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SEMIGROUPS ARISING FROM ASYNCHRONOUS AUTOMATA

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Abstract. We introduce a new class of semigroups arising from a restricted class of asynchronous automata. We call these semigroups "expanding automaton semigroups." We show that the class of synchronous automaton semigroups is strictly contained in the class of expanding automaton semigroups, and that the class of expanding automaton semigroups is strictly contained in the class of asynchronous automaton semigroups. We investigate the dynamics of expanding automaton semigroups acting on regular rooted trees, and show that undecidability arises in these actions. We show that this class is not closed under taking normal ideal extensions, but the class of asynchronous automaton semigroups is closed under taking these extensions. We construct every free partially commutative monoid as a synchronous automaton semigroup.

1. INTRODUCTION

Automaton groups were introduced in the 1980's as examples of groups with fascinating properties. For example, Grigorchuk's group is the first known group of intermediate growth and is also an infinite periodic group. Besides having interesting properties, many of these groups have deep connections with dynamical systems which were explored by Bartholdi and Nekrashevych in [3] and [9]. In particular, they use these groups to solve a longstanding problem in holomorphic dynamics (see [3]). For a general introduction to these groups, see [5] by Grigorchuk and Sunic or [9] by Nekrashevych.

Many generalizations of automaton groups have been studied. The most famous and well-studied generalization is the class self-similar groups. A good introduction to these groups can be found in [5] or [9]. More recently, Slupik and Sushchansky study semigroups arising from partial invertible synchronous automata in [14]. Cain, Reznikov, Silva, Sushchanskii, and Steinberg investigate automaton semigroups, which are semigroups that arise from (not necessarily invertible) synchronous automata in [1], [10], and [13]. Grigorchuk, Nekrashevich, and Sushchanskii study groups arising from asynchronous automata in [4].

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In all of the references listed above except for [4], the semigroups studied arose from synchronous automata. In [4], Grigorchuk et al. study groups arising from asynchronous automata. In particular, they give examples of automata generating Thompson groups and groups of shift automorphisms. This paper studies a class of semigroups that we call "expanding automaton semigroups." These semigroups arise from a restricted class of asynchronous automata that we call "expanding automata," and the class of expanding automata contains the class of synchronous automata. Thus the class of automaton semigroups is contained in the class of expanding automaton semigroups, and the class of expanding automaton semigroups is contained in the class of asynchronous automaton semigroups. As mentioned above, automaton semigroups and asynchronous automaton semigroups have been studied, but thus far a study of expanding automaton semigroups has not been done.

In Section 2 we give definitions of the different kinds of automata, and explain how the states of a given automaton act on a regular rooted tree. In particular, let Σ be a finite set, and let Σ^* denote the free monoid generated by Σ. Then the states of a given automaton act on Σ^* for some finite set Σ. Thus we can consider the semigroup generated by the states of an automaton as a semigroup of functions from Σ^* to Σ^*. Given a free monoid Σ^*, we associate a regular rooted tree $\mathcal{T}(\Sigma^*)$ with Σ^* by letting the vertices of $\mathcal{T}(\Sigma^*)$ be Σ^* and letting the edge set be $(w, w\sigma)$ for all $w \in \Sigma^*$ and $\sigma \in \Sigma$. The identity of Σ^* is the root of the tree. The action of a semigroup associated with an asynchronous automaton on Σ^* induces an action on the tree $\mathcal{T}(\Sigma^*)$. Let Σ^ω denote the set of right-infinite words over Σ. Then Σ^ω is the boundary of the tree $\mathcal{T}(\Sigma^*)$. The action of an asynchronous automaton semigroup on Σ^* induces an action of the semigroup on Σ^ω, and so an asynchronous automaton semigroup acts on the boundary of a regular rooted tree.

Section 2 also contains examples of expanding automaton semigroups that are not automaton semigroups (Proposition 2.3), as well as asynchronous automaton semigroups that are not expanding automaton semigroups (Proposition 2.5). Thus Propositions 2.3 and 2.5 combine to show the following.

Proposition. *The class of automaton semigroups is strictly contained in the class of expanding automaton semigroups, and the class of expanding automaton semigroups is strictly contained in the class of asynchronous automaton semigroups.*

We show the latter by proving that the bicyclic monoid (the monoid with monoid presentation $< a, b \mid ab = 1 >$) is not a submonoid of any expanding automaton semigroup (Proposition 2.4), and then we demonstrate an asynchronous automaton semigroup that contains the bicyclic monoid as a submonoid (Proposition 2.5).

In Section 3 we investigate the dynamics of expanding automaton semigroups and asynchronous automaton semigroups on the trees on which they act. Example 3.2 gives an example of an expanding automaton semigroup S acting on $\{0,1\}^*$ such that there are infinite words $\omega_1, \omega_2 \in \{0,1\}^\omega$ with $s(\omega_1) = \omega_1$ and $s(\omega_2) = (\omega_2)$ for all $s \in S$. Furthermore, if $\omega \in \{0,1\}^\omega$ is not equal to ω_1 or ω_2, then $s(\omega) \neq \omega$ for all $s \in S$. Proposition 3.1 shows that automaton semigroups cannot have this kind of dynamical behavior when acting on the boundary of a tree. Thus the boundary dynamics of expanding automaton semigroups is richer than the boundary dynamics of automaton semigroups.

Section 3 also investigates several algorithmic problems regarding the actions of expanding automaton semigroups on a tree. Proposition 3.3 gives an algorithm that solves the uniform word problem for expanding automaton semigroups. This result is already known, as Grigorchuk et al. show in Theorem 2.15 of [4] that the uniform word problem is solvable for asynchronous automaton semigroups. We give an algorithm with our terminology for completeness. Proposition 3.8 gives an algorithm which decides whether a state of an automaton over Σ induces an injective function from $\mathcal{T}(\Sigma^*)$ to $\mathcal{T}(\Sigma^*)$.

Since the uniform word problem is decidable for these semigroups, there is an algorithm that takes as input an expanding automaton over an alphabet Σ and states q_1, q_2 of the automaton and decides whether $q_1(w) = q_2(w)$ for all $w \in \Sigma^*$. On the other hand, Theorem 3.4 shows the following.

Theorem 3.4. *(1) There is no algorithm which takes as input an expanding automaton $\mathcal{A} = (Q, \Sigma, t, o)$ and states $q_1, q_2 \in Q$ and decides whether or not there is a word $w \in \Sigma^*$ with $q_1(w) = q_2(w)$.*

(2) There is no algorithm which takes as input an expanding automaton $\mathcal{A} = (Q, \Sigma, t, o)$ and states $q_1, q_2 \in Q$ and decides whether or not there is an infinite word $\omega \in \Sigma^\omega$ such that $q_1(\omega) = q_2(\omega)$.

The problem in part 1 of the above theorem is decidable for automaton semigroups: if $\mathcal{A} = (Q, \Sigma, t, o)$ is a synchronous automaton with $q_1, q_2 \in Q$, then (because q_1 and q_2 induce level producing functions $\Sigma^* \to \Sigma^*$) there is a word $w \in \Sigma^*$ such that $q_1(w) = q_2(w)$ if and only if there is a letter $\sigma \in \Sigma$ such that $q_1(\sigma) = q_2(\sigma)$.

We close Section 3 by applying Theorem 3.4 to study the dynamics of asynchronous automaton semigroups. Theorem 3.6 shows that there is no algorithm which takes as input an asynchronous automaton over an alphabet Σ, a subset $X \subseteq \Sigma$, and a state q of the automaton and decides whether there is a word $w \in X^*$ such that $q(w) = w$. Thus we cannot decide if q has a fixed point in X^*. Furthermore, Theorem 3.6 also shows that there is no algorithm which takes as input an asynchronous automaton over an alphabet Σ, a subset $X \subseteq \Sigma$, and a state q of the automaton and decides whether there is an infinite word $\omega \in X^\omega$ such that $q(\omega) = \omega$. Thus undecidability arises when trying to understand the fixed points sets of asynchronous automaton semigroups on the boundary of a tree.

In Section 4 we give the basic algebraic theory of expanding automaton semigroups. Recall that a semigroup S is *residually finite* if for all $s_1, s_2 \in S$ with $s_1 \neq s_2$ then there is a finite semigroup S' and a homomorphism $\phi : S \to S'$ such that $\phi(s_1) \neq \phi(s_2)$. Proposition 4.1 shows that expanding automaton semigroups are residually finite. It is already known that automaton groups are residually finite (see Proposition 2.2 of [5]) and automaton semigroups are residually finite (see Proposition 3.2 of [1]). Asynchronous automaton semigroups are not residually finite, as there is an asynchronous automaton generating Thompson's group F (see section 5.2 of [4]). This group is an infinite simple group, and so is not residually finite. Thus residual finiteness of expanding automaton semigroups also distinguishes this class from the class of asynchronous automaton semigroups. Recall that if S is a semigroup, an element $s \in S$ is said to be *periodic* if there are $m, n \in \mathbb{N}$ such that $a^m = a^n$. Proposition 4.2 shows that the periodicity structure of expanding automaton semigroups is restricted. In particular, let P_Σ denote the set of prime numbers that divide $|\Sigma|!$. If S is an expanding automaton semigroup and $s \in S$ is such that $s^m = s^n$ for some $m, n \in \mathbb{N}$ with $n > m$, then the prime factorization of $n - m$ contains only primes from P_Σ.

In Section 4.2 we provide information about subgroups of expanding automaton semigroups. Proposition 4.3 shows that an expanding automaton semigroup S is a group if and only if S is an automaton group. Proposition 3.1 of [1] shows that an automaton semigroup S is a group if and only if S is an automaton group; we use the idea of the proof of this proposition to obtain our result. Note that such a proposition does not apply to asynchronous automaton semigroups, as Thompson's group F can be realized with an asynchronous automaton. Proposition 4.4 shows that if H is a subgroup of an expanding automaton semigroup, then there is a self-similar group G such that H is a subgroup of G. Proposition 4.5 shows that if an expanding automaton semigroup S has a unique maximal subgroup H, then H is self-similar. In particular, this proposition implies that if S is the semigroup generated by the states of an invertible synchronous automaton, then the group of units of S is self-similar (Corollary 4.6).

In Section 5.1 we study closure properties of expanding automaton semigroups. Let S and T be semigroups. The *normal ideal extension* of S by T is the disjoint union of S and T with multiplication defined by $x \cdot y = xy$ if $x, y \in S$ or $x, y \in T$, $x \cdot y = y$ if $x \in S$ and $y \in T$, and $x \cdot y = x$ if $x \in T$ and $y \in S$. Note that if S is a semigroup, then adjoining a zero to S is an example of a normal ideal extension. Proposition 5.6 of [1] shows that the class of automaton semigroups is closed under normal ideal extensions. We show in Proposition 5.3 that the class of asynchronous automaton semigroups is closed under normal ideal extensions. On the other hand, we show in Proposition 5.2 that the free semigroup of rank 1 with a zero adjoined is not an expanding automaton semigroup. Example 3.2 shows that the free semigroup of rank 1 is an expanding automaton semigroup, and so we have that the class of expanding automaton semigroups is not closed under normal ideal extensions. Lastly, we show that the class of expanding automaton semigroups is closed under direct product (provided the direct product is finitely generated). In Proposition 5.5 of [1] Cain shows the same result for automaton semigroups, and our proof is similar.

