \frac{(\psi'_{a,b} * \mu)(d)}{d} = \Psi'(a,b),$$

which completes the proof. □

The proof of [**LB10**, Lemma 5] shows that we have the following result.

Lemma 3. — *Let $1 \leq t_1 < t_2$ and $I = [t_1, t_2]$. Let also g be a function having a piecewise continuous derivative on I whose sign changes at most $R_g(I)$ times on I. We have*

$$\sum_{n \in I \cap \mathbb{Z}_{>0}} \psi_{a,b}(n) g(n) = \Psi(a,b) \int_I g(t) \mathrm{d}t + O\left(\sigma_{-\delta}(ab) t_2^\delta M_I(g)\right),$$

and

$$\sum_{n \in I \cap \mathbb{Z}_{>0}} \psi'_{a,b}(n) g(n) = \Psi'(a,b) \int_I g(t) \mathrm{d}t + O\left(\sigma_{-\delta}(b) t_2^\delta M_I(g)\right),$$

where $M_I(g) = (1 + R_g(I)) \sup_{t \in I} |g(t)|$.

We also have the following estimation.

Lemma 4. — *With the same notations, if $2 \nmid b$ then*

$$\sum_{\substack{n \in I \cap \mathbb{Z}_{>0} \\ n \equiv 0 \pmod 2}} \psi_{a,b}(n) g(n) = \frac{1}{2}\Psi(a,b) \int_I g(t) \mathrm{d}t + O\left(\sigma_{-\delta}(ab) t_2^\delta M_I(g)\right).$$

In a similar way, if $2|a$ and $2 \nmid b$ then

$$\sum_{\substack{n \in I \cap \mathbb{Z}_{>0} \\ n \equiv 0 \pmod 2}} \psi'_{a,b}(n) g(n) = \frac{1}{2}\Psi'(a,b) \int_I g(t) \mathrm{d}t + O\left(\sigma_{-\delta}(b) t_2^\delta M_I(g)\right).$$

Proof. — Let us prove the statement for $\psi_{a,b}$, it suffices to notice that

$$\sum_{\substack{n\leq X\\ n\equiv 0 \pmod 2}} \psi_{a,b}(n) = \sum_{d=1}^{+\infty}(\psi_{a,b}*\mu)(d) \sum_{\substack{k\leq X/d\\ k\equiv 0 \pmod 2}} 1 + \sum_{\substack{d=1\\ d\equiv 0 \pmod 2}}^{+\infty} (\psi_{a,b}*\mu)(d) \sum_{\substack{k\leq X/d\\ k\equiv 1 \pmod 2}} 1,$$

and $(\psi_{a,b}*\mu)(d)=0$ for all $d\equiv 0 \pmod 2$ since $2\nmid b$ and therefore

$$\sum_{\substack{n\leq X\\ n\equiv 0 \pmod 2}} \psi_{a,b}(n) = \sum_{d=1}^{+\infty}(\psi_{a,b}*\mu)(d)\left(\frac{X}{2d}+O\left(\frac{X^\delta}{d^\delta}\right)\right).$$

We can conclude exactly as in the proof of lemma 1 and finally, as for lemma 3, use the proof of [**LB10**, Lemma 5]. The proof for $\psi'_{a,b}$ is strictly identical, it only uses the fact that $(\psi'_{a,b}*\mu)(d)=0$ for all $d\equiv 0 \pmod 2$ since $2|a$ and $2\nmid b$. □

3. The universal torsor

We now proceed to define a one-to-one function between the set of the points we want to count on U and certain integral points on the affine variety defined in the introduction. As explained above, the universal torsor of our problem is an affine variety of dimension 8 embedded in $\mathbb{A}^{11}$. It has five equations but we will only deal with ten of the eleven variables and will only make use of two equations among these five. Our choice of notation might be surprising but it is guided by our wish to adopt the notation used by Derenthal in [**Der**, Chapter 6]. Note that if $(x_0:x_1:x_2:x_3:x_4)\in V(\mathbb{Q})$ then we have $(x_0:x_1:x_2:x_3:x_4)\in U(\mathbb{Q})$ if and only if $x_0x_1x_2x_3\neq 0$. Let $(x_0,x_1,x_2,x_3,x_4)\in \mathbb{Z}^4_{\neq 0}\times\mathbb{Z}$ such that

$$x_0x_1-x_2^2=0,$$
$$(x_0+x_1+x_3)x_3-x_2x_4=0,$$

$\max\{|x_i|, 0\leq i\leq 4\}\leq B$ and $\gcd(x_0,x_1,x_2,x_3,x_4)=1$. Since $\mathbf{x}=-\mathbf{x}$ in $\mathbb{P}^4$, we can assume $x_0>0$ which gives $x_1>0$. Moreover, the application $(x_2,x_4)\mapsto(-x_2,-x_4)$ shows that we can also assume $x_2>0$ keeping in mind that we need to multiply our future result by 2. The first equation shows that there is a unique way to write $x_0=y_{01}x_0'^2$, $x_1=y_{01}x_1'^2$ and $x_2=y_{01}x_0'x_1'$ for $x_0',x_1',y_{01}>0$ and $\gcd(x_0',x_1')=1$. The second equation therefore gives

$$(y_{01}x_0'^2+y_{01}x_1'^2+x_3)x_3-y_{01}x_0'x_1'x_4 = 0.$$

We define $y_{01}'=\gcd(y_{01},x_3)>0$ and write $y_{01}=y_{01}'\eta_2$ and $x_3=y_{01}'x_3'$ with $\eta_2>0$ and $\gcd(\eta_2,x_3')=1$. We obtain

$$(\eta_2x_0'^2+\eta_2x_1'^2+x_3')y_{01}'x_3'-\eta_2x_0'x_1'x_4 = 0,$$

and thus $\eta_2|y_{01}'x_3'^2$ and it follows $\eta_2|y_{01}'$ since $\gcd(\eta_2,x_3')=1$. We can therefore write $y_{01}'=\eta_2y_{01}''$ for $y_{01}''>0$. The equation becomes

$$(\eta_2x_0'^2+\eta_2x_1'^2+x_3')y_{01}''x_3'-x_0'x_1'x_4 = 0.$$

We now see that $\gcd(x_0,x_1,x_2,x_3,x_4)=1$ implies $\gcd(y_{01}'',x_4)=1$ and thus $y_{01}''|x_0'x_1'$ and x_0', x_1' being coprime, we can write $y_{01}''=\eta_1\eta_3$, $x_0'=\eta_3x_0''$ and $x_1'=\eta_1x_1''$ for

$\eta_1, \eta_3, x_0'', x_1'' > 0$. Now we set $x_3' = \alpha_1 x_3''$, $x_4 = \alpha_1\alpha_4$ with $x_3'' > 0$ and $\gcd(x_3'', \alpha_4) = 1$ (we do not prescribe the sign of $\alpha_1 = \pm\gcd(x_3'', x_4)$). We finally get

$$(\eta_2\eta_3^2 x_0''^2 + \eta_2\eta_1^2 x_1''^2 + \alpha_1 x_3'')x_3'' - x_0'' x_1''\alpha_4 \quad = \quad 0.$$

