Ergodic Theory on Stationary Random Graphs

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Abstract

A stationary random graph is a random rooted graph whose distribution is invariant under re-rooting along the simple random walk. We adapt the entropy technique developed for Cayley graphs and show in particular that stationary random graphs of subexponential growth are almost surely Liouville, that is, admit no non constant bounded harmonic function. Applications include the uniform infinite planar quadrangulation and long-range percolation clusters.

1 Introduction

A stationary random graph (G, ρ) is a random rooted graph whose distribution is invariant under re-rooting along a simple random walk started at the root ρ (see Section 1.1 for a precise definition). The entropy technique and characterization of the Liouville property for groups of or homogeneous graphs [15, 17, 18] are adapted to this context. In particular we have

Theorem 1.1. *Let (G, ρ) be a stationary random graph of* subexponential growth *in the sense that*

$$n^{-1}\mathbf{E}\Big[\log\big(\#B_G(\rho, n)\big)\Big] \underset{n\to\infty}{\longrightarrow} 0, \tag{1}$$

then (G, ρ) is almost surely Liouville.

Recall that a function from the vertices of a graph to $\mathbb{R}$ is harmonic if and only if the value of the function at a vertex is the average of the value over its neighbors, for all vertices of the graph. We call graphs admitting no non constant bounded harmonic functions *Liouville*. In the case of graphs of bounded degree, Corollary 3.5 characterize stationary non-Liouville random graphs as those on which the simple random walk is *ballistic*.

The motivation of this work lies in the study of the Uniform Infinite Planar Quadrangulation (abbreviated by UIPQ) introduced in [19] (following the pionneer work of [2]). The UIPQ is a random infinite planar graph whose faces are all squares which is stationary. This object is very natural and of special interest for understanding two dimensional quantum gravity and has triggered a lot of work, see e.g. [2, 3, 11, 21, 24]. One of the fundamental questions regarding the UIPQ, is to decide recurrence or transience. Unfortunately, the degrees in the UIPQ are not bounded thus the techniques of [8] fail to apply. Nevertheless it has been conjectured in [2] that the UIPQ is a.s. recurrent. We come close by proving below, as an application of Theorem 1.1

Corollary 1.2. *The Uniform Infinite Planar Quadrangulation is almost surely Liouville.*

Another application concerns a question of Berger [9] proving that certain long range percolation clusters are Liouville (see Section 5.2).

The notion of stationary random graph generalizes the concept of Cayley and transitive graph where the homogeneity of the graph is replaced by a invariant distribution along the simple random walk. This notion is very closely related to the ergodic theory notions of unimodular random graphs (see [1] for a survey) and measured equivalence relations see e.g. [16]. Roughly speaking, unimodular random graphs of [1] correspond, after biasing by the degree of the root, to stationary and *reversible* random graphs (see Definition 1.3). Using ideas from measured equivalence relations theory we are able to prove (Theorem 4.4) that if a stationary random graph of bounded degree (G, ρ) is non reversible then the simple random walk on G is ballistic, thus improving Theorem A of [25] and extending [26] in the case of transitive graphs.

The paper is organized as follows. The remaining of this section is devoted to a formal definition of stationary and reversible random graphs. Section 2 links these concepts to previous works on unimodular random graphs and measured equivalence relations. The entropy technique is developed in Section 3. In Section 4 we explore under which conditions a stationary random graph is not reversible. The last section is devoted to applications and open problems.

Acknowledgments : We are grateful to Pierre Pansu, Frederic Paulin and Damien Gaboriau for many stimulating discussions on measured equivalence relations. We also deeply thank Russell Lyons for comments on a early version of this work.

1.1 Definitions

A graph $G = (\mathrm{V}(G), \mathrm{E}(G))$ is a pair of sets, $\mathrm{V}(G)$ representing the set of vertices and $\mathrm{E}(G)$ the set of (unoriented) edges. We allow multiple edges and loops. In the following all the graphs considered are connected and locally finite. Two vertices $x, y \in \mathrm{V}(G)$ linked by an edge are called *neighbors* in G and we write $x \sim y$. The *degree* $\deg(x)$ of x is the number of neighbors of x in G. For any pair $x, y \in G$, the *graph distance* $\mathrm{d}^G_{\mathrm{gr}}(x, y)$ is the minimal length of a path joining x and y in G. For every $r \in \mathbb{Z}_+$, the ball of radius r around x in G is the subgraph of G spanned by the vertices at distance less than or equal to r from x in G, it is denoted by $B_G(x, r)$.

A *rooted graph* is a pair (G, ρ) where $\rho \in \mathrm{V}(G)$ is called the root vertex. An isomorphism between two rooted graphs is a graph isomorphism that maps the roots of the graphs. Let $\mathcal{G}_\bullet$ be the set of isomorphism classes of locally finite rooted graphs (G, ρ), endowed with the distance $\mathrm{d}_{\mathrm{loc}}$ defined by

$$\mathrm{d}_{\mathrm{loc}}\left((G_1, \rho_1), (G_2, \rho_2)\right) = \inf\left\{ \frac{1}{r+1} : r \geqslant 0 \text{ and } B_{G_1}(\rho_1, r) \simeq B_{G_2}(\rho_2, r) \right\},$$

where $\simeq$ stands for the rooted graph equivalence. With this topology, $\mathcal{G}_\bullet$ is a Polish space (see [8]). Similarly, we define $\mathcal{G}_{\bullet\bullet}$ (resp. $\vec{\mathcal{G}}$) be the set of isomorphism classes of bi-rooted graphs (G, x, y) that are graphs with two distinguished ordered points (resp. graphs $(G, (x_n)_{n \geqslant 0})$ with an semi-infinite path), where the isomorphisms considered have to map the two distinguished points (resp. the path). These two sets are equipped with variants of the distance $\mathrm{d}_{\mathrm{loc}}$ and are Polish with the induced topology. Formally elements of $\mathcal{G}_\bullet, \mathcal{G}_{\bullet\bullet}$ and $\vec{\mathcal{G}}$ are equivalence classes of graphs, but we will not distinguish between graphs and their equivalence classes and we use the same terminology and notation. One way to bypass this identification is to choose once for all a canonical representant in each class, see [1, Section 2].

Let (G, ρ) be rooted graph, for $x \in \mathrm{V}(G)$ we denote the law of the *simple random walk* $(X_n)_{n \geqslant 0}$ on G starting from x by P^G_x and its expectation by E^G_ρ. It is easy to check that when (G, ρ) is an equivalence class of rooted graphs the distribution of $(G, (X_n)_{n \geqslant 0}) \in \vec{\mathcal{G}}$ is well-defined. We speak of "the simple random walk of law P^G_ρ conditionally on (G, ρ)". It is easy to check that all the quantities we will use in the paper do not depend of a choice of a representant of (G, ρ).

