

A Super Version of the Connes-Moscovici Hopf Algebra

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Dedicated, with much appreciation, to Henri Moscovici.

ABSTRACT. We define a super version of the Connes-Moscovici Hopf algebra, \mathcal{H}_1 . For that, we consider the supergroup $G^s = \text{Diff}^+(\mathbb{R}^{1,1})$ of orientation preserving diffeomorphisms of the superline $\mathbb{R}^{1,1}$ and define two (super) subgroups G_1^s and G_2^s of G^s where G_1^s is the supergroup of affine transformations. The super Hopf algebra \mathcal{H}_1^s is defined as a certain *bicrossproduct super Hopf algebra* of the super Hopf algebras attached to G_1^s and G_2^s . We also give an explicit description of \mathcal{H}_1^s in terms of generators and relations.

1. Introduction

In [2], Connes and Moscovici, among many other things, defined a Hopf algebra $\mathcal{H}(n)$, for any $n \geq 1$, and computed the periodic Hopf cyclic cohomology of $\mathcal{H}(n)$. Our main focus in this paper is $\mathcal{H}(1)$, to be denoted from now on by \mathcal{H}_1 , and its super analogue. It is by now clear that the Connes-Moscovici Hopf algebra \mathcal{H}_1 is a fundamental object of noncommutative geometry. An important feature of \mathcal{H}_1 , and in fact its *raison d'être*, is that it acts as quantum symmetries of various algebras of interest in noncommutative geometry, like the algebra of leaves of codimension one foliations and the algebra of modular forms modulo the action of Hecke correspondences [2, 3, 4, 5].

Our starting point, in fact the motivation to develop the super analogue of \mathcal{H}_1 , was to extend the results of [5] to cover the Rankin-Cohen brackets on super modular forms and super pseudodifferential operators as they are described in Section 7 of [1]. In [5] it is shown that the Rankin-Cohen brackets on modular forms [1] can be derived via the action of \mathcal{H}_1 on the modular Hecke algebras. In fact, more generally, it is shown how to obtain such brackets on any associative algebra endowed with an action of the Hopf algebra \mathcal{H}_1 , such that the derivation corresponding to the Schwarzian derivative is inner. To carry out this program in the SUSY case, as a first step one needs a super analogue of \mathcal{H}_1 .

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The Connes-Moscovici Hopf algebra \mathcal{H}_1 is isomorphic to a certain *bicrossproduct* Hopf algebra $F(G_2) \bowtie U(\mathfrak{g}_1)$ [2, 3, 8, 11, 12]. The actions and coactions involved in this bicrossproduct can be derived and understood by looking at the factorization of the group of orientation preserving diffeomorphisms of the real line, $G = Diff^+(\mathbb{R})$, into two subgroups G_1 and G_2 . Here G_1 is the group of affine transformations and G_2 is the subgroup of those diffeomorphisms ϕ with $\phi(0) = 0$ and $\dot{\phi}(0) = 1$, where $\dot{\phi}(x) = \frac{d}{dx}(\phi(x))$.

In this paper our goal is to define a super version of \mathcal{H}_1 , which we will denote by \mathcal{H}_1^s . For that we define a super version of the group $G = Diff^+(\mathbb{R})$, namely the supergroup $G^s = Diff^+(\mathbb{R}^{1,1})$ of orientation preserving diffeomorphisms of the superline $\mathbb{R}^{1,1}$. We define two (super) subgroups G_1^s and G_2^s of G^s , where G_1^s is the group of affine transformations. We show that the factorization $G^s = G_1^s G_2^s$ holds. We will use this factorization to define a bicrossproduct super Hopf algebra $F(G_2^s) \bowtie U(\mathfrak{g}_1^s)$, analogous to the non-super case. We will call this bicrossproduct super Hopf algebra the super version of \mathcal{H}_1 and denote it by \mathcal{H}_1^s .

One difficulty in working with super Hopf algebras is that they are not ‘honest’ Hopf algebras. In fact their multiplication map is not a morphism of coalgebras. It is so only up to sign, and this issue of signs can be quite confusing and demands a lot of care. Throughout this paper, for notations to be more consistent with the non-super case, we use the following conventions. To denote the comultiplication $\Delta : B \rightarrow B \otimes B$ of a bialgebra B we use Sweedler’s notation (summation understood) $\Delta^n(b) = b_{(1)} \otimes b_{(2)} \otimes \cdots \otimes b_{(n+1)}$, for any $n \geq 1$. Also for a coaction $\nabla : A \rightarrow B \otimes A$ of B on A we write $\nabla(a) = a^{(1)} \otimes a^{(2)}$.

One of the recent important developments regarding the Connes-Moscovici Hopf algebras $\mathcal{H}(n)$ is in [12], in which the authors generalize the Connes-Moscovici work, [2], to assign a Hopf algebra to any infinite primitive Lie pseudogroup. They also introduce an elaborate machinery, based on the bicrossproduct realization of those Hopf algebras, to compute their periodic and non-periodic Hopf cyclic cohomology.

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2. The Connes-Moscovici Hopf algebra \mathcal{H}_1

In this section we recall the definition of the standard (non-super) Connes-Moscovici Hopf algebra \mathcal{H}_1 and its description in terms of a bicrossproduct Hopf algebra [2, 3]. The following definition gives a description of \mathcal{H}_1 by generators and relations.

Definition 2.1. [2, 3] The Connes-Moscovici Hopf algebra \mathcal{H}_1 is generated by elements $X, Y, \delta_n, n \geq 1$ with relations:

$$\begin{aligned}
& [Y, X] = X, \quad [X, \delta_n] = \delta_{n+1}, \quad [Y, \delta_n] = n\delta_n, \quad [\delta_m, \delta_n] = 0, \quad \forall m, n \\
& \Delta(X) = X \otimes 1 + 1 \otimes X + Y \otimes \delta_1, \quad \Delta(Y) = Y \otimes 1 + 1 \otimes Y, \quad \Delta(\delta_1) = \delta_1 \otimes 1 + 1 \otimes \delta_1, \\
& \quad \varepsilon(X) = \varepsilon(Y) = \varepsilon(\delta_n) = 0, \quad \forall n, \\
(2.1) \quad & S(X) = Y\delta_1 - X, \quad S(Y) = -Y, \quad S(\delta_1) = -\delta_1.
\end{aligned}$$

Remark 2.2. The above definition follows the right-handed notation as in [8], in the sense that in the definition of $\Delta(X)$, the term $\delta_1 \otimes Y$ in [2, 3] is replaced by $Y \otimes \delta_1$. Or, alternatively, if we denote the original Connes-Moscovici Hopf algebra by \mathcal{H}_{1CM} , then one can say we are working in \mathcal{H}_{1CM}^{cop} , [12].

Lemma 2.3. [8, 9] *let A and H be two Hopf algebras such that A is a left H -module algebra, and H is a right A -comodule coalgebra. Let furthermore these structures satisfy the follow compatibility conditions:*

$$\begin{aligned}\Delta(h \triangleright a) &= h_{(1)}^{(1)} \triangleright a_{(1)} \otimes h_{(1)}^{(2)}(h_{(2)} \triangleright a_{(2)}), \\ \nabla_r(gh) &= g_{(1)}^{(1)}h^{(1)} \otimes g_{(1)}^{(2)}(g_{(2)} \triangleright h^{(2)}), \\ h_{(2)}^{(1)} \otimes (h_{(1)} \triangleright a)h_{(2)}^{(2)} &= h_{(1)}^{(1)} \otimes h_{(1)}^{(2)}(h_{(2)} \triangleright a), \\ \varepsilon(h \triangleright a) &= \varepsilon(h)\varepsilon(a), \quad \nabla_r(1) = 1 \otimes 1,\end{aligned}$$

for any a, b in A and g, h in H , where we have denoted the actions by $h \triangleright a$ and the coactions by $\nabla_r(h) = h^{(1)} \otimes h^{(2)}$. Then the vector space $A \otimes H$ can be equipped with a Hopf algebra structure as follows:

$$\begin{aligned}(a \otimes h)(b \otimes g) &= a(h_{(1)} \triangleright b) \otimes h_{(2)}g, \\ \Delta(a \otimes h) &= a_{(1)} \otimes h_{(1)}^{(1)} \otimes a_{(2)}h_{(1)}^{(2)} \otimes h_{(2)}, \\ \varepsilon(a \otimes h) &= \varepsilon(a)\varepsilon(h), \\ S(a \otimes h) &= (1 \otimes S(h^{(1)}))(S(ah^{(2)}) \otimes 1),\end{aligned}$$

for any a, b in A and g, h in H . It is called the left-right bicrossproduct Hopf algebra and is denoted by $A \blacktriangleright H$.

