ON THE VOLUME FORMULAS FOR A SPHERICAL TETRAHEDRON

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ABSTRACT. The present paper gives two concrete formulas for the volume of an arbitrary spherical tetrahedron, which is in a 3-dimensional spherical space of constant curvature $+1$. One formula is given in terms of edge lengths, and another one is given in terms of dihedral angles.

INTRODUCTION

The calculation of the volume of an arbitrary tetrahedron in a 3-space of non-zero constant curvature is rather hard, and the first result is given by [2] in 1999 for hyperbolic tetrahedra. The papers [6] and [5] gave another formulas for hyperbolic tetrahedra, which are implicitly based on the quantum $6j$-symbol. Moreover, it was stated in [6] that an adequate analytic continuation of the obtained formula also applicable for a spherical tetrahedron. But, the formula is given by multi-valued functions, and it is not described which stratum we should select for actual computation. On the other hand, volumes of spherical tetrahedra of special shapes are given by many people from old times, and the most recent work is [3], which gives a formula for a spherical tetrahedron having a small symmetry.

In the present paper, volume formulas for a spherical tetrahedron T of general shape are given in Theorems 1 and 2. The formula in Theorem 1 is given in terms of dihedral angles, and the formula in Theorem 2 is given in terms of edge lengths. These formulas are obtained by improving those in [6], [5], and, by using the Schläfli differential equality, it is shown that the new formulas actually give the volume of T modulo $2\,\pi^2$. Please note that $2\,\pi^2$ is the volume of S^3 with radius 1, which is the universal cover of any 3-dimensional spherical space of constant curvature $+1$. Since T can be included in a 3-dimensional hemisphere, the volume of T is less than π^2 and so we can compute the volume of T actually from the formulas in Theorems 1 and 2.

1. VOLUME FORMULAS

1.1. **Volume formula in terms of dihedral angles.** Let T be a spherical tetrahedron and θ_1, θ_2, $\cdots$, θ_6 be its dihedral angles at edges e_1, e_2, $\cdots$, e_6 given in Figure 1. We assume that $0 < \theta_j < \pi$ for $j = 1, 2, \cdots, 6$. Let $a_1 = e^{i\theta_1}$, $a_2 = e^{i\theta_2}$, $\cdots$, $a_6 = e^{i\theta_6}$, and

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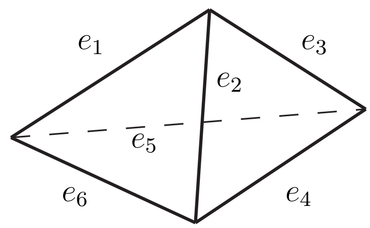

Figure 1. Edges of T

$$\begin{aligned}
&L(a_1, a_2, \cdots, a_6, z) = \\
&\frac{1}{2}\Big(\mathrm{Li}_2(z) + \mathrm{Li}_2(a_1^{-1}\,a_2^{-1}\,a_4^{-1}\,a_5^{-1}\,z) + \mathrm{Li}_2(a_1^{-1}\,a_3^{-1}\,a_4^{-1}\,a_6^{-1}\,z) + \mathrm{Li}_2(a_2^{-1}\,a_3^{-1}\,a_5^{-1}\,a_6^{-1}\,z) \\
&\qquad - \mathrm{Li}_2(-a_1^{-1}\,a_2^{-1}\,a_3^{-1}\,z) - \mathrm{Li}_2(-a_1^{-1}\,a_5^{-1}\,a_6^{-1}\,z) - \mathrm{Li}_2(-a_2^{-1}\,a_4^{-1}\,a_6^{-1}\,z) \\
&\qquad\qquad - \mathrm{Li}_2(-a_3^{-1}\,a_4^{-1}\,a_5^{-1}\,z) + \sum_{j=1}^{3} \log a_j\, \log a_{j+3}\Big),
\end{aligned}$$

where $\mathrm{Li}_2(z)$ is the dilogarithm function defined by the analytic continuation of the following integral:

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\,dt \qquad \text{for a real number } x < 1. \tag{1.1}$$

The analytic continuation of the right-hand side integral defines a multi-valued complex function $\mathrm{li}_2(z)$, and let $\mathrm{Li}_2(z)$ be the principal branch of $\mathrm{li}_2(z)$ which is the analytic continuation of (1.1) on the region $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \geq 1\}$. We also fix the principal branch of the log function as usual by the branch cut along the negative real axis.