A random rooted graph (G, ρ) is a random variable taking values in $\mathcal{G}_\bullet$. In this work we will use $\mathbf{P}$ and $\mathbf{E}$ for the probability and expectation referring to the underlying random graph.

If conditionally on (G,ρ), $(X_n)_{n\geqslant 0}$ is the simple random walk started at ρ, we denote the distribution of $(G,(X_n)_{n\geqslant 0}) \in \vec{\mathcal{G}}$ by $\mathbb{P}$, and by $\mathbb{E}$ the respective expectation.

Definition 1.3. *Let (G,ρ) be a random rooted graph. Conditionally on (G,ρ), let $(X_n)_{n\geqslant 0}$ be the simple random walk on G starting from ρ. The graph (G,ρ) is called* stationary *if*

$$(G,\rho) \quad = \quad (G,X_n) \quad \text{in distribution, for all } n \geqslant 1, \tag{2}$$

or equivalently for $n=1$. In words a stationary random graph is a random rooted graph whose distribution is invariant under re-rooting along a simple random walk on G. Furthermore, (G,ρ) is called reversible *if*

$$(G,X_0,X_1) \quad = \quad (G,X_1,X_0) \quad \text{in distribution.} \tag{3}$$

Clearly any reversible random graph is stationary.

Example 1. *Any Cayley graph rooted at any vertex is stationary and reversible. Any transitive graph G (i.e. whose isomorphism group is transitive on $\mathrm{V}(G)$) is stationary. For examples of transitive graphs which are not reversible, see [6, Examples 3.1 and 3.2]. E.g. the "grandfather" graph (see Fig. below) is a transitive (hence stationary) graph which is not reversible.*

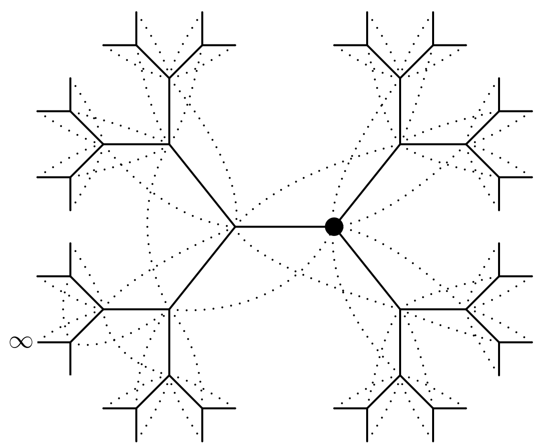

FIG.: *The "grandfather" graph is obtained from the 3-regular tree by choosing a point at Infinity that orientates the graph and adding all the edges from grand sons to grand-father.*

Example 2. *[8, Section 3.2] Let G be a finite connected graph. Pick a vertex $\rho \in \mathrm{V}(G)$ with a probability proportional to its degree (normalized by $\sum_{u\in \mathrm{V}(G)} \deg(u)$). Then (G,ρ) is a reversible random graph.*

Example 3 (Augmented Galton-Watson tree)**.** *Consider two independent Galton-Watson trees with offspring distribution $(p_k)_{k\geqslant 0}$. Link the roots vertices of the two trees by an edge and root the obtained graph at the root of the first tree. The resulting random rooted graph is stationary and reversible, see [22, 23, 16].*

2 Connections with other notions

As we will see, the concept of stationary random graph can be linked to various notions. In the context of bounded degree, stationary random graphs generalize unimodular random graphs [1]. Stationary random graphs are closely related to graphed equivalence relation with an harmonic measure, see [25]. We however think that the probabilistic Definition 1.3 is more natural and shed some additional light on the concept.

2.1 Ergodic theory

We formulate the notion of stationary random graphs in terms of ergodic theory. We can define the *shift* operator θ on $\vec{\mathcal{G}}$ by $\theta\big((G,(x_n)_{n\geqslant 0})\big) = \big(G,(x_{n+1})_{n\geqslant 0}\big)$, and the projection $\pi : \vec{\mathcal{G}} \to \mathcal{G}_\bullet$ by $\pi\big((G,(x_n)_{n\geqslant 0})\big) = (G, x_0)$.

Recall from the last section that if $\mathbf{P}$ is the law of (G,ρ) we write $\mathbb{P}$ for the distribution of $(G,(X_n)_{n\geqslant 0})$ where $(X_n)_{n\geqslant 0}$ is the simple random walk on G starting at ρ. The following proposition is a straightforward translation of the notion of stationary random graph into an θ-invariant probability measure on $\vec{\mathcal{G}}$.

Proposition 2.1. *Let $\mathbf{P}$ a probability measure on $\mathcal{G}_\bullet$ and $\mathbb{P}$ the associated probability measure on $\vec{\mathcal{G}}$. Then $\mathbf{P}$ is stationary if and only if $\mathbb{P}$ is invariant under θ.*

As usual, we will say that $\mathbb{P}$ (and by extension $\mathbf{P}$ or directly (G,ρ)) is ergodic if $\mathbb{P}$ is ergodic for θ. Proposition 2.1 enables us to use all the powerful machinery of ergodic theory in the context of stationary random graphs. For instance, the classical theorems on the range and speed of a random walk on a group are valid:

Theorem 2.2. *Let (G,ρ) be a stationary and ergodic random graph. Conditionally on (G,ρ) denote $(X_n)_{n\geqslant 0}$ the simple random walk on G starting from ρ. Set $R_n = \#\{X_0,\dots,X_n\}$ and $D_n = \mathrm{d}^G_{\mathrm{gr}}(X_0,X_n)$ for the range and distance from the root of the random walk at time n. There exists a constant $s \geqslant 0$ such that we have the following almost sure and $\mathbb{L}^1$ convergence for $\mathbb{P}$,*

$$\frac{R_n}{n} \xrightarrow[n\to\infty]{a.s.\ \mathbb{L}^1} \mathbb{P}\left(\bigcap_{i\geqslant 1}\{X_i \neq \rho\}\right), \tag{4}$$

$$\frac{D_n}{n} \xrightarrow[n\to\infty]{a.s.\ \mathbb{L}^1} s. \tag{5}$$

Remark 2.3. *In particular a stationary and ergodic random graph is transient if and only if the range of the simple random walk on it grows linearly.*

Proof. The two statements are straightforward adaptations of [12]. See also [1, Proposition 4.8]. □

2.2 Unimodular random graphs

The Mass-Transport Principle has been introduced by Häggström in [14] to study percolation and was further developed in [6]. A random rooted graph (G,ρ) obeys the *Mass-Transport principle* (abbreviated by MTP) if for every Borel positive function $F : \mathcal{G}_{\bullet\bullet} \to \mathbb{R}_+$ we have

$$\mathbf{E}\left[\sum_{x\in \mathrm{V}(G)} F(G,\rho,x)\right] = \mathbf{E}\left[\sum_{x\in \mathrm{V}(G)} F(G,x,\rho)\right]. \tag{6}$$

The name comes from the interpretation of F as an amount of mass sent from ρ to x in G: the mean amount of mass that ρ receives is equal to the mean quantity it sends. The MTP holds for a great variety of random graphs, see [1] were the MTP is extensively studied.

