Critical interfaces and duality in the Ashkin Teller model

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We report on the numerical measures on different spin interfaces and FK cluster boundaries in the Askhin-Teller (AT) model. For a general point on the AT critical line, we find that the fractal dimension of a generic spin cluster interface can take one of four different possible values. In particular we found spin interfaces whose fractal dimension is $d_f = 3/2$ all along the critical line. Further, the fractal dimension of the boundaries of FK clusters were found to satisfy all along the AT critical line a duality relation with the fractal dimension of their outer boundaries. This result provides a clear numerical evidence that such duality, which is well known in the case of the $O(n)$ model, exists in a extended CFT.

Introduction

These last years have seen the study of geometrical objects in two-dimensional critical statistical models as one of the most active areas in statistical and mathematical physics. The universality classes of a large variety of critical models (critical percolation, the self-avoiding walks, loop erased random walk, q-states Potts models) can be described in terms of one parameter family of loop models, the $O(n)$ loop models. clusters which appear in the Combining methods of conformal field theory (CFT), with a Coulomb-gas representation [1], all geometrical exponents characterizing the fractal geometry of the $O(n)$ critical loops can be computed [2]. The interest in these investigations has been certainly boosted by the more recent discovery that the continuum limit of certain boundary loops (i.e. loops created by imposing certain boundary conditions) can be described by conformally invariant stochastic growth, the so called Schramm-Loewner evolutions (SLE). The SLE approach, besides providing new formulas, paved also the way to put the results obtained from CFT methods on a more firm mathematical foundation.

According to these results, which concern mainly the $O(n)$ models in their critical dense phase, it is tantalizing to suggest a completely new interpretation of critical phases in terms of concepts of stochastic geometry. Still, this scenario is far to be established. The critical point of 2D systems can be classified according to the different families of CFTs. In this respect, the $O(n)$ model are described by the most simple of CFT family, i.e. the one constructed from the conformal symmetry alone. Other lattice statistical models, typically characterized by some symmetry in the internal degree of freedom, have critical points described by the so-called extended CFT, i.e. CFTs with additional symmetries. A representative example are the Z_N spin lattice models where the spins take N values and interact via a nearest-neighbor potential which is invariant under a Z_N cyclic permutation of the N states. These spin models have critical points which, for $N \geq 4$, are described by a family of extended CFT, the so-called parafermionic CFT [3]. If the role of the Z_N additional symmetries is under control as far as the operator algebra or the classification of the conformal boundary conditions is concerned, its effects on the geometrical features of the corresponding critical phases is still not understood. The study of the geometrical features of the extended CFTs turns out to be a very hard problem. Some progress in this direction has been done by defining loop models associated to some extended CFTs [4] or by proposing a possible extension of the SLE approach to these CFTs[5, 6]. But, so far, the most important insights into this problem come from numerical measurements of the fractal dimensions associated to the spin interfaces for Z_4 and Z_5 spin models [7]. By measuring systematically all the different bulk and boundary spin interfaces, it was found that there is a limited number of possible values for the fractal dimension which can be understood on the basis of the classification of Z_N conformal boundary conditions [8]. One of these value, corresponding to a certain interface, was found in agreement with the one proposed on the basis of CFT computation in [9]. This scenario, which establishes for the first time a connection between geometrical objects and extended CFT, has been made particularly clear in [8] where the spin cluster interfaces of the Z_4 model model were studied.

The critical point of the Z_4 is particularly interesting as it represents a particular point on the self-dual critical line of the well known Ashkin-Teller (AT) model [10]. The phase diagram of the AT model, defined below, presents a critical line which is described by the compactified free Gaussian boson along which the critical exponents change with the compactification radius.