We observe that since $\gcd(x_3'', \alpha_4) = 1$, we have $x_3'' | x_0'' x_1''$ and we can write $x_3'' = \eta_5\eta_7$, $x_0'' = \eta_5\eta_6$ and $x_1'' = \eta_4\eta_7$, for $\eta_4, \eta_5, \eta_6, \eta_7 > 0$. We have finally obtained

$$\begin{aligned} x_0 &= \eta_1\eta_2^2\eta_3^3\eta_5^2\eta_6^2, \\ x_1 &= \eta_1^3\eta_2^2\eta_3\eta_4^2\eta_7^2, \\ x_2 &= \eta_1^2\eta_2^2\eta_3^2\eta_4\eta_5\eta_6\eta_7, \\ x_3 &= \eta_1\eta_2\eta_3\eta_5\eta_7\alpha_1, \\ x_4 &= \alpha_1\alpha_4, \end{aligned}$$

and the equation is

$$\eta_2\eta_3^2\eta_5^2\eta_6^2 + \eta_1^2\eta_2\eta_4^2\eta_7^2 + \eta_5\eta_7\alpha_1 - \eta_4\eta_6\alpha_4 \quad = \quad 0.$$

Furthermore, it is easy to see that the coprimality conditions can be summed up by

$$\begin{aligned} &\gcd(\eta_3\eta_5\eta_6, \eta_1\eta_4\eta_7) = 1, \\ &\gcd(\eta_5\eta_7, \eta_2\alpha_4) = 1, \\ &\gcd(\eta_1\eta_2\eta_3, \alpha_1\alpha_4) = 1. \end{aligned}$$

Since η_6 and η_7 are coprime, we see that the equation is equivalent to the existence of $\alpha_2 \in \mathbb{Z}$ such that

$$\eta_1^2\eta_2\eta_4^2\eta_7 + \eta_5\alpha_1 - \eta_6\alpha_2 \quad = \quad 0, \tag{3.1}$$

$$\eta_2\eta_3^2\eta_5^2\eta_6 + \eta_7\alpha_2 - \eta_4\alpha_4 \quad = \quad 0. \tag{3.2}$$

In a similar way, since η_4 and η_5 are coprime, we can derive the existence of $\alpha_3 \in \mathbb{Z}$ such that

$$\begin{aligned} \eta_2\eta_3^2\eta_5\eta_6^2 + \eta_7\alpha_1 - \eta_4\alpha_3 &= 0, \\ \eta_1^2\eta_2\eta_4\eta_7^2 + \eta_5\alpha_3 - \eta_6\alpha_4 &= 0, \\ \eta_1^2\eta_2^2\eta_3^2\eta_4\eta_5\eta_6\eta_7 + \alpha_1\alpha_4 - \alpha_2\alpha_3 &= 0. \end{aligned}$$

As explained above, we will not use these three equations. We define $\mathcal{T}(B)$ as the set of $(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \alpha_1, \alpha_2, \alpha_4) \in \mathbb{Z}_{>0}^7 \times \mathbb{Z}^3$ satisfying the coprimality conditions above, the two equations (3.1) and (3.2) and finally the height conditions

$$\eta_1\eta_2^2\eta_3^3\eta_5^2\eta_6^2 \le B, \tag{3.3}$$

$$\eta_1^3\eta_2^2\eta_3\eta_4^2\eta_7^2 \le B, \tag{3.4}$$

$$\eta_1\eta_2\eta_3\eta_5\eta_7|\alpha_1| \le B, \tag{3.5}$$

$$|\alpha_1\alpha_4| \le B. \tag{3.6}$$

We have proved the following lemma.

Lemma 5. — *We have the equality*

$$N_{U,H}(B) \quad = \quad 2\#\mathcal{T}(B).$$

4. Calculation of Peyre's constant

We calculate the value of the constant $c_{V,H}$ predicted by Peyre. It is defined by

$$c_{V,H} = \alpha(\widetilde{V})\beta(\widetilde{V})\omega_H(\widetilde{V}),$$

where $\alpha(\widetilde{V}) \in \mathbb{Q}$ is the volume of a certain polytope in the cone of effective divisors of $\widetilde{V}$, $\beta(\widetilde{V}) = \# H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}_{\overline{\mathbb{Q}}}(\widetilde{V})) = 1$ since V is split over $\mathbb{Q}$ and finally

$$\omega_H(\widetilde{V}) = \omega_\infty \prod_p \left(1 - \frac{1}{p}\right)^6 \omega_p,$$

where ω_∞ and ω_p are respectively the archimedean and p-adic densities. The work of Derenthal [**Der07**] provides the value

$$\alpha(\widetilde{V}) = \frac{1}{4320}.$$

Furthermore, using [**Lou10**, Lemma 2.3], we get

$$\omega_p = 1 + \frac{6}{p} + \frac{1}{p^2}.$$

To calculate ω_∞, we set $f_1(x) = x_0x_1 - x_2^2$, $f_2(x) = (x_0 + x_1 + x_3)x_3 - x_2x_4$ and we parametrize the points of V by x_0, x_2 and x_3. We have

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_4} \end{pmatrix} = \begin{vmatrix} x_0 & 0 \\ x_3 & -x_2 \end{vmatrix} = -x_0x_2.$$

Moreover, $x_1 = x_2^2/x_0$ and $x_4 = (x_0^2 + x_2^2 + x_0x_3)x_3/(x_0x_2)$. Since $\mathbf{x} = -\mathbf{x}$ in $\mathbb{P}^4$, we have

$$\omega_\infty = 2 \int\int\int_{x_0,x_2>0, x_0, x_2^2/x_0, |x_3|, \left|x_0^2+x_2^2+x_0x_3\right||x_3|/|x_0x_2| \leq 1} \frac{\mathrm{d}x_0\mathrm{d}x_2\mathrm{d}x_3}{x_0x_2}.$$

Define the function

$$h : (u_2, t_7, t_6) \mapsto \max\{t_6, t_7, t_7|t_7 - t_6u_2|, |t_7 - t_6u_2||t_6 + t_7u_2|\}. \tag{4.1}$$

The change of variables given by $x_0 = t_6^2$, $x_2 = t_6t_7$ and $x_3 = -t_7(t_7 - t_6u_2)$ yields

$$\omega_\infty = 4 \int\int\int_{t_6,t_7>0, h(u_2,t_7,t_6)\leq 1} \mathrm{d}u_2\mathrm{d}t_7\mathrm{d}t_6.$$