Definition 2.4. *[1, Definition 2.1] If (G,ρ) satisfies (6) it is called* unimodular *(See [1] for explanation of the terminology).*

Let us detail the link between unimodular random graphs and reversible random graphs. Suppose that $F : \mathcal{G}_{\bullet\bullet} \to \mathbb{R}_+$ is a Borel positive function such that

$$F(G,x,y) = F(G,x,y)\mathbf{1}_{x\sim y}. \tag{7}$$

Applying the MTP to a unimodular random graph (G, ρ) with the function F we get

$$\mathbf{E}\left[\sum_{x\sim\rho} F(G,\rho,x)\right] = \mathbf{E}\left[\sum_{x\sim\rho} F(G,x,\rho)\right],$$

or equivalently

$$\mathbf{E}\left[\deg(\rho)\frac{1}{\deg(\rho)}\sum_{x\sim\rho} F(G,\rho,x)\right] = \mathbf{E}\left[\deg(\rho)\frac{1}{\deg(\rho)}\sum_{x\sim\rho} F(G,x,\rho)\right].$$

In other words, if $(\tilde{G},\tilde{\rho})$ is distributed according to (G,ρ) biased by $\deg(\rho)$ (assuming that $\mathbf{E}[\deg(\rho)] < \infty$) and if conditionally on $(\tilde{G},\tilde{\rho})$, X_1 is a one-step simple random walk starting on $\tilde{\rho}$ in $\tilde{G}$ then we have the following equality in distribution

$$(\tilde{G},\tilde{\rho},X_1) \overset{(d)}{=} (\tilde{G},X_1,\tilde{\rho}). \tag{8}$$

The graph $(\tilde{G},\tilde{\rho})$ is thus reversible hence stationary. Reciprocally, if $(\tilde{G},\tilde{\rho})$ is reversible we deduce that the graph (G,ρ) obtained after biasing by $\deg(\rho)^{-1}$ obeys the MTP with functions of the form $F(G,x,y)\mathbf{1}_{x\sim y}$. By [1, Proposition 2.2] this is sufficient to imply the full mass transport principle. Let us sum-up.

Proposition 2.5. *There is a correspondence between unimodular random graphs such that the expectation of the degree of the root is finite and reversible random graphs:*

$$(G,\rho) \textit{ unimodular and } \mathbf{E}[\deg(\rho)] < \infty \quad \underset{\textit{bias by } \deg(\rho)^{-1}}{\overset{\textit{bias by } \deg(\rho)}{\rightleftarrows}} \quad (G,\rho) \textit{ reversible.}$$

2.3 Measured equivalence relations

Let (B,μ) be a standard Borel space with a probability measure μ and let $E \subset B^2$ be a symmetric Borel set. We denote the smallest equivalence relation containing E by $\mathcal{R}$. Under mild assumptions (see below) the triple (B,μ,E) is called *a measured graphed equivalence relation.* The set E induces a graph structure on B by setting $x \sim y \in B$ if $(x,y) \in E$ or $(y,x) \in E$. For $x \in B$, one can interpret the equivalence class of x as a graph with the edge set given by E, which we root at the point x. If x is sampled according to μ, any measured graphed equivalence relation can be seen as a random rooted graph.
Here are the mild conditions to require for (B,μ,E) to be a measured graphed equivalence relation. We suppose that $\mathcal{R} \subset B^2$ is Borel, that each equivalence class is at most countable and that the $\mathcal{R}$-satured of any Borel set of μ measure zero is still of μ measure zero. We also assume that the applications $o : (x,y) \in E \mapsto x \in B$ and $r : (x,y) \in E \mapsto (y,x) \in E$ are Borel and that $\#o^{-1}(x)$ is finite for μ-almost every x. We can define a probability measure ν on E by $\nu(f) = \int_B d\mu(x)\frac{1}{\#o^{-1}(x)}\sum_{x\sim y} f((x,y))$. If ν is in the same class as its push-forward $r_*\nu$ by r then the triple (B,μ,E) is called *a measured graphed equivalence relation.*

Reciprocally, the set $\mathcal{G}_\bullet$ can be equipped with a natural symmetric Borel set E where $((G,\rho),(G',\rho')) \in E$ if (G,ρ) and (G',ρ') represent the same isomorphism class of non-rooted graph but rooted at two different neighbor vertices. Denote $\mathcal{R}$ the smallest equivalence relation on $\mathcal{G}_\bullet$ that contains E. Thus a random rooted graph (G,ρ) of distribution $\mathbf{P}$ gives rise to $(\mathcal{G}_\bullet,\mathbf{P},E)$ which, under slight assumptions on (G,ρ) is a MGEQ.

Remark however that the measured graphed equivalence relation we obtain with this procedure can have a graph structure on equivalence classes very different from the graph (G,ρ).

Consider for example the (random) graph $\mathbb{Z}^2$ rooted at $(0,0)$. Since $\mathbb{Z}^2$ is a transitive graph, the measure obtained on $\mathcal{G}_\bullet$ by the above procedure is concentrated on the singleton corresponding to the isomorphism class of $(\mathbb{Z}^2,(0,0))$. There are two ways to bypass this difficulty: considering *rigid* graphs (that are graphs without non trivial isomorphisms see [16, Section 1E]) or add independent uniform labels $\in [0,1]$ on the graphs (see [1, Example 9.9]). Both procedures yield a MGEQ whose graph structure is that of (G,ρ).

In particular we have the following dictionary between the notions of harmonic MGEQ [25, Definition 1.11], totally invariant MGEQ [25, Definition 1.12], measure preserving MGEQ [13, Section 8] or [1, Example 9.9] and the corresponding analogous for random rooted graphs.

measured graphed equivalence relation	random rooted graph
harmonic	stationary
totally invariant	reversible
measure preserving	unimodular

3 Liouville property

In this section, we extend a well-known result on groups first proved in [4] relating Poisson boundary to entropy of a group. Here we adapt the proof of [17, Theorem 1] in the case of group (see also [18] in the case of homogeneous graphs). We basically follow the argument using *expectation* of entropy. The stationarity of the underlying random graph together with the Markov property of the simple random walk will replace homogeneity of the graph. We introduce the *mean entropy* of the random walk and prove some useful lemmas. Then we derive the main results of this Section.