In this paper we report on the numerical measures of the fractal dimension of different spin interfaces of the AT model by showing how the results found in [8] generalise all along the critical line. In particular we found a spin interface whose fractal dimension is $d_f = 3/2$ all along the critical line suggesting this interface to be described by an SLE_4 process. Moreover, we measured the fractal dimension of particular FK cluster defined bellow. Interestingly, these values seem to satisfy all along the critical line a duality relation with the fractal dimension of one particular spin interface. This is a clear numerical evidence that such duality, which is well known in the case of the $O(n)$ model, exists in a extended CFT. Note that the same duality was also found by studying spins and FK interfaces in the Potts model at the random conformal critical point [11] which are also believed to be described by (non-unitary) extended CFTs. In the case of Z_4 spin model, a duality was predicted on the basis of a CFT computation but the proposed FK cluster was not the same as the one studied here.

The model

The Ashkin-Teller model is usually defined in terms of two coupled Ising models. On each site i of a square lattice one associates a pair of spins, denoted by σ_i and τ_i, which take two values, say up (+) and down (-). The Hamiltonian is defined by

$$H_{AT}(\{\sigma_i,\tau_i\}) = -\sum_{<ij>} \left(K(\sigma_i\sigma_j+\tau_i\tau_j)+K_4\sigma_i\sigma_j\tau_i\tau_j\right) \ . \tag{1}$$

The AT model presents a rich phase diagram which has been very well studied [12, 13]: there is a critical line defined by the self-dual condition $\sinh 2K = \exp(-2K_4)$ and terminating at $K = K_4 = K^P$ where $K^P = \ln 3/4$. The point K^P corresponds to the 4-states Potts model critical point. Other two special points are known: i) the point ($K_4 = K_4^I = 0$,$K = K^I = \ln\left(1+\sqrt{2}\right)/2$) corresponding to two decoupled critical Ising models and ii) the so called Fateev Zamolodchikov (FZ) point, located between the K^P and K^I, and described in the continuum limit by the Z_4 parafermionic theory mentioned above.

For our purposes, it is convenient to rewrite the model (1) in terms of the spin variables S_i which take four values, $S_i = 1,2,3,4$ and defined via the correspondence:

$$\begin{aligned} &S_i = 1 : (\sigma_i,\tau_i) = (+,+); S_i = 2 : (\sigma_i,\tau_i) = (+,-)\\ &S_i = 3 : (\sigma_i,\tau_i) = (-,-); S_i = 4 : (\sigma_i,\tau_i) = (-,+) \end{aligned} \tag{2}$$

With this formulation, it is easy to see that the Z_4 symmetry of the model (1) is completely explicit.

Spin cluster interfaces

In this paper we investigated numerically the spin cluster interfaces of the AT model which are defined, in the representation (2), as boundaries of cluster of spins S_i. A spin cluster indicates a connected set of spins which can take a given set of values.

In analogy with the chordal SLE interfaces defined in the critical $O(n)$ model, we consider the model on a bounded domain and we define spin interfaces generated by imposing certain spin configurations on the boundary. More in detail, we simulate finite square lattices of size $L \times L$ with certain boundary conditions $(A_1 + A_2\cdots|B_1 + B_2\cdots)$. By this notation, we mean that we set one half of the boundary spins to take the values $A_1, A_2, ..$ and the other half the values $B_1, B_2..$ with equal probabilities. Moreover, we impose that the change from one condition to the other is on the middle of the two opposite borders of the square lattice. Then, for each spin configuration, there is at least one interface defined as the line on the dual lattice separating the A_i spins connected to one boundary from the B_i spins connected to the other boundary. We present one example in Fig. 1 which correspond to the boundary condition (1|2). In this figure there is two interfaces, one separating the red spins corresponding to $S = 1$ and connected to one boundary from other colors, the second interface separating the green spins corresponding to $S = 2$ and connected to another boundary from other colors. These interfaces are shown as thin black line. The common part of these interfaces is shown as a thick black line.