5. Proof of the main theorem

5.1. First steps of the proof. — The idea of the proof is to see the equations (3.1) and (3.2) as congruences respectively modulo η_5 and η_4 and then to count the number of α_2 satisfying these two congruences. In order to do so, we replace the height conditions (3.5) and (3.6) by

$$\eta_1\eta_2\eta_3\eta_7 \left|\eta_1^2\eta_2\eta_4^2\eta_7 - \eta_6\alpha_2\right| \leq B,$$
$$\eta_4^{-1}\eta_5^{-1} \left|\eta_1^2\eta_2\eta_4^2\eta_7 - \eta_6\alpha_2\right| \left|\eta_2\eta_3^2\eta_5^2\eta_6 + \eta_7\alpha_2\right| \leq B,$$

and we carry on denoting them the same way. We note that the equation (3.1) proves that we necessarily have $\gcd(\eta_1\eta_2, \alpha_2\eta_6) = 1$ since $\gcd(\eta_1\eta_2, \eta_5\alpha_1) = 1$. Exactly the same way we get $\gcd(\alpha_2, \eta_3\eta_5) = 1$ thanks to the equation (3.2) and $\gcd(\eta_3\eta_5, \eta_4\alpha_4) = 1$. We also have $\gcd(\eta_2, \eta_4) = 1$, indeed, if $p|\eta_2, \eta_4$ then (3.2) gives $p|\alpha_2$ since $\gcd(\eta_2, \eta_7) = 1$ and then (3.1) gives $p|\alpha_1$ since $\gcd(\eta_2, \eta_5) = 1$, which is impossible because $\gcd(\eta_2, \alpha_1) = 1$. This new coprimality condition also yields

$\gcd(\alpha_2, \eta_4) = 1$ since we have $\gcd(\eta_2\eta_3\eta_5\eta_6, \eta_4) = 1$. In a similar way, we finally obtain $\gcd(\alpha_1, \eta_4\eta_6) = 1$ and also $\gcd(\eta_4, \eta_7) = 1$ and $\gcd(\eta_5, \eta_6) = 1$. We can therefore rewrite the coprimality conditions as

$$\gcd(\alpha_1, \eta_1\eta_2\eta_3\eta_4\eta_6) = 1, \tag{5.1}$$

$$\gcd(\alpha_4, \eta_1\eta_2\eta_3\eta_5\eta_7) = 1, \tag{5.2}$$

$$\gcd(\alpha_2, \eta_1\eta_2\eta_3\eta_4\eta_5) = 1, \tag{5.3}$$

$$\gcd(\eta_7, \eta_2\eta_3\eta_4\eta_5\eta_6) = 1, \tag{5.4}$$

$$\gcd(\eta_6, \eta_1\eta_2\eta_4\eta_5) = 1, \tag{5.5}$$

$$\gcd(\eta_1\eta_4, \eta_3\eta_5) = 1, \tag{5.6}$$

$$\gcd(\eta_2, \eta_4\eta_5) = 1. \tag{5.7}$$

From now on, we set $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) \in \mathbb{Z}_{>0}^5$ and $\boldsymbol{\eta}' = (\boldsymbol{\eta}, \eta_6, \eta_7) \in \mathbb{Z}_{>0}^7$. Consider that $\boldsymbol{\eta}' \in \mathbb{Z}_{>0}^7$ is fixed and is subject to the height condition (3.3), (3.4) and the coprimality conditions (5.4), (5.5), (5.6) and (5.7). Let $N(\boldsymbol{\eta}', B)$ be the number of $(\alpha_1, \alpha_2, \alpha_4) \in \mathbb{Z}$ satisfying the equations (3.1), (3.2), the height conditions (3.5) and (3.6) and finally the coprimality conditions (5.1), (5.2) and (5.3). For $(r_1, r_2, r_3, r_4, r_5) \in \mathbb{Q}^5$, we define

$$\boldsymbol{\eta}^{(r_1,r_2,r_3,r_4,r_5)} = \eta_1^{r_1}\eta_2^{r_2}\eta_3^{r_3}\eta_4^{r_4}\eta_5^{r_5},$$

and we adopt the following notations in order to help in the understanding of the height conditions,

$$\begin{aligned} A_2 &= \boldsymbol{\eta}^{(1,1,1,1,1)}, \\ Y_6 &= \frac{B^{1/2}}{\boldsymbol{\eta}^{(1/2,1,3/2,0,1)}}, \\ Y_7 &= \frac{B^{1/2}}{\boldsymbol{\eta}^{(3/2,1,1/2,1,0)}}, \end{aligned}$$

and recalling the definition (4.1) of the function h, we can sum up all the height conditions as

$$h\left(\frac{\alpha_2}{A_2}, \frac{\eta_7}{Y_7}, \frac{\eta_6}{Y_6}\right) \leq 1.$$

We also introduce the real-valued functions

$$\begin{aligned} g_1 &: (t_7, t_6) \mapsto \int_{h(u_2,t_7,t_6)\leq 1} \mathrm{d}u_2, \\ g_2 &: (t_6; \boldsymbol{\eta}, B) \mapsto \int_{t_7Y_7\geq 1} g_1(t_7, t_6)\mathrm{d}t_7, \\ g_3 &: (\boldsymbol{\eta}, B) \mapsto \int_{t_6Y_6\geq 1} g_2(t_6; \boldsymbol{\eta}, B)\mathrm{d}t_6. \end{aligned}$$

We obviously have

$$g_3(\boldsymbol{\eta}, B) = \int\int\int_{t_6Y_6\geq 1, t_7Y_7\geq 1, h(u_2,t_7,t_6)\leq 1} \mathrm{d}u_2\mathrm{d}t_7\mathrm{d}t_6. \tag{5.8}$$

Lemma 6. — *We have the bounds*

$$\begin{aligned} g_1(t_7, t_6) &\ll t_6^{-1/2}t_7^{-1/2}, \\ g_2(t_6; \boldsymbol{\eta}, B) &\ll t_6^{-1/2}. \end{aligned}$$

Proof. — The condition $|t_7 - t_6 u_2||t_6 + t_7 u_2| \leq 1$ implies that u_2 runs over a range whose length is $\ll t_6^{-1/2} t_7^{-1/2}$ which gives the first bound. The second bound is an immediate consequence of the first since $t_7 \leq 1$. □

We have the following result.

Lemma 7. — *The following estimate holds*

$$N(\boldsymbol{\eta}', B) = \frac{A_2}{\eta_4 \eta_5} g_1 \left(\frac{\eta_7}{Y_7}, \frac{\eta_6}{Y_6} \right) \theta(\boldsymbol{\eta}') + R(\boldsymbol{\eta}', B),$$

where θ is a certain arithmetic function given in (5.9) *and*

$$\sum_{\boldsymbol{\eta}'} R(\boldsymbol{\eta}', B) \ll B \log(B)^2.$$

Let us remove the coprimality conditions (5.1) and (5.2) employing two Möbius inversions, we get

$$N(\boldsymbol{\eta}', B) = \sum_{k_1 | \eta_1 \eta_2 \eta_3 \eta_4 \eta_6} \mu(k_1) \sum_{k_4 | \eta_1 \eta_2 \eta_3 \eta_5 \eta_7} \mu(k_4) S_{k_1, k_4},$$

where, with the notations $\alpha_1 = k_1 \alpha_1'$ and $\alpha_4 = k_4 \alpha_4$,

$$\begin{aligned} S_{k_1,k_4} &= \# \left\{ (\alpha_1', \alpha_4', \alpha_2) \in \mathbb{Z}^3, \begin{array}{r} \eta_1^2 \eta_2 \eta_4^2 \eta_7 + \eta_5 k_1 \alpha_1' - \eta_6 \alpha_2 = 0 \\ \eta_2 \eta_3^2 \eta_5^2 \eta_6 + \eta_7 \alpha_2 - \eta_4 k_4 \alpha_4' = 0 \\ (3.5), (3.6), (5.3) \end{array} \right\} \\ &= \# \left\{ \alpha_2 \in \mathbb{Z}, \begin{array}{r} \eta_6 \alpha_2 \equiv \eta_1^2 \eta_2 \eta_4^2 \eta_7 \pmod{k_1 \eta_5} \\ \eta_7 \alpha_2 \equiv -\eta_2 \eta_3^2 \eta_5^2 \eta_6 \pmod{k_4 \eta_4} \\ (3.5), (3.6), (5.3) \end{array} \right\}. \end{aligned}$$