In the following (G,ρ) is a stationary random graph. Recall that conditionally on (G,ρ), P_x^G is the law of the simple random walk $(X_n)_{n\geqslant 0}$ on G starting from $x\in \mathrm{V}(G)$. For every integer $0\leqslant a\leqslant b<+\infty$, the *entropy* of the simple random walk started at $x\in\mathrm{V}(G)$ between time a and b is

$$H_a^b(G,x)=\sum_{x_a,x_{a+1},\dots,x_b}\varphi\left(\mathrm{P}_x^G(X_a=x_a,\dots,X_b=x_b)\right),$$

where $\varphi(t)=-t\log(t)$. To simplify notation we write $H_a(G,x)=H_a^a(G,x)$. Since (G,ρ) is a random graph we set

$$h_a^b=\mathbf{E}\left[H_a^b(G,\rho)\right] \text{ and } h_a=\mathbf{E}\left[H_a(G,\rho)\right].$$

Proposition 3.1. *If (G,ρ) is a stationary random graph then $(h_n)_{n\geqslant 0}$ is a subadditive sequence.*

Proof. Let $n,m\geqslant 0$. We have

$$H_{n+m}(G,\rho) \;=\; -\sum_{x_{n+m}}\mathrm{P}_\rho^G(X_{n+m}=x_{n+m})\log\left(\mathrm{P}_\rho^G(X_{n+m}=x_{n+m})\right).$$

Applying Markov property at time n, it comes

$$H_{n+m}(G,\rho) \;=\; \sum_{x_{n+m}}\varphi\left(\sum_{x_n}\mathrm{P}_\rho^G(X_n=x_n)\mathrm{P}_{x_n}^G(X_m=x_{n+m})\right).$$

Since φ is concave and $\varphi(0)=0$ we deduce that for every $x,y\geqslant 0$ we have $\phi(x+y)\leqslant\phi(x)+\phi(y)$, hence we obtain

$$\begin{aligned}H_{n+m}(G,\rho) &\leqslant \sum_{x_{n+m}}\sum_{x_n}\varphi\left(\mathrm{P}_\rho^G(X_n=x_n)\mathrm{P}_{x_n}^G(X_m=x_{n+m})\right)\\ &= H_n(G,\rho)+\sum_{x_n}\mathrm{P}_\rho^G(X_n=x_n)H_m(G,x_n).\end{aligned}$$

Taking expectations one has using (2)

$$\begin{aligned} h_{n+m} &\leqslant& h_n + \mathbf{E}\left[\sum_{x_n} \mathrm{P}_\rho^G(X_n = x_n) H_m(G, x_n)\right] \\ &=& h_n + \mathbf{E}\left[H_m(G, X_n)\right] \quad = \quad h_n + h_m. \end{aligned}$$

□

The subadditive lemma then implies that

$$\frac{h_n}{n} \underset{n\to\infty}{\longrightarrow} h \geqslant 0. \tag{9}$$

This limit is called the *mean entropy* of the stationary random graph (G, ρ). It plays the role of the (deterministic) entropy of a random walk on a group. The following theorem generalizes the well-known connection between Liouville property and entropy.

Theorem 3.2. *Let (G, ρ) be a stationary random graph. The following conditions are equivalent:*

- *the tail σ-algebra associated to the simple random walk on G started from ρ is almost surely trivial (in particular (G, ρ) is almost surely Liouville),*
- *the mean entropy h of (G, ρ) is null.*

Before doing the proof, we start with a few lemmas.

Lemma 3.3. *For every $0 \leqslant a \leqslant b < \infty$ we have $h_a^b = h_a + (b - a)h_1$.*

Proof. Let $0 \leqslant a \leqslant b < \infty$. An application of Markov's property at time a leads to

$$\begin{aligned} H_a^b(G, \rho) &=& -\sum_{x_a,\dots,x_b} \mathrm{P}_\rho^G(X_a = x_a, \dots, X_b = x_b) \log\left(\mathrm{P}_\rho^G(X_a = x_a, \dots, X_b = x_b)\right) \\ &=& -\sum_{x_a} \mathrm{P}_\rho^G(X_a = x_a) \log\left(\mathrm{P}_\rho^G(X_a = x_a)\right) + \sum_{x_a} \mathrm{P}_\rho^G(X_a = x_a) H_1^{b-a}(G, x_a) \end{aligned}$$

Taking expectations we get $h_a^b = h_a + h_1^{b-a}$. An iteration of the argument proves the lemma. □

If (G, ρ) is fixed and $(X_n)_{n\geqslant 0}$ is distributed according to P_ρ^G, we denote

$$\begin{aligned} \mathcal{F}_n(G, \rho) &=& \sigma(X_0, \dots, X_n), \\ \mathcal{F}^n(G, \rho) &=& \sigma(X_n, \dots), \\ \mathcal{F}^\infty(G, \rho) &=& \bigcap_{n\geqslant 0} \mathcal{F}^n(G, \rho). \end{aligned}$$

The last σ-algebra consists of tail events. By classical results on entropy theory, for all $k \geqslant 0$, the conditional entropy $H(\mathcal{F}_k(G, \rho)|\mathcal{F}^n(G, \rho))$ increases as $n \to \infty$ and converges to $H(\mathcal{F}_k(G, \rho)|\mathcal{F}^\infty(G, \rho))$. Furthermore, we have

$$H(\mathcal{F}_k(G, \rho)|\mathcal{F}^\infty(G, \rho)) \leqslant H(\mathcal{F}_k(G, \rho)),$$

with equality if and only if $\mathcal{F}_k(G, \rho)$ and $\mathcal{F}^\infty(G, \rho)$ are independent.

Lemma 3.4. *For $0 \leqslant k \leqslant n < +\infty$ we have $\mathbf{E}\left[H(X_1, \dots, X_k|X_n, \dots, X_m)\right] = kh_1 + h_{n-k} - h_n$.*

Proof. We have by definition

$$\begin{aligned}
&H(X_1,\ldots,X_k|X_n,\ldots,X_m)\\
= &\sum_{\substack{x_1,\ldots,x_k\\x_n,\ldots,x_m}} \mathrm{P}^G_\rho(X_i=x_i, 1\leqslant i\leqslant k \text{ and } n\leqslant i\leqslant m)\log\left(\frac{\mathrm{P}^G_\rho(X_i=x_i, 1\leqslant i\leqslant k \text{ and } n\leqslant i\leqslant m)}{\mathrm{P}^G_\rho(X_i=x_i, n\leqslant i\leqslant m)}\right).
\end{aligned}$$