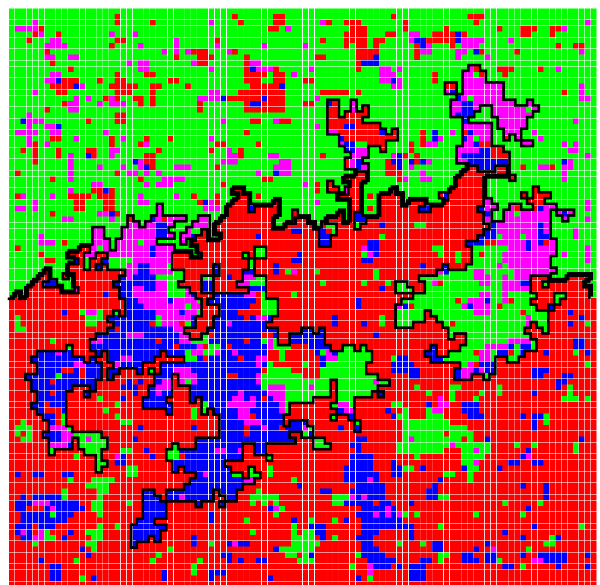

FIG. 1. Spin interfaces for one sample of size 100×100 with the boundary condition (1|2). Red color corresponds to $S = 1$; Green color corresponds to $S = 2$; Blue color corresponds to $S = 3$; Magenta color corresponds to $S = 4$

One can easily observe that, using the S_i spin degree of freedom, the Z_4 symmetry is explicit in the definition of the interfaces and is crucial to understand the properties of these interfaces. In this respect, the FZ point plays a special role: this is the only point of the AT critical line where, in the continuum limit, the Z_4 symmetry conserved currents form a chiral algebra. The correspondent CFT (i.e. the Z_4 parafermionic CFT) enjoys the properties i) to have a finite number of primary operators which close under operator algebra and ii) the Hilbert space can be classified on the basis of the Z_4 transformation properties of the states. Using these properties we were able to almost fully characterize the conformal boundary conditions in terms of spin configurations. Despite the fact we considered a great number of different boundary spin interfaces, we found a limited number of fractal dimensions and this could be understood on the basis of such classification. These results were confirmed by simulating also for bulk spin interfaces i.e. closed interfaces surrounding spin clusters as opposed to open interfaces connecting opposite boundaries.

We present here how the numerical results obtained for the FZ point generalize to the entire critical line. The first main observation is that, for a general point on this line, the fractal dimension of many different boundary interfaces take only one of four different values. In Fig. 2 we show the values of the fractal dimensions of four representative spin interfaces computed for different points on the critical line. The interface $(12|34)$ has been already considered in [14]. In the σ, τ variables this conditions is equivalent to impose $\sigma = +$ ($\sigma = -$) on the left (right) border of the lattice while $\tau = +, -$ is free. At the Ising decoupling point, the $(12|34)$ interface is thus equivalent to the SLE_3 Ising interface. Its fractal dimension $d_{(12|34)}$ has to be equal to $d_{(12|34)} = 1+3/8$, in perfect agreement with the value shown in Fig. 2. Note that at the Ising decoupling point, we found three other non-trivial fractal dimensions, $d_{(1|234)}$, $d_{(1|2)}$ and $d_{(13|24)}$. From the classification done in [15], one can show that the boundary operator generating the $(13|24)$ conditions has conformal dimension $1/4$. The corresponding interface is thus expected [16–18] to be described by an SLE_4 process and therefore to have fractal dimension dimension $3/2$, again in agreement with the value shown in Fig. 2. On the contrary, we do not have any theoretical argument to explain the other two values $d_{(1|234)}$, $d_{(1|2)}$ at the Ising decoupling point. These boundary conditions are indeed highly non trivial in terms of the Ising σ and τ variables.