in [2, 3] a Hopf subalgebra of \mathcal{H}_1 is defined as the unital commutative subalgebra of \mathcal{H}_1 generated by $\{\delta_n, n \geq 1\}$. This Hopf algebra, which we denote by $F(G_2)$, is isomorphic to the so-called *comeasuring* Hopf algebra of the real line, generated by $\{a_n, n \geq 1\}$ with $a_1 = 1$ and with the following relations [7, 8]:

$$\begin{aligned}\Delta(a_n) &= \sum_{k=1}^n \left(\sum_{i_1 + \dots + i_k = n} a_{i_1} \dots a_{i_k} \right) \otimes a_k, \\ \varepsilon(a_n) &= \delta_{n,1}, \\ S(a_{n+1}) &= \sum_{(c_1, \dots, c_{n+1}) \in \Lambda} (-1)^{n-c_1} \frac{(2n-c_1)!c_1!}{(n+1)!} \frac{a_1^{c_1} a_2^{c_2} \dots a_{n+1}^{c_{n+1}}}{c_1!c_2! \dots c_{n+1}!},\end{aligned}$$

where

$$\Lambda = \{(c_1, \dots, c_{n+1}) \mid \sum_{j=1}^{n+1} c_j = n, \sum_{j=1}^{n+1} jc_j = 2n\}.$$

If instead of generators a_n we work with na_n we get the relations for the so called *Faà di Bruno* Hopf algebra which is isomorphic to $F(G_2)$. One can also define another Hopf algebra, denoted by $U(\mathfrak{g}_1)$, as follows. Let \mathfrak{g}_1 be the Lie algebra generated by two elements X and Y as in \mathcal{H}_1 (i.e., $[Y, X] = X$), and let $U(\mathfrak{g}_1)$ denote the universal enveloping algebra of \mathfrak{g}_1 .

Lemma 2.4. [8] $F(G_2)$ is a left $U(\mathfrak{g}_1)$ -module algebra via the actions

$$X \triangleright a_n = (n+1)a_{n+1} - 2a_2a_n, \quad Y \triangleright a_n = (n-1)a_n,$$

and $U(\mathfrak{g}_1)$ is a right $F(G_2)$ -comodule coalgebra via the coactions

$$\nabla_r(X) = X \otimes 1 + Y \otimes 2a_2, \quad \nabla_r(Y) = Y \otimes 1.$$

The actions and coactions are compatible in the sense of Lemma (2.3).

Theorem 2.5. [2, 3, 8, 11, 12] \mathcal{H}_1 is isomorphic to the bicrossproduct Hopf algebra $F(G_2) \blacktriangleright \blacktriangleleft U(\mathfrak{g}_1)$.

A good way to understand the actions, coactions and even the notations introduced above is to look at the *factorization* of the group $Diff^+(\mathbb{R})$ [2, 3]. Let us recall that given any group G with two subgroups G_1 and G_2 , we say we have a *group factorization* $G = G_1G_2$ if for any $g \in G$ there is a unique decomposition $g = ab$ where $a \in G_1$ and $b \in G_2$. Given any group factorization $G = G_1G_2$, one has always a left action of G_2 on G_1 and a right action of G_1 on G_2 defined in the following way. First one defines two maps $\pi_1 : G \rightarrow G_1$, $\pi_2 : G \rightarrow G_2$ by $\pi_1(g) = a$ and $\pi_2(g) = b$, for any $g = ab$ in G , with $a \in G_1$ and $b \in G_2$. Next, one can define the aforementioned actions by $g_2 \triangleright g_1 = \pi_1(g_2g_1)$ and $g_2 \triangleleft g_1 = \pi_2(g_2g_1)$ for any g_1 in G_1 and g_2 in G_2 .

Let

$$G = Diff^+(\mathbb{R}) = \{\Phi \in Diff(\mathbb{R}) \mid \dot{\Phi}(x) > 0, \forall x \in \mathbb{R}\},$$

where $\dot{\Phi}(x) = \frac{d}{dx}(\Phi(x))$, be the group of orientation preserving homeomorphisms of the real line and

$$G_1 = \{\psi = (a, b) \in G \mid \psi(x) = ax + b, a, b \in \mathbb{R}, a > 0\},$$

be the affine subgroup of G . The following representation of G_1 as a subgroup of $GL(2)$ is very useful:

$$(2.2) \quad G_1 = \left\{ (a, b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in GL(2) \mid a > 0 \right\}.$$

Let also

$$G_2 = \{\phi \in G \mid \phi(0) = 0, \dot{\phi}(0) = 1\}.$$

The factorization $Diff^+(\mathbb{R}) = G_1G_2$ is as follows. For any Φ in G we have $\Phi = \psi\phi$ with $\psi \in G_1$ and $\phi \in G_2$, where

$$(2.3) \quad \psi = (\dot{\Phi}(0), \Phi(0)), \quad \phi(x) = \frac{\Phi(x) - \Phi(0)}{\dot{\Phi}(0)}, \quad \forall x \in \mathbb{R}.$$

The actions and coactions in lemma (2.4) are induced from the factorization of the group $G = Diff^+(\mathbb{R})$ as follows:

$$\phi \triangleright \psi = (a\dot{\phi}(b), \phi(b)), \quad (\phi \triangleleft \psi)(x) = \frac{\phi(ax + b) - \phi(b)}{a\dot{\phi}(b)},$$

for any $\psi = (a, b)$ in G_1 and ϕ in G_2 .

Next, using the matrix representation of G_1 in $GL(2)$ as in (2.2), we get a basis for its Lie algebra $\mathfrak{g}_1 = Lie(G_1)$. It turns out that \mathfrak{g}_1 is generated by two elements

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

with the relation $[Y, X] = X$ as in Definition (2.1). On the other hand, in the Hopf algebra $F(G_2)$ defined by relations (2), the generators a_n can be considered as the following functions on G_2 :

$$a_n(\phi) = \frac{1}{n!} \phi^{(n)}(0), \quad \forall \phi \in G_2.$$

The actions defined in Lemma (2.4) can be realized in the following way. The exponentials e^{tX} and e^{tY} , as elements of the affine group G_1 , are given by

$$e^{tX} = (1, t), \quad e^{tY} = (e^t, 0).$$

The action $X \triangleright a_n$ in lemma (2.4) can be identified as:

$$\begin{aligned} (X \triangleright a_n)(\phi) &= \frac{d}{dt} \Big|_{t=0} a_n(\phi \triangleleft e^{tX}) \\ &= \frac{d}{dt} \Big|_{t=0} a_n\left(\frac{\phi(x+t) - \phi(t)}{\dot{\phi}(t)}\right) \\ &= \frac{d}{dt} \Big|_{t=0} \left(\frac{\phi^{(n)}(t)}{n! \dot{\phi}(t)}\right) \\ &= \frac{d}{dt} \Big|_{t=0} \left(\frac{\phi^{(n)}(0) + t\phi^{(n+1)}(0)}{n!(\dot{\phi}(0) + t\ddot{\phi}(0))}\right) \\ &= \frac{d}{dt} \Big|_{t=0} (1/n!)(\phi^{(n)}(0) - t\ddot{\phi}(0)\phi^{(n)}(0) + t\phi^{(n+1)}(0)) \\ &= (1/n!)(-\ddot{\phi}(0)\phi^{(n)}(0) + \phi^{(n+1)}(0)) \\ &= [(n+1)a_{n+1} - 2a_2a_n](\phi), \end{aligned}$$

for any ϕ in G_2 . This implies

$$X \triangleright a_n = (n+1)a_{n+1} - 2a_2a_n,$$

as in Lemma (2.4). A similar computation will give the action $Y \triangleright a_n$ as in Lemma (2.4). The coactions defined in Lemma (2.4) can also be realized using the factorization $G = G_1G_2$ [8].

Note that the above realization of actions and coactions is not necessary for the proof of Lemma (2.4). It rather gives a good intuition about where those formulas for actions and coactions come from.

3. The supergroup $G^s = Diff^+(\mathbb{R}^{1,1})$ and its factorization

In this section, by replacing \mathbb{R} by the supermanifold $\mathbb{R}^{1,1}$, we define a super version of the group $G = Diff^+(\mathbb{R})$, namely the supergroup $G^s = Diff^+(\mathbb{R}^{1,1})$. Analogous to the non-super case, we consider two sub supergroups G_1^s and G_2^s of G^s , where G_1^s is the affine part of G^s . We establish the factorization $G^s = G_1^s G_2^s$. This factorization will result in a left action of G_2^s on G_1^s and a right action of G_1^s on G_2^s .

For the general theory of supermanifolds we refer to [6, 13, 14]. The supermanifold $\mathbb{R}^{1,1}$, the superline, is a super ringed space $S = (\mathbb{R}^1, \mathfrak{D}_S)$, where \mathfrak{D}_S is a sheaf of supercommutative \mathbb{R} -algebras over \mathbb{R} defined, for each $U \subset \mathbb{R}$ open, by

$$\mathfrak{D}_S(U) = C^\infty(U) \otimes \Lambda^\bullet(\mathbb{R}) = C^\infty(U)[\theta],$$

where θ , the generator of the exterior algebra $\Lambda^\bullet(\mathbb{R})$, is called the odd generator. We also normally denote the even indeterminate by x . Supermanifolds form a

category where a morphism $f : S_1 \rightarrow S_2$ is a morphism of the underlying super ringed spaces [6, 13, 14].