Section 5.2 contains further constructions of expanding automaton semigroups. A *free partially commutative monoid* is a monoid generated by a set $X = \{x_1, ..., x_n\}$ with relation set R such that $R \subseteq \{(x_i x_j, x_j x_i) \mid 1 \leq i, j \leq n\}$, i.e. a monoid in which the only relations are commuting relations between generators. We show the following.

Theorem 5.6. *Every free partially commutative monoid is an automaton semigroup.*

2. DEFINITIONS AND EXAMPLES

Given a set X, let X^+ denote the free semigroup generated by X. In the free monoid X^*, let $\emptyset$ denote the identity. As defined in [4], an *asynchronous automaton* is a quadruple (Q, Σ, t, o) where Q is a finite set of states, Σ is a finite alphabet of symbols, $t : Q \times \Sigma \to Q$ is a transition function, and $o : Q \times \Sigma \to \Sigma^*$ is an output function. A *synchronous automaton* is defined analogously, the difference being that $o : Q \times \Sigma \to \Sigma$ (the range of the output function is Σ rather than Σ^*). In this paper, we study a restricted class of asynchronous automata.

An *expanding automaton* is a quadruple $\mathcal{A} = (Q, \Sigma, t, o)$ where Q is a finite set of states, Σ is a finite alphabet of symbols, $t : Q \times \Sigma \to Q$ is a transition function, and $o : Q \times \Sigma \to \Sigma^+$ is an output function. We view an expanding automaton $\mathcal{A}$ as a directed labeled graph with vertex set Q and an edge from q_1 to q_2 labeled by $\sigma|w$ if and only if $t(q_1, \sigma) = q_2$ and $o(q_1, \sigma) = w$. Given an edge $\sigma|w$ in the graph, we refer to σ as the *input* of the edge, and w as the *output* of the edge. The interpretation of this graph is that if the automaton A is in state q_1 and reads symbol σ, then it changes to state q_2 and outputs the word w. Thus, if we fix $q_0 \in Q$, the automaton can read a sequence of symbols $\sigma_1\sigma_2...\sigma_n$ and output a sequence $w_1w_2...w_n$ where $t(q_{i-1}, \sigma_i) = q_i$ and $o(q_{i-1}, \sigma_i) = w_i$ for $i = 1, ..., n$.

Each state $q \in Q$ induces a function $\Sigma^* \to \Sigma^*$ in the following way: q acting on β, denoted $q(\beta)$, is defined to be the sequence that the automaton outputs when the automaton starts in state q and reads the sequence β. We also insist that $q(\emptyset) = \emptyset$. This action of q on Σ^* induces an action of q on $\mathcal{T}(\Sigma^*)$. The state q induces a function $f_q : \mathcal{T}(\Sigma^*) \to \mathcal{T}(\Sigma^*)$ by $f_q(w) = q(w)$ if $w \in \Sigma^*$, and if e is an edge in $\mathcal{T}(\Sigma^*)$ with endpoints w and $w\sigma$ then $f_q(e) = e_1e_2...e_n$ where $e_1...e_n$ is the unique geodesic sequence of edges in $\mathcal{T}(\Sigma^*)$ connecting $q(w)$ and $q(w\sigma)$. By abuse of notation, we identify f_q with q, as context should eliminate confusion. Considering the states of an automaton as functions leads to the following definition:

Definition 2.1. Given an expanding automaton $\mathcal{A}$, we say that the *expanding automaton semigroup* (respectively monoid) corresponding to $\mathcal{A}$, denoted $S(\mathcal{A})$, is the semigroup (respectively monoid) generated by the states of $\mathcal{A}$.

An *invertible synchronous automaton* (or *invertible automaton*) is a quadruple $\mathcal{A} = (Q, \Sigma, t, o)$ where $o : Q \times \Sigma \to \Sigma$ and, for any $q \in Q$, the restricted function $o_q : \{q\} \times \Sigma \to \Sigma$ is a permutation of Σ. The states of an invertible automaton (Q, Σ, t, o) induce bijections on $\mathcal{T}(\Sigma^*)$. Furthermore, these functions are *level-preserving*, i.e. $|w| = |q(w)|$ for all $w \in \Sigma^*$ and $q \in Q$ (where $|\cdot|$ is the length function on Σ^*). Thus, given an invertible automaton $\mathcal{A} = (Q, \Sigma, t, o)$, we define the *automaton group* associated with $\mathcal{A}$ to be the group generated by the states of $\mathcal{A}$. An *automaton semigroup* is a semigroup generated by the states of a synchronous automaton. Thus the generators of an automaton semigroup over the alphabet Σ induce level-preserving functions on $\mathcal{T}(\Sigma^*)$, but these functions are not necessarily bijective. Finally, an *asynchronous automaton semigroup* is a semigroup generated by the states of an asynchronous automaton.

A *self-similar group* is a group generated by the states of an invertible synchronous automaton with possibly infinitely states. A *self-similar semigroup* is defined analogously. Thus we define an *expanding self-similar semigroup* to be a semigroup generated by the states of an expanding automaton with possibly infinitely many states.

Note that if $s \in S$ where S is an expanding automaton semigroup acting on $\mathcal{T}(\Sigma^*)$, then s need not induce a level-preserving function $\mathcal{T}(\Sigma^*) \to \mathcal{T}(\Sigma^*)$. Thus elements of expanding automaton semigroups are not necessarily graph morphisms. If $\mathcal{A} = (Q, \Sigma, t, o)$ is an expanding automaton, then the output function mapping into Σ^+ implies that $|w| \leq |s(w)|$ for all $s \in S(\mathcal{A})$, $w \in \Sigma^*$. We say that a function $f : \mathcal{T}(\Sigma^*) \to \mathcal{T}(\Sigma^*)$ is *prefix-preserving* if $f(v)$ is a prefix of $f(w)$ in Σ^* whenever v is a prefix of w in Σ^*. We call a function $f : \mathcal{T}(\Sigma^*) \to \mathcal{T}(\Sigma^*)$ *length-expanding* if $|w| \leq |f(w)|$ for all $w \in \Sigma^*$ and $f(\emptyset) = \emptyset$. If we topologize the tree $\mathcal{T}(\Sigma^*)$ by making each edge isometric to $[0, 1]$ and imposing the path metric, then an element of an expanding automaton semigroup acting on $\mathcal{T}(\Sigma^*)$ will induce a prefix-preserving, length-expanding endomorphism of the tree. We call $f : \mathcal{T}(\Sigma^*) \to \mathcal{T}(\Sigma^*)$ an *expanding endomorphism* if f is prefix-preserving and length-expanding.

Let f be an expanding endomorphsm of a tree $\mathcal{T}(\Sigma^*)$, where $\Sigma = \{\sigma_1, ..., \sigma_n\}$. Then f induces a function $\Sigma \to \Sigma^+$; for the rest of the paper we denote this function by τ_f. Note that for any $w \in \Sigma^*$, the tree $w\mathcal{T}(\Sigma^*)$ is isomorphic (as a graph or a metric space) to $\mathcal{T}(\Sigma^*)$. Now for each $\sigma \in \Sigma$, f induces an expanding endomorphism $\sigma\mathcal{T}(\Sigma^*) \to f(\sigma)\mathcal{T}(\Sigma^*)$; for the rest of the paper we denote this induced endomorphism by f_σ. For any $\sigma \in \Sigma$ and $w \in \Sigma^*$, f_σ is characterized by the equation

$$f(\sigma w) = \tau_f(\sigma) f_\sigma(w).$$

The function f_σ is called the *section of f at σ*. Inductively, given $w \in \Sigma^*$, there exists an expanding endomorphism f_w such that $f(wv) = f(w)f_w(v)$ for every $v \in \Sigma^*$. We call f_w the *section of f at w*. To completely describe an expanding automorphism f, we need only know the induced function τ_f and the sections $f_{\sigma_1}, ..., f_{\sigma_n}$. Thus, in keeping with the notation for automaton groups and semigroups in [1] and [9], any expanding endomorphism f can be written as

$$f = (f_{\sigma_1}, ..., f_{\sigma_n})\tau_f$$

where each f_{σ_i} is the section of f at σ_i.

We denote a function $\tau : \Sigma \to \Sigma^+$ by $[a_1, ..., a_n]$ where $\tau(\sigma_i) = a_i$. If f and g are expanding endomorphisms with $f = (f_1, ..., f_n)[w_1, ..., w_n]$ and $g = (g_1, ..., g_n)[v_1, ..., v_n]$, then their composition (our functions act on the left) is given by the formula

$$f \circ g = (f_{v_1}g_1, ..., f_{v_n}g_n)[f(v_1), ..., f(v_n)]. \tag{1}$$

Let $\mathcal{A} = (Q, \Sigma, t, o)$ be an asynchronous automaton and $q \in Q$. If $w \in \Sigma^*$, then q_w is obtained by viewing the word w as a path in $\mathcal{A}$ starting at q. The terminal vertex of this path is the section of q at w. Thus any section of a state of $\mathcal{A}$ is itself a state of $\mathcal{A}$.

Let $\mathcal{A} = (Q, \Sigma, t, o)$ be an expanding automaton, and let $s \in Q^*$ be an element of $S(\mathcal{A})$. Equation (1) allows us to build an expanding automaton $\mathcal{A}_s$ that contains s as a state. Write $s = q_1...q_n$. Using the original automaton $\mathcal{A}$, compute τ_s. If we iteratively use Equation (1) and $\mathcal{A}$, we can compute the section of s at σ for any $\sigma \in \Sigma$ in terms of the sections of the q_i's. Furthermore, a straightforward induction on word length in Q^* shows that if t is a section of s, then the word length of t in Q^* is less than or equal to the word length of s in Q^*. Thus we will compute an expanding automaton $\mathcal{A}_s$ whose set of states has cardinality less than the cardinality of the set $\{w \in Q^* \ : \ |w| \leq |s|\}$.

Before giving examples, we mention that we use the word "action" when describing the functions arising from these semigroups on regular rooted trees. In general, if one says that a monoid M has an action on a set, one assumes that the identity of the monoid fixes each element of the set. In this case, however, we can have expanding automaton monoids (and indeed automaton monoids) in which the identity element of the monoid does not fix each vertex of the tree, so we do not include that assumption as part of the definition of "action". Consider Example 2.2 below.

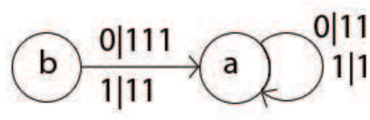

FIGURE 1. Example 2.2

Example 2.2. Consider the expanding automaton $\mathcal{A}$ over the alphabet $\{0,1\}$ given by $a = (a,a)[11,1], b = (a,a)[111,11]$. See Figure 1 for the graphical representation of $\mathcal{A}$. We claim that a is an identity element of $S(\mathcal{A})$ even though a does not fix every element of $\mathcal{T}(\{0,1\}^*)$. To see this, first note that the range of a is $\{1\}^*$. Since a fixes this set, $a^2 = a$. Now the range of b is $\{1\}^* - \{1\}$ and a fixes this set, so $ab = b$. Now let $w \in \{0,1\}^*$, and let $w_0 \in \mathbb{N}$ denote the number of 0's that occur in w; define w_1 analogously. Then $a(w) = 1^{2w_0+w_1}$, and therefore $ba(w) = 1^{2w_0+w_1+1}$. Let w' be the word obtained from w by deleting the first letter of w. If 0 is the first letter of w, then

$$b(w) = 1111^{2(w')_0+(w')_1} = 1^{2w_0+w_1+1} = ba(w).$$

Similarly, if w starts with a 1 we have $b(w) = ba(w)$. Hence $ab = b = ba$, and a is an identity element. Thus the action of $S(\mathcal{A})$ on $\{0,1\}^*$ includes the action of a semigroup identity that is not the identity function on $\mathcal{T}(\Sigma^*)$.