We define an auxiliary parameter z_0 as follows:

$$z_0 = \frac{-q_1 + \sqrt{q_1^2 - 4\,q_0\,q_2}}{2\,q_2}, \tag{1.2}$$

where

$$\begin{aligned}
q_0 &= a_1\,a_4 + a_2\,a_5 + a_3\,a_6 + a_1\,a_2\,a_6 + a_1\,a_3\,a_5 + a_2\,a_3\,a_4 + a_4\,a_5\,a_6 + a_1\,a_2\,a_3\,a_4\,a_5\,a_6, \\
q_1 &= -(a_1 - a_1^{-1})(a_4 - a_4^{-1}) - (a_2 - a_2^{-1})(a_5 - a_5^{-1}) - (a_3 - a_3^{-1})(a_6 - a_6^{-1}), \\
q_2 &= a_1^{-1}a_4^{-1} + a_2^{-1}a_5^{-1} + a_3^{-1}a_6^{-1} + a_1^{-1}a_2^{-1}a_6^{-1} + a_1^{-1}a_3^{-1}a_5^{-1} + \\
&\qquad a_2^{-1}a_3^{-1}a_4^{-1} + a_4^{-1}a_5^{-1}a_6^{-1} + a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1}a_5^{-1}a_6^{-1}.
\end{aligned}$$

Then z_0 is a solutions of

$$\exp\left(2\,z\,\frac{\partial L}{\partial z}\right) = 1, \tag{1.3}$$

where

$$\exp\left(2\,z\,\frac{\partial L}{\partial z}\right) = \frac{(a_1\,a_2\,a_3 + z)\,(a_1\,a_5\,a_6 + z)\,(a_2\,a_4\,a_6 + z)\,(a_3\,a_4\,a_5 + z)}{(1-z)\,(a_1\,a_2\,a_4\,a_5 - z)\,(a_1\,a_3\,a_4\,a_6 - z)\,(a_2\,a_3\,a_5\,a_6 - z)}.$$

Now we state the main result of this paper.

Theorem 1. *Let T be a spherical tetrahedron with edge lengths θ_1, θ_2, $\cdots$, θ_6 for edges e_1, e_2, $\cdots$ e_6 given in Figure 1. Let $a_j = e^{i\theta_j}$ for $j = 1, 2, \cdots, 6$ and let $\mathrm{Vol}(T)$ be the volume of T. Then*

$$\mathrm{Vol}(T) = -\mathrm{Re}(L(a_1, a_2, \cdots, a_6, z_0)) - \pi \left(\arg(-q_2) - \frac{1}{2}\sum_{j=1}^{6} \theta_j\right) - \frac{3}{2}\pi^2 \mod 2\pi^2,$$

where $\mathrm{Re}(z)$ is the real part of z and q_2 is given in (1.2).

1.2. **Volume formula in terms of edge lengths.** Let T be a spherical tetrahedron with edge lengths l_1, l_2, $\cdots$, l_6 at the edges e_1, e_2, $\cdots$ e_6 given in Figure 1. Let $b_j = e^{i\,l_j}$ for $j = 1, 2, \cdots, 6$ and $\widetilde{L}(b_1, b_2, b_3, b_4, b_5, b_6, z) = L(-b_4^{-1}, -b_5^{-1}, -b_6^{-1}, -b_1^{-1}, -b_2^{-1}, -b_3^{-1}, z)$. Then the following formula holds.

Theorem 2. *For a spherical tetrahedron T as above,*

$$\mathrm{Vol}(T) = \mathrm{Re}\Big(\widetilde{L}(b_1, b_2, \cdots, b_6, \widetilde{z}_0)\Big) + \pi \arg(-\widetilde{q}_2) - \sum_{j=1}^{6} l_j \left. \frac{\partial\, \mathrm{Re}\big(\widetilde{L}(b_1, b_2, \cdots, b_6, \widetilde{z}_0)\big)}{\partial l_j}\right|_{z=\widetilde{z}_0} - \frac{1}{2}\pi^2 \mod 2\pi^2,$$

where $\widetilde{z}_0$ and $\widetilde{q}_2$ are obtained from z_0 and q_2 in (1.2) by substituting $-b_{j\pm 3}^{-1}$ to a_j for $j = 1$, $2, \cdots, 6$.