A super Lie group is a group object in the category of supermanifolds. Alternatively, a super Lie group can be defined as a representable functor from the category of supermanifolds to the category of groups. A typical example is $GL(p, q) = GL(\mathbb{R}^{p,q})$, the super general linear group of automorphisms of $\mathbb{R}^{p,q}$. This supergroup has a matrix representation as follows. It is formed by matrices

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right),$$

where A, D are, respectively, $p \times p$ and $q \times q$ invertible matrices consisting of even elements, and C, B are, respectively, $p \times q$ and $q \times p$ matrices consisting of odd elements. As a special case, $GL(2, 1)$ is formed by matrices

$$\left(\begin{array}{cc|c} a & b & x \\ c & d & y \\ \hline z & w & e \end{array} \right),$$

where a, b, c, d, e are even elements, $ad - bc \neq 0$, $e \neq 0$, and x, y, z, w are odd elements.

Definition 3.1. The supergroup of orientation preserving diffeomorphisms of the super real line, $\mathbb{R}^{1,1}$, is defined as follows:

$$G^s = Diff^+(\mathbb{R}^{1,1}) = \{\Phi(x, \theta) = (A(x) + B(x)\theta, C(x) + D(x)\theta)\},$$

such that $A(x), D(x)$ are even, $A(x), D(x) \in Diff(\mathbb{R})$, $\dot{A}(x) > 0$, $\dot{D}(x) > 0$, and $B(x), C(x)$ are odd.

Definition 3.2. The affine part of G^s , denoted by G_1^s and also denoted by $Aff(\mathbb{R}^{1,1})$, is defined by:

$$G_1^s = \{\psi(x, \theta) = (ax + b\theta + e, cx + d\theta + f) \in G^s\},$$

such that a, e, d are even, $a, d > 0$, and b, c, f are odd. An element $\psi(x, \theta) = (ax + b\theta + e, cx + d\theta + f)$ of G_1^s can also be represented in the following way:

$$\psi(x, \theta) = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array} \right) \begin{pmatrix} x \\ \theta \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}.$$

Therefore there exists the following representation of G_1^s in $GL(2, 1)$, which will be very useful for our purpose:

$$G_1^s = \left\{ \psi = \left(\begin{array}{cc|c} a & e & b \\ 0 & 1 & 0 \\ \hline c & f & d \end{array} \right) \mid a, e, d \text{ are even, } b, c, f \text{ are odd, and } a, d > 0 \right\}.$$

Definition 3.3. A second super subgroup G_2^s of G^s is defined by:

$$G_2^s = \{\phi = (A(x) + B(x)\theta, C(x) + D(x)\theta) \in G^s \mid \phi(0, 0) = 0, J\phi(0, 0) = 1\}.$$

In other words

$$(3.1) \quad G_2^s = \{\phi = (A(x) + B(x)\theta, C(x) + D(x)\theta) \in G^s\},$$

where $A(0) = B(0) = C(0) = \dot{C}(0) = 0$, and $\dot{A}(0) = D(0) = 1$.

In order to proceed to the factorization $G^s = G_1^s G_2^s$, for any $\Phi \in Diff^+(\mathbb{R}^{1,1})$ we define

$$(3.2) \quad \pi_1(\Phi) = (J\Phi(0,0), \Phi(0,0)) = \left(\frac{\dot{A}(0) \mid B(0)}{\dot{C}(0) \mid D(0)} \right) \begin{pmatrix} x \\ \theta \end{pmatrix} + \begin{pmatrix} A(0) \\ C(0) \end{pmatrix} \in G_1^s,$$

and

$$(3.3) \quad \begin{aligned} \pi_2(\Phi) &= (J\Phi(0,0))^{-1}(\Phi(x,\theta) - \Phi(0,0)) \\ &= \left(\frac{\dot{A}(0) \mid B(0)}{\dot{C}(0) \mid D(0)} \right)^{-1} (\Phi(x,\theta) - \Phi(0,0)) \in G_2^s. \end{aligned}$$

Here we have used the definition

$$J\Phi(x,\theta) := \left(\frac{\frac{\partial \Phi_1}{\partial x} \mid -\frac{\partial \Phi_1}{\partial \theta}}{\frac{\partial \Phi_2}{\partial x} \mid \frac{\partial \Phi_2}{\partial \theta}} \right) = \left(\frac{\dot{A}(x) + \dot{B}(x)\theta \mid B(x)}{\dot{C}(x) + \dot{D}(x)\theta \mid D(x)} \right),$$

where $\Phi_1 = A(x) + B(x)\theta$ and $\Phi_2 = C(x) + D(x)\theta$ are, respectively, the even and odd components of $\Phi \in Diff^+(\mathbb{R}^{1,1})$. The operator $\frac{\partial}{\partial x}$ is even and the operator $\frac{\partial}{\partial \theta}$ is odd. Also, the formula for the inverse supermatrix is

$$\left(\frac{a \mid b}{c \mid d} \right)^{-1} = \frac{1}{da} \left(\frac{d + \frac{bc}{a} \mid -b}{-c \mid a + \frac{cb}{d}} \right).$$

Now if we let $\psi(x,\theta) = \pi_1(\Phi)$ and $\phi(x,\theta) = \pi_2(\Phi)$, it is clear that for any $\Phi \in Diff^+(\mathbb{R}^{1,1})$

$$\Phi = \psi\phi,$$

which proves the factorization

$$G^s = G_1^s G_2^s.$$

Therefore, we have the following two natural actions. The left action of G_2^s on G_1^s , $G_2^s \times G_1^s \rightarrow G_1^s$, defined by $\phi \triangleright \psi = \pi_1(\phi\psi)$, and the right action of G_1^s on G_2^s , $G_2^s \times G_1^s \rightarrow G_2^s$, defined by $\phi \triangleleft \psi = \pi_2(\phi\psi)$.

4. The super Hopf algebras $U(\mathfrak{g}_1^s)$ and $F(G_2^s)$

In this section we assign to supergroups G_1^s and G_2^s , the super Hopf algebras $U(\mathfrak{g}_1^s)$ and $F(G_2^s)$, respectively. The super Hopf algebra $U(\mathfrak{g}_1^s)$ is just the universal enveloping algebra of the super Lie algebra $\mathfrak{g}_1^s = Lie(G_1^s)$. As for $F(G_2^s)$, we define the coordinate functions a_n, b_n, c_n and d_n on G_2^s . Then $F(G_2^s)$ would be the corresponding *Faà di Bruno* or rather *comeasuring* super Hopf algebra generated, as a supercommutative superalgebra, by a_n, b_n, c_n and d_n , for which we will define the Hopf algebra structure.

4.1. The super Hopf algebra $U(\mathfrak{g}_1^s)$. The super Lie algebra $\mathfrak{g}_1^s = Lie(G_1^s)$ is generated by three even generators:

$$X = \left(\frac{0 \ 1 \mid 0}{0 \ 0 \mid 0} \right), \quad Y = \left(\frac{1 \ 0 \mid 0}{0 \ 0 \mid 0} \right), \quad Z = \left(\frac{0 \ 0 \mid 0}{0 \ 0 \mid 1} \right),$$

and three odd generators

$$U = \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad V = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{array} \right), \quad W = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{array} \right).$$

It is easy to check that the following *super bracket* relations hold:

$$\begin{aligned} [Y, X] &= X, & [X, Z] &= 0, & [Y, Z] &= 0, \\ [X, U] &= 0, & [X, V] &= -W, & [X, W] &= 0, \\ [Y, U] &= U, & [Y, V] &= -V, & [Y, W] &= 0, \\ [Z, U] &= -U, & [Z, V] &= V, & [Z, W] &= W, \\ [U, V] &= -(Y + Z), & [U, W] &= -X, & [V, W] &= 0. \end{aligned}$$

(4.1) $[X, X] = [Y, Y] = [Z, Z] = [U, U] = [V, V] = [W, W] = 0.$

The super Hopf algebra $U(\mathfrak{g}_1^s)$ is the universal enveloping algebra of \mathfrak{g}_1^s .