Applying Markov property at time k one gets

$$= \quad H_1^k(G,\rho) - H_n^m(G,\rho) + \sum_{x_k}\mathrm{P}^G_\rho(X_k=x_k)H^{m-k}_{n-k}(G,x_k),$$

and taking expectations the RHS becomes $h_1^k - h_n^m + h_{n-k}^{m-k}$. Lemma 3.3 concludes the proof. $\square$

In particular we see that the expected value of $H(X_1,\ldots,X_k|X_n,\ldots,X_m)$ does not depend upon m (this is also true without taking expectation and follows from Markov property at time n). If we let $m\to\infty$ in the statement of the last lemma, we get by monotone convergence

$$\mathbf{E}\left[H(\mathcal{F}_k(G,\rho)|\mathcal{F}^n(G,\rho))\right] = kh_1 + h_{n-k} - h_n. \tag{10}$$

Proof of Theorem 3.2. The equality (10) for $k=1$ and the monotonicity of conditional entropy allow us to deduce that $(h_{n+1}-h_n)_{n\geqslant 0}$ is decreasing and converges towards $\tilde{h}\geqslant 0$. By (9) and Cesaro's Theorem, we deduce that $\tilde{h}=h$. Thus sending $n\to\infty$ in (10) we get by monotone convergence

$$\mathbf{E}\left[H(\mathcal{F}_k(G,\rho)|\mathcal{F}^\infty(G,\rho))\right] = k(h_1-h).$$

Comparing the last display with Lemma 3.3 with $a=1$ and $b=k$, it follows by monotonicity of conditional entropy that $h=0$ if and only if almost surely, for all $k\geqslant 0$, $\mathcal{F}^\infty(G,\rho)$ is independent of $\mathcal{F}_k(G,\rho)$. In this case the classical Kolmogorov's $0-1$ law implies that $\mathcal{F}^\infty(G,\rho)$ is almost surely trivial, in particular (G,ρ) is Liouville. This completes the proof of Theorem 3.2. $\square$

Proof of Theorem 1.1. Let (G,ρ) be a stationary random graph of *subexponential growth* that is $\mathbf{E}[\log(\#B_G(\rho,n))]=o(n)$, as $n\to\infty$. Thanks to Theorem 3.2, we only have to prove that the mean entropy of G is zero. But by a classical inequality we have $H_n(G,\rho)\leqslant\log(\#B_G(\rho,n))$, taking expectation and using (9) yields the result. $\square$

In the preceding theorem we saw that subexponential growth plays a crucial role. In the case of transitive or Cayley graphs, all the graphs considered have at most an exponential growth. But that there are stationary graphs with superexponential growth, here is an example.

Example 4. *Considerer an augmented Galton-Watson tree (see Example 3) with offspring distribution $(p_k)_{k\geqslant 1}$ such that $\sum_{k\geqslant 1}kp_k=\infty$. We have*

$$\liminf_{n\to\infty}\frac{\mathbb{E}[\log\left(B_G(\rho,n)\right)]}{n}=\infty.$$

Corollary 3.5. *Let (G,ρ) be a stationary and ergodic random graph of degree almost surely bounded by $M>0$. Conditionally on (G,ρ) let $(X_n)_{n\geqslant 0}$ be the simple random walk on G starting from ρ. We denote* the speed *of the random walk by s and* the exponential volume growth *of G by v, namely*

$$\begin{aligned}
s &= \limsup_{n\to\infty} n^{-1}\mathbb{E}\left[\mathrm{d}^G_{\mathrm{gr}}(X_0,X_n)\right],\\
v &= \limsup_{n\to\infty} n^{-1}\mathbf{E}\left[\log(\#B_G(\rho,n))\right].
\end{aligned}$$

Then the mean entropy h of (G,ρ) satisfies

$$\frac{s^2}{2} \;\leqslant\; h \;\leqslant\; vs.$$

In particular $h=0 \iff s=0$ and if s or v is null then (G,ρ) is almost surely Liouville.

Remark 3.6. *This is an extension of the "fundamental inequality" for groups [27, Theorem 1], see also [18, Theorem 5.3.] for homogeneous graphs.*

Proof. For (G,ρ) is ergodic, we know from Theorem 2.2(5) that $n^{-1}\,\mathrm{d}^G_{\mathrm{gr}}(X_0,X_n)$ converges almost surely and in $\mathbb{L}^1(\mathbb{P})$ towards $s \geqslant 0$. In particular for every $\varepsilon > 0$ we have

$$\mathbb{P}\big((s-\varepsilon)n \leqslant \mathrm{d}^G_{\mathrm{gr}}(X_0,X_n) \leqslant (s+\varepsilon)n\big) \underset{n\to\infty}{\longrightarrow} 1. \tag{11}$$

Lower bound. We have

$$\begin{aligned} H_n(G,\rho) &\geqslant \sum_{\substack{x_n \\ \mathrm{d}^G_{\mathrm{gr}}(\rho,x_n)\geqslant(s-\varepsilon)n}} \varphi(\mathrm{P}^G_\rho(X_n=x_n)) \\ &= \sum_{\substack{x_n \\ \mathrm{d}^G_{\mathrm{gr}}(\rho,x_n)\geqslant(s-\varepsilon)n}} -\mathrm{P}^G_\rho(X_n=x_n)\log\big(\mathrm{P}^G_\rho(X_n=x_n)\big) \end{aligned}$$

At this point we use the Varopoulos-Carne estimates (see [23, Theorem 12.1]), for the probability inside the logarithm. Hence,

$$\begin{aligned} H_n(G,\rho) &\geqslant \sum_{\substack{x_n \\ \mathrm{d}^G_{\mathrm{gr}}(\rho,x_n)\geqslant(s-\varepsilon)n}} \mathrm{P}^G_\rho(X_n=x_n)\log\left(2\sqrt{M}\exp\left(-\frac{(s-\varepsilon)^2 n}{2}\right)\right) \\ &= \log\left(2\sqrt{M}\exp\left(-\frac{(s-\varepsilon)^2 n}{2}\right)\right)\mathrm{P}^G_\rho\big(\mathrm{d}^G_{\mathrm{gr}}(X_0,X_n)\geqslant(s-\varepsilon)n\big). \end{aligned} \tag{12}$$

Now, we take expectation with respect to $\mathbf{E}$, divide by n and let $n\to\infty$. Using (11) and (9) we have $h \geqslant \frac{(s-\varepsilon)^2}{2}$.