At the FZ point, on the basis of our boundary classification, we conjectured [8] that the value $d_{(12|34)}$ should be equal to the one of the $(1|234)$ interfaces which was predicted to be $d_{(1|234)} = 17/12$ in [9], in agreement with the numerical findings [8]. From Fig. 2 one can notice that these two fractal dimension are slightly different, but this difference can be explained by finite size effects, as shown in Fig. 2. In this figure, we represent by thin plots the fractal dimensions obtained by fiting the numerical data for sizes $L = 20 - 320$. This can be compared with the data obtained for sizes $L = 80 - 1280$ for the $(1|2)$ and $(12|34)$ boundary condition cases and represented as thick lines. For the boundary condition $(1|234)$, we were able to simulate only up to size $L = 640$, the thick line for this case corresponds then to a fit in the range $L = 40 - 640$. For the boundary condition $(13|24)$, we were able to simulate only up to size $L = 320$.

The finite size effects become more and more important when approaching the four states Potts model. At this point, the model has a permutational S_4 symmetry (and thus larger than a Z_4 one) and the Boltzmann weight associated to an interface can only take two values depending if the spin at the interface are equal or not. It is well known that the critical point of a $q \leq 4$-states Potts model is in the same universality class as the $O(\sqrt{q})$ model in its dense phase. The 4-states Potts model in particular is described by the point separating the dense and dilute critical phases of the $O(2)$ loop model. The spin boundary interfaces and the Fortuin-Kasteleyn (FK) cluster interfaces (discussed below) of the 4-states Potts models are predicted to have the same fractal dimension $d_f = 3/2$. At the Potts model we expect the fractal dimension of all the spin interfaces to take this value. But it is well known that strong logarithmic corrections exist for the 4-states Potts model [19] and thus it is not a surprise that we obtain numerical values not in very good agreement with this prediction. In a direct measurement of interfaces of spin clusters for the 4-states Potts model in [20], a fractal dimension in very good agreement with our value $d_{(1|234)}$ was obtained.

The interface $(13|24)$ is particularly interesting: from our numerical analysis, a fractal dimension with the value $d_{(13|24)} = 3/2$ is obtained all along the critical line (apart close to the limit of the 4-states Potts model where strong finite size effect are expected as explained above). To support this result, we have verified, by a transfer matrix numerical analysis of the boundary states that the conformal dimension of the operator associated to the conditions $(13|24)$ is compatible with the dimension $1/4$ all along the critical line. Moreover, one can easily show that the corresponding bulk interfaces can be defined as the boundaries of the spin clusters which appear in the high-temperature expansion of the AT model [1, 13, 21]. Very recently, theoretical arguments to prove that these interfaces have dimension $3/2$ have been proposed in [22].

<u>FK interfaces</u> Although the spin interfaces are a very natural object, it is often very difficult if not impossible to tackle the study of these interfaces by some exact theoretical method. This is the case of the $q = 3, 4$ Potts model where the geometric features of spin interfaces boundaries are by far less understood than the FK cluster boundaries [23]. The FK cluster are at the basis of Fortuin-Kasteleyn and their boundaries are directly related to the critical loops of the $O(n = \sqrt{q})$ model. The FK cluster are generally obtained by a random walk on a spin cluster and therefore the relation between these two kinds of clusters is highly nontrivial. The main insight into the properties of spin interfaces comes from the observation that the critical exponents extracted by studying a spin cluster in a q Potts model characterize the universality class of the FK cluster of correspondent tri-critical Potts model [24, 25]. According to this, the fractal dimension of the spin cluster boundary d_s and of the associated FK cluster d_{FK} boundary are related by

