ON THIN-COMPLETE IDEALS OF SUBSETS OF GROUPS

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ABSTRACT. Let $\mathcal{F} = \bigcup_{F\in\mathcal{F}} \mathcal{P}_G \subset \mathcal{P}_G$ be a left-invariant lower family of subsets of a group G. A subset $A \subset G$ is called $\mathcal{F}$-*thin* if $xA \cap yA \in \mathcal{F}$ for any distinct elements $x, y \in G$. The family of all $\mathcal{F}$-thin subsets of G is denoted by $\tau(\mathcal{F})$. If $\tau(\mathcal{F}) = \mathcal{F}$, then $\mathcal{F}$ is called *thin-complete*. The *thin-completion* $\tau^*(\mathcal{F})$ of $\mathcal{F}$ is the smallest thin-complete subfamily of $\mathcal{P}_G$ that contains $\mathcal{F}$.

Answering questions of Lutsenko and Protasov, we prove that a set $A \subset G$ belongs to $\tau^*(G)$ if and only if for any sequence $(g_n)_{n\in\omega}$ of non-zero elements of G there is $n \in \omega$ such that

$$\bigcap_{i_0,\dots,i_n\in\{0,1\}} g_0^{i_0}\cdots g_n^{i_n} A \in \mathcal{F}.$$

Also we prove that for an additive family $\mathcal{F} \subset \mathcal{P}_G$ its thin-completion $\tau^*(\mathcal{F})$ is additive. If the group G is countable and torsion-free, then the completion $\tau^*(\mathcal{F}_G)$ of the ideal $\mathcal{F}_G$ of finite subsets of G is coanalytic and not Borel in the power-set $\mathcal{P}_G$ endowed with the natural compact metrizable topology.

1. Introduction

This paper was motivated by problems posed by Ie. Lutsenko and I.V. Protasov in the preliminary version of the paper [5] devoted to relatively thin sets in groups.

Following [4], we say that a subset A of a group G is *thin* if for any distinct points $x, y \in G$ the intersection $xA \cap yA$ is finite. In [5] (following the approach of [1]) Lutsenko and Protasov generalized the notion of a thin set to that of $\mathcal{F}$-thin set where $\mathcal{F}$ is a family of subsets of G. By $\mathcal{P}_G$ we shall denote the Boolean algebra of all subsets of the group G.

We shall say that a family $\mathcal{F} \subset \mathcal{P}_G$ is

- *left-invariant* if $xF \in \mathcal{F}$ for all $F \in \mathcal{F}$ and $x \in G$, and
- *lower* if $\bigcup_{F\in\mathcal{F}} \mathcal{P}_F \subset \mathcal{F}$;
- *additive* if $A \cup B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$;
- *an ideal* if $\mathcal{F}$ is lower and additive.

Let $\mathcal{F} \subset \mathcal{P}_G$ be a left-invariant lower family of subsets of a group G. A subset $A \subset G$ is defined to be $\mathcal{F}$-*thin* if for any distinct points $x, y \in G$ we get $xA\cap yA \in \mathcal{F}$. The family of all $\mathcal{F}$-thin subsets of G will be denoted by $\tau(\mathcal{F})$. It is clear that $\tau(\mathcal{F})$ is a left-invariant lower family of subsets of G and $\mathcal{F} \subset \tau(\mathcal{F})$. If $\tau(\mathcal{F}) = \mathcal{F}$, then the family $\mathcal{F}$ will be called *thin-complete*.

Let $\tau^*(\mathcal{F})$ be the intersection of all thin-complete families $\tilde{\mathcal{F}} \subset \mathcal{P}_G$ that contain $\mathcal{F}$. It is clear that $\tau^*(\mathcal{F})$ is the smallest thin-complete family containing $\mathcal{F}$. This family is called the *thin-completion* of $\mathcal{F}$.

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The family $\tau^*(\mathcal{F})$ has an interesting hierarchic structure that can be described as follows. Let $\tau^0(\mathcal{F}) = \mathcal{F}$ and for each ordinal α put $\tau^\alpha(\mathcal{F})$ be the family of all sets $A \subset G$ such that for any distinct points $x, y \in G$ we get $xA \cap yA \in \bigcup_{\beta<\alpha} \tau^\beta(\mathcal{F})$. So,

$$\tau^\alpha(\mathcal{F}) = \tau(\tau^{<\alpha}(\mathcal{F})) \quad \text{where} \quad \tau^{<\alpha}(\mathcal{F}) = \bigcup_{\beta<\alpha} \tau^\beta(\mathcal{F}).$$

By Proposition 3 of [5], $\tau^*(\mathcal{F}) = \bigcup\limits_{\alpha<|G|^+} \tau^\alpha(\mathcal{F})$.

The following theorem (that will be proved in Section 3) answers the problem of combinatorial characterization of the family $\tau^*(\mathcal{F})$ posed by Ie. Lutsenko and I.V. Protasov. Below by e we denote the neutral element of the group G.