Upper bound. Fix $\varepsilon > 0$. To simplify notation, we write B_s for $B_G(\rho,(s+\varepsilon)n)$ and B_s^c for $B_G(\rho,n)\backslash B_G(\rho,(s+\varepsilon)n)$. We decompose the entropy $H_n(G,\rho)$ as follows

$$\begin{aligned} H_n(G,\rho) &= \sum_{x_n\in B_s}\varphi(\mathrm{P}^G_\rho(X_n=x_n)) + \sum_{x_n\in B_s^c}\varphi(\mathrm{P}^G_\rho(X_n=x_n)) \\ &\leqslant \left(\sum_{x_n\in B_s}\mathrm{P}^G_\rho(X_n=x_n)\right)\log\left(\frac{\#B_s}{\sum_{x_n\in B_s}\mathrm{P}^G_\rho(X_n=x_n)}\right) \\ &+ \left(\sum_{x_n\in B_s^c}\mathrm{P}^G_\rho(X_n=x_n)\right)\log\left(\frac{\#(B_s^c)}{\sum_{x_n\in B_s^c}\mathrm{P}^G_\rho(X_n=x_n)}\right). \end{aligned}$$

Where we used the concavity of φ for the inequalities on the sums of the RHS. Using the uniform bound on the degree, we deduce the crude upper bound $\#(B_s^c) \leqslant \#B_G(\rho,n) \leqslant M^n$. Taking expectation we obtain using the easy fact that $-x\log(x) \leqslant e^{-1}$, for $x\in[0,1]$

$$h_n \leqslant 2e^{-1} + \mathbf{E}\left[\log\left(\#B_G(\rho,(s+\varepsilon)n)\right)\right] + \mathbb{P}\left(\mathrm{d}^G_{\mathrm{gr}}(X_0,X_n)\geqslant(s+\varepsilon)n\right) n\log(M).$$

Divide the last quantities by n and let $n\to\infty$, then (9) and (11) show that $h \leqslant (s+\varepsilon)v$. $\square$

4 Radon-Nikodym Cocycle

In this part we borrow and reinterpret in probabilistic terms a notion coming from the measured equivalence relation theory, the Radon-Nikodym cocycle, in order to deduce several properties of stationary non reversible graphs, (see e.g. [16] for another application). This notion will play the role of the modular function in transitive graphs, see [26]. In the remaining of this section, (G, ρ) is a stationary random graph whose degree is almost surely bounded by a constant $M > 0$.

Conditionally on (G, ρ) of law $\mathbf{P}$, let $(X_n)_{n\geqslant 0}$ be a simple random walk of law P_ρ^G. We define two random variables taking values in $\mathcal{G}_{\bullet\bullet}$: (G, X_0, X_1) of law $\mu_\rightarrow$ and (G, X_1, X_0) of law $\mu_\leftarrow$. It is easy to see that under the stationarity assumption, these two random variables are equivalent, to be more precise, the stationarity implies that (G, X_0, X_1) and (G, X_1, X_0) have the same distribution as (G, X_1, X_2) and (G, X_2, X_1) respectively. But $X_2 = X_0$ with probability at least M^{-1} by the bounded degree hypothesis. Thus the Radon-Nikodym derivative of (G, X_1, X_0) with respect to (G, X_0, X_1), given for any $(g, x, y) \in \mathcal{G}_{\bullet\bullet}$ such that $x \sim y$ by

$$\Delta(g, x, y) := \frac{d\mu_\leftarrow}{d\mu_\rightarrow}(g, x, y),$$

can be chosen such that

$$M^{-1} \leqslant \Delta(g, x, y) \leqslant M. \tag{13}$$

Note that the function Δ is defined up to a set of $\mu_\rightarrow$-measure zero, and in the following we fix an arbitrary representative satisfying (13) and we keep the notation Δ for this function. Since Δ is a Radon-Nikodym derivative we obviously have $\mathbb{E}[\Delta(G, X_0, X_1)] = 1$ and Jensen's inequality yields

$$\mathbb{E}\Big[\log\big(\Delta(G, X_0, X_1)\big)\Big] \quad \leqslant \quad 0, \tag{14}$$

with equality if and only if $\Delta(G, X_0, X_1) = 1$ almost surely. In this latter case the two random variables (G, X_0, X_1) and (G, X_1, X_0) have the same law, that is (G, ρ) is reversible.

Lemma 4.1. *With the above notation. Let A be a Borel set of $\mathcal{G}_{\bullet\bullet}$ of $\mu_\rightarrow$-measure zero. Then for* $\mathbf{P}$ *almost every rooted graph (g, ρ) and every $x, y \in \mathrm{V}(g)$ such that $x \sim y$ we have $(g, x, y) \notin A$.*

Proof. By stationary, for any $n \geqslant 0$ the variable (G, X_n, X_{n+1}) has the same distribution as (G, X_0, X_1). Thus we have

$$\begin{aligned} 0 &= \sum_{n\geqslant 0} \mathbb{E}\left[\mathbf{1}_{(G,X_n,X_{n+1})\in A}\right] = \mathbb{E}\left[\sum_{n\geqslant 0} \mathbf{1}_{(G,X_n,X_{n+1})}\right] \\ &= \mathbf{E}\left[\sum_{x\sim y\in G} \mathbf{1}_{(G,x,y)\in A}\left(\sum_{n\geqslant 0} \mathrm{P}_\rho^G(X_n = x, X_{n+1} = y)\right)\right]. \end{aligned}$$

But for any $x \sim y$ in G, since G is connected, for n big enough the probability that $X_n = x$ and $X_{n+1} = y$ is positive, thus the sum between parentheses in the last display is positive. This proves the lemma. □

Note that we have $\Delta(g, x, y) = \Delta(g, y, x)^{-1}$ for $\mu_\rightarrow$-almost every bi-rooted graphs in $\mathcal{G}_{\bullet\bullet}$, so by the above lemma we also have $\Delta(g, x, y) = \Delta(g, y, x)^{-1}$ for $\mathbf{P}$-almost every rooted graph (g, ρ) and every vertices $x, y \in \mathrm{V}(g)$.

Lemma 4.2. *For* $\mathbf{P}$*-almost every* (g,ρ)*, an every cycle* $\rho = x_0 \sim x_1 \sim \ldots \sim x_n = \rho$ *in* g *we have*

$$\prod_{i=0}^{n-1} \Delta(g, x_i, x_{i+1}) \quad = \quad 1. \tag{15}$$

Proof. In the measured equivalence relation theory this proposition is known as the property of cocycle of the so called Radon-Nikodym derivative of the equivalence relation, see [25, Lemme 1.16]. However we give a probabilistic proof of this fact.
By reversibility of the simple random walk, conditionally on (G,ρ) and on $\{\rho = X_0 = X_n\}$, the path $(X_0, X_1, \ldots, X_{n-1}, X_n)$ has the same distribution as the reversed one $(X_n, X_{n-1}, \ldots, X_1, X_0)$. In other words, for any Borel positive function $F : \mathbb{R}_+ \to \mathbb{R}_+$ we have

$$\begin{aligned}\mathbb{E}\left[F\left(\prod_{i=0}^{n-1} \Delta(G, X_i, X_{i+1})\right)\mathbf{1}_{X_n = X_0}\right] \quad &= \quad \mathbb{E}\left[F\left(\prod_{i=0}^{n-1} \Delta(G, X_{i+1}, X_i)\right)\mathbf{1}_{X_n = X_0}\right] \\ &= \quad \mathbb{E}\left[F\left(\prod_{i=0}^{n-1} \Delta(G, X_i, X_{i+1})^{-1}\right)\mathbf{1}_{X_n = X_0}\right].\end{aligned}$$

