QUANTIZED OPEN CHAOTIC SYSTEMS

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ABSTRACT. Two different "wave chaotic" systems, involving complex eigenvalues or resonances, can be analyzed using common semiclassical methods. In particular, one obtains fractal Weyl upper bounds for the density of resonances/eigenvalues near the real axis, and a classical dynamical criterion for a spectral gap.

1. INTRODUCTION

These notes present a sketch of semiclassical methods which can be used to describe the spectral properties (and as a consequence, the long time properties) of a certain class of 1-particle quantum chaotic systems. Here are two examples:

- damped waves $\psi(x,t)$ on a compact riemannian manifold X of negative sectional curvature. The dynamics is described by the damped wave equation

$$\big(\partial_t^2 - \Delta_X + 2b(x)\partial_t\big)\psi(x,t) = 0\,, \tag{1}$$

  and the *damping function* $b(x) \geq 0$ is assumed to be smooth.
- quantum scattering on $\mathbb{R}^d$, described by the Schrödinger equation

$$i\hbar\partial_t\psi(x,t) = P(\hbar)\psi(x,t)\,,\quad P(\hbar) = -\frac{\hbar^2\Delta}{2} + V(x)\,, \tag{2}$$

  where the potential $V(x)$ has compact support and consists of 3 peaks centered on an equilateral triangle; $\hbar$ is Planck's constant.

These two systems seem very different. The configuration spaces are respectively compact and of infinite volume, the wave equation does not depend on $\hbar$; the damped wave equation can be rewritten in terms of a contracting semigroup, while the propagator $e^{-itP(\hbar)/\hbar}$ is unitary.

In the damped wave situation, Eq. (1) can be diagonalized by a discrete set of metastable modes $e^{-ik_jt}\psi_j(x)$, where $\psi_j(x) \in L^2(X)$ satisfies the generalized eigenvalue equation

$$\big(\Delta + k_j^2 + 2i\,b(x)\,k_j\big)\psi_j(x) = 0\,. \tag{3}$$

The eigenvalues k_j are complex, and lie in the strip $-2\max(b) \leq \Im k_j \leq 0$. We are interested in the distribution of these eigenvalues in the *high-frequency limit* $\Re k_j \to \infty$.

In the scattering situation (2), the Hamiltonian $P(\hbar)$ is selfadjoint on $L^2(\mathbb{R})$, with absolutely continuous spectrum on $\mathbb{R}_+$. However, the Green's function $(P(\hbar) - z)^{-1}(x,y)$, well-defined for $\Im z > 0$, admits a meromorphic continuation through $\mathbb{R}_+$ to the lower half-plane, with discrete poles $\{z_j(\hbar)\}$ of finite multiplicities (the *resonances* of $P(\hbar)$). We will investigate the distribution of these resonances, in the vicinity of a fixed energy $E > 0$, in the *semiclassical limit* $\hbar \to 0$. To each resonance is associated a metastable state $\psi_j(x)$, which is not in L^2 but formally decays with time as $e^{-itz_j(\hbar)/\hbar}\psi_j(x)$.

The *lifetime* of a metastable state is given by the *imaginary part* of the eigenvalue, $\tau_j = \frac{1}{2|\Im k_j|}$, resp. $\tau_j = \frac{\hbar}{2|\Im z_j(\hbar)|}$.

Here are some common features of the two systems. In §2.1 we show that the high-frequency limit $\Re k_j \to \infty$ is similar with the semiclassical limit $\hbar \to 0$. It is then relevant to study the corresponding *classical dynamics*: in the damped wave situation, it is the geodesic flow on X, while in the scattering case it is the Hamiltonian flow generated by the Hamiltonian $p(x,\xi) = \frac{|\xi|^2}{2} + V(x)$, in some energy interval $[E-\delta, E+\delta]$. Our assumptions on X or $V(x)$ imply that these classical flows are both "strongly chaotic".

In both cases, the long time properties of our quantum system involves a spectrum of complex "eigenvalues" associated with metastable states. Our main aim is to understand the distribution of the lifetimes τ_j in the semiclassical/high energy limit, especially the ones which are not infinitesimally small when $\hbar \to 0$: we will thus focus on resonances such that $|\Im z_j(\hbar)| = \mathcal{O}(\hbar)$.

2. Transformation to nonselfadjoint spectral problems

Each of these two quantum systems can be recast into a spectral problem for an associated nonselfadjoint differential operator on L^2, with discrete spectrum near the real axis.