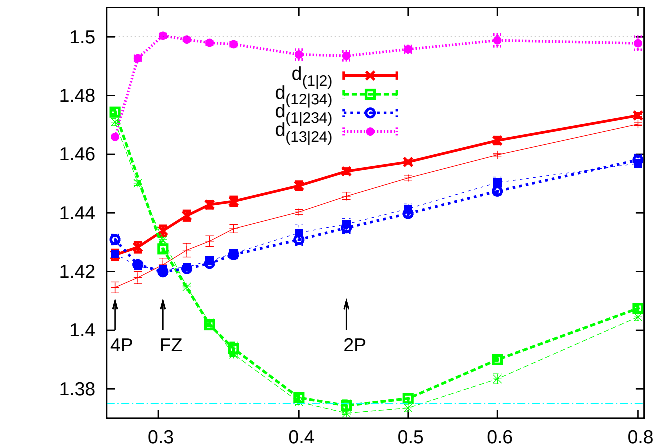

FIG. 2. Fractal dimensions for spin interfaces vs. K. We show the fractal dimensions for four types of interface with two sizes ranges. A thin plot is used for a fit with sizes $L = 20 - 320$. For (1|2) and (12|34), the thick plot corresponds to the fit with sizes $L = 80 - 1280$. For (1|234) the thick plot is for $L = 40 - 640$. For (13|24) we were able to simulate only up to the size $L = 320$. The critical points corresponding to the 4-states Potts (4P), the FZ and to the two decoupled Ising (2P) models are indicated.

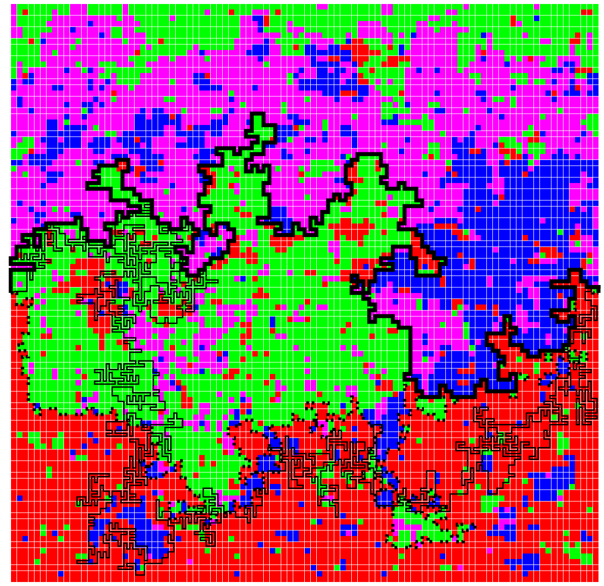

FIG. 3. One sample of size 100×100 with the boundary condition (1|234). The different colors correspond to different values of spin as in Fig. 1. The thin interface corresponds to the FK interface constructed as described in the text, the dotted interface corresponds to the spin interface (1|234) and the thick interface corresponds to the outer boundary. Note that the outer boundary of the FK cluster separates the spin $S = 1, 2$ from the spins $S = 3, 4$: it's fractal dimension is then given by $d_{(12|34)}$

the Duplantier duality relation:

$$(1 - d_s)(1 - d_{FK}) = \frac{1}{4} \,. \tag{3}$$

In the Coulomb gas formulation of the Potts model, this duality can be expressed in terms of an electric-magnetic duality transformation. This transformation also relates the descriptions of the dilute and dense phase of the correspondent $O(n)$ model [17].

At our knowledge, the first numerical evidence that a duality relation relating FK cluster to spin cluster exists for extended CFTs has been presented in [11] by studying a 3-states Potts model at the critical random fixed point. A theoretical argument that Duplantier duality still holds for extended CFTs has been proposed in [9] on the basis of CFT results. In [26] we attempted to define generalised FK clusters for the Z_4 and Z_5 spin models: we observed that these clusters do not percolate at the critical point of the model under consideration.

There exist in fact another type of FK clusters for the AT model. These clusters are constructed by considering one of the two coupled Ising models appearing in (1). By rewriting the Hamiltonian (1) in the following way:

$$\begin{aligned} H_{AT}(\{\sigma_i, \tau_i\}) &= -\sum_{<ij>} ((K + K_4 \tau_i \tau_j)\sigma_i \sigma_j + K \tau_i \tau_j) \\ &= -\sum_{<ij>} (K_{ij}(\tau)\sigma_i \sigma_j + K \tau_i \tau_j) \\ &= -\sum_{<ij>} \left(\tilde{K}_{ij}(\sigma)\tau_i \tau_j + K \sigma_i \sigma_j \right) \,. \end{aligned} \tag{4}$$

one can build an ordinary FK cluster for the Ising model defined by the σ (or τ) spins which interact via the local couplings $K_{ij}(\tau)$ (or $\tilde{K}_{ij}(\sigma)$). This is just the basis of the cluster algorithm for the AT model employed in numerical simulations [27].