4.2. The super Hopf algebra $F(G_2^s)$. The super Hopf algebra $F(G_2^s)$, as a supercommutative superalgebra, is the super polynomial algebra $\mathbb{R}[a_n, b_n, c_n, d_n]$, generated by two sets of even generators $a_n, d_n, n \geq 0, a_0 = 0, a_1 = d_0 = 1$, and two sets of odd generators $b_n, c_n, n \geq 0, b_0 = c_0 = c_1 = 0$, where for any $\phi(x, \theta) = (A(x) + B(x)\theta, C(x) + D(x)\theta)$ in G_2^s we have:

$$\begin{aligned} a_n(\phi) &= (1/n!)A^{(n)}(0), \\ b_n(\phi) &= (1/n!)B^{(n)}(0), \\ c_n(\phi) &= (1/n!)C^{(n)}(0), \\ d_n(\phi) &= (1/n!)D^{(n)}(0). \end{aligned}$$

(4.2)

To define the coproduct on $F(G_2^s)$, analogous to the non-super case, we use the following formalism:

$$(4.3) \quad m\Delta(a_n)(\phi \otimes \phi') = a_n(\phi \circ \phi'), \quad \phi, \phi' \in G_2^s,$$

and similar formulas for b_n, c_n and d_n . Note that using this formalism provides us with the coassociativity property of Δ .

We need to study the composition of two elements of G_2^s . Let $\phi(x, \theta) = (A(x) + B(x)\theta, C(x) + D(x)\theta)$ and $\phi'(x, \theta) = (A'(x) + B'(x)\theta, C'(x) + D'(x)\theta)$ be two elements of G_2^s . Let $f = A'(x) + B'(x)\theta$ and $g = C'(x) + D'(x)\theta$. It is easy to check that the composition is given by

$$\phi \circ \phi'(x, \theta) = (A(f) + B(f)g, C(f) + D(f)g),$$

where

$$\begin{aligned} A(f) + B(f)g &= [A(A'(x) + B(A'(x))C'(x)) + \\ & \quad [\dot{A}(A'(x))B'(x) + B(A'(x))D'(x) - \dot{B}(A'(x))B'(x)C'(x)]\theta, \end{aligned}$$

and

$$\begin{aligned} C(f) + D(f)g &= [C(A'(x) + D(A'(x))C'(x)) + \\ & \quad [\dot{C}(A'(x))B'(x) + D(A'(x))D'(x) - \dot{D}(A'(x))B'(x)C'(x)]\theta. \end{aligned}$$

Now we proceed to comultiplications, starting with $\Delta(a_n)$. It is not hard to show that

$$a_1(\phi \circ \phi') = 1 = m(1 \otimes 1)(\phi \otimes \phi'),$$

$$a_2(\phi \circ \phi') = a'_2 + a_2 = m(1 \otimes a_2 + a_2 \otimes 1)(\phi \otimes \phi'),$$

$$a_3(\phi \circ \phi') = a'_3 + 2a_2a'_2 + a_3 + b_1c'_2 = m(1 \otimes a_3 + 2a_2 \otimes a_2 + a_3 \otimes 1 + b_1 \otimes c_2)(\phi \otimes \phi').$$

Therefore by (4.3) we have:

$$\Delta(a_1) = \Delta(1) = 1 \otimes 1,$$

$$\Delta(a_2) = 1 \otimes a_2 + a_2 \otimes 1,$$

$$\Delta(a_3) = 1 \otimes a_3 + a_3 \otimes 1 + 2a_2 \otimes a_2 + b_1 \otimes c_2,$$

and more generally for $n \geq 1$,

$$\begin{aligned} \Delta(a_n) = & \sum_{k=1}^n a_k \otimes \sum_{l_1+l_2+\dots+l_k=n} a_{l_1}a_{l_2}\dots a_{l_k} + \\ & \sum_{i=1}^n \sum_{k=1}^i b_k \otimes \left(\sum_{l_1+l_2+\dots+l_k=i} a_{l_1}a_{l_2}\dots a_{l_k} \right) c_{n-i}. \end{aligned}$$

With a similar method we have, for $\Delta(b_n)$,

$$\Delta(b_1) = 1 \otimes b_1 + b_1 \otimes 1,$$

$$\Delta(b_2) = 1 \otimes b_2 + b_2 \otimes 1 + 2a_2 \otimes b_1 + b_1 \otimes d_1 + b_1 \otimes a_2,$$

$$\begin{aligned} \Delta(b_3) = & 1 \otimes b_3 + b_3 \otimes 1 + 2a_2 \otimes b_2 + 2a_2 \otimes a_2b_1 + 3a_3 \otimes b_1 + b_1 \otimes d_2 \\ & + b_1 \otimes a_2d_1 + b_2 \otimes d_1 + b_1 \otimes a_3 + 2b_2 \otimes a_2 - b_1 \otimes b_1c_2, \end{aligned}$$

and in general

$$\begin{aligned} \Delta(b_n) = & 1 \otimes b_n + \sum_{i=1}^n \sum_{k=1}^i (k+1)a_{k+1} \otimes \left(\sum_{l_1+l_2+\dots+l_k=i} a_{l_1}a_{l_2}\dots a_{l_k} \right) b_{n-i} \\ & + \sum_{i=1}^n \sum_{k=1}^i b_k \otimes \left(\sum_{l_1+l_2+\dots+l_k=i} a_{l_1}a_{l_2}\dots a_{l_k} \right) d_{n-i} \\ - & \sum_{i=1}^n \left(b_1 \otimes b_i c_{n-i} + \sum_{j=1}^i \sum_{k=1}^j (k+1)b_{k+1} \otimes \left(\sum_{l_1+l_2+\dots+l_k=j} a_{l_1}a_{l_2}\dots a_{l_k} \right) b_{i-j} c_{n-i} \right). \end{aligned}$$

For $\Delta(c_n)$, $n \geq 1$, we have:

$$\begin{aligned} \Delta(c_n) = & 1 \otimes c_n + \sum_{k=1}^n c_k \otimes \sum_{l_1+l_2+\dots+l_k=n} a_{l_1}a_{l_2}\dots a_{l_k} + \\ & \sum_{i=1}^n \sum_{k=1}^i d_k \otimes \left(\sum_{l_1+l_2+\dots+l_k=i} a_{l_1}a_{l_2}\dots a_{l_k} \right) c_{n-i}. \end{aligned}$$

In particular for $n = 2, 3$:

$$\Delta(c_2) = 1 \otimes c_2 + c_2 \otimes 1,$$

$$\Delta(c_3) = 1 \otimes c_3 + c_3 \otimes 1 + 2c_2 \otimes a_2 + d_1 \otimes c_2.$$

For $\Delta(d_n)$, $n \geq 1$, we have:

$$\begin{aligned} \Delta(d_n) &= 1 \otimes d_n + \sum_{i=1}^n \sum_{k=1}^i (k+1)c_{k+1} \otimes \left(\sum_{l_1+l_2+\dots+l_k=i} a_{l_1} a_{l_2} \cdots a_{l_k} \right) b_{n-i} \\ &\quad + \sum_{i=1}^n \sum_{k=1}^i d_k \otimes \left(\sum_{l_1+l_2+\dots+l_k=i} a_{l_1} a_{l_2} \cdots a_{l_k} \right) d_{n-i} \\ &- \sum_{i=1}^n \left(d_1 \otimes b_i c_{n-i} + \sum_{j=1}^i \sum_{k=1}^j (k+1)d_{k+1} \otimes \left(\sum_{l_1+l_2+\dots+l_k=j} a_{l_1} a_{l_2} \cdots a_{l_k} \right) b_{i-j} c_{n-i} \right). \end{aligned}$$

In particular for $n = 1, 2, 3$:

$$\Delta(d_1) = 1 \otimes d_1 + d_1 \otimes 1,$$

$$\Delta(d_2) = 1 \otimes d_2 + d_2 \otimes 1 + 2c_2 \otimes b_1 + d_1 \otimes d_1 + d_1 \otimes a_2,$$

$$\begin{aligned} \Delta(d_3) &= 1 \otimes d_3 + d_3 \otimes 1 + 2c_2 \otimes b_2 + 2c_2 \otimes a_2 b_1 + 3c_3 \otimes b_1 + d_1 \otimes d_2 \\ &\quad + d_1 \otimes a_2 d_1 + d_2 \otimes d_1 + d_1 \otimes a_3 + 2d_2 \otimes a_2 - d_1 \otimes b_1 c_2. \end{aligned}$$

We extend Δ linearly to $F(G_2)$ via the relation

$$\Delta(ab) := \Delta(a)\Delta(b), \forall a, b \in F(G_2),$$

where multiplication on the right hand side operates in the super sense. We also set $\varepsilon(1) = 1$, and define ε to be equal to zero on all other generators. This defines a super bialgebra structure on $F(G_2)$.

To prove that $F(G_2^s)$ is a super Hopf algebra, the only missing data is the antipode map. Analogous to the non-super case, it suffices to define an anti-algebra and coalgebra map $S : F(G_2^s) \rightarrow F(G_2^s)$ satisfying

$$S(a)(\phi) = a(\phi^{-1}), \quad \phi \in G_2^s, \quad a = a_n, b_n, c_n, d_n.$$

We shall do this in an inductive fashion at the end of the next section, where we define the actions and coactions between $F(G_2^s)$ and $U(\mathfrak{g}_1^s)$. The reason behind this is that we want to use the fact that the antipode, if it exists, should interact, *in a nice way*, with those actions and coactions, in the sense of relation (5.8) below [10].