2. Proof of the formulas

2.1. **Gram matrices.** Let T be a spherical tetrahedron with dihedral angles $\theta_1, \cdots, \theta_6$ as before. Let G be the Gram matrix of T defined by

$$G = \begin{pmatrix} 1 & -\cos\theta_1 & -\cos\theta_2 & -\cos\theta_6 \\ -\cos\theta_1 & 1 & -\cos\theta_3 & -\cos\theta_5 \\ -\cos\theta_2 & -\cos\theta_3 & 1 & -\cos\theta_4 \\ -\cos\theta_6 & -\cos\theta_5 & -\cos\theta_4 & 1 \end{pmatrix}.$$

An actual computation shows that the discriminant in (1.2) is give by

$$q_1^2 - 4\,q_0\,q_2 = 16\,\det G, \tag{2.1}$$

which is positive since T is spherical. It is known (see, e.g. [1]) that

$$\cos l_j = \frac{c_{pq}}{\sqrt{c_{pp}\,c_{qq}}}$$

and so we have

$$\exp(2\,i\,l_j) = \frac{2\,c_{pq}^2 - c_{pp}\,c_{qq} + 2\,i\,c_{pq}\sqrt{\det G}\,\sin\theta_j}{c_{pp}\,c_{qq}} \tag{2.2}$$

by using the formula (5.1) in [7] that is $c_{pq}^2 - c_{pp}\,c_{qq} = -\det G\,\sin^2\theta_j$. Here p and q denote the row and column of $G = (g_{ab})$ such that $g_{pq} = -\cos\theta_j$, and c_{ab} is the cofactor of G, i.e. $c_{ab} = (-1)^{a+b}\det G_{ab}$ where G_{ab} is the submatrix obtained from G by deleting its a-th row and b-th column.

2.2. **Some functions and their properties.** Before proving the formulas, we introduce some functions and investigate their properties. Let T be a tetrahedron, θ_1, θ_2, $\cdots$, θ_6 be its dihedral angles at edges e_1, e_2, $\cdots$, e_6 as before, and

$$D_s = \{(\theta_1, \theta_2, \cdots, \theta_6) \in (0,\pi)^6 \subset \mathbb{R}^6 \mid \theta_1,\ \theta_2,\ \cdots,\ \theta_6 \text{ correspond to the dihedral angles of a spherical tetrahedron}\}.$$

Let $a_j = e^{i\theta_j}$ for $j = 1, 2, \cdots, 6$,

$$\Delta_0(x,y,z) = -\frac{1}{4}\left(\mathrm{Li}_2(-x\,y^{-1}\,z^{-1}) + \mathrm{Li}_2(-x^{-1}\,y\,z^{-1}) + \mathrm{Li}_2(-x^{-1}\,y^{-1}\,z) + \mathrm{Li}_2(-x\,y\,z)\right),$$

$$\Delta(a_1, a_2, \cdots, a_6) = \Delta_0(a_1,a_2,a_3) + \Delta_0(a_1,a_5,a_6) + \Delta_0(a_2,a_4,a_6) + \Delta_0(a_3,a_4,a_5) - \frac{1}{2}\sum_{j=1}^{6}\big(\log a_j\big)^2,$$

$$U(a_1, a_2, \cdots, a_6, z) = L(a_1, a_2, \cdots, a_6, z) + \Delta(a_1, a_2, \cdots, a_6),$$

and

$$V(a_1,a_2,a_3,a_4,a_5,a_6) = -U(a_1,a_2,a_3,a_4,a_5,a_6,z_0) + \pi\,i\left(\log z_0 - \sum_{j=1}^{6}\log a_j\right) - \frac{13}{6}\pi^2.$$

Lemma 2.1. *The function $\Delta(a_1, a_2, \cdots, a_6)$ is analytic on D_s and the imaginary part of $4a_j\frac{\partial\Delta}{\partial a_j}$ is given by*

$$\mathrm{Im}\left(4\,a_j\,\frac{\partial \Delta}{\partial a_j}\right) = -2\,\pi.$$

Proof. We show for the case $j=1$. For the function Δ,

$$a_1\,\frac{\partial\Delta}{\partial a_1} = a_1\,\frac{\partial\Delta_0(a_1,a_2,a_3)}{\partial a_1} + a_1\,\frac{\partial\Delta_0(a_1,a_5,a_6)}{\partial a_1} - \log a_1$$

and

$$a_1\,\frac{\partial\Delta_0(a_1,a_p,a_q)}{\partial a_1} = \frac{1}{4}\left(\log(1+\frac{a_1}{a_p\,a_q}) - \log(1+\frac{a_p}{a_1\,a_q}) - \log(1+\frac{a_q}{a_1\,a_p}) + \log(1+a_1\,a_p\,a_q)\right)$$

for $\{p,q\} = \{2,3\}$, $\{5,6\}$. The imaginary part $\mathrm{Im}\log(1+e^{il})$ is given by

$$\mathrm{Im}\log(1+e^{i\theta}) = \begin{cases} \dfrac{\theta}{2} & \text{if } -\pi < \theta < \pi, \\[2ex] \dfrac{\theta}{2} - \pi & \text{if } \pi < \theta < 3\,\pi. \end{cases}$$