We note that we necessarily have $\gcd(k_1, \eta_6) = 1$ since $\gcd(\eta_6, \eta_1 \eta_2 \eta_4 \eta_7) = 1$ and $\gcd(k_1, \eta_1 \eta_2 \eta_4) = 1$ since $\gcd(\eta_1 \eta_2 \eta_4, \eta_6 \alpha_2) = 1$. In a similar way, we also have $\gcd(k_4, \eta_2 \eta_3 \eta_5 \eta_7) = 1$. In particular, we see that η_6 and η_7 are respectively invertible modulo $k_1 \eta_5$ and $k_4 \eta_4$. We therefore get

$$N(\boldsymbol{\eta}', B) = \sum_{\substack{k_1 | \eta_3 \\ \gcd(k_1, \eta_1 \eta_2 \eta_4 \eta_6) = 1}} \mu(k_1) \sum_{\substack{k_4 | \eta_1 \\ \gcd(k_4, \eta_2 \eta_3 \eta_5 \eta_7) = 1}} \mu(k_4) S_{k_1, k_4},$$

and

$$S_{k_1,k_4} = \# \left\{ \alpha_2 \in \mathbb{Z}, \begin{array}{l} \alpha_2 \equiv \eta_6^{-1} \eta_1^2 \eta_2 \eta_4^2 \eta_7 \pmod{k_1 \eta_5} \\ \alpha_2 \equiv -\eta_7^{-1} \eta_2 \eta_3^2 \eta_5^2 \eta_6 \pmod{k_4 \eta_4} \\ (3.5), (3.6), (5.3) \end{array} \right\}.$$

Furthermore, $k_1 \eta_5$ and $k_4 \eta_4$ are coprime since $\eta_3 \eta_5$ and $\eta_1 \eta_4$ are coprime thus the Chinese remainder theorem gives

$$S_{k_1,k_4} = \# \left\{ \alpha_2 \in \mathbb{Z}, \begin{array}{l} \alpha_2 \equiv a \pmod{k_1 k_4 \eta_4 \eta_5} \\ (3.5), (3.6), (5.3) \end{array} \right\},$$

for a certain integer a coprime to $k_1 k_4 \eta_4 \eta_5$ since $\gcd(k_1 k_4 \eta_4 \eta_5, \alpha_2) = 1$. A Möbius inversion yields

$$\begin{aligned} S_{k_1,k_4} &= \sum_{k_2 | \eta_1 \eta_2 \eta_3 \eta_4 \eta_5} \mu(k_2) \# \left\{ \alpha_2' \in \mathbb{Z}, \begin{array}{l} k_2 \alpha_2' \equiv a \pmod{k_1 k_4 \eta_4 \eta_5} \\ (3.5), (3.6) \end{array} \right\} \\ &= \sum_{\substack{k_2 | \eta_1 \eta_2 \eta_3 \\ \gcd(k_2, k_1 k_4 \eta_4 \eta_5) = 1}} \mu(k_2) \# \left\{ \alpha_2' \in \mathbb{Z}, \begin{array}{l} \alpha_2' \equiv k_2^{-1} a \pmod{k_1 k_4 \eta_4 \eta_5} \\ (3.5), (3.6) \end{array} \right\}, \end{aligned}$$

since $\gcd(k_1k_4\eta_4\eta_5, a) = 1$. Using the elementary estimate

$$\#\{n \in \mathbb{Z} \cap [t_1,t_2], n \equiv a \pmod q\} = \frac{t_2-t_1}{q} + O(1),$$

and the change of variable $u_2 \mapsto u_2A_2/k_2$, we get

$$\#\left\{\alpha_2' \in \mathbb{Z}, \begin{array}{l}\alpha_2' \equiv k_2^{-1}a \pmod{k_1k_4\eta_4\eta_5}\\ (3.5),(3.6)\end{array}\right\} = \frac{A_2}{k_2k_1k_4\eta_4\eta_5} g_1\left(\frac{\eta_7}{Y_7},\frac{\eta_6}{Y_6}\right) + O(1).$$

We see that the main term is equal to

$$\frac{A_2}{\eta_4\eta_5} g_1\left(\frac{\eta_7}{Y_7},\frac{\eta_6}{Y_6}\right)\theta(\boldsymbol{\eta}'),$$

where

$$\begin{aligned}\theta(\boldsymbol{\eta}') &= \sum_{\substack{k_1|\eta_3\\ \gcd(k_1,\eta_1\eta_2\eta_4\eta_6)=1}} \frac{\mu(k_1)}{k_1} \sum_{\substack{k_4|\eta_1\\ \gcd(k_4,\eta_2\eta_3\eta_5\eta_7)=1}} \frac{\mu(k_4)}{k_4} \sum_{\substack{k_2|\eta_1\eta_2\eta_3\\ \gcd(k_2,k_1k_4\eta_4\eta_5)=1}} \frac{\mu(k_2)}{k_2}\\ &= \varphi^*(\eta_1\eta_2\eta_3\eta_4\eta_5) \sum_{\substack{k_1|\eta_3\\ \gcd(k_1,\eta_2\eta_6)=1}} \frac{\mu(k_1)}{k_1\varphi^*(k_1\eta_5)} \sum_{\substack{k_4|\eta_1\\ \gcd(k_4,\eta_2\eta_7)=1}} \frac{\mu(k_4)}{k_4\varphi^*(k_4\eta_4)}.\end{aligned}$$

We have removed $\eta_1\eta_4$ from the condition over k_1 and $\eta_3\eta_5$ from the condition over k_4 respectively because $\gcd(\eta_3, \eta_1\eta_4) = 1$ and $\gcd(\eta_1, \eta_3\eta_5) = 1$. A straightforward calculation yields, for $a, b, c \geq 1$,

$$\sum_{\substack{k|a\\ \gcd(k,c)=1}} \frac{\mu(k_1)}{k_1\varphi^*(k_1b)} = \frac{\varphi^*(\gcd(a,b))}{\varphi^*(b)\varphi^*(\gcd(a,b,c))} \prod_{p|a, p\nmid bc}\left(1-\frac{1}{p-1}\right).$$