5. Actions and coactions

In this section we prove that $F(G_2^s)$ is a super left $U(\mathfrak{g}_1^s)$ -module algebra and $U(\mathfrak{g}_1^s)$ is a super right $F(G_2^s)$ -comodule coalgebra.

For $Y = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \in \mathfrak{g}_1^s$ we have

$$e^{tY} = \left(\begin{array}{cc|c} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) = (e^t x, \theta) \in G_1^s, \quad (t \text{ even}).$$

The action $Y \triangleright a_n$ can be realized by

$$(Y \triangleright a_n)(\phi) = \frac{d}{dt} \Big|_{t=0} a_n(\phi \triangleleft e^{tY}),$$

for any $\phi = (A(x) + B(x)\theta, C(x) + D(x)\theta) \in G_2^s$. Thus, by computing $\phi \triangleleft e^{tY} = \pi_2(\phi e^{tY})$, we obtain

$$a_n(\phi \triangleleft e^{tY}) = (1/n!)e^{(n-1)t}A^{(n)}(0),$$

and

$$(Y \triangleright a_n)(\phi) = \frac{d}{dt}\Big|_{t=0} a_n(\phi \triangleleft e^{tY}) = (n-1)a_n(\phi).$$

This implies that

$$Y \triangleright a_n = (n-1)a_n.$$

In the same way we have

$$\begin{aligned} Y \triangleright b_n &= (n-1)b_n, \\ Y \triangleright c_n &= nc_n, \\ Y \triangleright d_n &= nd_n. \end{aligned}$$

Using the same method we can derive all other actions. In fact we have the following lemma:

Lemma 5.1. *Let us define the actions of X, Y, Z, U, V, W on a_n, b_n, c_n, d_n by the following relations:*

$$\begin{aligned} X \triangleright a_n &= (n+1)a_{n+1} - 2a_n a_2 - b_1 c_n, & X \triangleright b_n &= (n+1)b_{n+1} - 2b_n a_2 - b_1 d_n, \\ X \triangleright c_n &= -2c_2 a_n + (n+1)c_{n+1} - c_n d_1, & X \triangleright d_n &= -2c_2 b_n + (n+1)d_{n+1} - d_n d_1, \\ Y \triangleright a_n &= (n-1)a_n, & Y \triangleright b_n &= (n-1)b_n, & Y \triangleright c_n &= nc_n, & Y \triangleright d_n &= nd_n, \\ Z \triangleright a_n &= 0, & Z \triangleright b_n &= b_n, & Z \triangleright c_n &= -c_n, & Z \triangleright d_n &= 0, \\ U \triangleright a_n &= c_n, & U \triangleright b_n &= -(n+1)a_{n+1} + d_n, & U \triangleright c_n &= 0, & U \triangleright d_n &= (n+1)c_{n+1}, \\ V \triangleright a_n &= b_{n-1}, & V \triangleright b_n &= 0, & V \triangleright c_n &= a_n - d_{n-1}, & V \triangleright d_n &= +b_n, \\ W \triangleright a_n &= -b_1 a_n + b_n, & W \triangleright b_n &= -b_1 b_n, \\ (5.1) \quad W \triangleright c_n &= d_1 a_n - d_n, & W \triangleright d_n &= d_1 b_n, \end{aligned}$$

and extend those actions to an action of $U(\mathfrak{g}_1^s)$ on $F(G_2^s)$ such that

$$(5.2) \quad (gh) \triangleright a = g \triangleright (h \triangleright a),$$

$$(5.3) \quad h \triangleright (ab) := (-1)^{|a||h_{(2)}|} (h_{(1)} \triangleright a)(h_{(2)} \triangleright b),$$

for a, b in $F(G_2^s)$ and g, h in $U(\mathfrak{g}_1^s)$. Then $F(G_2^s)$ is a super left $U(\mathfrak{g}_1^s)$ -module algebra.

PROOF. It is enough to show that this action is consistent with the bracket relations (4.1). We check this just for some of the bracket relations. The rest would be the same:

$$\begin{aligned} (YX) \triangleright a_n &\stackrel{(5.2)}{=} Y \triangleright (X \triangleright a_n) \stackrel{(5.1)}{=} Y \triangleright ((n+1)a_{n+1} - 2a_n a_2 - b_1 c_n) \\ &= (n+1)(Y \triangleright a_{n+1}) - 2Y \triangleright (a_n a_2) - Y \triangleright (b_1 c_n) \\ &\stackrel{(5.3)}{=} n(n+1)a_{n+1} - 2(a_n(Y \triangleright a_2) + (Y \triangleright a_n)a_2) - (b_1(Y \triangleright c_n) + (Y \triangleright b_1)c_n) \\ &= (n^2 + n)a_{n+1} - 2na_n a_2 - nb_1 c_n, \\ (XY) \triangleright a_n &= X \triangleright (Y \triangleright a_n) = X \triangleright ((n-1)a_n) \\ &= (n-1)((n+1)a_{n+1} - 2a_n a_2 - b_1 c_n) \\ &= (n^2 - 1)a_{n+1} - 2(n-1)a_n a_2 - (n-1)b_1 c_n. \end{aligned}$$

Therefore,

$$[Y, X] \triangleright a_n = (YX - XY) \triangleright a_n = (n+1)a_{n+1} - 2a_n a_2 - b_1 c_n \stackrel{(5.1)}{=} X \triangleright a_n,$$

which is consistent with the relation $[Y, X] = X$ of (4.1). Next we check a bracket involving odd generators:

$$\begin{aligned} (UW) \triangleright d_n &= U \triangleright (W \triangleright d_n) = U \triangleright (d_1 b_n) \\ &= (d_1(U \triangleright b_n) + (U \triangleright d_1)b_n) = d_1(-(n+1)a_{n+1} + d_n) + 2c_2 b_n \\ &= -(n+1)d_1 a_{n+1} + d_1 d_n + 2c_2 b_n, \\ (WU) \triangleright d_n &= W \triangleright (U \triangleright d_n) = W \triangleright ((n+1)c_{n+1}) = (n+1)d_1 a_{n+1} - (n+1)d_{n+1}. \end{aligned}$$

Therefore,

$$[U, W] \triangleright d_n = (UW + WU) \triangleright d_n = 2c_2 b_n - (n+1)d_{n+1} + d_1 d_n = (-X) \triangleright d_n,$$

which agrees with the relation $[U, W] = -X$. \square

By using almost the same method we can find the coactions and prove the following lemma. Let us denote the right coaction of $F(G_2^s)$ on $U(\mathfrak{g}_1^s)$, by $\nabla_r : U(\mathfrak{g}_1^s) \rightarrow U(\mathfrak{g}_1^s) \otimes F(G_2^s)$.

Lemma 5.2. *Let us define the coactions of $F(G_2^s)$ on generators X, Y, Z, U, V, W of $U(\mathfrak{g}_1^s)$ by*

$$\begin{aligned} \nabla_r(X) &= 2Y \otimes a_2 + X \otimes 1 + Z \otimes d_1 + U \otimes b_1 + 2V \otimes c_2, & \nabla_r(Y) &= Y \otimes 1, \\ \nabla_r(Z) &= Z \otimes 1, & \nabla_r(U) &= U \otimes 1, & \nabla_r(V) &= V \otimes 1, \\ (5.4) \quad \nabla_r(W) &= Y \otimes b_1 + V \otimes d_1 + W \otimes 1, \end{aligned}$$

and extend them to $U(\mathfrak{g}_1^s)$ such that

$$(5.5) \quad \nabla_r(gh) = (-1)^{|h^{(1)}|(|g_{(1)}^{(2)}|+|g_{(2)}|)} g_{(1)}^{(1)} h^{(1)} \otimes g_{(1)}^{(2)} (g_{(2)} \triangleright h^{(2)}),$$

for all g and h in $U(\mathfrak{g}_1^s)$. Then $U(\mathfrak{g}_1^s)$ is a super right $F(G_2^s)$ -comodule coalgebra.