Since θ_1, θ_p, θ_q are dihedral angles at the three edges having a vertex in common, they satisfy $0 < \theta_1 + \theta_p - \theta_q,\ \theta_1 - \theta_p + \theta_q,\ -\theta_1 + \theta_p + \theta_q < \pi$ and $\pi < \theta_1 + \theta_p + \theta_q < 3\,\pi$. Hence $\Delta_0(a_1, a_p, a_q)$ is analytic on D_s and we have

$$\operatorname{Im}\left(a_1\,\frac{\partial \Delta_0(a_1, a_p, a_q)}{\partial a_1}\right) = \frac{\theta_1}{2} - \frac{\pi}{4}, \qquad \operatorname{Im}\left(4\,a_1\,\frac{\partial \Delta}{\partial a_1}\right) = -2\,\pi.$$

Moreover, Δ is analytic on D_s because none of the imaginary parts of the log terms of Δ attains π nor $-\pi$ on D_s. □

Lemma 2.2. *The function $L(a_1, a_2, \cdots, a_6, z_0(a_1, a_2, \cdots, a_6))$ is analytic on D_s, and so $U(a_1, a_2, \cdots, a_6, z_0(a_1, a_2, \cdots, a_6))$ is analytic on D_s.*

Proof. We know that $|z_0| < 1$ because, for q_0, q_1, q_2 in (1.2), q_1 and $q_0 q_2$ are positive real numbers, and $q_1^2 - 4q_0q_2$ is also a positive real number by (2.1). This means that, for $w \in \mathbb{C}$ with $|w| = 1$ and $w \neq 1$, $|w\,z_0| < 1$ and $w\,z_0$ does not meet the branch cut $\{x \in \mathbb{R} \mid x \geq 1\}$ of Li_2. Therefore, all the dilog terms of L are analytic on D_s. □

Lemma 2.3. *The differential $\frac{\partial U}{\partial z}$ satisfies $z_0 \frac{\partial U}{\partial z}\big|_{z=z_0} = \pi\,i$.*

Proof. Since $\frac{\partial U}{\partial z} = \frac{\partial L}{\partial z}$ and z_0 is a solution of the equation (1.3), $z_0\,\frac{\partial U}{\partial z}\big|_{z=z_0} = k\,\pi\,i$ for some integer constant k because U is analytic on D_s by the above lemma. Let $T_{\frac{\pi}{2}}$ be the regular spherical tetrahedron with edge lengths $\pi/2$. Then $a_j = i$, $z_0 = (i+1)/2$ and

$$z_0\,\frac{\partial U}{\partial z}\bigg|_{z=z_0} = \frac{1}{2}\left(-4\,\log\frac{1-i}{2} + 4\,\log\frac{1+i}{2}\right) = \pi\,i.$$

Hence $z_0 \frac{\partial U}{\partial z}\big|_{z=z_0} = \pi i$ for all the spherical tetrahedron. □

Now, we show the following proposition for V corresponding to the Schläfli differential equality.

Proposition 2.4. *The function V satisfies $\frac{\partial V}{\partial \theta_j} = l_j/2$ for $j = 1, 2, \cdots, 6$.*

Proof. Let $\varphi = \exp\left(4\,a_1\,\frac{\partial \Delta}{\partial a_1}\right)$ and $\psi = \exp\left(\,2\,a_1\,\frac{\partial L}{\partial a_1}\Big|_{z=z_0}\right)$, then

$$\varphi = \frac{(a_1 + a_2\,a_3)(a_1\,a_2\,a_3 + 1)(a_1 + a_5\,a_6)(a_1\,a_5\,a_6 + 1)}{(a_1\,a_2 + a_3)(a_1\,a_3 + a_2)(a_1\,a_5 + a_6)(a_1\,a_6 + a_5)},$$

$$\psi = \frac{(a_1\,a_2\,a_4\,a_5 - z_0)(a_1\,a_3\,a_4\,a_6 - z_0)}{a_4\,(a_1\,a_2\,a_3 + z_0)(a_1\,a_5\,a_6 + z_0)}.$$