In order to build an FK interfaces, we proceeded as follows. For sake of clarity, let us consider the case with spin boundary conditions (1|234). In Fig. 3, a snapshot of a configuration with (1|234) boundary conditions together with the correspondent FK is shown. These conditions correspond to having $\sigma_i = +$ and $\tau_i = +$ on half of the border. In this case it is equivalent to consider the Ising model in the σ or in τ variables. Once we have chosen one of the two Ising model, we impose wired boundary conditions on one half of the border, i.e. the equal spins sitting on this half are linked by an FK bond with probability one. After imposing these boundary conditions, we can build the FK cluster connected to the border and study its boundary. The fractal dimensions $d^{FK}_{(1|234)}$ of this FK cluster is shown in Fig. 4. The results of similar construction for the boundary conditions (1|2) and (12|34) is also shown in Fig. 4. For the (1|2) it is possible

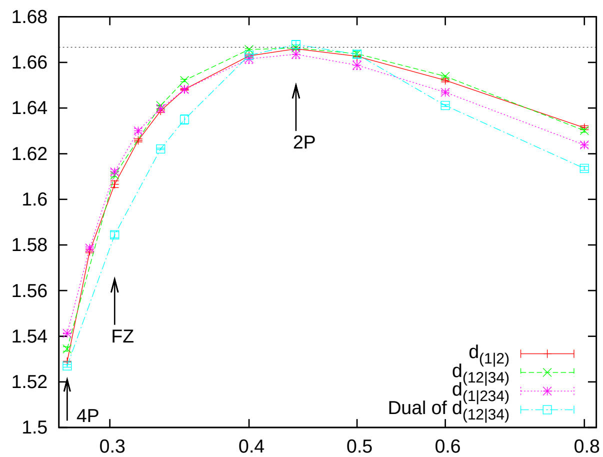

FIG. 4. Fractal dimensions for the FK interfaces (defined in the text). We also show the dual of the fractal dimension for the spin interface with (12|34) boundary condition.

to build two types of interface, since the boundary conditions breaks the symmetry under the exchange of the two Ising models. We have checked numerically that the two types of interface produce the same fractal dimension.

The results shown in Fig. 4 strongly support the fact that the values of the fractal dimension of the FK cluster are dual, see Eq. (3), to the fractal dimension $d_{(12|34)}$. At the FZ point the value of $d^{FK}_{(1|234)}$ is in strong agreement with the value 8/5 which have been proposed on the basis of a CFT computation [9]. At the Ising decoupling point this can be completely understood. Indeed remember that, at this point, the interface $d_{(12|34)}$ corresponds to the well known SLE_3 interface of the Ising model which is also the spin cluster boundary associated to the FK cluster. The boundary of this cluster is described by an $\mathrm{SLE}_{16/3}$ process and its fractal dimension is related to the spin interfaces one by the duality Eq.(3). In general, the spin cluster boundary associated to all the FK cluster considered here in the interface whose fractal dimension is given by $d_{(12|34)}$. For the (1|234) boundary condition for instance, this can be clearly seen in Fig. 3 where the FK cluster and its outer boundary are shown. Note in particular that the interface of the spin cluster on which th FK cluster is built and the spin interface (1|234) are not the same.

Our numerical findings show then that the fractal dimension of the FK ans spin cluster boundary are dual all along the critical line. This is remarkable as it clearly shows that the Duplantier duality, which seems to be valid also for extended CFTs [11], has a deep geometrical origin.