PROOF. It is straightforward to check the coaction property, $(id \otimes \Delta)\nabla_r(h) = (\nabla_r \otimes id)\nabla_r(h)$, for all h in $U(\mathfrak{g}_1^s)$, in other words:

$$h^{(1)} \otimes h^{(2)}_{(1)} \otimes h^{(2)}_{(2)} = h^{(1)(1)} \otimes h^{(1)(2)} \otimes h^{(2)}.$$

We prove that this coaction is consistent with the bracket relations (4.1). We verify this only for one of the purely odd cases. The rest are similar. By formula (5.5) we have

$$\begin{aligned} \nabla_r(VW) &= (-1)^{|W^{(1)}|(|V_{(1)}^{(2)}|+|V_{(2)}|)} V_{(1)}^{(1)} W^{(1)} \otimes V_{(1)}^{(2)} (V_{(2)} \triangleright W^{(2)}) \\ &= [(-1)^{|W^{(1)}||V|} W^{(1)} \otimes (V \triangleright W^{(2)})] \\ &\quad + [(-1)^{|W^{(1)}||V^{(2)}|} V^{(1)} W^{(1)} \otimes V^{(2)} W^{(2)}] \\ &= [(-1)^{|W^{(1)}|} W^{(1)} \otimes (V \triangleright W^{(2)})] + [V^{(1)} W^{(1)} \otimes W^{(2)}] \\ &= [Y \otimes (V \triangleright b_1) - V \otimes (V \triangleright d_1) - W \otimes (V \triangleright 1)] \\ &\quad + [VY \otimes b_1 + V^2 \otimes d_1 + VW \otimes 1] \\ &= [-V \otimes b_1] + [VY \otimes b_1 + V^2 \otimes d_1 + VW \otimes 1, \end{aligned}$$

and

$$\begin{aligned}
\nabla_r(WV) &= (-1)^{|V^{(1)}|(|W_{(1)}^{(2)}|+|W_{(2)}|)} W_{(1)}^{(1)}V^{(1)} \otimes W_{(1)}^{(2)}(W_{(2)} \triangleright V^{(2)}) \\
&= [(-1)^{|V^{(1)}||W|} V^{(1)} \otimes (W \triangleright V^{(2)})] \\
&\quad + [(-1)^{|V^{(1)}||W^{(2)}|} W^{(1)}V^{(1)} \otimes W^{(2)}V^{(2)}] \\
&= [-V \otimes (0)] + [(-1)^{|W^{(2)}|} W^{(1)}V \otimes W^{(2)}] \\
&= -YV \otimes b_1 + V^2 \otimes d_1 + WV \otimes 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
\nabla_r([V, W]) &= \nabla_r(VW + WV) = \nabla_r(VW) + \nabla_r(WV) \\
&= [V, Y] \otimes b_1 + [V, V] \otimes d_1 + [V, W] \otimes 1 - v \otimes b_1 = 0,
\end{aligned}$$

which agrees with $[V, W] = 0$ of relations (4.1).

We also leave it to the reader to check that $U(\mathfrak{g}_1^s)$ is a right $F(G_2^s)$ -comodule coalgebra, i.e., for all h in $U(\mathfrak{g}_1^s)$,

$$h^{(1)}_{(1)} \otimes h^{(1)}_{(2)} \otimes h^{(2)} = (-1)^{|h_{(2)}^{(1)}||h_{(1)}^{(2)}|} h_{(1)}^{(1)} \otimes h_{(2)}^{(1)} \otimes h_{(1)}^{(2)} h_{(2)}^{(2)}.$$

□

Antipode for $F(G_2^s)$. Now we prove that $F(G_2^s)$ is a super Hopf algebra, by defining an anti-algebra and coalgebra map $S : F(G_2^s) \rightarrow F(G_2^s)$ satisfying the antipode property or the following identity:

$$(5.6) \quad S(a)(\phi) = a(\phi^{-1}), \quad \phi \in G_2^s, \quad a = a_n, b_n, c_n, d_n.$$

We do this, inductively, by defining S on generators a_n, b_n, c_n, d_n , and then extend it linearly to $F(G_2^s)$ via the following relations:

$$S(ab) := (-1)^{|a||b|} S(b)S(a),$$

$$\Delta(S(a)) := (-1)^{|a_{(1)}||a_{(2)}|} S(a_{(2)}) \otimes S(a_{(1)}).$$

Let us define, first, $S(1) = 1$, and $S(a) = -a$, for $a = a_2, b_1, c_2, d_1$. It is obvious that the antipode property holds for these elements. Now suppose relation (5.6) is true for all elements a_i, b_i, c_i, d_i , $i \leq n$. The actions of X defined in relations (5.1) in Lemma 5.1 allow us to write the higher degree elements, $a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}$, in terms of some other elements of lower degree, a_i, b_i, c_i, d_i , $i \leq n$. Therefore, to finish the process it is enough to prove that

$$(5.7) \quad S(X \triangleright a)(\phi) = (X \triangleright a)(\phi^{-1}), \quad \phi \in G_2^s, \quad a = a_i, b_i, c_i, d_i, \quad i \leq n.$$

To prove this identity we need the following three lemmas.

Lemma 5.3. [10] *The antipode $S : F(G_2^s) \rightarrow F(G_2^s)$, if it exists, should satisfy the following relation:*

$$(5.8) \quad S(g \triangleright a) = (g^{(1)} \triangleright S(a))S(g^{(2)}), \quad g \in U(\mathfrak{g}_1^s), a \in F(G_2^s)$$

Lemma 5.4. *For any $\phi = (A(x) + B(x)\theta, C(x) + D(x)\theta)$ in G_2^s , one can prove*

$$(5.9) \quad \frac{d}{dt}\Big|_{t=0} \left(e^{tX^{(1)}} X^{(2)}(\phi) \right) = \frac{d}{dt}\Big|_{t=0} (\phi \triangleright e^{tX}).$$

PROOF. From the coactions defined in Lemma 5.2 we have:

$$\begin{aligned}
\text{LHS} &= \frac{d}{dt}\Big|_{t=0} (e^{tY} 2a_2(\phi) + e^{tX} 1(\phi) + e^{tZ} d_1(\phi) - e^{tU} b_1(\phi) - e^{tV} 2c_2(\phi)) \\
&= \frac{d}{dt}\Big|_{t=0} ((e^t X, \theta) 2a_2 + (X + t, \theta) + (X, e^t \theta) d_1 \\
&\quad - (X + t\theta, \theta) b_1 - (X, -tX + \theta) 2c_2) \\
&= (X, 0) 2a_2 + (1, 0) + (0, \theta) d_1 - (\theta, 0) b_1 - (0, -X) 2c_2 \\
&= (2a_2 X + 1 + b_1 \theta, 2c_2 X + d_1 \theta)
\end{aligned}$$

From the discussion in Section 3 and formula (3.2), we have

$$\begin{aligned}
\text{LHS} &= \frac{d}{dt}\Big|_{t=0} (\pi_1(\phi(X + t, \theta))) = \frac{d}{dt}\Big|_{t=0} \left(\left(\frac{\dot{A}(t)}{C(t)} \middle| \frac{B(t)}{D(t)} \right) \begin{pmatrix} x \\ \theta \end{pmatrix} + \begin{pmatrix} A(t) \\ C(t) \end{pmatrix} \right) \\
&= \frac{d}{dt}\Big|_{t=0} (\dot{A}(t)X + A(t) + B(t)\theta, \dot{C}(t)X + C(t) + D(t)\theta) \\
&= (\ddot{A}(0)X + \dot{A}(0) + \dot{B}(0)\theta, \ddot{C}(0)X + \dot{C}(0) + \dot{D}(0)\theta) \\
&= (2a_2 X + 1 + b_1 \theta, 2c_2 X + d_1 \theta) \\
&= \text{RHS}
\end{aligned}$$

□

Lemma 5.5. *If $G = G_1 G_2$ is a factorisation of the group G , then, for any φ_1, φ_2 in G_2 and ψ in G_1 , one has*

$$(5.10) \quad (\varphi_1 \varphi_2) \triangleleft \psi = (\varphi_1 \triangleleft (\varphi_2 \triangleright \psi)) (\varphi_2 \triangleleft \psi)$$

Now we prove the identity (5.7), $S(X \triangleright a)(\phi) = (X \triangleright a)(\phi^{-1})$.

PROOF.

$$\begin{aligned}
S(X \triangleright a)(\phi) &\stackrel{\text{Lemma 5.3}}{=} (X^{(1)} \triangleright S(a))(\phi) S(X^{(2)})(\phi) \\
&= \frac{d}{dt}\Big|_{t=0} (S(a)(\phi \triangleleft e^{tX^{(1)}})) S(X^{(2)})(\phi) \\
&\stackrel{\text{Induction}}{=} \frac{d}{dt}\Big|_{t=0} (a((\phi \triangleleft e^{tX^{(1)}})^{-1})) X^{(2)}(\phi^{-1}) \\
&\stackrel{\text{Lemma 5.5}}{=} \frac{d}{dt}\Big|_{t=0} (a(\phi^{-1} \triangleleft (\phi \triangleright e^{tX^{(1)}}))) X^{(2)}(\phi^{-1}) \\
&\stackrel{\text{Lemma 5.4}}{=} \frac{d}{dt}\Big|_{t=0} (a(\phi^{-1} \triangleleft e^{tX})) \\
&= (X \triangleright a)(\phi^{-1})
\end{aligned}$$

□

6. Compatibilities and the super Hopf algebra \mathcal{H}_1^s

The following proposition is the super analogue of Lemma (2.3). It gives the compatibility conditions to construct a bicrossproduct super Hopf algebra. To complete the construction of the bicrossproduct super Hopf algebra $F(G_2^s) \blacktriangleright \blacktriangleleft U(\mathfrak{g}_1^s)$ one needs to check these compatibility conditions between the actions and coactions introduced in Lemmas (5.1) and (5.2) in the last section.