An actual computation and (2.2) shows that

$$\exp\left(4\,a_1\,\frac{\partial U}{\partial a_1}\right)\bigg|_{z=z_0} = \varphi\,\psi^2 = \frac{2\,c_{12}^2 - c_{11}\,c_{22} + 2\,i\,c_{12}\sqrt{\det G}\,\sin\theta_1}{c_{11}\,c_{22}} = \exp(2\,l_1\,i).$$

Hence we get $a_1 \frac{\partial U}{\partial a_1}\Big|_{z=z_0} = i\,(l_1 + k\,\pi)/2$ for some integer constant k because U is analytic on D_s by Lemma 2.2. For the tetrahedron $T_{\frac{\pi}{2}}$ given in the proof of Lemma 2.3, $l_1 = \pi/2$

and $a_1 \frac{\partial U}{\partial a_1}\Big|_{z=z_0} = -3\,\pi\, i/4$, which means that $k = -2$ and $a_1\, \frac{\partial U}{\partial a_1}\Big|_{z=z_0} = i\,(l_1 - 2\,\pi)/2$. According to $\frac{\partial U}{\partial \theta_1} = i a_1 \frac{\partial U}{\partial a_1}$, we have

$$\frac{\partial U}{\partial \theta_1}\bigg|_{z=z_0} = \frac{1}{2}\,(2\,\pi - l_1). \tag{2.3}$$

Therefore

$$\frac{\partial}{\partial \theta_1}\left(-U\big(a_1, \cdots, a_6, z_0(a_1, \cdots, a_6)\big) + \pi\, i\left(\log z_0 - \sum_{j=1}^{6} \log a_j\right)\right) = \frac{l_1}{2} - \frac{\partial z_0}{\partial \theta_1}\,\frac{\partial U}{\partial z}\bigg|_{z=z_0} + \pi\, i\,\frac{\partial z_0}{\partial \theta_1}\,\frac{1}{z_0}.$$

Since $\frac{\partial U}{\partial z}\big|_{z=z_0} = i\,\pi/z_0$ by Lemma 2.3, we get $\frac{\partial V}{\partial \theta_1} = l_1/2$. $\square$

2.3. **Proof of the formula in terms of dihedral angles.** We first give a formula by complex analytic functions.

Proposition 2.5. *Let T be a spherical tetrahedron with dihedral angles θ_1, θ_2, $\cdots$, θ_6 for edges e_1, e_2, $\cdots$ e_6 as in Figure 1. Let $a_j = e^{i\theta_j}$ for $j = 1, 2, \cdots, 6$ as before and let $\mathrm{Vol}(T)$ be the volume of T. Then*

$$\mathrm{Vol}(T) = V(a_1, a_2, \cdots, a_6) \quad \mod \ 2\,\pi^2.$$

Proof. For the tetrahedron $T_{\frac{\pi}{2}}$ in the proof of Lemma 2.3, we have $a_j = i$, $z_0 = \frac{1+i}{2}$ and $V(i,i,i,i,i,i) = \pi^2/8 = \mathrm{Vol}(T_{\frac{\pi}{2}})$. Because V is analytic on some neighborhood N of $T_{\frac{\pi}{2}}$ in D_s, two functions V and Vol are identical on N by Proposition 2.4 and Schäfli's formlua. Moreover, $\mathrm{Vol}(T)$ is analytic on D_s and so it is given by an adequate analytic continuation of V. We already showed in previous lemmas that all the terms in V except $\pi i \log z_0$ are analytic on D_s, and the analytic continuation of $\pi i \log z_0$ is $\pi i \log z_0 + 2k\pi^2$ for some integer k. Hence we get the proposition. $\square$

Proof of Theorem 1. We prove Theorem 1 by investigating the real part of V. For $\theta \in [0, 2\,\pi] \subset \mathbb{R}$, the real part of $\mathrm{Li}_2(e^{i\theta})$ is given by $\mathrm{Re}\big(\mathrm{Li}_2(e^{i\theta})\big) = \mathrm{Re}\big(\mathrm{Li}_2(e^{-i\theta})\big) = \theta^2/4 - \pi\,\theta/2 + \pi^2/6$. Substituting this to each dilog functions of $\mathrm{Re}(\Delta(a_1, a_2, \cdots, a_6))$, we get $\mathrm{Re}(\Delta(a_1, a_2, \cdots, a_6)) = -2\,\pi^2/3 + \sum_{j=1}^{6} \pi\theta_j/2$. We also know that $\mathrm{Im} \log z_0 = -\arg(-q_2)$ since the numerator of z_0 in (1.2) is a negative real number. Hence we get Theorem 1 from Proposition 2.5. $\square$

Remark 2.6. The function V is non-continuous at the points where the values of q_2 are positive real numbers.