We now show that there are semigroups which are expanding automaton semigroups but not automaton semigroups.

Proposition 2.3. *The class of automaton semigroups is strictly contained in the class of expanding automaton semigroups.*

Proof. Let $m,n \in \mathbb{N} - \{1\}$, and let $S_{m,n}$ denote the semigroup with semigroup presentation $< a,b \mid b^m = b^n, ab = b >$. We show that $S_{m,n}$ is not an automaton semigroup for any m,n, but $S_{m,n}$ is an expanding automaton semigroup for any m,n.

Note that for any distinct $m,n \in \mathbb{N}$ with $m < n$, the rewriting system defined by the rules $ab \to a$ and $b^n \to b^m$ is terminating and confluent. Thus $\{b^j a^n \mid j = 1,...,n-1,\ n \in \mathbb{N}\}$ is a set of normal forms for $S_{m,n}$, and so a is not periodic in $S_{m,n}$.

We begin by showing $S_{m,n}$ is not an automaton semigroup. Fix $1 < m < n$. Let $\mathcal{A}_{m,n} = (Q, \Sigma, t, o)$ be a synchronous automaton such that $S(\mathcal{A}_{m,n})$ is generated by two elements a and b with $b^m = b^n$ and $ab = b$. We show that a is periodic in $S(\mathcal{A}_{m,n})$. Note that a and b must both be states of $\mathcal{A}_{m,n}$ as higher powers of a and b cannot multiply to obtain a, and powers of a cannot multiply to obtain b. Let $\sigma_1 \in \Sigma$ be such that there exists a minimal number $n > 0$ with $a^n(\sigma_1) = \sigma_1$. Since the action of a is length-preserving, there must exist such a σ_1. Let $\{\sigma_1, ..., \sigma_{n-1}\}$ be the orbit of σ_1 under the action of a where $a(\sigma_i) = \sigma_{i+1}$ for $1 \leq i \leq n-2$ and $a(\sigma_{n-1}) = \sigma_m$.

First suppose that $a_{\sigma_j} = a^{m_j}$ for each $1 \leq j \leq n-1$. If $m_j > 1$ for some j, then $(a^{m_j})_{\sigma_j} = a^{n_1}$ where $n_1 > m_j$, $(a^{n_1})_{\sigma_j} = a^{n_2}$ where $n_2 > n_1$, and so on. In this case, a will have infinitely many sections, which cannot happen since a is a state of a finite automaton. Thus $m_j = 1$ for all j. Note that if $a^k(\sigma) = \sigma_1$ for some $k > 0$ and $\sigma \in \Sigma$, then the same logic implies that if $a_\sigma = a^r$ for some r then $r = 1$. Thus we see that if $\sigma \in \Sigma$ and the section of a at $a^k(\sigma)$ is a power of a for all k, then the section of a at $a^k(\sigma)$ is a for all $k > 0$. Suppose that $a_\sigma = a$ for all $\sigma \in \Sigma$. Since the action of a is length-preserving, there exist distinct $r, s \in \mathbb{N}$ such that $\tau_a^r = \tau_a^s$. Then, as the only section of a is a, we have $a^r = a^s$.

Suppose now that there is a letter $\sigma \in \Sigma$ such that there exists σ' in the forward orbit of σ under the action of a where $a_{\sigma'} \notin < a >$. Since $ab = b$ and b is periodic, there exist distinct $m_\sigma, n_\sigma \in \mathbb{N}$ with $n_\sigma > m_\sigma$ such that $(a^{m_\sigma})_\sigma = (a^{m_\sigma + k(n_\sigma - m_\sigma)})_\sigma$ for any $k \in \mathbb{N}$. To see that this is true, let t be the minimal number such that the orbit of $a^t(\sigma)$ under the action of a is a cycle. Since the action of a is length-preserving, there must exist such a t. Suppose that there is a $k \in \mathbb{N}$ such that $k \geq t$ and the section of a at $a^k(\sigma)$ is $b^i a^j$ for some $i \in \mathbb{N}$ and $j \in \mathbb{N} \cup \{0\}$. Then the relation $ab = b$ implies that for any $k' \geq k$ we have $(a^{k'})_\sigma = b^{i'} a^j$ for some i'. Periodicity of b then implies that there are $m_\sigma, n_\sigma \geq k$ as desired. Suppose, on the other hand, that the section of a at $a^r(\sigma)$ is in $< a >$ for all $r \geq t$. Let c be the maximal number such that the section of a at $a^c(\sigma)$ is not in $< a >$ and let $p \in \mathbb{N}$. Then $(a^{c+p})_\sigma = a^{n_p}(a^c)_\sigma$ for some $n_p \in \mathbb{N}$ and the relation $ab = b$ implies that $(a^{c+p})_\sigma = (a^c)_\sigma$. In this case we let $m_\sigma = c$ and $n_\sigma = c + 1$.

Let $\hat{\Sigma} = \{\sigma \in \Sigma \mid (a^r)_\sigma \notin < a > \text{ for some } r\}$. By the preceding paragraph, for each $\sigma \in \hat{\Sigma}$ choose $m_\sigma, n_\sigma \in \mathbb{N}$ such that $(a^{m_\sigma})_\sigma = (a^{m_\sigma + k(n_\sigma - n_\sigma)})_\sigma$. Since a acts in a length-preserving fashion, there exist distinct t_1, t_2 such that $\tau_a^{t_1} = \tau_a^{t_1 + k(t_2 - t_1)}$ for all $k \in \mathbb{N}$. Thus we can choose distinct $s, t \in \mathbb{N}$

such that $\tau_{a^s} = \tau_{a^{s+k(t-s)}}$ and $(a^s)_\sigma = (a^{s+k(t-s)})_\sigma$ for all $\sigma \in \hat{\Sigma}$ and $k \in \mathbb{N}$. We claim that $a^s = a^t$. To see this, let $\delta \in \Sigma$. If $\eta \in \hat{\Sigma}$, then the choice of s and t implies that $(a^s)_\eta = (a^t)_\eta$. Fix $\delta \notin \hat{\Sigma}$. Then $(a^s)_\delta = a^s$ and $(a^t)_\delta = a^t$, so the choice of s and t implies that $\tau_{(a^s)_\delta} = \tau_{(a^t)_\delta}$. If $\eta \in \hat{\Sigma}$, then

$$(a^s)_{\delta\eta} = (a^s)_\eta = (a^t)_\eta = (a^t)_{\delta\eta}.$$

If $\eta \notin \hat{\Sigma}$ then $(a^s)_{\delta\eta} = a^s$ and $(a^t)_{\delta\eta} = a^t$, and so $\tau_{(a^s)_{\delta\eta}} = \tau_{(a^t)_{\delta\eta}}$. Similarly, let $w \in \Sigma^*$ and write $w = \sigma_1 ... \sigma_n$. Suppose there is an $i \in \mathbb{N}$ such that $\sigma_i \in \hat{\Sigma}$ and $\sigma_1, ..., \sigma_{i-1} \in \Sigma - \hat{\Sigma}$. Then

$$(a^s)_w = (a^s)_{\sigma_i ... \sigma_n} = (a^t)_{\sigma_i ... \sigma_n} = (a^t)_w.$$

On the other hand, if $w \in (\Sigma - \hat{\Sigma})^*$ then $\tau_{(a^s)_w} = \tau_{a^s} = \tau_{a^t} = \tau_{(a^t)_w}$. Thus $a^s = a^t$, and so $S(\mathcal{A}_{m,n})$ is not $S_{m,n}$.

Fix $1 < m < n$, and let $\Sigma = \{\sigma_1, ..., \sigma_n\}$ be an alphabet. Let $\mathcal{A}_{m,n}$ be the automaton with states a and b (which depend on m, n) defined by

$$a = (a, ..., a)[\sigma_1\sigma_1, \sigma_2, ..., \sigma_n], \quad b = (b, ..., b)\tau_b$$

where

$$\tau_b(\sigma_i) = \begin{cases} \sigma_{i+1} & 1 \leq i < n \\ \sigma_m & i = n \end{cases}.$$

Then $b^m = b^n$ in $S(\mathcal{A}_{m,n})$. Note also that the range of b is $\{\sigma_2, ..., \sigma_n\}^*$, and a fixes this set. So $ab = b$. Now fix $i, j \in \mathbb{N}$ such that $i < n$. Then $b^i a^j(\sigma_1) = b^i(\sigma_1^{2^j}) = \sigma_i^{2^j}$. Thus $b^i a^j = b^k a^l$ in $S(\mathcal{A}_{m,n})$ if and only if $i = k$ and $j = l$, and we have $S(\mathcal{A}_{m,n}) \cong S_{m,n}$. □

Recall that the bicyclic monoid is the monoid with monoid presentation $B := < a, b \mid ab = 1 >$. Clifford and Preston show in Corollary 1.32 of [2] that $ba \neq 1$ in B. Furthermore, the same corollary shows that if S is a semigroup and $a, b, c \in S$ such that $c^2 = c$, $ca = ac = c$, $cb = bc = b$, and $ba = c$, then the submonoid generated by a,b, and c is the bicyclic monoid if and only if $ba \neq c$.

Proposition 2.4. *Let S be an expanding automaton semigroup. If M is a submonoid of S, then M is not isomorphic to the bicyclic monoid.*

Proof. Let S be an expanding automaton semigroup over an alphabet Σ. Suppose $a, b, c \in S$ such that $c^2 = c$, $ca = ac = c$, $cb = bc = b$, and $ba = c$. We show that $ab = c$.

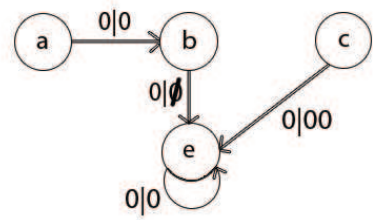

FIGURE 2. The automaton from Proposition 2.5

If $s \in S$, let range(s) denote $s(\Sigma^*)$. Since c is idempotent, c fixes range(c). The equations $ca = a$ and $cb = b$ imply that c fixes range(a) and range(b). Thus range(a), range$(b) \subseteq$ range(c). So we see that a must act injectively on range(c): if $x, y \in$ range(c) and $a(x) = a(y) = z$, then $b(z) = x$ and $b(z) = y$ and so $x = y$. Furthermore, because b cannot reduce word length, a must act in a length-preserving fashion on range(c). Thus range(a)=range(c). Now the equation $ba = c$ implies that b maps range(a) onto range(c), and hence b acts injectively and in a length-preserving fashion on range(a). Thus if $w \in$ range(c) then $bc(w) = w = a(w)$. Suppose $w \notin$ range(c). Then $bab(w) = b(w) = bc(w)$. Since $ab(w), c(w) \in$ range(c) and b acts injectively on range(c), $ab(w) = c(w)$. Thus $ab = c$. □

We now distinguish the class of expanding automaton semigroups from the class of asynchronous automaton semigroups.