Therefore, we have obtained

$$\theta(\boldsymbol{\eta}') = \theta_1(\boldsymbol{\eta},\eta_6) \prod_{p|\eta_1, p\nmid\eta_2\eta_4\eta_7}\left(1-\frac{1}{p-1}\right), \tag{5.9}$$

where

$$\theta_1(\boldsymbol{\eta},\eta_6) = \varphi^*(\eta_1\eta_2\eta_3\eta_4\eta_5)\frac{\varphi^*(\gcd(\eta_1,\eta_4))}{\varphi^*(\eta_4)}\frac{\varphi^*(\gcd(\eta_3,\eta_5))}{\varphi^*(\eta_5)} \prod_{p|\eta_3, p\nmid\eta_2\eta_5\eta_6}\left(1-\frac{1}{p-1}\right).$$

We see that the overall contribution of the error term is

$$\begin{aligned}\sum_{\boldsymbol{\eta},\eta_6,\eta_7} 2^{\omega(\eta_3)}2^{\omega(\eta_1)}2^{\omega(\eta_1\eta_2\eta_3)} &\ll \sum_{\boldsymbol{\eta}} 2^{\omega(\eta_3)}2^{\omega(\eta_1)}2^{\omega(\eta_1\eta_2\eta_3)}Y_6Y_7\\ &= \sum_{\boldsymbol{\eta}} 2^{\omega(\eta_3)}2^{\omega(\eta_1)}2^{\omega(\eta_1\eta_2\eta_3)}\frac{B}{\boldsymbol{\eta}^{(2,2,2,1,1)}}\\ &\ll B\log(B)^2,\end{aligned}$$

which completes the proof of lemma 7.

5.2. Summation over η_7. — Our next task is to sum over η_7, that is why we have isolated η_7 in $\theta(\boldsymbol{\eta}')$. In order to do so, we have to distinguish two cases. Let us define

$$\mathcal{N} = \{(\eta_1,\eta_2,\eta_4) \in \mathbb{Z}^3_{>0}, 2 \nmid \eta_1 \text{ or } 2|\eta_2\eta_4\}.$$

It is plain to see that if $(\eta_1,\eta_2,\eta_4) \in \mathcal{N}$ or $2|\eta_7$ then

$$\prod_{p|\eta_1, p\nmid\eta_2\eta_4\eta_7}\left(1-\frac{1}{p-1}\right) = \prod_{\substack{p|\eta_1, p\nmid\eta_2\eta_4\eta_7\\ p\neq 2}}\left(1-\frac{1}{p-1}\right),$$

and this product is equal to 0 otherwise. Furthermore, since $\eta_2\eta_4$ and η_7 are coprime, we see that

$$\prod_{\substack{p\mid\eta_1, p\nmid\eta_2\eta_4\eta_7\\ p\neq 2}} \left(1-\frac{1}{p-1}\right) = \frac{\varphi^{\circ}(\eta_1)}{\varphi^{\circ}(\gcd(\eta_1,\eta_2\eta_4))\varphi^{\circ}(\gcd(\eta_1,\eta_7))}.$$

We call $N'(\boldsymbol{\eta},\eta_6,B)$ the sum of the main term of $N(\boldsymbol{\eta}',B)$ over η_7, η_7 being subject to the conditions (3.4) and (5.4). We also use $N_1'(\boldsymbol{\eta},\eta_6,B)$ and $N_2'(\boldsymbol{\eta},\eta_6,B)$ to denote the sums over η_7 respectively for $(\eta_1,\eta_2,\eta_4)\in\mathcal{N}$ and $(\eta_1,\eta_2,\eta_4)\notin\mathcal{N}$ (in the latter case, the main term vanishes if $2\nmid\eta_7$). We now proceed to prove the following lemma.

Lemma 8. — *We have the estimate*

$$N'(\boldsymbol{\eta},\eta_6,B) = \frac{A_2Y_7}{\eta_4\eta_5} g_2\left(\frac{\eta_6}{Y_6};\boldsymbol{\eta},B\right)\theta_1'(\boldsymbol{\eta})\theta_2'(\boldsymbol{\eta},\eta_6) + R'(\boldsymbol{\eta},\eta_6,B),$$

where $\theta_1'(\boldsymbol{\eta})$ *and* $\theta_2'(\boldsymbol{\eta},\eta_6)$ *are arithmetic functions defined in* (5.10) *and* (5.11) *and*

$$\sum_{\boldsymbol{\eta},\eta_6} R'(\boldsymbol{\eta},\eta_6,B) \ll B\log(B)^4.$$

First, we estimate the contribution of $N_1'(\boldsymbol{\eta},\eta_6,B)$, we use lemma 3 to deduce that

$$\begin{aligned} N_1'(\boldsymbol{\eta},\eta_6,B) &= \frac{A_2Y_7}{\eta_4\eta_5} g_2\left(\frac{\eta_6}{Y_6};\boldsymbol{\eta},B\right)\theta_1(\boldsymbol{\eta},\eta_6)\frac{\varphi^{\circ}(\eta_1)}{\varphi^{\circ}(\gcd(\eta_1,\eta_2\eta_4))}\Psi(\eta_1,\eta_2\eta_3\eta_4\eta_5\eta_6) \\ &\quad + O\left(\frac{A_2}{\eta_4\eta_5}Y_7^{\delta}\sigma_{-\delta}(\eta_1\eta_2\eta_3\eta_4\eta_5\eta_6)\sup_{t_7Y_7\geq 1} g_1\left(t_7,\frac{\eta_6}{Y_6}\right)\right). \end{aligned}$$

Let us estimate the overall contribution of this error term. Use the bound of lemma 6 for g_1 and choose $\delta=1/4$. The average order of $\sigma_{-\delta}$ is $O(1)$ so we see that this contribution is

$$\begin{aligned} \sum_{\boldsymbol{\eta},\eta_6}\sigma_{-1/4}(\eta_1\eta_2\eta_3\eta_4\eta_5\eta_6)\frac{A_2Y_6^{1/2}Y_7^{3/4}}{\eta_4\eta_5\eta_6^{1/2}} &\ll \sum_{\boldsymbol{\eta}}\sigma_{-1/4}(\eta_1\eta_2\eta_3\eta_4\eta_5)\frac{A_2Y_6Y_7^{3/4}}{\eta_4\eta_5} \\ &\ll \sum_{\eta_1,\eta_2,\eta_3,\eta_5}\sigma_{-1/4}(\eta_1\eta_2\eta_3\eta_5)\frac{B}{\boldsymbol{\eta}^{(1,1,1,0,1)}} \\ &\ll B\log(B)^4, \end{aligned}$$

which is satisfactory. Concerning the main term, we have

$$\Psi(\eta_1,\eta_2\eta_3\eta_4\eta_5\eta_6) = \varphi^*(\eta_2\eta_3\eta_4\eta_5\eta_6)\frac{\varphi^{\flat}(\eta_1)}{\varphi^{\flat}(\gcd(\eta_1,\eta_2\eta_4))},$$

and since $(\eta_1,\eta_2,\eta_4)\in\mathcal{N}$, we also have

$$\frac{\varphi^{\circ}(\eta_1)}{\varphi^{\circ}(\gcd(\eta_1,\eta_2\eta_4))}\frac{\varphi^{\flat}(\eta_1)}{\varphi^{\flat}(\gcd(\eta_1,\eta_2\eta_4))} = \frac{\varphi^*(\eta_1)}{\varphi^*(\gcd(\eta_1,\eta_2\eta_4))}.$$