2.4. **Proof of the formula in terms of edge lengths.** We use the notations in Subsection 2.2.

Proof of Theorem 2. Let $\theta_1, \theta_2, \cdots, \theta_6$ be the dihedral angles at the edges $e_1, e_2, \cdots, e_6$ of T and let T^* be the dual tetrahedron of T given in p.294 of [4]. Then the dihedral angles of T^* are $\pi - l_4$, $\pi - l_5$, $\pi - l_6$, $\pi - l_1$, $\pi - l_2$, $\pi - l_3$ and the edge length of T^* are $\pi - \theta_4$, $\pi - \theta_5$, $\pi - \theta_6$, $\pi - \theta_1$, $\pi - \theta_2$, $\pi - \theta_3$. The relation of volumes of T and T^* is give in p.294 of [4] as follows:

$$\mathrm{Vol}(T) + \mathrm{Vol}(T^*) + \frac{1}{2}\sum_{j=1}^{6} l_j\,(\pi - \theta_j) = \pi^2.$$

By Theorem 1, we have

$$\mathrm{Vol}(T^*) = -\mathrm{Re}(\widetilde{L}(b_1, b_2, \cdots, b_6, \widetilde{z}_0)) - \pi\left(\arg(-\widetilde{q}_2) - \frac{1}{2}\sum_{j=1}^{6}(\pi - l_j)\right) - \frac{3}{2}\pi^2 \mod 2\pi^2.$$

Because $\frac{\partial}{\partial(\pi - l_j)} U(-b_4^{-1}, -b_5^{-1}, -b_6^{-1}, -b_1^{-1}, -b_2^{-1}, -b_3^{-1}, z)\Big|_{z=\widetilde{z}_0} = \big(2\pi - (\pi - \theta_j)\big)/2$ by (2.3) and $\frac{\partial}{\partial(\pi - l_j)}\Delta(-b_4^{-1}, -b_5^{-1}, -b_6^{-1}, -b_1^{-1}, -b_2^{-1}, -b_3^{-1}, z) = \pi/2$ by Lemma 2.1, we know that $\frac{\partial \widetilde{L}}{\partial l_j}\Big|_{z=\widetilde{z}_0} = -\theta_j/2$. Hence

$$\mathrm{Vol}(T) = \mathrm{Re}\big(\widetilde{L}(b_1, b_2, \cdots, b_6, \widetilde{z}_0)\big) + \pi\,\arg(-\widetilde{q}_2) - \sum_{j=1}^{6} l_j\,\frac{\partial\,\mathrm{Re}\big(\widetilde{L}(b_1, b_2, \cdots, b_6, \widetilde{z}_0)\big)}{\partial l_j}\Bigg|_{z=\widetilde{z}_0} - \frac{1}{2}\pi^2 \mod 2\pi^2,$$

and we get Theorem 2. $\square$

References

1. D. V. Alekseevskij, B. Vinberg and A. S. Solodovnikov, Geometry of spaces of constant curvature, *Geometry, II*, Encyclopaedia Math. Sci., **29**, Springer, Berlin, 1993, pp. 1–138.
2. Y. Cho and H. Kim, On the volume formula for hyperbolic tetrahedra, *Discrete Comput. Geom.* **22** (1999), no. 3, 347–366.
3. A. Kolpakov, A. Mednykh and M. Pashkevich, Volume formula for a $\mathbb{Z}_2$-symmetric spherical tetrahedron through its edge lengths, preprint, arXiv:1007.3948.
4. J. Milnor, The Schläfli differential equality. *John Milnor Collected Papers Volume 1, Geometry,* Publish or Perish Inc., Houston and Texas, 1994, pp. 281–295.
5. J. Murakami and A. Ushijima, A volume formula for hyperbolic tetrahedra in terms of edge lengths. *J. Geom.* **83** (2005), 153–163.
6. J. Murakami and M. Yano, On the volume of a hyperbolic and spherical tetrahedron. *Comm. Anal. Geom.* **13** (2005), 379–400.
7. A. Ushijima, A volume formula for generalized hyperbolic tetrahedra. In *Non-Euclidean geometries*, Math. Appl. (N. Y.) **581**, Springer, New York 2006, pp. 249–265.

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