Proposition 2.5. *The class of expanding automaton semigroups is strictly contained in the class of asynchronous automaton semigroups.*

Proof. Let $\mathcal{A}$ be the asynchronous automaton over the alphabet $\{0\}$ with four states defined by

$$a = (b)[0], \quad b = (e)[\emptyset], \quad c = (e)[00], \quad e = (e)[0].$$

Figure 2 gives the graphical representation of $\mathcal{A}$. Note that $e(0^n) = 0^n$ for all $n \in \mathbb{N}$, so e is an identity element of $S(\mathcal{A})$. Note also that by construction $ac(0^n) = 0^n = e(0^n)$ for any n, but $ca(0) = 00$. Thus $ac = e$ but $ca \neq e$ in $S(\mathcal{A})$. So Corollary 1.32 of [2] implies that the submonoid generated by a and c is the bicyclic monoid, and Proposition 2.4 implies that $S(\mathcal{A})$ is not an expanding automaton semigroup. □

3. Decision Properties and Dynamics

We begin this section by showing that expanding automaton semigroups have richer boundary dynamics than automaton semigroups. Proposition 3.1 restricts the kind of action that an automaton semigroup can have on the boundary of a tree, and Example 3.2 gives an expanding automaton semigroup which shows that this restriction does not extend to the dynamics of these semigroups. Example 3.2 also provides a realization of the free semigroup of rank 1 as an expanding automaton semigroup. Proposition 4.3 of [1] shows that the free semigroup of rank 1 is not an automaton semigroup, so Example 3.2 provides another example of an expanding automaton semigroup that is not an automaton semigroup. Let S be a semigroup acting on a set X, and $s \in S$. We say that $x \in X$ is a *fixed point of* s if $s(x) = x$.

Proposition 3.1. *Let S be an automaton semigroup with corresponding automaton $\mathcal{A} = (Q, \Sigma, t, o)$. If every state of $\mathcal{A}$ has at least two fixed points in Σ^ω, then every state of $\mathcal{A}$ has infinitely many fixed points in Σ^ω.*

Proof. We begin with some terminology. We call a path p in $\mathcal{A}$ an *inactive path* if each edge on p has the form $\sigma|\sigma$ for some $\sigma \in \Sigma$.

Let $q \in Q$. Since $\mathcal{A}$ is a synchronous automaton, q acts in a length-preserving fashion. Since q has a fixed point in Σ^ω, in the finite automaton $\mathcal{A}$ there must exist an inactive circuit c_1 accessible from q via an inactive path p. Let q_1 be a state on c_1. As q_1 must also have two fixed points in Σ^ω, either there is another inactive circuit containing q_1 or there is another inactive circuit accessible from q_1 via an inactive path. In either case, q has infinitely many fixed points in Σ^ω by "pumping" the two inactive circuits. □

Example 3.2. (Thue-Morse Automaton): This example is constructed to model the substitution rules which give the Thue-Morse sequence. This infinite binary sequence, denoted (T_i), is the limit of 0 under iterations of the substitution rules $0 \to 01, 1 \to 10$. The complement of the Thue-Morse sequence, denoted $(\overline{T_i})$, is the limit of 1 under iterations of these substitution rules. For more information on these sequences, see Section 2.2 of [8] by Lothaire.

Consider the expanding automaton $\mathcal{A}$ given by $a = (a, a)[01, 10]$ over the alphabet $\Sigma = \{0, 1\}$. First note that $S(\mathcal{A})$ is the free semigroup of rank 1. To see this, by construction of $\mathcal{A}$ we have $|a^n(0)| = 2^n$ for all n, and thus $a^m \neq a^n$ for any $m \neq n$.

Also by construction of $\mathcal{A}$, the action of $S(\mathcal{A})$ has exactly two fixed points in $\{0,1\}^\omega$: (T_i) and $(\overline{T_i})$. To see this, first notice that (T_i) and $(\overline{T_i})$ are the fixed points of a (see section 2.1 of [8]). Thus (T_i) and $(\overline{T_i})$ are fixed points of a^n for any n. Furthermore, $a^n = (a^n, a^n)\tau_{a^n}$ where τ_{a^n} maps 0 to the prefix of length 2^n of (T_i) and maps 1 to the prefix of length 2^n of $(\overline{T_i})$. Thus section 2.1 of [8] implies that a^n has exactly two fixed points for all n.

The following proposition gives an algorithm for solving the uniform word problem in the class of expanding automaton semigroups. This proposition is a special case of Theorem 2.15 of [4], which shows that asynchronous automaton semigroups have solvable uniform word problem. We include a proof for completeness.

Proposition 3.3. *Expanding automaton semigroups have solvable uniform word problem.*

Proof. Let $A = (Q, \Sigma, t, o)$ be an expanding automaton, and let $S = S(A)$. Let $s = q_1 ... q_m$ and $t = q'_1 ... q'_n$ be elements of S. If $\tau_s \neq \tau_t$, then $s \neq t$. If $\tau_s = \tau_t$, then use Equation (1) to calculate s_σ and t_σ for all $\sigma \in \Sigma$. If $\tau_{s_\sigma} \neq \tau_{t_\sigma}$ for some $\sigma \in \Sigma$, then $s \neq t$. If $\tau_{s_\sigma} = \tau_{t_\sigma}$ $\forall\ \sigma \in \Sigma$, then calculate τ_{s_w}, τ_{t_w} for each $w \in \Sigma^+$ with $|w| = 2$, and continue the process. Since $|\{s_w : w \in \Sigma^*\}| \leq |\{w' \in Q^* : |w'| \leq m\}|$ and $|\{t_w : w \in \Sigma^*\}| \leq |\{w' \in Q^* : |w'| \leq n\}|$, this process stops in finite time. □

We now turn to showing that undecidability arises in the dynamics of these semigroups.

Theorem 3.4.
(1) *There is no algorithm which takes as input an expanding automaton $\mathcal{A} = (Q, \Sigma, t, o)$ and states $q_1, q_2 \in Q$ and decides whether or not there is a word $w \in \Sigma^*$ with $q_1(w) = q_2(w)$.*
(2) *There is no algorithm which takes as input an expanding automaton $\mathcal{A} = (Q, \Sigma, t, o)$ and states $q_1, q_2 \in Q$ and decides whether or not there is an infinite word $\omega \in \Sigma^\omega$ such that $q_1(\omega) = q_2(\omega)$.*

Proof. We show undecidability by embedding the Post Correspondence Problem. Let $X = \{x_1, ..., x_m\}$ be an alphabet, and let $V = (v_1, ..., v_n)$ and $W = (w_1, ..., w_n)$ be two lists of words over X. Let $Y = \{1, ..., n\} \subseteq \mathbb{N}$ and $Z = \{z_1, z_2\}$ be alphabets such that $X \cap Y \cap Z = \emptyset$. Undecidability of the Post Correspondence Problem implies that, in general, we cannot decide if there is a sequence $(y_1, ..., y_t)$ of elements of Y such that $v_{y_1} v_{y_2} ... v_{y_t} = u_{y_1} u_{y_2} ... u_{y_t}$.

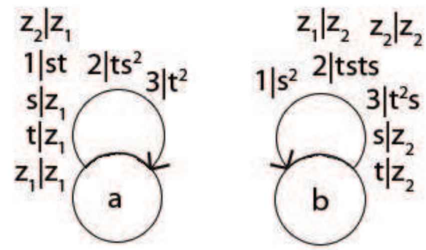

FIGURE 3. The automaton $\mathcal{A}_{X,U,W}$ where $X = \{s, t\}$, $V = (st, ts^2, t^2)$, and $W = (s^2, tsts, t^2s)$

We build an expanding automaton $\mathcal{A}_{X,V,W}$ over the alphabet $\Sigma := X \cup Y \cup Z$ as follows. Let the state set Q of $\mathcal{A}_{X,V,W}$ be $\{a, b\}$, and let

$$t(q, \sigma) = q \text{ for all } q \in Q, \sigma \in \Sigma$$

$$o(a, i) = v_i \text{ for } 1 \leq i \leq n, \quad o(a, \sigma) = z_1 \text{ for } \sigma \in \Sigma - Y$$

$$o(b, i) = w_i \text{ for } 1 \leq i \leq n, \quad o(b, \sigma) = z_2 \text{ for } \sigma \in \Sigma - Y$$

Figure 3 shows $\mathcal{A}_{X,U,W}$ where $X = \{s, t\}$, $V = (st, ts^2, t^2)$, and $W = (s^2, tsts, t^2s)$.

Note that for any $w \in \Sigma^*$, $a(w)$ does not contain the letter z_2; similarly, $b(w)$ does not contain the letter z_1. Now if $w \in \Sigma^*$ contains a letter of $X \cup Z$, then we know $a(w) \neq b(w)$ since $a(w)$ contains the letter z_1 and $b(w)$ contains the letter z_2. Thus if there is a word $w \in \Sigma^*$ such that $a(w) = b(w)$, then $w \in Y^*$. By construction of $\mathcal{A}_{X,V,W}$, if $y = y_1 y_2 ... y_n \in Y^*$ and $a(y) = b(y)$, then $v_{y_1} v_{y_2} ... v_{y_t} = u_{y_1} u_{y_2} ... u_{y_t}$. Thus the expanding automaton $\mathcal{A}_{X,V,W}$ simulates Post's problem, and since we cannot decide the Post Correspondence Problem, we cannot decide if there is a word $w \in Y^*$ with $a(w) = b(w)$. This proves part (1).

It is shown by Rouhonen in [12] that the infinite Post Correspondence Problem is undecidable. That is, there is no algorithm that takes as input two lists of words $v_1, ..., v_n$ and $w_1, ..., w_n$ over an alphabet X and decides if there is an infinite sequence $(i_k)_{k=1}^{\infty}$ such that $v_{i_1} v_{i_2} ... = w_{i_1} w_{i_2} ....$ Thus, using the same expanding automata and logic as above, (2) is proven. $\square$

We now show that undecidability arises when trying to understand the fixed point sets of elements of asynchronous automaton semigroups. If $w \in A^*$ for a set A, let $\text{Pref}_k(w)$ denote the prefix of w of length k.

Definition 3.5. Let A^* be a free monoid. A subset $C \subseteq A^*$ is a **prefix code** if

(1) C is the basis of a free submonoid of A^*
(2) If $c \in C$, then $\text{Pref}_k(c) \notin C$ for all $1 \leq k \leq |C|$

The *prefix code Post correspondence problem* is a stronger form of the Post Correspondence Problem. The input of the prefix code Post Correspondence Problem is two lists of words $v_1, ..., v_n$ and $w_1, ..., w_n$ over an alphabet X such that $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_n\}$ are prefix codes. A solution to the problem is a sequence of indices $(i_k)_{1 \leq k \leq m}$ with $1 \leq i_k \leq n$ such that $v_{i_1}...v_{i_m} = w_{i_1}...w_{i_m}$. Rouhonen also shows in [12] that this form of Post's problem is undecidable. We use the prefix code Post problem to prove the following:

Theorem 3.6. *(1) There is no algorithm that takes as input an asynchronous automaton $\mathcal{A}$ over an alphabet X, a subset $Y \subseteq X$, and a state q of $\mathcal{A}$ and decides whether or not q has a fixed point in Y^*, i.e. decides if there is a word $w \in Y^*$ such that $q(w) = w$.*

(2) There is no algorithm that takes as input an asynchronous automaton $\mathcal{A}$ over an alphabet X, a subset $Y \subseteq X$, and a state q of $\mathcal{A}$ and decides whether or not q has a fixed point in Y^ω, i.e. decides if there is an infinite word $\omega \in Y^\omega$ such that $q(\omega) = \omega$.