2.1. **Damped quantum mechanics.**

Following [1], let us start from the damped wave system. The generalized eigenvalue equation (3) for $\Re k_j \gg 1$ can be rewritten using an effective "Planck's constant" $\hbar \approx (\Re k_j)^{-1}$, and replacing k_j by the "energy" $z_j = \frac{(\hbar k_j)^2}{2} = 1/2 + \mathcal{O}(\hbar)$:

$$P_{dw}(\hbar)\psi_j = z_j\psi_j + \mathcal{O}(\hbar^2), \qquad P_{dw}(\hbar) = -\frac{\hbar^2\Delta}{2} - i\hbar b(x)\,. \tag{4}$$

The principal symbol of the operator $P_{dw}(\hbar)$, $p_0(x,\xi) = |\xi|^2/2$, is real and generates the geodesic flow on X. The skew-adjointness of $P_{dw}(\hbar)$ only appears in the subprincipal symbol $-i\hbar b(x)$: the latter does not influence the classical dynamics, but is responsible for the decay of probability along the flow. Indeed, for ψ_0 a wavepacket microlocalized on $\rho_0 = (x_0,\xi_0) \in T^*X$, its evolution $\psi(t) = e^{-itP_{dw}(\hbar)/\hbar}\psi_0$ will be another wavepacket microlocalized at $\rho_t = (x_t,\xi_t) = \Phi^t(\rho_0)$, with total probability reduced by a finite factor

$$\frac{\|\psi(t)\|^2}{\|\psi_0\|^2} \approx \exp\Big(-2\int_0^t b(x_s)\,ds\Big)\,. \tag{5}$$

In the limit $\hbar \to 0$, one can speak of a *damped classical dynamics*: each point ρ_t evolves according to the geodesic flow, and carries a weight which gets reduced by the above factor along the flow.

The horizontal spectral density of $P_{dw}(\hbar)$ is given by Weyl's law, which (to lowest order) does not depend on the damping [1]: for any $c > 0$,

$$\sharp\{\mathrm{Spec}(P_{dw}(\hbar)) \cap ([1/2 - c\hbar, 1/2 + c\hbar] + i\mathbb{R})\} = \hbar^{-d+1}\big(c\,C_X + \mathcal{O}(1)\big)\,, \tag{6}$$

where $C_X > 0$ only depends on X. On the other hand, we will see that the distribution of the imaginary parts $\Im z_j(\hbar)$ strongly depends on the *interplay* between the geodesic flow and the damping $b(x)$.

2.2. **Complex scaled scattering Hamiltonian and open dynamics.** The scattering operator $P(\hbar)$ in (2) can be transformed to a nonselfadjoint one through the complex scaling method [2]. One deforms the configuration space $\mathbb{R}^d$ into a complex contour $\Gamma_\theta = \{x + i\theta f(x)\}$, where $f(x) = 0$ for x in a ball $B(0,R)$ containing the support of $V(x)$ (the "interaction region"), while $f(x) = x$ for $|x| \geq 2R$. We take an angle $\theta = M\hbar\log(1/\hbar)$, $M > 0$ fixed. This leads to a "scaled" operator $P_\theta(\hbar)$, which is no more selfadjoint: in the sector $-2\theta < \arg(z) \leq 0$ it admits discrete eigenvalues, which correspond to the resonances of $P(\hbar)$.

Our quest for resonances has turned into the spectral study of $P_\theta(\hbar)$. This operator admits the symbol

$$p_\theta(x,\xi) = p(x,\xi) - i\theta\langle \xi, df(x)^t\xi\rangle + \mathcal{O}(\theta^2)|\xi|^2\,, \qquad p(x,\xi) = \frac{|\xi|^2}{2} + V(x)\,.$$

The operator $P_\theta(\hbar)$ presents similarities with (4): its principal symbol $p(x,\xi)$ is real and generates the Hamiltonian flow on $p^{-1}(E)$, while its imaginary part is of higher order $\hbar\log(1/\hbar)$, and generates a damping outside $B(0,R)$. One difference with P_{dw} lies the strength of this damping: for an initial wavepacket localized on a point $\rho_0 \in p^{-1}(E)$, the full probability after time t will be reduced by a factor $\hbar^{2M\int_0^t\langle\xi_s, df(x_s)^t\xi_s\rangle ds}$: the probability is semiclassically strongly suppressed as soon as the trajectory enters the zone where $\langle\xi, df(x)^t\xi\rangle > 0$ (this "absorbing zone" contains the exterior of $B(0,2R)$).