Proposition 6.1. Let A and H be two super Hopf algebras such that A is a left H -module algebra, and H is a right A -comodule coalgebra. Let furthermore these structures satisfy the follow compatibility conditions:

$$(6.1) \quad \Delta(h \triangleright a) = (-1)^{|\alpha_{(1)}|(|h_{(1)}^{(2)}|+|h_{(2)}|)} h_{(1)}^{(1)} \triangleright a_{(1)} \otimes h_{(1)}^{(2)}(h_{(2)} \triangleright a_{(2)}),$$

$$\varepsilon(h \triangleright a) = \varepsilon(h)\varepsilon(a),$$

$$(6.2) \quad \nabla_r(gh) = (-1)^{|h^{(1)}|(|g_{(1)}^{(2)}|+|g_{(2)}|)} g_{(1)}^{(1)}h^{(1)} \otimes g_{(1)}^{(2)}(g_{(2)} \triangleright h^{(2)}),$$

$$\nabla_r(1) = 1 \otimes 1,$$

(6.3)

$$(-1)^{|h_{(1)}||h_{(2)}^{(1)}|+|a||h_{(2)}^{(2)}|} h_{(2)}^{(1)} \otimes (h_{(1)} \triangleright a)h_{(2)}^{(2)} = h_{(1)}^{(1)} \otimes h_{(1)}^{(2)}(h_{(2)} \triangleright a),$$

for any a, b in A and g, h in H , where we have denoted the actions by $h \triangleright a$ and the coactions by $\nabla_r(h) = h_{(0)} \otimes h_{(1)}$. Then the super vector space $A \otimes H$ can be equipped with a super Hopf algebra structure as follows:

$$(6.4) \quad (a \otimes h)(b \otimes g) = (-1)^{|h_{(2)}||b|} a(h_{(1)} \triangleright b) \otimes h_{(2)}g,$$

$$(6.5) \quad \Delta(a \otimes h) = (-1)^{|h_{(1)}^{(1)}||a_{(2)}|} a_{(1)} \otimes h_{(1)}^{(1)} \otimes a_{(2)}h_{(1)}^{(2)} \otimes h_{(2)},$$

$$\varepsilon(a \otimes h) = \varepsilon(a)\varepsilon(h),$$

$$S(a \otimes h) = (-1)^{|h^{(1)}||a|} (1 \otimes S(h^{(1)}))(S(ah^{(2)}) \otimes 1),$$

for any a, b in A and g, h in H . We call this super Hopf algebra the left-right bicrossproduct super Hopf algebra $A \blacktriangleright \blacktriangleleft H$.

PROOF. We prove that Δ , defined in (6.5), is an algebra map. First we have:

(6.6)

$$\begin{aligned} \Delta((a \otimes h) \cdot (b \otimes g)) &\stackrel{(6.4)}{=} (-1)^{\alpha_0} \Delta(a(h_{(1)} \triangleright b) \otimes h_{(2)}g) \\ &\stackrel{(6.5)}{=} (-1)^{\alpha_0+\alpha_1} (a(h_{(1)} \triangleright b))_{(1)} \otimes (h_{(2)}g)_{(1)}^{(1)} \otimes (a(h_{(1)} \triangleright b))_{(2)}(h_{(2)}g)_{(1)}^{(2)} \\ &\quad \otimes (h_{(2)}g)_{(2)} \\ &= (-1)^{\alpha_0+\alpha_1+\alpha_2+\alpha_3} a_{(1)}(h_{(1)} \triangleright b)_{(1)} \otimes (h_{(2)(1)}g_{(1)})^{(1)} \\ &\quad \otimes a_{(2)}(h_{(1)} \triangleright b)_{(2)}(h_{(2)(1)}g_{(1)})^{(2)} \otimes h_{(2)(2)}g_{(2)} \\ &\stackrel{(6.1),(6.2)}{=} (-1)^{\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5} a_{(1)}(h_{(1)(1)}^{(1)} \triangleright b_{(1)}) \otimes (h_{(2)(1)(1)}^{(1)}g_{(1)}^{(1)}) \\ &\quad \otimes a_{(2)}h_{(1)(1)}^{(2)}(h_{(1)(2)} \triangleright b_{(2)})(h_{(2)(1)(1)}^{(2)}(h_{(2)(1)(2)} \triangleright g_{(1)}^{(2)})) \otimes h_{(2)(2)}g_{(2)} \\ &\stackrel{coassociativity}{=} (-1)^{\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5} a_{(1)}(h_{(1)}^{(1)} \triangleright b_{(1)}) \otimes (h_{(3)}^{(1)}g_{(1)}^{(1)}) \\ &\quad \otimes a_{(2)}h_{(1)}^{(2)}(h_{(2)} \triangleright b_{(2)})(h_{(3)}^{(2)}(h_{(4)} \triangleright g_{(1)}^{(2)})) \otimes h_{(5)}g_{(2)}, \end{aligned}$$

where

$$\begin{aligned}
\alpha_0 &= |h_{(2)}||b| = (|h_{(2)(1)(1)}| + |h_{(2)(1)(2)}| + |h_{(2)(2)}|)(|b_{(1)}| + |b_{(2)}|) \\
&= (|h_{(3)}| + |h_{(4)}| + |h_{(5)}|)(|b_{(1)}| + |b_{(2)}|) \\
\alpha_1 &= |(h_{(2)}g)_{(1)}|^{(1)}|(a(h_{(1)} \triangleright b)_{(2)})| \\
&= |(h_{(2)(1)(1)}|^{(1)}g_{(1)}|^{(1)}||a_{(2)}h_{(1)(1)}|^{(2)}(h_{(1)(2)} \triangleright b_{(2)})| \\
&= (|h_{(2)(1)(1)}|^{(1)} + |g_{(1)}|^{(1)})(|a_{(2)}| + |h_{(1)(1)}|^{(2)} + |h_{(1)(2)}| + |b_{(2)}|) \\
&= (|h_{(3)}|^{(1)} + |g_{(1)}|^{(1)})(|a_{(2)}| + |h_{(1)}|^{(2)} + |h_{(2)}| + |b_{(2)}|) \\
&= |g_{(1)}|^{(1)}|a_{(2)}| + |g_{(1)}|^{(1)}||h_{(1)}|^{(2)} + |g_{(1)}|^{(1)}|h_{(2)}| + |g_{(1)}|^{(1)}|b_{(2)}| \\
&\quad + |h_{(3)}|^{(1)}|a_{(2)}| + |h_{(3)}|^{(1)}||h_{(1)}|^{(2)} + |h_{(3)}|^{(1)}|h_{(2)}| + |h_{(3)}|^{(1)}|b_{(2)}| \\
\alpha_2 &= |a_{(2)}||h_{(1)} \triangleright b_{(1)}| = |a_{(2)}||h_{(1)(1)}|^{(1)} \triangleright b_{(1)}| \\
&= |a_{(2)}|(|h_{(1)(1)}|^{(1)} + |b_{(1)}|) \\
&= |a_{(2)}|(|h_{(1)}|^{(1)} + |b_{(1)}|) = |a_{(2)}||h_{(1)}|^{(1)} + |a_{(2)}||b_{(1)}| \\
\alpha_3 &= |h_{(2)(2)}||g_{(1)}| = |h_{(5)}||g_{(1)}| \\
\alpha_4 &= |b_{(1)}|(|h_{(1)(2)}| + |h_{(1)(1)}|^{(2)}) \\
&= |b_{(1)}|(|h_{(2)}| + |h_{(1)}|^{(2)}) = |b_{(1)}|(|h_{(2)}| + |b_{(1)}||h_{(1)}|^{(2)}) \\
\alpha_5 &= |g_{(1)}|^{(1)}(|h_{(2)(1)(2)}| + |h_{(2)(1)(1)}|^{(2)}) \\
&= |g_{(1)}|^{(1)}(|h_{(4)}| + |h_{(3)}|^{(2)}) = |g_{(1)}|^{(1)}|h_{(4)}| + |g_{(1)}|^{(1)}|h_{(3)}|^{(2)}
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\Delta(a \otimes h) \cdot \Delta(b \otimes g) \\
&\stackrel{(6.5)}{=} (-1)^{\beta_0} (a_{(1)} \otimes h_{(1)}|^{(1)} \otimes a_{(2)}h_{(1)}|^{(2)} \otimes h_{(2)}) \cdot \\
&\quad (b_{(1)} \otimes g_{(1)}|^{(1)} \otimes b_{(2)}g_{(1)}|^{(2)} \otimes g_{(2)}) \\
&= (-1)^{\beta_0 + \beta_1} (a_{(1)} \otimes h_{(1)}|^{(1)}) \cdot \\
&\quad (b_{(1)} \otimes g_{(1)}|^{(1)} \otimes (a_{(2)}h_{(1)}|^{(2)} \otimes h_{(2)}) \cdot (b_{(2)}g_{(1)}|^{(2)} \otimes g_{(2)}) \\
&\stackrel{(6.4)}{=} (-1)^{\beta_0 + \beta_1 + \beta_2} a_{(1)}(h_{(1)}|^{(1)}_{(1)} \triangleright b_{(1)}) \otimes h_{(1)}|^{(1)}_{(2)}g_{(1)}|^{(1)} \\
&\quad \otimes a_{(2)}h_{(1)}|^{(2)}(h_{(2)(1)} \triangleright b_{(2)}g_{(1)}|^{(2)}) \otimes h_{(2)(2)}g_{(2)}
\end{aligned}$$