Proof. Let X be an alphabet, and let $C, D \subseteq X^*$ be prefix codes where $C = \{c_1, ..., c_m\}$ and $D = \{d_1, ..., d_m\}$. Let $\mathcal{A}_{X,C,D}$ be the expanding automaton with states c, d that we constructed in the proof of Proposition 3.4. Then $\mathcal{A}_{X,C,D}$ is an expanding automaton over the alphabet $\Sigma := \{1, ..., m\} \cup X \cup \{z_1, z_2\}$ such that $o(c, i) = c_i$ and $o(d, i) = d_i$. We build an asynchronous automaton $\mathcal{B}$ over the alphabet Σ with a state c' such that $c'c$ is the identity function from $\{1, ..., m\}^*$ to $\{1, ..., m\}^*$. We know that there is a function $c' : X^* \to \{1, ..., m\}^*$ such that $c'c$ is the identity because $\{c_1, ..., c_m\}$ generates a free monoid, so c induces an injection from $\{1, ..., m\}^*$ to X^*.

We begin construction of $\mathcal{B}$ by starting with a single state c', and then attaching a loop based at c' such that the input letters of the loop read the word c_1 when read starting at c'. The corresponding output word, when read starting at c', we define to be $(\emptyset)^{|c_1|-1}1$. In other words, the first $|c_1| - 1$

edges of the loop have the form $x|\emptyset$, and the last edge of the loop has the form $x|1$. Next, we attach a loop at c' such that the input letters of the loop when read starting at c' read the word c_2, and the corresponding output word is $(\emptyset)^{|c_2|-1}2$. If c_1 ad c_2 have a non-trivial common prefix, then the resulting automaton with two loops is not deterministic. In this case, we "fold" the maximum length common prefixes together, resulting in a deterministic automaton. We iteratively continue this process until we can read the words $c_1, ..., c_m$ as input words starting at c', and $c'(c_i) = i$ for all i. Note that we can do this process since c_i is not a prefix of c_j for any $i \neq j$. At this step in the construction of $\mathcal{B}$, $\mathcal{B}$ is a partial asynchronous automaton, i.e. given a state of q of $\mathcal{B}$, the domain of q is not all of Σ^*. However, we do have $c'c$ is the identity function $\{1, ..., m\}^* \to \{1, ..., m\}^*$ by construction of $\mathcal{B}$. In order to make $\mathcal{B}$ an asynchronous automaton, we add a sink state i such that $t(i, \sigma) = i$ and $o(i, \sigma) = \sigma$ for all $\sigma \in \Sigma$. Now for each q a state in $\mathcal{B}$ and $\sigma \in \Sigma$ such that there there is no edge out of q with σ as an input letter, define $t(q, \sigma) = i$ and $o(q, \sigma) = z_1$.

Recall that in the proof of Theorem 3.4, in general we cannot find $w \in \{1, ..., m\}^*$ such that $c(w) = d(w)$ because such a w is a solution to the Post Correspondence Problem. By construction of $\mathcal{B}$, any $w \in \{1, ..., m\}^*$ such that $c'd(w) = w = c'c(w)$ is a solution to the prefix code Post Correspondence Problem. Now $c'd$ is an element of the asynchronous automaton semigroup generated by the states of $\mathcal{A}_{X,C,D}$ and $\mathcal{B}$. Thus, undecidability of the prefix code Post Correspondence Problem implies part (1).

In [12], Ruohonen shows that the that there is no algorithm which takes as input two lists of words $v_1, ..., v_n$ and $w_1, ..., w_n$ over an alphabet X such that $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_n\}$ are prefix codes and decides whether there is an infinite sequence of indices $(i_k)_{k=1}^{\infty}$ such that $v_{i_1}v_{i_2}... = w_{i_1}w_{i_2}...$. Thus the same logic and automata as above implies part (2). □

We now give an algorithm which determines whether or not an element of an expanding automaton semigroup induces an injection $\mathcal{T}(\Sigma^*) \to \mathcal{T}(\Sigma^*)$. Before we do this, we must recall some basic automata theory which can be found in more detail in Chapters 1 and 2 of [7] by Hopcroft and Ullman. In order to avoid ambiguity of language in this paper, we will use the phrase "deterministic finite state automaton" to denote a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where Q is a state set, Σ is an alphabet, δ is a partial function from $Q \times \Sigma$ to Q, q_0 is an initial state, and F is a set of final states. Let "nondeterministic finite state automaton" denote a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where Q is a state set, Σ is an alphabet, δ is a partial relation from $Q \times \Sigma$ to Q that is not a partial function, q_0 is an initial state, and F is a set of final states. We view a finite state automaton (deterministic or nondeterministic) as a finite directed graph with vertex set Q and an arrow from q_1 to q_2 labeled by σ if and only if $\delta(q_1, \sigma) = q_2$. Given a finite state automaton $M = (Q, \Sigma, \delta, q_0, F)$, call a directed edge path p an *acceptable path* in M if p begins at q_0 and ends at a final state. If M is a deterministic or a nondeterministic finite state automaton, let $\phi_M : \{\text{acceptable paths in M}\} \to L(M)$ (where $L(M)$ denotes the language accepted by M) denote the map which sends a path p to the word in $L(M)$ that labels the path p. If M is deterministic then ϕ_M is injective. We show in the following lemma that we can decide if ϕ_M is injective for a nondeterministic finite state automaton M.

Lemma 3.7. *Let M be a nondeterministic finite state automaton. Then there is algorithm that decides whether or not ϕ_M is injective.*

Proof. Let $M = (Q, \Sigma, \delta, q_o, F)$ be a nondeterministic finite state automaton. We build a deterministic finite state automaton $M' = (Q', \Sigma, \delta', q_0', F')$ from M using a construction of Hopcroft and Ullman from chapter 1 of [7] as follows. The state set Q' is the power set of Q, $q_0' = \{q_0\}$, and $F' = \{S \in Q' \mid \text{there exists } q \in S \text{ such that } q \in F\}$. Lastly, $\delta'(\{q_1, ..., q_k\}, \sigma) = \{\delta(q_1, \sigma), ..., \delta(q_k, \sigma)\}$. Then, by construction of M', ϕ_M is not injective if and only if in M' there is an edge $\{r_1, ..., r_t\} \xrightarrow{\sigma} \{s_1, ..., s_v\}$ accessible from $\{q_0\}$ such that there exist distinct r_{i_1}, r_{i_2} with $\delta(r_{i_j}, \sigma) \in F$ or there is an r_j such that in M there are two edges coming out of r_j labeled by σ whose terminal vertices are final states. □

Proposition 3.8. *Let $\mathcal{A} = (Q, \Sigma, t, o)$ be an expanding automaton. Given $q \in Q$, there is an algorithm to decide if $q : \Sigma^* \to \Sigma^*$ is injective.*

Proof. Fix $q \in Q$. First we build a finite state automaton $M = (Q', \Sigma, \delta, q_0, F)$ from $\mathcal{A}$. Begin with state set Q' in bijection with Q. Whenever $q_1 \xrightarrow{\sigma|w} q_2$ in $\mathcal{A}$ with $w = v_1...v_k$ where $v_i \in \Sigma$, add enough states in M so that there is a path labeled by $v_1...v_k$ from q_1' to q_2'. Intuitively, M is the finite state automaton we get from $\mathcal{A}$ by dropping the inputs off of the edges in $\mathcal{A}$, then making each edge into a path so that every edge in M is labeled by an element of Σ. Let $F = Q'$ and $q_0 = q'$. Note that q is not injective if and only if there exist distinct paths in $\mathcal{A}$ such that the outputs read along each path give the same element of Σ^*. Now M is constructed so that for each $w \in \text{range}(q)$ there exists an acceptable path p in M such that $\phi_M(p) = w$, and given an acceptable path p' in M we have $\phi_M(p') \in \text{range}(q)$. Furthermore, each acceptable path in M corresponds to an input path in $\mathcal{A}$. Thus q is not injective if and only if there exist two distinct paths p_1 and p_2 in M such that $\phi_M(p_1) = \phi_M(p_2)$. By lemma 3.7, there is an algorithm to decide this property of M. □

The set of semigroups that can be realized by expanding automata such that the states induce injective functions is very restricted. Let S be an expanding automaton semigroup with corresponding automaton $\mathcal{A} = (Q, \Sigma, t, o)$ such that each state $q \in Q$ induces an injection $\mathcal{T}(\Sigma^*) \to \mathcal{T}(\Sigma^*)$. Then any element of Q^* also induces an injection $\mathcal{T}(\Sigma^*) \to \mathcal{T}(\Sigma^*)$. Let $e \in S$, and suppose that e is idempotent. Since e is idempotent, e fixes range(e). If $w \in \Sigma^*$ is such that $e(w) \neq w$, then w and $e(w)$ are both preimages of $e(w)$ under e. Since e induces an injection, we have that e is the identity function on $\mathcal{T}(\Sigma^*)$. Let e_Σ denote the identity function on $\mathcal{T}(\Sigma^*)$. Then S can contain at most one idempotent, namely e_Σ. If $e_\Sigma \in S$, then Proposition 4.5 implies that the group of units of S is self-similar. Suppose that $e_\Sigma \notin S$. Then S contains no idempotents and hence any $s \in S$ is non-periodic.

Suppose that there is an $s \in S$ such that there exists a word $w \in \Sigma^*$ with $|w| < |s(w)|$. Then, because each element of S is injective and elements of S cannot shorten word length when acting on Σ^*, there cannot be an element $s' \in S$ such that $ss's = s$. A semigroup T is said to be *von Neumann regular* if for each $t \in T$ there is a $t' \in T$ with $tt't = t$. Then S is not von Neumann regular. Thus we have shown the following.

Proposition 3.9. *Let S be an expanding automaton semigroup over an expanding automaton $\mathcal{A} = (Q, \Sigma, t, o)$ such that each q induces an injective function $\mathcal{T}(\Sigma^*) \to \mathcal{T}(\Sigma^*)$. Then*

(1) The group of units of S is self-similar.
(2) S is von Neumann regular if and only if $\mathcal{A}$ is an invertible automaton and S is a group.
(3) If $e \in S$ is idempotent then $e = e_\Sigma$.
(4) If $e_\Sigma \notin S$, then S does not contain any periodic elements.

4. Algebraic Properties

4.1. **Residual Finiteness and Periodicity.** In this section we show that expanding automaton semigroups are residually finite and that the periodicity structure of these semigroups is restricted.