Classically, this corresponds to an "open dynamics": the point ρ_t evolves according to the Hamiltonian flow, but it gets "absorbed", or "killed" as soon as it enters the absorbing zone.

Trapped set. For any energy $E > 0$, the forward (resp. backward) trapped set K_E^- (resp. K_E^+) is defined as the set of initial points which remain bounded for all positive (resp. negative) times:

$$K_E^\mp = \{\rho_0 = (x_0,\xi_0) \in p^{-1}(E),\ |x_t| \leq R,\ \forall t \gtrless 0\}\,,$$

while the trapped set is made of their intersection $K_E = K_E^- \cap K_E^+$. Notice that K_E is a compact flow-invariant set. The above remark shows that any wavepacket localized on a point $\rho_0 \notin K_E^-$ will be absorbed after a finite time, namely the time it takes to enter the absorbing zone. On the other hand, a point $\rho \in p^{-1}(E)\setminus K_E^-$ will converge to the trapped set K_E as $t \to \infty$. This argument shows that, in some sense, long time quantum mechanics (at energy $\approx E$) takes place on K_E.

3. Fractal Weyl laws

The above argument can be made precise when estimating the number of resonances of $P(\hbar)$ near E. In the case we are interested in, K_E is a hyperbolic repeller (that is, there is no fixed point on K_E, and all trajectories are hyperbolic), this number is bounded above by a *fractal Weyl law* directly related with the geometry of K_E [4].

Theorem 1. *Assume that the trapped set K_E at some energy $E > 0$ is a hyperbolic repeller, and write its Minkowski dimension $\dim_M(K_E) = 1 + 2\nu$. Then for any $c, \alpha > 0$, one has in the semiclassical limit*

$$\sharp\{\mathrm{Res}(P(\hbar)) \cap ([E - c\hbar, E + c\hbar] - i\hbar[0,\alpha])\} = \mathcal{O}(\hbar^{-\nu-0})\,. \tag{7}$$

This theorem is proved by conjugating $P_\theta(\hbar)$ by a suitable "weight" $G(\hbar)$, so that the symbol $p_{\theta,G}$ of the conjugated operator $e^{-G(\hbar)}P_\theta e^{G(\hbar)}$ satisfies $\Im p_{\theta,G} < -2C\hbar$ outside the $\sqrt{\hbar}$-neighbourhood of K_E. Estimating the volume of this neighbourhood, and some involved pseudodifferential calculus on $p_{\theta,G}$, lead to the above upper bound.

A similar argument can be used to study the distribution of decay rates $\Im z_j/\hbar$ for the operator (4). For any time $T > 0$, one can construct a weight $G_T(\hbar)$, such that the conjugate operator $P_{dw,G_T}(\hbar) = e^{-G_T(\hbar)}P_{dw}(\hbar)e^{G_T(\hbar)}$ admits the symbol

$$p_{dw,G_T}(x,\xi) = \frac{|\xi|^2}{2} - i\hbar b_T(x,\xi) + \mathcal{O}(\hbar^2)\,, \tag{8}$$

where the subprincipal symbol $b_T(\rho) = T^{-1}\int_{-T/2}^{T/2} b(x_t)\,dt$ is the average of the damping along the flow. The geometric assumption of negative curvature implies that the geodesic flow on X is *Anosov*, in particular it is ergodic. This implies that, on the energy shell $p^{-1}(1/2)$, the time average $b_T(\rho)$ converges almost everywhere to the microcanonical average $\bar{b} = \mathrm{Vol}(X)^{-1}\int_X b(x)dx$ when $T \to \infty$. From there one can deduce that most of the eigenvalues $\Re z_i(\hbar) \in [1/2, c\hbar, 1/2 + c\hbar]$ concentrate near the "typical line" $\Im z = -\hbar\bar{b}$ [1].

Using finer properties of the Anosov geodesic flow, one can estimate the number of eigenvalues away from this "typical line" [3]. To state the result, we need to introduce the extremal ergodic averages of the damping, $b_- = \lim_{T\to\infty}\min_{p^{-1}(1/2)} b_T$, and similarly for b_+.

Theorem 2. *Assume X is a compact surface of negative curvature. Then, there exists a function $H : \mathbb{R} \to \mathbb{R}$, strictly concave on $[b_-, b_+]$ and equal to $-\infty$ outside, with maximum $H(\bar{b}) = d - 1$, such that for any $c > 0$ and any $\alpha \in [0, \bar{b}]$,*

$$\sharp\{\mathrm{Spec}(P_{dw}(\hbar)) \cap ([1/2 - c\hbar, 1/2 + c\hbar] - i\hbar[0,\alpha])\} = \mathcal{O}(\hbar^{-H(\alpha)-0})\,. \tag{9}$$

A similar estimate holds for the range $-\Im z \geq \hbar\alpha > \hbar\bar{b}$, $\alpha > \bar{b}$.