Where we used the fact that for almost every (g,ρ) and for any $x, y \in \mathrm{V}(G)$ neighbors, we have $\Delta(G, x, y) = \Delta(G, y, x)^{-1}$. Since every cycle $x_0 \sim x_1 \sim \ldots \sim x_n = x_0$ has a probability bigger than M^{-n} to be realized by the simple random walk and because for any fixed graph there are only countably many cycles we deduce (15). □

Thank to the preceding lemma, we can define Δ for an arbitrary (isomorphism class of) bi-rooted graph (g, x, y) (see [25, Proof of Theoreme 1.15]): let $x = x_0 \sim x_1 \sim \ldots \sim x_n = y$ be a path in g between x and y, finally set

$$\Delta(g, x, y) \quad := \quad \prod_{i=0}^{n-1} \Delta(g, x_i, x_{i+1}), \tag{16}$$

and by convention $\Delta(g, x, x) = 1$. This definition does not depend on the representative bi-rooted graph nor on the path chosen to go from x to y by the last Lemma and is well founded for $\mathbf{P}$-almost every graph (g,ρ) and every $x, y \in \mathrm{V}(g)$. We can now rephrase Theorem 1.15 of [25].

Theorem 4.3 ([25])**.** *Let* (G,ρ) *be a stationary ergodic random graph. Assume that* (G,ρ) *is non reversible then for almost surely the function*

$$x \in \mathrm{V}(G) \mapsto \Delta(G, \rho, x),$$

is positive harmonic and non constant.

Proof. We follow the proof of [25]. By stationarity of (G,ρ), for any Borel function $\mathcal{G}_\bullet \to \mathbb{R}_+$ we have

$$\mathbb{E}\left[F(G, X_0)\right] = \mathbb{E}\left[F(G, X_1)\right] = \mathbb{E}\left[F(G, X_0)\Delta(G, X_0, X_1)\right].$$

We deduce that for almost surely we have $\deg(\rho)^{-1} \sum_{\rho \sim x} \Delta(G, \rho, x) = 1$. By Lemma 4.1, we deduce that almost surely, for any $x \in \mathrm{V}(G)$ we have

$$\frac{1}{\deg(x)} \sum_{x \sim y} \Delta(G, x, y) = 1.$$

We deduce from the previous display and the definition of Δ, that $x \mapsto \Delta(G,\rho,x)$ is almost surely harmonic. By ergodicity if $x \mapsto \Delta(G,\rho,x)$ has a positive probability to be non constant then it is almost surely constant, and this constant equals 1. This case is excluded because (G,ρ) is non reversible. $\square$

Theorem 4.4. *Let (G,ρ) be a stationary and ergodic random graph of degree almost surely bounded by $M > 0$. If (G,ρ) is non reversible, then the speed (see (5)) of the simple random walk on (G,ρ) is positive $s > 0$.*

Proof. For (G,ρ) is not reversible, the inequality (14) is not saturated and $\mathbb{E}[\log(\Delta(G,X_0,X_1))]$ is strictly negative. We consider the random process $(\log(\Delta(G,X_0,X_n)))_{n\geqslant 0}$. By Proposition 4.2 we almost surely have for all $n \geqslant 0$

$$\log\big(\Delta(G,X_0,X_n)\big) \quad = \quad \sum_{i=0}^{n-1} \log\big(\Delta(G,X_i,X_{i+1})\big). \tag{17}$$

By (13) we have $\mathbb{E}[|\log(\Delta(G,X_0,X_1))|] < \infty$ and the ergodic theorem implies the following almost sure and $\mathbb{L}^1$ convergence with respect to $\mathbb{P}$

$$\frac{\log\big(\Delta(G,X_0,X_n)\big)}{n} \quad \xrightarrow[n\to\infty]{} \quad \mathbb{E}[\log(\Delta(G,X_0,X_1))]. \tag{18}$$

By computing $\Delta(G,X_0,X_n)$ using (16) along a geodesic path from X_0 to X_n in G and using (13) we deduce that a.s. for every $n \geqslant 0$

$$|\log(\Delta(G,X_0,X_n))| \leqslant \log(M)\, \mathrm{d}^G_{\mathrm{gr}}(X_0,X_n).$$

Thus the convergence (18) shows that the speed of the random walk $(X_n)_{n\geqslant 0}$ (see Theorem 2.2) is positive $s \geqslant \mathbb{E}[\log(\Delta(G,X_0,X_1))]\log(M)^{-1} > 0$, which is the desired result. $\square$

Remark 4.5. *By Corollary 3.5, subexponential growth in the sense of (1) implies $s = 0$ for stationary and ergodic random graphs of bounded degree, so in particular such random graphs are reversible. This fact should also hold without bounded degree assumption (Russell Lyons, personal communication).*

5 Applications

5.1 Uniform planar quadrangulation

A planar map is an embedding of a planar graph into the two-dimensional sphere seen up to continuous deformations. A quadrangulation is a planar map whose faces all have degree four. The Uniform Infinite Planar Quadrangulation (UIPQ) denoted $(\mathrm{Q}_\infty, \vec{e})$ introduced by Krikun in [19] is the weak local limit (in a sense related to $\mathrm{d}_{\mathrm{loc}}$) of uniform quadrangulations with n faces with a distinguished oriented edge (see Angel and Schramm [2] for previous work on triangulations). We will not entered the subtilities of planar maps nor in the details of the construction of the UIPQ and refer to [19, 21, 24] for more details.

The UIPQ is an random infinite map Q_∞ of the plane (roughly speaking a embedding of a planar graph seen up to continuous deformation) given with a distinguished oriented edge $\vec{e}$. We can forget the planar structure of the UIPQ and get a random graph that we root at the origin of $\vec{e}$, that we denote (Q_∞,ρ). One of the main open question about this random infinite graph is its conformal type, namely is it (almost surely) recurrent or transient. It has been conjectured in [2] (for the related Uniform Infinite Planar Triangulation) that Q_∞ is almost surely recurrent. It is still an open problem. We provide a step towards recurrence.