Proposition 4.1. *Expanding automaton semigroups are residually finite.*

Proof. Let S be an expanding automaton semigroup over the alphabet Σ and let $a, b \in S$ with $a \neq b$. For each $m \in \mathbb{N}$, let $L(m) = \{w \in \Sigma^* \ : \ |w| = m\}$, i.e. $L(m)$ is the mth level of the tree Σ^*. Since a and b are distinct, there is $n \in \mathbb{N}$ such that a and b act differently on $L(n)$. Let

$$n' = \max\{|a(w)|, |b(w)| \ : \ w \in L(n)\}$$

and let $\mathcal{L} = \left(\bigcup_{i=1}^{n'} L(i)\right) \cup \{\$\}$. Finally, let $T(\mathcal{L})$ denote the semigroup of transformations $\mathcal{L} \to \mathcal{L}$. Since $\mathcal{L}$ is finite, $T(\mathcal{L})$ is a finite semigroup. Define a homomorphism $\rho : S \to T(\mathcal{L})$ by $\rho(s) = f$ where $f(\$) = \$$ and

$$f(x) = \begin{cases} s(x) & s(x) \in \mathcal{L} \\ \$ & s(x) \notin \mathcal{L} \end{cases}$$

Since a and b act differently on $L(n)$, construction of ρ ensures that $\rho(a)$ and $\rho(b)$ are distinct in $T(\mathcal{L})$. □

Let G be an automaton group over an alphabet Σ and let P_Σ denote the set of prime numbers that divide $|\Sigma|!$. If $g \in G$ has finite order, then the order of g must have only primes from P_Σ in its prime factorization. One can see this by considering g as a level-preserving automorphism on a tree of degree $|\Sigma|$, and thus the cardinality of any orbit under the action of g must have only prime numbers dividing $|\Sigma|!$ in its prime factorization. We show an analogous proposition for the periodicity structure of expanding automaton semigroups. First, we define a *partial invertible automaton* to be a quadruple (Q, Σ, t, o) where t is a partial function from $Q \times \Sigma$ to Q and o is a partial function from $Q \times \Sigma$ to Σ such that the restricted partial function o_q from $\{q\} \times \Sigma$ to Σ is a partial permutation of Σ. It is straightforward to show that any partial invertible automaton can be "completed" to an invertible automaton, i.e. given a partial invertible automaton $\mathcal{B}$ there is an invertible automaton $\mathcal{A}$ (not necessarily unique) such that $\mathcal{B}$ embeds (via a labeled graph homomorphism) into $\mathcal{A}$.

Proposition 4.2. *Let S be an expanding automaton semigroup over an alphabet Σ, and let P_Σ be as above. If $s \in S$ is periodic with $s^m = s^n$, $m < n$, and $s, ..., s^{n-1}$ distinct, then $n - m$ has only primes from P_Σ in its prime factorization.*

Proof. Let $\mathcal{A} = (Q, \Sigma, t, o)$ be an expanding automaton with $S = S(\mathcal{A})$. Suppose $s \in S$ is periodic with $s^m = s^n$, $m < n$, and $s, ..., s^{n-1}$ distinct. Fix $w \in s^m(\Sigma^*)$. Then $R_w := \{s^k(w) \mid k \geq m\}$ is a finite set, and the cardinality of R_w divides $n - m$. Note that for any $w' \in R_w$, s acts like a cycle on w' as $s^m(w') = s^n(w')$. Furthermore, if $v, v' \in R_w$ then $|v| = |v'|$ because s cannot shorten word length. Thus the paths in $\mathcal{A}$ corresponding to the input words $s^m(w), ..., s^{n-1}(w)$ form a partial invertible subautomaton of $\mathcal{A}$. Denote this partial invertible subautomaton by β_w. Consider the partial invertible subautomaton β of $\mathcal{A}$ given by $\beta = \cup_{w \in \Sigma^*}(\beta_w)$. Complete β to an invertible automaton β'. Then R_w is an orbit under the action of an element of an automaton group for all $w \in \Sigma^*$, and the result follows. □

4.2. **Subgroups.** Let $\mathcal{A} = (Q, \Sigma, t, o)$ be a synchronous invertible automaton. As in [9], construct an automaton $\mathcal{A}^{-1} = (Q^{-1}, \Sigma, t^{-1}, o^{-1})$ where Q^{-1} is a set in bijection with Q under the mapping $q \to q^{-1}$, $t^{-1}(q_1^{-1}, \sigma) = q_2^{-1}$ if and only if $t(q_1, \sigma) = q_2$, and $o(q^{-1}, \sigma_1) = \sigma_2$ if and only if $o(q, \sigma_2) = \sigma_1$. Then $qq^{-1} = q^{-1}q$ is the identity function $\Sigma^* \to \Sigma^*$, and so $\mathcal{A}^{-1}$ is called the *inverse automaton for* $\mathcal{A}$ (we always denote the inverse automaton for $\mathcal{A}$ by $\mathcal{A}^{-1}$).

Proposition 4.3. *A group G is an automaton group (respectively self-similar group) if and only if G is an expanding automaton semigroup (respectively expanding self-similar semigroup).*

Proof. Let G be an automaton group corresponding to the automaton $\mathcal{A} := (Q, \Sigma, t, o)$. Since G is an automaton group, $\mathcal{A}$ is invertible and synchronous. Construct a new automaton $\mathcal{B} = \mathcal{A} \cup \mathcal{A}^{-1}$. Then $S(\mathcal{B}) = G$ and $\mathcal{B}$ is an expanding automaton. Thus G is an expanding automaton semigroup.

Conversely, let the group G be an expanding automaton semigroup corresponding to the expanding automaton $\mathcal{A} = (Q, \Sigma, t, o)$. Let e be the identity of G and $g \in G$. Then

$$e(\Sigma^*) = g(g^{-1}(\Sigma^*)) \subseteq g(\Sigma^*)$$

and

$$g(\Sigma^*) = e(g(\Sigma^*)) \subseteq e(\Sigma^*)$$

Hence $e(\Sigma^*) = g(\Sigma^*)$. Now e is idempotent and thus fixes $e(\Sigma^*)$, so (as in the proof of 2.4) g is bijective and length-preserving on $g(\Sigma^*) = e(\Sigma^*)$. Thus G is isomorphic to the semigroup generated by $\{g|_{e(\Sigma^*)} \ : \ g \in G\}$.

Construct an invertible automaton $\mathcal{B} = (\overline{Q} \cup \{1\}, \Sigma = \{\sigma_1, ..., \sigma_n\}, \overline{t}, \overline{o})$ as follows. The state set is $\overline{Q} \cup \{i\}$ where $\overline{Q}$ is a set in bijection with Q and $i = (i, ..., i)[\sigma_1, ..., \sigma_n]$, i.e. i is a sink state that pointwise fixes Σ^*. The transition function is given by

$$\overline{t(q,\sigma)} = \begin{cases} t(q,\sigma) & \text{if } \sigma \in e(\Sigma) \\ i & \text{if } \sigma \notin e(\Sigma) \end{cases}$$

and the output function is given by

$$\overline{o(q,\sigma)} = \begin{cases} o(q,\sigma) & \text{if } \sigma \in e(\Sigma) \\ \sigma & \text{if } \sigma \notin e(\Sigma) \end{cases}$$

Let $g \in G$ and let $w \in \Sigma^* - e(\Sigma^*)$ be of minimal length. Write $w = v\sigma$ where $v \in e(\Sigma^*)$. Then the above conditions imply that, for any $w' \in \Sigma^*$, $\hat{q}(ww') = q(w)\sigma w'$. In other words, each state $\overline{q}$ of $\mathcal{B}$ will mimic the action of q on words that are in the image of e, but will enter the state i and act identically on the suffix of a word w following the largest prefix of w lying in $e(\Sigma^*)$. So the part of the action which does not act bijectively and in a length-preserving fashion collapses to the identity, and we have an invertible automaton giving G.

None of the above uses that the automata have finitely many states, so the same logic shows that G is a self-similar group if and only if G is an expanding self-similar semigroup. □

The idea in the last proof allows us to prove the following:

Proposition 4.4. *Let S be an expanding automaton semigroup and H a subgroup of S. Then there is a self-similar group G with $H \leq G$.*

Proof. Let S be an expanding automaton semigroup and H a subgroup of S. Let e denote the identity of H. Let $\mathcal{A} = (Q, \Sigma, t, o)$ be the expanding automaton associated with S. As in the proof of Proposition 4.3, H is isomorphic to the semigroup generated by $\{h|_{e(\Sigma^*)} : h \in H\}$ and each element of H acts injectively and in length-preserving fashion on $e(\Sigma^*)$. Then we can again collapse the "non-group" part of the action to the state which fixes the tree to get a length-preserving and invertible action of H. Thus we can construct an invertible (and possibly infinite state) synchronous automaton containing the elements of H as states. The states generated by this automaton is a self-similar group G with $H \leq G$. □

If S is an expanding automaton semigroup and H is a subgroup of S, then S is a subgroup of a self-similar group, but H is not necessarily self-similar. If H is the unique maximal subgroup of S, then we show below that H is self-similar.

Proposition 4.5. *Let S be an expanding automaton semigroup with a unique maximal subgroup G. Then G is self-similar.*

Proof. Let $\mathcal{A} = (Q, \Sigma, t, o)$ be the automaton associated with S. Let $g \in G$ and write $g = (g_1, ...g_n)\tau_g$ where $n = |\Sigma|$. Let e be the identity of G, and write $e = (e_1, ..., e_n)\tau_e$. Since e is idempotent, e fixes range(e), and thus the set $\hat{\Sigma} := \{\sigma \in \Sigma \mid e(\sigma) = \sigma\}$ is non-empty. To see this, let $\sigma \in \Sigma$ and suppose that $e(\sigma) = \sigma' w$ for some $\sigma' \in \Sigma$. Then e fixes $\sigma' w$, and since e is length-expanding $e(\sigma') = \sigma'$. Since e is idempotent, $e_{\hat{\sigma}}$ is idempotent for all $\hat{\sigma} \in \hat{\Sigma}$. This is true because $(e^n)_{\hat{\sigma}} = (e_{\hat{\sigma}})^n$. Since G is the unique maximal subgroup of S, there is only one idempotent element of S. Thus $e_{\hat{\sigma}} = e$ for all $\hat{\sigma} \in \hat{\Sigma}$.

Let $\sigma \in \hat{\Sigma}$. Then $\tau_g(\sigma) \in \hat{\Sigma}$ and so $e_{\tau_g(\sigma)} = e$. Thus Equation (1) implies

$$g_\sigma = (eg)_\sigma = e_{\tau_g(\sigma)} g_\sigma$$

and, as e stabilizes σ,

$$g_\sigma = (ge)_\sigma = g_\sigma e_\sigma.$$

Hence $eg_\sigma = g_\sigma e = g_\sigma$ for any $\sigma \in \hat{\Sigma}$.

Let $h = g^{-1}$, $\sigma \in \hat{\Sigma}$, and write $h = (h_1, ..., h_n)\tau_h$. By the same logic as above, $eh_\sigma = h_\sigma e = h_\sigma$. Since $hg = e$ we have

$$(hg)_\sigma = h_{\tau_g(\sigma)} g_\sigma = e_\sigma = e$$

Since g_σ is left-invertible, Proposition 2.4 implies that g_σ is invertible. Therefore $g_\sigma \in G$ for all $\sigma \in \hat{\Sigma}$.

Continuing inductively, we see that $g_w \in G$ for all $w \in$ range(e). Similar to the proof of Proposition 4.3, if $w \notin$ range(e) then for all $g \in G$ we can replace g_w with e and the resulting group will still be isomorphic to G. This is because, as in Proposition 4.3, the action of G on range(e) captures all of the group information. Thus G is an expanding self-similar semigroup, and Proposition 4.3 implies that G is a self-similar group. ∎

If $\mathcal{A}$ is an invertible synchronous automaton, then $S(\mathcal{A})$ has at most one idempotent, namely the identity function on the tree. Thus Proposition 4.5 has the following corollary.

Corollary 4.6. *Let $\mathcal{A}$ be an invertible synchronous automaton. Then the group of units of $S(\mathcal{A})$ is self-similar.*

5. Closure Properties and further examples

5.1. **Closure Properties.** We begin this section by showing that the class of expanding automaton semigroups is not closed under taking normal ideal extensions. In particular, we show that the free semigroup of rank 1 with a zero adjoined is not an expanding automaton semigroup. We then show that the class of asynchronous automaton semigroups is closed under taking normal ideal extensions.