These equalities and a short calculation prove that

$$\theta_1(\boldsymbol{\eta},\eta_6)\frac{\varphi^{\circ}(\eta_1)}{\varphi^{\circ}(\gcd(\eta_1,\eta_2\eta_4))}\Psi(\eta_1,\eta_2\eta_3\eta_4\eta_5\eta_6)$$

can be rewritten $\theta_1'(\boldsymbol{\eta})\theta_2'(\boldsymbol{\eta},\eta_6)$ for

$$\theta_1'(\boldsymbol{\eta}) = \varphi^*(\eta_1\eta_2\eta_3\eta_4\eta_5)\varphi^*(\eta_2\eta_3\eta_4\eta_5)\frac{\varphi^*(\eta_1\eta_2)}{\varphi^*(\eta_2\eta_4)}\frac{\varphi^*(\gcd(\eta_3,\eta_5))}{\varphi^*(\eta_5)}, \tag{5.10}$$

$$\theta_2'(\boldsymbol{\eta},\eta_6) = \frac{\varphi^*(\eta_6)}{\varphi^*(\gcd(\eta_6,\eta_3))}\prod_{p\mid\eta_3, p\nmid\eta_2\eta_5\eta_6}\left(1-\frac{1}{p-1}\right). \tag{5.11}$$

We now turn to the estimation of $N_2'(\boldsymbol{\eta}, \eta_6, B)$. We only need to sum on the even η_7 and so, given the coprimality condition (5.4), $\eta_2\eta_3\eta_4\eta_5\eta_6$ is odd and thus we can apply lemma 4. The error term is the same as the previous one and, in the main term, there are exactly two differences with the case of $N_1'(\boldsymbol{\eta}, \eta_6, B)$. The first is the factor 1/2 and the second is the fact that here, since $(\eta_1, \eta_2, \eta_4) \notin \mathcal{N}$,

$$\frac{\varphi^\circ(\eta_1)}{\varphi^\circ(\gcd(\eta_1, \eta_2\eta_4))} \frac{\varphi^\flat(\eta_1)}{\varphi^\flat(\gcd(\eta_1, \eta_2\eta_4))} = 2\frac{\varphi^*(\eta_1)}{\varphi^*(\gcd(\eta_1, \eta_2\eta_4))},$$

and thus we find exactly the same main term, which completes the proof of lemma 8.

5.3. Summation over η_6. — We set

$$\mathcal{M} = \{(\eta_3, \eta_2, \eta_5) \in \mathbb{Z}^3_{>0}, 2 \nmid \eta_3 \text{ or } 2|\eta_2\eta_5\}.$$

As for the summation over η_7, it is clear that if $(\eta_3, \eta_2, \eta_5) \in \mathcal{M}$ or $2|\eta_6$ then

$$\prod_{p|\eta_3, p\nmid\eta_2\eta_5\eta_6} \left(1 - \frac{1}{p-1}\right) = \prod_{\substack{p|\eta_3, p\nmid\eta_2\eta_5\eta_6 \\ p\neq 2}} \left(1 - \frac{1}{p-1}\right),$$

and this product is equal to 0 otherwise. Furthermore, since $\eta_2\eta_5$ and η_6 are coprime, we have

$$\prod_{\substack{p|\eta_3, p\nmid\eta_2\eta_5\eta_6 \\ p\neq 2}} \left(1 - \frac{1}{p-1}\right) = \frac{\varphi^\circ(\eta_3)}{\varphi^\circ(\gcd(\eta_3, \eta_2\eta_5))\varphi^\circ(\gcd(\eta_3, \eta_6))}.$$

We need to treat two cases separately depending on whether $(\eta_3, \eta_2, \eta_5) \in \mathcal{M}$ or not (if not, note that the main term vanishes if $2 \nmid \eta_6$). Let $\mathbf{N}(\boldsymbol{\eta}, B)$ be the sum of the main term of $N'(\boldsymbol{\eta}, \eta_6, B)$ over η_6, η_6 satisfying the conditions (3.3) and (5.5) and let also $\mathbf{N}_1(\boldsymbol{\eta}, B)$ and $\mathbf{N}_2(\boldsymbol{\eta}, B)$ be the sums over η_6 respectively for $(\eta_3, \eta_2, \eta_5) \in \mathcal{M}$ and $(\eta_3, \eta_2, \eta_5) \notin \mathcal{M}$.

Lemma 9. — *We have the estimate*

$$\mathbf{N}(\boldsymbol{\eta}, B) = \zeta(2)^{-1}\frac{B}{\boldsymbol{\eta}^{(1,1,1,1,1)}} g_3(\boldsymbol{\eta}, B)\Theta(\boldsymbol{\eta}) + \mathbf{R}(\boldsymbol{\eta}, B),$$

where

$$\Theta(\boldsymbol{\eta}) = \frac{\varphi^*(\eta_1\eta_2\eta_3\eta_4\eta_5)}{\varphi^\dagger(\eta_1\eta_2\eta_3\eta_4\eta_5)}\varphi^*(\eta_2\eta_3\eta_4\eta_5)\varphi^*(\eta_1\eta_2\eta_4\eta_5)\frac{\varphi^*(\eta_1\eta_2)}{\varphi^*(\eta_2\eta_4)}\frac{\varphi^*(\eta_2\eta_3)}{\varphi^*(\eta_2\eta_5)},$$

and

$$\sum_{\boldsymbol{\eta}} \mathbf{R}(\boldsymbol{\eta}, B) \ll B\log(B)^4.$$

We first treat the contribution of $\mathbf{N}_1(\boldsymbol{\eta}, B)$, we use lemma 3 to deduce that

$$\begin{aligned}\mathbf{N}_1(\boldsymbol{\eta}, B) = {} & \frac{A_2Y_7Y_6}{\eta_4\eta_5} g_3(\boldsymbol{\eta}, B)\,\theta_1'(\boldsymbol{\eta})\frac{\varphi^\circ(\eta_3)}{\varphi^\circ(\gcd(\eta_3, \eta_2\eta_5))}\Psi'(\eta_3, \eta_1\eta_2\eta_4\eta_5) \\ & + O\left(\frac{A_2Y_7}{\eta_4\eta_5}Y_6^\delta\sigma_{-\delta}(\eta_1\eta_2\eta_4\eta_5)\sup_{t_6Y_6\geq 1} g_2(t_6; \boldsymbol{\eta}, B)\right).\end{aligned}$$