Comparing this "fractal Weyl upper bound" with the Weyl law (6) confirms that most resonances are on the typical line. The above theorem is obtained by studying the *large deviations* of the value distribution of b_T in the limit $T \to \infty$: roughly speaking, for $\alpha < \bar{b}$, the volume of the points $\rho \in p^{-1}(1/2)$ such that $b_T(\rho) \leq \alpha$ decays like $e^{T(H(\alpha)-(d-1))}$. This volume estimate is then used to get (9).

In both situations, the upper bounds on counting resonances/eigenvalues were obtained by deforming the operator P_θ (resp. P_{dw}) by an appropriate microlocal weight, and studying the imaginary part of the resulting operator by phase space volume arguments, where the classical dynamics plays a prominent rôle.

3.1. **Spectral gaps.** We now present a complementary type of spectral information, which can be obtained by a similar method for these two systems. Namely, we want to understand if the lifetimes $\tau_j(\hbar)$ can be arbitrarily large in the semiclassical limit; or on the opposite, if there exists a *gap* of size $\propto \hbar$ between the real axis and the eigenvalues/resonances. The presence of such a gap has important consequences on the long time properties of the system.

Let us start with the scattering problem. The gap question can be rephrased as: "Are the metastable states able to concentrate on K_E when $\hbar \to 0$?" The answer will result from a *competition* between, one one side, the fast dispersion of wavepackets due to the hyperbolic classical flow, on the other side the "thickness" of the trapped set allowing the state to reconstruct itself through constructive interferences.

A dynamical quantity reflecting this competition is of statistical nature, it is a *topological pressure* associated with the flow on K_E:

$$\mathcal{P}(-\varphi_u/2) = \lim_{T\to\infty} \frac{1}{T} \log \sum_{\gamma: T_\gamma \leq T} \exp\Big(- \int_0^{T_\gamma} \varphi_u(\rho_t)/2 \, dt \Big) .$$

Each γ is a periodic orbit in K_E with period T_γ, and $\varphi_u(\rho)$ is the unstable Jacobian of the flow. The above mentioned competition lies in the fact that each exponential weight gets very small when $T_\gamma \to \infty$, while the number of terms grows exponentially.

The following gap criterion was first obtained [5] for the case of hard obstacles, and then generalized in [6] to smooth potentials.

Theorem 3. *Assume the trapped set K_E is a hyperbolic repeller. If the pressure $\mathcal{P}(-\varphi_u/2) < 0$, then for any $c > 0$ and any small enough $\hbar$, the strip $[E - c\hbar, E + c\hbar] + i\hbar[\mathcal{P}(-\varphi_u/2) + 0, 0]$ does not contain any resonance of $P(\hbar)$.*

In dimension $d = 2$, the condition $\mathcal{P}(-\varphi_u/2) < 0$ is equivalent with a purely geometric statement, namely the fact that the Hausdorff dimension $\dim_H(K_E) < 2$ (notice that $\dim_H p^{-1}(E) = 3$).

In the case of damped waves, the gap question is nontrivial if $b_- = 0$, that is, if there exists a flow-invariant subset of $p^{-1}(1/2)$ with no damping. A result similar to the one above was obtained in [7]. In this case, the local decay of probability is due to both hyperbolic dispersion and damping.

Theorem 4. *Assume that X has negative curvature, and that the topological pressure $\mathcal{P}(-\varphi_u/2 - b) < 0$. Then, for any $c > 0$, and $\hbar > 0$ small enough, the strip $[1/2 - c\hbar, 1/2 + c\hbar] + i\hbar[\mathcal{P}(-b - \varphi_u/2) + 0, 0]$ does not contain eigenvalues of $P_{dw}(\hbar)$.*

4. Open questions

Most of the above results are upper or lower bounds. The natural question is: "Are these bounds sharp?" The fractal Weyl bound (7) is conjectured to be sharp for $\alpha > 0$ large enough, a fact which has been tested numerically on a number of examples, but could be proved only for a very specific toy model [8]. On the opposite, the bounds (9) for eigenmodes of the damped wave equation are not expected to be sharp for all values of α. The size of the gap itself is believed to be larger than the topological pressure bound we gave above. Such an "extra gap" was proved for the 3-disk scattering, using advanced estimates on classical mixing [9].

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