Summing up, we have presented numerical results on spin cluster and FK cluster interfaces of the AT model which are generated by imposing certain boundary conditions. The AT model is particularly interesting in this respect as it presents a critical line in which the Z_4 symmetry, together with the conformal one, plays a central role. We have computed the fractal dimensions of different spin interfaces all along the AT critical line. The results are shown in Fig. 2 : the main observation is that, for a general point on the critical line, the fractal dimension of a generic spin cluster interface takes one of four different possible values. These results interpolate between different interesting special critical models which can be found on the critical line, namely the 4-states Potts, the FZ and the two-decoupled Ising point. At the FZ point the different values for the fractal dimensions were associated to the classification of conformal boundary conditions [8]. Another important finding is the existence of critical interfaces whose fractal dimension take the value 3/2 all along the critical line: this result suggests the existence of interfaces described by SLE_4 processes in the AT model, as it has been recently discussed in [22]. Finally, we have computed the fractal dimension of the boundary of certain FK clusters, and the results are shown in Fig. 4. All along the AT critical line, this value has been found to be the dual, see Eq. (3), to the fractal dimension of the boundary of the associated spin cluster, thus showing that Duplantier duality exists also for extended CFTs.

The main motivation behind this work was to provide new insights into the geometrical properties of extended CFTs. We believe our results shed some light on the general problem of finding a systematic description of two dimensional critical phases in terms of stochastic geometry concepts.

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[1] B. Nienhuis, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, London, 1987), Vol. 11, p.1.
[2] H. Saleur and B. Duplantier, Phys. Rev. Lett. **58**, 2325 (1987).
[3] V. A. Fateev and A. B. Zamolodchikov, Sov. Phys. JETP **62**, 215 (1985).
[4] P. Fendley, J. Phys. A **39**:15445 (2006).
[5] J. Rasmussen, arXiv:hep-th/0409026.
[6] E. Bettelheim, I. A. Gruzberg,A. W. W. Ludwig and P. Wiegmann Phys. Rev. Lett. 95, 251601 (2005).
[7] M. Picco and R. Santachiara, Phys. Rev. Lett. **100**, 015704 (2008).
[8] M. Picco and R. Santachiara, J. Stat. Mech. P07027 (2010).
[9] R. Santachiara, Nucl. Phys. **B793**, 396 (2008).

[10] J. Ashkin and E. Teller, Phys. Rev. **64**, 178 (1943).
[11] J. Jacobsen, P. Le Doussal, M. Picco, R. Santachiara and K. Wiese, Phys. Rev. Lett. **102**, 070601 (2009).
[12] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Associated Press, London, 1982.
[13] B. Nienhuis, J. Stat. Phys. **34**, 731 (1984).
[14] M. Caselle, S. Lottini and M. A. Rajabpour, arXiv:0907.5094.
[15] M. Oshikawa and I. Affleck, Phys. Rev. Lett. **77**, 2604 (1996); Nucl. Phys. **B 495**, 533 (1997).
[16] J. Cardy, arXiv:math-ph/0412033.
[17] I. Rushkin, E. Bettelheim, I. A. Gruzberg and P. Wiegmann J. Phys. A : Math. Theor. **40**, 2164 (2007).
[18] C. Hagendorf, D. Bernard and M. Bauer J. Stat. Phys. **140**, 1 (2010)
[19] J. L. Cardy, N. Nauenberg and D. J. Scalapino, Phys. Rev. **B 22**, 2560 (1980).
[20] A. Zatelepin and L. Shchur, arXiv:1008.3573.
[21] H. Saleur, J. Phys. A **20**, L1127 (1987).
[22] Y. Ikhlef and M. A. Rajabpour, arXiv:1009.3374.
[23] J. Dubail, J. Jacobsen and H. Saleur, arXiv:1008.1216 and arXiv:1010.1700.
[24] A. L. Stella and C. Vanderzande, Phys. Rev. Lett. **62**, 1067 (1989).
[25] C. Vanderzande, J. Phys. **A 25**, L75 (1992).
[26] M. Picco, R. Santachiara and A. Sicilia, J. Stat. Mech. P04013 (2009).
[27] S. Wiseman and E. Domany, Phys. Rev. **E 48**, 4080 (1993).