Lemma 5.1. *The free semigroup of rank 1 with a zero adjoined is not an automaton semigroup.*

Proof. Let S be an automaton semigroup over an alphabet $\Sigma = \{\sigma_1, ..., \sigma_n\}$ such that S is generated by two elements a and b with $ab = ba = b$ and $b^2 = b$. We use the same idea of the proof of Proposition 2.3 to show that a is periodic.

Let $\sigma \in \Sigma$. Suppose that the section of a at $a^n(\sigma) = b$ for some n. Then $(a^n)_\sigma = (a^{n+k})_\sigma$ for all $k \in \mathbb{N}$. If the section of a at $a^n(\sigma)$ is a power of a for all n, then (as in the proof of Proposition 2.3) the section of a at $a^n(\sigma)$ is a for all n.

Let $\hat{\Sigma} = \{\sigma \in \Sigma \mid (a^r)_\sigma = b \text{ for some } r\}$. As in the previous proof, we can choose s and t such that $\tau_{a^s} = \tau_{a^t}$ and $(a^s)_\sigma = (a^t)_\sigma$ for all $\sigma \in \hat{\Sigma}$. Then the same logic of the previous proof shows that $a^s = a^t$. □

We now apply Lemma 5.1 to show the following.

Proposition 5.2. *The class of expanding automaton semigroups is not closed under taking normal ideal extensions. In particular, the free semigroup of rank 1 with a zero adjoined is not an expanding automaton semigroup.*

Proof. Let $S =< a, b \mid b^2 = b,\ ab = ba = b >$ be the free semigroup of rank 1 with a zero adjoined, and suppose S were an expanding automaton semigroup corresponding to the automaton

$\mathcal{A} = (Q, \Sigma, t, o)$. Since b is idempotent, b fixes range(b). Hence the set $\hat{\Sigma} = \{\sigma \in \Sigma \mid b(\sigma) = \sigma\}$ is non-empty. Since b is the only idempotent of S, $b_{\hat{\sigma}} = b$ for all $\hat{\sigma} \in \hat{\Sigma}$.

Let $\sigma \in \Sigma - \hat{\Sigma}$, and suppose that $b_\sigma = a^n$ for some $n > 0$. Let $w \in \Sigma^*$. Then $b(\sigma w) = b(\sigma)a^n(w)$. Since b fixes range(b), we have that $b(b(\sigma w)) = b(\sigma)a^n(w)$. We also have that b fixes $b(\sigma)$ and the section of b at $b(\sigma)$ is b. Thus b fixes $a^n(w)$, and (as w is arbitrary) $ba^n = a^n$ in S. But $ba^n = b$, which implies that a^n is idempotent. Since a^n is not idempotent in S, we have $b_\sigma = b$ for all $\sigma \in \Sigma$. Note that b must be a state of $\mathcal{A}$ as powers of a cannot multiply to obtain b. Thus, in the graphical representation of $\mathcal{A}$, all edges going out of b are loops based at b. Note also that a must be a state of $\mathcal{A}$.

Let $\Gamma = \{\sigma \in \Sigma \ : \ |a^m(\sigma)| = 1 \text{ for all } m\}$. The equation $ab = b$ implies that a fixes range(b), and so Γ is nonempty. In $\mathcal{A}$, for each state q in $< a >$ and $\gamma \in \Gamma$ there is an arrow labeled by $\gamma|\hat{\gamma}$ coming out of q where $\hat{\gamma} \in \Gamma$. Let $w \in \Gamma^*$ with $w = \gamma_1 ... \gamma_k$. Suppose that $|a(w)| > 1$. Then w, as a path in $\mathcal{A}$ based at a, must enter the state b. Choose i maximal so that $\gamma_1 ... \gamma_{i-1}$ is a path such that the initial vertex of each edge is not the state b. Then $a(w) = \gamma'_1 ... \gamma'_k$ where $\gamma'_m \in \Gamma$ for $1 \leq m \leq i - 1$ and $\gamma'_m \in \hat{\Sigma}^*$ for $i \leq m \leq k$. Since a fixes $\hat{\Sigma}^*$, $|a^n(w)| = |a^2(w)|$ for all $n \geq 2$. Thus for any $w \in \Gamma^*$, $|a^{|\Sigma|}(w)| = |a^k(w)|$ for any $k \geq |\Sigma|$.

Suppose that $|a(\sigma)| = 1$ for all $\sigma \in \Sigma$. Then the same logic as in the proof of Proposition 2.3 shows that either a is periodic or has infinitely many sections (note that the proof does not use that the periodic element acts in a length-preserving fashion). So the set $\Sigma' = \{\sigma \in \Sigma \ : \ |a(\sigma)| > 1\}$ is nonempty. Let $\sigma' \in \Sigma'$, and write $a(\sigma') = \sigma_1 ... \sigma_m$ where $\sigma_i \in \Sigma$. Suppose that $\sigma_i = \sigma'$ for some i. Then $b(a(\sigma')) = b(\sigma_1 ... \sigma_n) = b(\sigma_1) ... b(\sigma_m) = b(\sigma')$, and so $|b(a(\sigma'))| > |b(\sigma')|$, a contradiction. Thus σ' is not a letter of $a(\sigma')$. The same calculation also shows that σ' is not a letter of $a^n(\sigma')$ for any n and that σ' is not a letter of $a(\sigma_i)$ for any i.

Let $w \in \Sigma^*$ and write $w = \sigma_1 ... \sigma_k$. Suppose that $\sigma_i \notin \Gamma$ for some i. Then every edge in $\mathcal{A}$ with input label σ_i has an output label without σ_i as a letter. Thus $a^n(w)$ does not contain σ_i as a letter for any n. If $a(w) \in \Gamma^*$, then as mentioned above a will act in a length-preserving fashion on $a^{|\Sigma|}(w)$. Suppose that $a(w) \notin \Gamma^*$ where $\sigma_j \notin \Gamma$ is a letter of $a(w)$. Then $a^2(w)$ does not contain σ_i or σ_j as a letter. Continuing inductively, we see that $a^{|\Sigma|}(w) \in \Gamma^*$. Thus there is an $m \in \mathbb{N}$ such that a acts in a length-preserving fashion on $a^m(w)$ for any $w \in \Sigma^*$, i.e. $|a^m(w)| = |a^k(w)|$ for

$k \geq m$ and any $w \in \Sigma^*$. This induces a length-preserving action of S on Γ^*, contradicting Lemma 5.1. □

Proposition 5.3. *Let S and T be asynchronous automaton semigroups. Then the normal ideal extension of S by T is an asynchronous automaton semigroup.*

Proof. Let $\mathcal{A} = (Q_1, \Sigma, t_1, o_1)$ and $\mathcal{B} = (Q_2, \Gamma, t_2, o_2)$ be asynchronous automata with $S(\mathcal{A}) = S$ and $S(\mathcal{B}) = T$. Construct a new automaton $\mathcal{C} = (Q_1 \cup Q_2, \Sigma \cup \Gamma, t, o)$ with transition and output functions as follows:

$$t(q_1, \sigma) = t_1(q_1, \sigma) \text{ for all } q_1 \in Q_1 \text{ and } \sigma \in \Sigma$$

$$t(q_1, \gamma) = q_1 \text{ for all } q_1 \in Q_1 \text{ and } \gamma \in \Gamma$$

$$t(q_2, \sigma) = q_2 \text{ for all } q_2 \in Q_2 \text{ and } \sigma \in \Sigma$$

$$t(q_2, \gamma) = t_2(q_2, \gamma) \text{ for all } q_2 \in Q_2 \text{ and } \gamma \in \Gamma$$

$$o(q_1, \sigma) = o_1(q_1, \sigma) \text{ for all } q_1 \in Q_1 \text{ and } \sigma \in \Sigma$$

$$o(q_1, \gamma) = \gamma \text{ for all } q_1 \in Q_1 \text{ and } \gamma \in \Gamma$$

$$o(q_2, \sigma) = \emptyset \text{ for all } q_2 \in Q_2 \text{ and } \sigma \in \Sigma$$

$$o(q_2, \gamma) = o_2(q_2, \gamma) \text{ for all } q_2 \in Q_2 \text{ and } \gamma \in \Gamma$$

By construction of $\mathcal{C}$, the subsemigroup of $S(\mathcal{C})$ generated by Q_1 is S and the subsemigroup of $S(\mathcal{C})$ generated by Q_2 is T.

Now let $w \in (\Sigma \cup \Gamma)^*$. Write $w = \sigma_1\gamma_1\sigma_2\gamma_2...\sigma_n\gamma_n$ with $\sigma_i \in \Sigma^*$ and $\gamma_j \in \Gamma^*$. Let $s \in Q_1^*$ and $t \in Q_2^*$. Then

$$ts(w) = t(s(\sigma_1)\gamma_1 s(\sigma_2)\gamma_2...s(\sigma_n)\gamma_n) = \emptyset t(\gamma_1)\emptyset t(\gamma_2)...t(\gamma_n) = t(\gamma_1)t(\gamma_2)...t(\gamma_n)$$

and

$$st(w) = s(t(\gamma_1)t(\gamma_2)...t(\gamma_n)) = t(\gamma_1)t(\gamma_2)...t(\gamma_n)$$

Thus both $st(w)$ and $ts(w)$ equal $t(w)$, so $st = ts = t$. □

We close this section by showing that the class of expanding automaton semigroups is closed under direct product, provided the direct product is finitely generated.

Proposition 5.4. *Let S and T be expanding automaton semigroups. Then $S \times T$ is an expanding automaton semigroup if and only if $S \times T$ is finitely generated.*

Proof. An expanding automaton semigroup must be finitely generated, so the forward direction is clear. Suppose that $S \times T$ is finitely generated. Then $S \times T$ is generated by $A \times B$ for some finite $A \subseteq S$ and $B \subseteq T$. Let $\mathcal{A}_S$ and $\mathcal{A}_T$ be expanding automata with state sets P and Q respectively such that $S = S(\mathcal{A}_S)$ and $T = S(\mathcal{A}_T)$. Furthermore, choose m, n so that $A \subseteq P^m$ and $B \subseteq Q^n$, and add enough states to each expanding automaton so that we obtain new automata $\mathcal{A}'_S$ and $\mathcal{A}'_T$ with $S = S(\mathcal{A}'_S)$, $T = S(\mathcal{A}'_T)$, and P^m is contained in the state set of $\mathcal{A}'_S$; likewise for Q^n and $\mathcal{A}'_T$. Write $\mathcal{A}'_S = (X', C, t', o')$ and $\mathcal{A}'_T = (Y', D, \hat{t}, \hat{o})$. with C and D disjoint.