where $\beta_0 = |h_{(1)}|^{(1)}|a_{(2)}| + |g_{(1)}|^{(1)}|b_{(2)}|$, $\beta_1 = |(b_{(1)} \otimes g_{(1)}|^{(1)})|(a_{(2)}h_{(1)}|^{(2)} \otimes h_{(2)})|$, and $\beta_2 = |b_{(1)}||h_{(1)}|^{(1)}_{(2)}| + |b_{(2)}g_{(1)}|^{(2)}||h_{(2)(2)}|$.

Continuing this computation, using the standard sign rules for superalgebras, coalgebras and Hopf algebras and coassociativity of Δ_H , we obtain

$$\begin{aligned}
(6.7) \quad \Delta(a \otimes h) \cdot \Delta(b \otimes g) &= (-1)^{\beta_0 + \beta_1 + \beta_2 + \beta_3 - \beta_4 - \beta_5 - \beta_6 + \beta_7 - \beta_8 + \beta_9 - \beta_{10} + \beta_{11} - \beta_{12} + \beta_{13} + \beta_{14} + \beta_{15}} \\
&\quad (a_{(1)}h_{(1)}|^{(1)} \triangleright b_{(1)} \otimes h_{(3)}|^{(1)}g_{(1)}|^{(1)}) \\
&\quad \otimes a_{(2)}h_{(1)}|^{(2)}(h_{(2)} \triangleright b_{(2)})h_{(3)}|^{(2)}(h_{(4)} \triangleright g_{(1)}|^{(2)}) \otimes h_{(5)}g_{(2)}
\end{aligned}$$

where at the end

$$\begin{aligned}
& \beta_0 + \beta_1 + \beta_2 + \beta_3 - \beta_4 - \beta_5 - \beta_6 + \beta_7 - \beta_8 + \beta_9 - \beta_{10} + \beta_{11} - \beta_{12} + \beta_{13} + \beta_{14} + \beta_{15} \\
&= (|h_{(3)}| + |h_{(4)}| + |h_{(5)}|)(|b_{(1)}| + |b_{(2)}|) \\
&\quad + |g_{(1)}^{(1)}||a_{(2)}| + |g_{(1)}^{(1)}||h_{(1)}^{(2)}| + |g_{(1)}^{(1)}||h_{(2)}| + |g_{(1)}^{(1)}||b_{(2)}| \\
&\quad + |h_{(3)}^{(1)}||a_{(2)}| + |h_{(3)}^{(1)}||h_{(1)}^{(2)}| + |h_{(3)}^{(1)}||h_{(2)}| + |h_{(3)}^{(1)}||b_{(2)}| \\
&\quad + |a_{(2)}||h_{(1)}^{(1)}| + |a_{(2)}||b_{(1)}| \\
&\quad + |h_{(5)}||g_{(1)}| + |b_{(1)}||h_{(2)}| + |b_{(1)}||h_{(1)}^{(2)}| + |g_{(1)}^{(1)}||h_{(4)}| + |g_{(1)}^{(1)}||h_{(3)}^{(2)}|
\end{aligned}$$

Therefore,

$$(6.8) \quad \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = \beta_0 + \beta_1 + \beta_2 + \beta_3 - \beta_4 - \beta_5 - \beta_6 + \beta_7 - \beta_8 + \beta_9 - \beta_{10} + \beta_{11} - \beta_{12} + \beta_{13} + \beta_{14} + \beta_{15}.$$

By (6.6), (6.7) and (6.8) we have

$$\Delta((a \otimes h) \cdot (b \otimes g)) = \Delta(a \otimes h) \cdot \Delta(b \otimes g).$$

The rest of the proof can be done by a similar method. \square

Remark 6.2. The above lemma is actually true in any symmetric monoidal category. The proof involves braiding diagrams.

We skip the lengthy, but computational, proof of the following theorem which is the main result of this paper.

Theorem 6.3. *The actions and coactions defined in Lemmas (5.1) and (5.2) satisfy the conditions (6.1)-(6.3) in Proposition (6.1). Therefore we have a bicrossproduct super Hopf algebra $F(G_2^s) \blacktriangleright U(\mathfrak{g}_1^s)$ with the following structures:*

$$\begin{aligned}
(a \otimes h)(b \otimes g) &= (-1)^{|h_{(2)}||b|} a(h_{(1)} \triangleright b) \otimes h_{(2)}g, \\
\Delta(a \otimes h) &= (-1)^{|h_{(1)}^{(1)}||a_{(2)}|} a_{(1)} \otimes h_{(1)}^{(1)} \otimes a_{(2)}h_{(1)}^{(2)} \otimes h_{(2)}, \\
\varepsilon(a \otimes h) &= \varepsilon(a)\varepsilon(h), \\
S(a \otimes h) &= (-1)^{|h_{(1)}^{(1)}||a|} (1 \otimes S(h^{(1)}))(S(ah^{(2)}) \otimes 1),
\end{aligned}$$

for any a, b in $F(G_2^s)$ and g, h in $U(\mathfrak{g}_1^s)$.

The bicrossproduct super Hopf algebra $\mathcal{H}_1^s := F(G_2^s) \blacktriangleright U(\mathfrak{g}_1^s)$ is the super version of the Connes-Moscovici Hopf algebra \mathcal{H}_1 .

References

- [1] P. Beazley Cohen, Y. Manin, and D. Zagier, *Automorphic pseudodifferential operators*, Algebraic aspects of integrable systems, 17–47, Progr. Nonlinear Differential Equations Appl., **26**, Birkhäuser Boston, Boston, MA, 1997.
- [2] A. Connes and H. Moscovici, *Hopf algebras, Cyclic cohomology and the transverse index theorem*, Comm. Math. Phys. **198** (1998), no. 1, 199–246.
- [3] A. Connes, and H. Moscovici, *Cyclic cohomology and Hopf algebra symmetry*, Lett. Math. Phys. **52** (2000), no. 1, 1–28.
- [4] A. Connes, and H. Moscovici, *Modular Hecke algebras and their Hopf symmetry*, Mosc. Math. J. **4** (2004), no. 1, 67–109, 310.
- [5] A. Connes, and H. Moscovici, *Rankin-Cohen brackets and the Hopf algebra of transverse geometry*, Mosc. Math. J. **4** (2004), no. 1, 111–130, 311.

- [6] P. Deligne, and J. W. Morgan, *Notes on supersymmetry (following Joseph Bernstein)*, *Quantum fields and strings: a course for mathematicians 1*, 41-97, American Mathematical Society, Providence, RI, 1999.
- [7] H. Figueroa, and J. Gracia-Bonda, *Combinatorial Hopf algebras in quantum field theory, I*, *Rev. Math. Phys.* **17** (2005), no. 8, 881–976.
- [8] T. Hadfield and S. Majid, *Bicrossproduct Approach to the Connes-Moscovici Hopf Algebra*, *J. Algebra* **312** (2007), 228-256.
- [9] S. Majid, *Foundations of quantum group theory*, Cambridge University Press, Cambridge, 1995.
- [10] H. Moscovici and B. Rangipour, *Hopf cyclic cohomology and transverse characteristic classes*, To appear.
- [11] H. Moscovici and B. Rangipour, *Cyclic cohomology of Hopf algebras of transverse symmetries in codimension 1*, *Adv. Math.* **210** (2007), no. 1, 323–374.
- [12] H. Moscovici and B. Rangipour, *Hopf algebras of primitive Lie pseudogroups and Hopf cyclic cohomology*, *Adv. Math.* **220** (2009), no. 3, 706–790.
- [13] A. Rogers, *Supermanifolds. Theory and applications*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.
- [14] V. S. Varadarajan, *Supersymmetry for mathematicians: an introduction*, Courant Lecture Notes in Mathematics, 11. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2004.

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