We use the bound of lemma 6 for g_2 and choose $\delta = 1/4$ to estimate the overall contribution of the error term. Since the average order of $\sigma_{-\delta}$ is $O(1)$, we obtain that this contribution is

$$\begin{aligned}\sum_{\boldsymbol{\eta}} \sigma_{-1/4}(\eta_1\eta_2\eta_4\eta_5)\frac{A_2Y_7Y_6^{3/4}}{\eta_4\eta_5} & \ll \sum_{\eta_1,\eta_2,\eta_3,\eta_4} \sigma_{-1/4}(\eta_1\eta_2\eta_4)\frac{B}{\boldsymbol{\eta}^{(1,1,1,1,0)}} \\ & \ll B\log(B)^4,\end{aligned}$$

which is satisfactory. Let us turn to the main term. First, note that

$$\frac{A_2 Y_7 Y_6}{\eta_4\eta_5} = \frac{B}{\boldsymbol{\eta}^{(1,1,1,1,1)}}.$$

In addition, we have

$$\Psi'(\eta_3, \eta_1\eta_2\eta_4\eta_5) = \varphi^*(\eta_1\eta_2\eta_4\eta_5)\frac{\zeta(2)^{-1}}{\varphi^\dagger(\eta_1\eta_2\eta_3\eta_4\eta_5)}\frac{\varphi^\flat(\eta_3)}{\varphi^\flat(\gcd(\eta_3, \eta_2\eta_5))},$$

and since $(\eta_3, \eta_2, \eta_5) \in \mathcal{M}$, we also have

$$\frac{\varphi^\circ(\eta_3)}{\varphi^\circ(\gcd(\eta_3, \eta_2\eta_5))}\frac{\varphi^\flat(\eta_3)}{\varphi^\flat(\gcd(\eta_3, \eta_2\eta_5))} = \frac{\varphi^*(\eta_3)}{\varphi^*(\gcd(\eta_3, \eta_2\eta_5))}.$$

An easy calculation now yields

$$\theta_1'(\boldsymbol{\eta})\frac{\varphi^\circ(\eta_3)}{\varphi^\circ(\gcd(\eta_3, \eta_2\eta_5))}\Psi'(\eta_3, \eta_1\eta_2\eta_4\eta_5) = \zeta(2)^{-1}\Theta(\boldsymbol{\eta}).$$

We now deal with the estimation of $\mathbf{N}_2(\boldsymbol{\eta}, B)$. We only need to sum on the even η_6 and so, given the coprimality condition (5.5), $\eta_1\eta_2\eta_4\eta_5$ is odd and moreover since $(\eta_3, \eta_2, \eta_5) \notin \mathcal{M}$, we have $2|\eta_3$ and thus we can apply lemma 4. The error term is the same as the previous one and, in the main term, there are exactly two differences with the case of $\mathbf{N}_1(\boldsymbol{\eta}, B)$. The first is the factor $1/2$ and the second is that here, since $(\eta_3, \eta_2, \eta_5) \notin \mathcal{N}$,

$$\frac{\varphi^\circ(\eta_3)}{\varphi^\circ(\gcd(\eta_3, \eta_2\eta_5))}\frac{\varphi^\flat(\eta_3)}{\varphi^\flat(\gcd(\eta_3, \eta_2\eta_5))} = 2\frac{\varphi^*(\eta_3)}{\varphi^*(\gcd(\eta_3, \eta_2\eta_5))},$$

and we finally obtain the same main term, which concludes the proof of lemma 9.

5.4. Conclusion. — The aim of the following lemma is to remove the conditions $t_6Y_6 \geq 1$ and $t_7Y_7 \geq 1$ from the expression (5.8) of g_3 in the main term (i. e. replace them respectively by $t_6 > 0$ and $t_7 > 0$).

Lemma 10. — *For $Z_6, Z_7 > 0$, we have*

(5.12) $$\operatorname{meas}\{t_6, t_7 > 0, h(u_2, t_7, t_6) \leq 1, t_6Z_6 < 1\} \ll Z_6^{-1/2},$$

(5.13) $$\operatorname{meas}\{t_6, t_7 > 0, h(u_2, t_7, t_6) \leq 1, t_7Z_7 < 1\} \ll Z_7^{-1/2}.$$

Proof. — These two bounds follow from the bound of lemma 6 for g_1 and the fact that $h(u_2, t_7, t_6) \leq 1$ implies $t_6, t_7 \leq 1$. □

Now, using the bound (5.12), we see that removing the condition $t_6Y_6 \geq 1$ from the expression of g_3 in the main term yields an error term whose overall contribution is

$$\sum_{\boldsymbol{\eta}} \frac{A_2 Y_7 Y_6^{1/2}}{\eta_4\eta_5} \ll \sum_{\eta_1,\eta_2,\eta_3,\eta_4} \frac{B}{\boldsymbol{\eta}^{(1,1,1,1,0)}} \ll B\log(B)^4,$$

which is satisfactory. The bound (5.13) shows that we have exactly the same conclusion for the condition $t_7Y_7 \geq 1$. Finally, we see that we can replace $g_3(\boldsymbol{\eta}, B)$ in our main term by

$$\int\int\int_{t_6,t_7>0, h(u_2,t_7,t_6)\leq 1} \mathrm{d}u_2\mathrm{d}t_7\mathrm{d}t_6 = \frac{\omega_\infty}{4}.$$

Using lemma 5, we obtain the following lemma.

Lemma 11. — *We have the estimate*

$$N_{U,H}(B) = \zeta(2)^{-1}\frac{\omega_\infty}{2}B\sum_{\boldsymbol{\eta}}\frac{\Theta(\boldsymbol{\eta})}{\boldsymbol{\eta}^{(1,1,1,1,1)}} + O\left(B\log(B)^4\right),$$

where the sum is taken over the $\boldsymbol{\eta}$ *subject to* $Y_6 \geq 1$ *and* $Y_7 \geq 1$.

We redefine Θ as being equal to zero if the remaining coprimality conditions (5.6) and (5.7) are not satisfied and we carry on denoting it by Θ. Set $\mathbf{k} = (k_1, k_2, k_3, k_4, k_5)$ and define, for $s \in \mathbb{C}$ such that $\Re(s) > 1$,

$$\begin{aligned} F(s) &= \sum_{\boldsymbol{\eta}\in\mathbb{Z}_{>0}^5}\frac{|(\Theta * \boldsymbol{\mu})(\boldsymbol{\eta})|}{\boldsymbol{\eta}^{(s,s,s,s,s)}} \\ &= \prod_p\left(\sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^5}\frac{\left|(\Theta * \boldsymbol{\mu})\left(p^{k_1},p^{k_2},p^{k_3},p^{k_4},p^{k_5}\right)\right|}{p^{k_1 s}p^{k_2 s}p^{k_3 s}p^{k_4 s}p^{k_5 s}}\right). \end{aligned}$$