Let $\mathcal{Y} = (X' \cup Y', C \cup D, t, o)$ be the expanding automaton defined by

$$t(q,\sigma) = \begin{cases} t'(q,\sigma) & q \in X' \text{ and } \sigma \in C \\ q & q \in X' \text{ and } \sigma \in D \\ \hat{t}(q,\sigma) & q \in Y' \text{ and } \sigma \in D \\ q & q \in Y' \text{ and } \sigma \in C \end{cases} \quad \text{and} \quad o(q,\sigma) = \begin{cases} o'(q,\sigma) & q \in X' \text{ and } \sigma \in C \\ \sigma & q \in X' \text{ and } \sigma \in D \\ \hat{o}(q,\sigma) & q \in Y' \text{ and } \sigma \in D \\ \sigma & q \in Y' \text{ and } \sigma \in C \end{cases}.$$

Then the subsemigroup of $S(\mathcal{Y})$ generated by X' is S and the subsemigroup of $S(\mathcal{Y})$ generated by Y' is T, and construction of $\mathcal{Y}$ implies that $x'y' = y'x'$ for all $x' \in X'$ and $y' \in Y'$. Thus $S(\mathcal{Y}) \cong S \times T$. □

Let S and T be finitely generated semigroups such that T is infinite. Robertson, Ruskuc, and Wiegold show in [11] that if S is finite then $S \times T$ is finitely generated if and only if $S^2 = S$. If S is infinite, then $S \times T$ is finitely generated if and only if $S^2 = S$ and $T^2 = T$. Let $\mathbb{N}$ denote the free semigroup of rank 1. Then $\mathbb{N}^2 \neq \mathbb{N}$, and thus $\mathbb{N} \times \mathbb{N}$ is not an expanding automaton semigroup (even though $\mathbb{N}$ is an expanding automaton semigroup).

5.2. **Free Partially Commutative Monoids.** In this section we show that any free partially commutative monoid is an expanding automaton semigroup. Let M be a free partially commutative monoid with monoid presentation $< X|R >$. We begin by defining the *shortlex normal form* on M. First, if $v \in X^*$, $|v|$ will always denote the length of v in X^*. Order the set X by $x_i < x_j$ whenever $i < j$. If $v, w \in X^*$, let $v < w$ if and only if $|v| < |w|$ or, if $|v| = |w|$, v comes before w in the dictionary order induced by the order on X. This is called the *shortlex ordering* on X^*. To obtain the set of shortlex normal forms of M, for each $w \in M$ choose a word $w' \in X^*$ such that $w = w'$ in M and w' is minimal in X^* with respect to the shortlex ordering. We remark that it is immediate from this definition that a word $w \in X^*$ is in shortlex normal form in M if and only if for all factorizations $x = ybuaz$ in M where $y, u, z \in X^*$, a and b commute, and $a < b$, there is a letter of u which does not commute with a.

For any $w \in X^*$, let $w(x_i, x_j)$ denote the word obtained from w by erasing all letters except x_i and x_j. We write $w(x_i)$ to denote the word obtained from w by deleting all occurrences of the letter x_i. We will need the following lemma regarding free partially commutative monoids.

Lemma 5.5. *Let M be a free partially commutative monoid generated by $X = \{x_1, ..., x_n\}$, and let $v, w \in X^*$ such that v and w are in shortlex normal form in M. Suppose that*

(1) $|v(x_i)| = |w(x_i)|$ for $1 \leq i \leq n$ and

(2) $v(x_i, x_j) = w(x_i, x_j)$ in X^* whenever $1 \leq i, j \leq n$ and x_i and x_j do not commute.

Then $v = w$ in M.

Proof. Let $v, w \in M$ be words satisfying $|v(x_i)| = |w(x_i)|$ for all i. This implies that the number of occurrences of x_i as a letter of v equals the number of occurrences of x_i as a letter of w. In particular, $|v| = |w|$. Write $v = x_{i_1}...x_{i_k}$ and $w = x_{j_1}...x_{j_k}$ with v, w in shortlex normal form. Suppose that $x_{i_1} < x_{j_1}$. Then $v(x_{i_1}, x_{j_1}) \neq w(x_{i_1}, x_{j_1})$ in X^*, and condition (2) in the hypotheses implies that x_{i_1} and x_{j_1} commute. Condition (1) implies that x_{j_1} is a letter of v and x_{i_1} is a letter of w, and so we write $v = x_{i_1} v_1 x_{j_1} v_2$ where v_1 does not contain x_{j_1} as a letter. Similarly, write $w = x_{j_1} w_1 x_{i_1} w_2$. Condition (2) implies that x_{i_1} commutes with every letter of w_1. Since $x_{i_1} < x_{j_1}$, we have that w was not in lexicographic normal form. Thus $x_{i_1} \not< x_{j_1}$, and symmetry implies $x_{j_1} \not< x_{i_1}$. So $x_{i_1} = x_{j_1}$. Inductively continuing the argument implies that $x_{i_t} = x_{j_t}$ $\forall$ $1 \leq t \leq k$. □

Theorem 5.6. *Every free partially commutative monoid is an automaton semigroup.*

Proof. Let M be a partially commutative monoid generated by $X = \{x_1, ..., x_n\}$. Let $N = \{\{i, j\} \mid x_i$ and x_j do not commute$\}$. Let $A = \{a_1, ..., a_n\}$, $B = \{b_1, ..., b_n\}$, $C = \{c_{ij} \mid i < j$ and $\{i, j\} \in N\}$, and $D = \{d_{ij} \mid i < j$ and $\{i, j\} \in N\}$ be four alphabets where C, D are in bijective correspondence with N. We construct an automaton $\mathcal{A}_M$ with state set $Q := \{y_1, ..., y_n, 1\}$ over the alphabet $\Sigma = A \cup B \cup C \cup D$ such that $S(\mathcal{A}_M) \cong M$ as follows. Let 1 be the sink state that pointwise fixes Σ^*. For each i, define

$$t(y_i, a_j) = 1 \text{ for all } j, \quad t(y_i, b_j) = \begin{cases} y_i & i = j \\ 1 & i \neq j \end{cases}$$

and

$$o(y_i, a_j) = \begin{cases} b_j & i = j \\ a_j & i \neq j \end{cases}, \quad o(y_i, b_j) = \begin{cases} a_j & i = j \\ b_j & i \neq j \end{cases}.$$

By construction, the subautomaton consisting of the states y_i and 1 over the alphabet $\{a_i, b_i\}$ is the *adding machine* automaton (see Figure 1.3 of [9]) for all i. Note that for any $k > j$, $y_i^j(a_i^{2j}) \neq y_i^k(a^{2j})$, and so the semigroup corresponding to this subautomaton is the free monoid of rank 1 for all i. Thus each y_i acts non-periodically on $\{a_i, b_i\}^*$ for all i. Furthermore, if $i \neq j$ then y_j induces the identity function from $x\Sigma^*$ to $x\Sigma^*$ where $x \in \{a_i, b_i\}$.

We now complete the construction of $\mathcal{A}$. Fix $i < j$ with $\{i, j\} \in N$, and let $k \in \mathbb{N}$ such $1 \leq k \leq n$ and $k \neq i, j$. Define

$$t(y_i, c_{ij}) = y_j, \quad t(y_i, d_{ij}) = y_i, \quad t(y_j, c_{ij}) = y_i, \quad t(y_j, d_{ij}) = y_j$$

$$o(y_i, c_{ij}) = d_{ij}, \quad o(y_i, d_{ij}) = c_{ij}, \quad o(y_j, c_{ij}) = c_{ij}, \quad o(y_j, d_{ij}) = d_{ij}$$

$$t(y_k, c_{ij}) = t(y_k, d_{ij}) = 1$$

$$o(y_k, c_{ij}) = c_{ij}, \quad t(y_k, d_{ij}) = d_{ij}.$$

For all other i', j' such that $\{i', j'\} \subseteq N$ and $i' < j'$, define the output and transition function analogously. Figure 5.2 gives the automaton $\mathcal{A}_M$ where M is the free partially commutative monoid $< y_1, y_2, y_3 \mid y_1 y_2 = y_2 y_1, \ y_1 y_3 = y_3 y_1 >$ (we omit the arrow on the sink state).

For each $\{i, j\} \in N$, the subautomaton of $\mathcal{A}_M$ corresponding to the states y_i and y_j over the alphabet $\{c_{ij}, d_{ij}\}$ is the "lamplighter automaton" (see Figure 1.1 of [6]). Grigorchuk and Zuk show in Theorem 2 of [6] that this automaton generates the lamplighter group, and in particular in Lemma 6 of [6] they show that the states of this automaton generate a free semigroup of rank 2. Thus y_i and y_j generate a free semigroup of rank 2 when acting on $\{c_{ij}, d_{ij}\}^*$, and hence the semigroup generated by y_i and y_j in $S(\mathcal{A}_M)$ is free of rank 2.

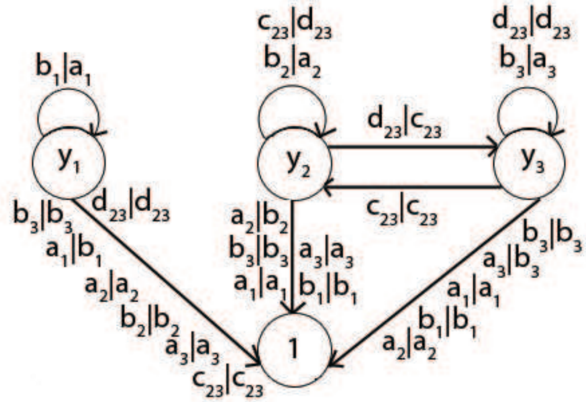

FIGURE 4. An automaton generating the monoid $< y_1, y_2, y_3 \mid y_1y_2 = y_2y_1,\ y_1y_3 = y_3y_1 >$

Let $1 \leq i, j \leq n$ be such that $\{i, j\} \not\subseteq N$. By construction of $\mathcal{A}_M$, y_i and y_j have disjoint support, i.e. the sets $\{w \in \Sigma^* \mid y_i(w) \neq w\}$ and $\{w \in \Sigma^* \mid y_j(w) \neq w\}$ are disjoint. Thus if x_i and x_j commute in M, then y_i and y_j commute in $S(\mathcal{A}_M)$. So $S(\mathcal{A}_M)$ is a quotient of M.

Let $v, w \in Q^*$ such that v and w are written in shortlex normal form when considered as elements of M. Suppose that $w(y_i) \neq v(y_i)$ for some i. By construction of $\mathcal{A}_M$, for any $i \neq j$ we have y_j acts as the identity function on $\{a_i, b_i\}^*$. Thus the action of v and w on $\{a_i, b_i\}^*$ is the same as the action of $v(y_i)$ and $w(y_i)$, respectively, on $\{a_i, b_i\}^*$. So $w(y_i) \neq v(y_i)$ implies that $v \neq w$ in $S(\mathcal{A}_M)$. Hence $v = w$ in $S(\mathcal{A}_M)$ implies that $w(y_i) = v(y_i)$ for all i.

Suppose now that there exist $\{r, s\} \in N$ such that $v(y_r, y_s) \neq w(y_r, y_s)$. If $t \neq r, s$, then y_t acts like the identity function on $\{c_{rs}, d_{rs}\}^*$. Thus the action of v and w on $\{c_{rs}, d_{rs}\}^*$ is the same as the action of $v(y_r, y_s)$ and $w(y_r, y_s)$, respectively, on $\{c_{rs}, d_{rs}\}^*$. So $v(y_r, y_s) \neq w(y_r, y_s)$ implies that $v \neq w$ in $S(\mathcal{A}_\mathcal{M})$. Thus if $v = w$ in $S(\mathcal{A}_\mathcal{M})$ then $v(y_r, y_s) = w(y_r, y_s)$ in Q^* for all $\{r, s\} \in N$.

The last two paragraphs have shown that if $v = w$ in $S(\mathcal{A}_\mathcal{M})$, then v and w satisfy the hypotheses of Lemma 5.5. Hence $v = w$ in M, and the result follows. □

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