Proof of Corollary 1.2. The random rooted graph (Q_∞, ρ) is a stationary random graph. A proof of this fact can be found in [20, Section 1.3]. By virtue of Theorem 1.1, we just have to show that (Q_∞, ρ) is of subexponential growth. Thanks to [24], we know that the random infinite quadrangulation investigated in [11] has the same distribution as the UIPQ. Hence, the main result of [11] can be translated into

$$\mathbf{E}\left[\#B_{Q_\infty}(\rho, n)\right] = \Theta(n^4).$$

Hence Jensen's inequality proves that the UIPQ is of subexponential growth in the sense of (1) which finishes the proof of the corollary. $\square$

This corollary does not use the planar structure of UIPQ but only the invariance with respect to SRW and the subexponential growth. It is a very robust result, and could be extend e.g. to the uniform infinite planar triangulation.

5.2 Long range percolation cluster

Consider the graph obtained from $\mathbb{Z}^d$ by adding edges between vertices $x, y \in \mathbb{Z}^d$ with probability $p_{x,y}$ independent of all other pairs, such that

$$p_{x,y} = \beta |x - y|^{-s}.$$

This model is called *long range percolation.* Berger [9] proved in dimensions $d = 1$ or $d = 2$ that if $d < s < 2d$, and if there exists an infinite cluster, then this cluster is almost surely transient. In the same paper the following question (6.3) is addressed:

Question 1. *Are there nontrivial harmonic functions on the infinite cluster of long range percolation with $d < s < 2d$?*

We answer negatively this question for *bounded* harmonic functions.

Proof. First we remark that by a general result (see [1, Example 9.4]), the cluster $\mathcal{C}_\infty$ containing 0 conditionally on the 0 belonging to an infinite open cluster is a unimodular random graph. Furthermore, since $s > d$ the expected degree of 0 is finite. Hence, by Proposition 2.5, the random graph $(\tilde{\mathcal{C}}_\infty, \tilde{0})$ obtained by biasing $(\mathcal{C}_\infty, 0)$ with the degree of 0 is stationary. By Theorem 1.1 it suffices to show that the graph $\tilde{\mathcal{C}}_\infty$ is of subexponential growth in the sense of (1). For that purpose, we use the estimates given in [10, Theorem 3.1]. If $x \in \mathbb{Z}$ belongs to the same infinite cluster as the origin, recall that we denote its graph distance from the origin 0 by $\mathrm{d}_{\mathrm{gr}}^{\mathcal{C}_\infty}(0, x)$. Then for each $s' \in (d, s)$ there are constants $c_1, c_2 \in (0, +\infty)$ such that, for $\delta' = 1/\log_2(2d/s')$,

$$\mathbf{P}\left(\mathrm{d}_{\mathrm{gr}}^{\mathcal{C}_\infty}(0, x) \leqslant n\right) \leqslant c_1 \left(\frac{e^{c_2 n^{1/\delta'}}}{|x|}\right)^{s'}.$$

In particular, we deduce that

$$\mathbf{E}\left[\#B_{\mathcal{C}_\infty}(0, n)\right] \leqslant \kappa_1 \exp\left(\kappa_2 n^{1/\delta'}\right), \tag{19}$$

where κ_1 and κ_2 are positive constants. Remark that $\delta' > 1$. Thus we have

$$\begin{aligned}
\mathbf{E}\left[\log(\#B_{\tilde{\mathcal{C}}_\infty}(\tilde{0}, n))\right] &= \frac{1}{\mathbf{E}[\deg(0)]} \mathbf{E}\left[\deg(0)\log(\#B_{\mathcal{C}_\infty}(0, n))\right] \\
&\leqslant \frac{1}{\mathbf{E}[\deg(0)]} \sqrt{\mathbf{E}[\deg(0)^2]\mathbf{E}[\log^2(\#B_{\mathcal{C}_\infty}(0, n))]}, \qquad (20)
\end{aligned}$$

by Cauchy-Schwarz inequality. Since $s > d$ it is easy to check that the second moment of deg(0) is finite. Furthermore, the function $x \mapsto \log^2(x)$ is concave on $]e, \infty[$ so by Jensen's inequality we have

$$\mathbf{E}\left[\log^2(\#B_{\mathcal{C}_\infty}(0,n))\right] \leqslant \log^2\left(\mathbf{E}\left[\#B_{\mathcal{C}_\infty}(0,n)\right] + 2\right).$$

Hence, combining the last display with (19) and (20) we deduce that $(\mathcal{C}_\infty, 0)$ is of subexponential growth in the sense of (1). $\square$

Note that by similar considerations, clusters of any invariant percolation on a group, in which the clusters have subexponential volume growth are Liouville, see [6] for many examples. In particular Bernoulli percolation on Cayley graphs of subexponential growth e.g. $\mathbb{Z}^d$.

5.3 Planarity

Simply connected planar Riemannian surfaces are either conformal to the Euclidean or to the hyperbolic plane. Thus are either recurrent for Brownian motion or admit non constant bounded harmonic functions. The same alternative holds for planar graphs of *bounded degree*. They are either recurrent for the simple random walk or admit non constant bounded harmonic function. The combination of Theorem 1.1 with these results coming from planarity yields:

Corollary 5.1. *Let (G, ρ) be a stationary random graph with subexponential growth in the sense of (1). Suppose furthermore that almost surely (G, ρ) is planar and has bounded degree. Then (G, ρ) is almost surely recurrent.*

Proof. We already know by Theorem 1.1 that (G, ρ) is almost surely Liouville. In [7] it is shown that a transient planar graph with bounded degree admits non constant bounded harmonic functions, which ends the proof of the Corollary. $\square$

Recurrence. Note that without the bounded degree assumption it is easy to construct planar transient Liouville graphs, see [7]. However these graphs are not stationary, that leads us to the following conjecture.

Question 1. *We conjecture that the bounded degree assumption in the above corollary can be removed. That is, a planar stationary random graph with subexponential growth in the sense of (1) is almost surely recurrent.*

Local limit. If a sequence (G_n, ρ_n) of finite stationary random graphs converge weakly for $\mathrm{d_{loc}}$ towards (G, ρ), we say that (G, ρ) is a *local limit* of finite stationary graph or local limit in short. Here are a few questions concerning this class of graphs.

Question 2. *Let (G, ρ) be a local limit of finite* planar *stationary graphs. Is it the case that (G, ρ) is almost surely amenable? Liouville? has zero speed for SRW? recurrent?*

Remark 5.2. *There are local limits of finite planar graphs with exponential growth. For example local limit of full binary trees up to level n with uniform root vertex.*

Question 3. *Can we characterize the graphs (G, ρ) that are local limit of finite planar graphs? Are they the reversible, amenable (in the sense of [1, Section 8]) planar graphs?*

Extensions.

Question 4. *In [5] a generalization of local limits of finite planar graphs to graphs presented by spheres in $\mathbb{R}^d$ was studied. Extend Question 3 to these graphs.*

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