It is easy to see that if $\mathbf{k} \notin \{0,1\}^5$ then $(\Theta*\boldsymbol{\mu})\left(p^{k_1},p^{k_2},p^{k_3},p^{k_4},p^{k_5}\right) = 0$ and moreover if exactly one of the k_i is equal to 1, then $(\Theta * \boldsymbol{\mu})\left(p^{k_1},p^{k_2},p^{k_3},p^{k_4},p^{k_5}\right) \ll 1/p$, so the local factors F_p of F satisfy

$$F_p(s) = 1 + O\left(\frac{1}{p^{\min(\Re(s)+1,2\Re(s))}}\right).$$

This proves that F actually converges in the half-plane $\Re(s) > 1/2$, which implies that Θ satifies the assumption of lemma [**LB10**, Lemma 7]. Applying this lemma, we get

$$\sum_{\substack{\boldsymbol{\eta} \\ Y_6,Y_7\geq 1}}\frac{\Theta(\boldsymbol{\eta})}{\boldsymbol{\eta}^{(1,1,1,1,1)}} = \alpha\left(\sum_{\boldsymbol{\eta}\in\mathbb{Z}_{>0}^5}\frac{(\Theta * \boldsymbol{\mu})(\boldsymbol{\eta})}{\boldsymbol{\eta}^{(1,1,1,1,1)}}\right)\log(B)^5 + O\left(\log(B)^4\right), \tag{5.14}$$

where α is the volume of the polytope defined by $t_1, t_2, t_3, t_4, t_5 \geq 0$ and

$$\begin{aligned} t_1 + 2t_2 + 3t_3 + 2t_5 &\leq 1, \\ 3t_1 + 2t_2 + t_3 + 2t_4 &\leq 1. \end{aligned}$$

A computation using Franz's additional *Maple* package [**Fra09**] provides

$$\begin{aligned} \alpha &= \frac{1}{2160} \\ &= 2\alpha(\widetilde{V}) \end{aligned} \tag{5.15}$$

and moreover

$$\begin{aligned} \sum_{\boldsymbol{\eta}\in\mathbb{Z}_{>0}^5}\frac{(\Theta * \boldsymbol{\mu})(\boldsymbol{\eta})}{\boldsymbol{\eta}^{(1,1,1,1,1)}} &= \prod_p\left(\sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^5}\frac{(\Theta * \boldsymbol{\mu})\left(p^{k_1},p^{k_2},p^{k_3},p^{k_4},p^{k_5}\right)}{p^{k_1}p^{k_2}p^{k_3}p^{k_4}p^{k_5}}\right) \\ &= \prod_p\left(1-\frac{1}{p}\right)^5\left(\sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^5}\frac{\Theta\left(p^{k_1},p^{k_2},p^{k_3},p^{k_4},p^{k_5}\right)}{p^{k_1}p^{k_2}p^{k_3}p^{k_4}p^{k_5}}\right). \end{aligned}$$

The remaining coprimality conditions greatly simplify the calculation and we obtain

$$\sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^5}\frac{\Theta\left(p^{k_1},p^{k_2},p^{k_3},p^{k_4},p^{k_5}\right)}{p^{k_1}p^{k_2}p^{k_3}p^{k_4}p^{k_5}} = \left(1-\frac{1}{p^2}\right)^{-1}\left(1-\frac{1}{p}\right)\left(1+\frac{6}{p}+\frac{1}{p^2}\right),$$

which gives

$$\sum_{\boldsymbol{\eta}\in\mathbb{Z}_{>0}^5} \frac{(\Theta * \boldsymbol{\mu})(\boldsymbol{\eta})}{\boldsymbol{\eta}^{(1,1,1,1,1)}} = \zeta(2)\prod_p \left(1-\frac{1}{p}\right)^6 \omega_p. \tag{5.16}$$

We complete the proof of theorem 1 putting together the equalities (5.14), (5.15), (5.16) and lemma 11.

References

[BB07] R. de la Bretèche and T. D. Browning. On Manin's conjecture for singular del Pezzo surfaces of degree 4. I. *Michigan Math. J.*, 55(1):51–80, 2007.

[BD09] T. D. Browning and U. Derenthal. Manin's conjecture for a quartic del Pezzo surface with A_4 singularity. *Ann. Inst. Fourier (Grenoble)*, 59(3):1231–1265, 2009.

[Bro07] T. D. Browning. An overview of Manin's conjecture for del Pezzo surfaces. In *Analytic number theory*, volume 7 of *Clay Math. Proc.*, pages 39–55. Amer. Math. Soc., Providence, RI, 2007.

[BT98] V. V. Batyrev and Y. Tschinkel. Manin's conjecture for toric varieties. *J. Algebraic Geom.*, 7(1):15–53, 1998.

[CLT02] A. Chambert-Loir and Y. Tschinkel. On the distribution of points of bounded height on equivariant compactifications of vector groups. *Invent. Math.*, 148(2):421–452, 2002.

[CT88] D. F. Coray and M. A. Tsfasman. Arithmetic on singular Del Pezzo surfaces. *Proc. London Math. Soc. (3)*, 57(1):25–87, 1988.

[Der] U. Derenthal. *Geometry of universal torsors.* PhD thesis.

[Der06] U. Derenthal. Singular del Pezzo surfaces whose universal torsors are hypersurfaces. *Preprint*, 2006.

[Der07] U. Derenthal. On a constant arising in Manin's conjecture for del Pezzo surfaces. *Math. Res. Lett.*, 14(3):481–489, 2007.

[Der09] U. Derenthal. Counting integral points on universal torsors. *Int. Math. Res. Not. IMRN*, (14):2648–2699, 2009.

[DL10] U. Derenthal and D. Loughran. Singular del Pezzo surfaces that are equivariant compactifications. *Zapiski Nauchnykh Seminarov (POMI)*, 377:26–43, 2010.

[DT07] U. Derenthal and Y. Tschinkel. Universal torsors over del Pezzo surfaces and rational points. In *Equidistribution in number theory, an introduction*, volume 237 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 169–196. Springer, Dordrecht, 2007.

[FMT89] J. Franke, Y. I. Manin, and Y. Tschinkel. Rational points of bounded height on Fano varieties. *Invent. Math.*, 95(2):421–435, 1989.

[Fra09] M. Franz. Convex - a Maple package for convex geometry, version 1.1, 2009.

[LB10] P. Le Boudec. Manin's conjecture for two quartic del Pezzo surfaces with $3\mathbf{A}_1$ and $\mathbf{A}_1 + \mathbf{A}_2$ singularity types. *To appear in Acta Arithmetica*, 2010.

[Lou10] D. Loughran. Manin's conjecture for a singular sextic del Pezzo surface. *To appear in J. Théorie Nombres Bordeaux*, 2010.

[Pey95] E. Peyre. Hauteurs et mesures de Tamagawa sur les variétés de Fano. *Duke Math. J.*, 79(1):101–218, 1995.

Pierre Le Boudec, Université Denis Diderot (Paris VII), Institut de Mathématiques de Jussieu, UMR 7586, Case 7012 - Bâtiment Chevaleret, Bureau 7C14, 75205 Paris Cedex 13

E-mail : pleboude@math.jussieu.fr