Theorem 1.1. *Let $\mathcal{F} \subset \mathcal{P}_G$ be a left-invariant lower family of subsets of a group G. A subset $A \subset G$ belongs to the family $\tau^*(\mathcal{F})$ if and only if for any sequence $(g_n)_{n\in\omega} \in (G \setminus \{e\})^{\mathbb{N}}$ there is a number $n \in \omega$ such that*

$$\bigcap_{k_0,\dots,k_n \in \{0,1\}} g_0^{k_0} \cdots g_n^{k_n} A \in \mathcal{F}.$$

We recall that a family $\mathcal{F} \subset \mathcal{P}_G$ is called if $\{A \cup B : A, B \in \mathcal{F}\} \subset \mathcal{F}$. It is clear that the family $\mathcal{F}_G$ of finite subsets of a group G is additive. If G is an infinite Boolean group, then the family $\tau^*(\mathcal{F}_G) = \tau(\mathcal{F}_G)$ is not additive, see Remark 2 in [5]. For torsion-free groups the situation is totally diferent:

Theorem 1.2. *For a torsion-free group G and a left-invariant ideal $\mathcal{F} \subset \mathcal{P}_G$ the family $\tau^{<\alpha}(\mathcal{F})$ is additive for any limit ordinal α. In particular, the thin-completion $\tau^*(\mathcal{F})$ of $\mathcal{F}$ is an ideal in $\mathcal{P}_G$.*

We define a subset of a group G to be $*$-*thin* if its belongs to the thin-completion $\tau^*(\mathcal{F}_G)$ of the family $\mathcal{F}_G$ of all finite subsets of the group G. By Proposition 3 of [5], for each countable group G we get $\tau^*(\mathcal{F}_G) = \tau^{<\omega_1}(\mathcal{F}_G)$. It is natural to ask if the equality $\tau^*(\mathcal{F}_G) = \tau^{<\alpha}(\mathcal{F}_G)$ can happen for some cardinal $\alpha < \omega_1$. If the group G is Boolean, then the answer is affirmative: $\tau^*(\mathcal{F}) = \tau^1(\mathcal{F})$ according to Theorem 1 of [5]. The situation is different for non-torsion groups:

Theorem 1.3. *If an infinite group G contains an abelian torsion-free subgroup H of cardinality $|H| = |G|$, then $\tau^*(\mathcal{F}_G) \neq \tau^\alpha(\mathcal{F}_G) \neq \tau^{<\alpha}(\mathcal{F}_G)$ for each ordinal $\alpha < |G|^+$.*

Theorems 1.2 and 1.3 will be proved in Sections 4 and 6, respectively. In Section 7 we shall study the Borel complexity of the family $\tau^*(\mathcal{F}_G)$ for a countable group G. In this case the power-set $\mathcal{P}_G$ carries a natural compact metrizable topology, so we can talk about topological properties of subsets of $\mathcal{P}_G$.

Theorem 1.4. *For a countable group G and a countable ordinal α the family $\tau^\alpha(\mathcal{F}_G)$ is Borel while the family $\tau^*(\mathcal{F}_G) = \tau^{<\omega_1}(\mathcal{F}_G)$ is coanalytic. If G contains an element of infinite order, then the space $\tau^*(\mathcal{F}_G)$ is coanalytic but not analytic.*

2. Preliminaries on well-founded posets and trees

In this section we collect the neccessary information on well-founded posets and trees. A *poset* is an abbreviation from a *partially ordered set.* A poset $(X, \leq)$ is *well-founded* if each subset $A \subset X$ has a maximal element $a \in A$ (this means that each element $x \in A$ with $x \geq a$ is equal to a). In a well-founded poset $(X, \leq)$

to each point $x \in X$ we can assign the ordinal $\mathrm{rank}_X(x)$ defined by the recursive formula:

$$\mathrm{rank}_X(x) = \sup\{\mathrm{rank}_X(y) + 1 : y > x\}$$

where $\sup \emptyset = 0$. The ordinal $\mathrm{rank}(X) = \sup\{\mathrm{rank}_X(x) + 1 : x \in X\}$ is called the rank of the poset X.

A *tree* is a poset $(T, \le)$ with the smallest element $\emptyset_T$ such that for each $t \in T$ the lower set ${\downarrow}t = \{s \in T : s \le t\}$ is well-ordered in the sense that each subset $A \subset {\downarrow}t$ has the smallest element. A *branch* of a tree T is any maximal linearly ordered subset of T. If a tree is well-founded, then all its branches are finite.

A subset $S \subset T$ of a tree is called a *subtree* if it is a tree with respect to the induced partial order. A subtree $S \subset T$ is *lower* if $S = {\downarrow}S = \{t \in T : \exists s \in T \ \ t \le s\}$.

All trees that appear in this paper are (lower) subtrees of the tree $X^{<\omega} = \bigcup_{n\in\omega} X^n$ of finite sequences of a set X. The tree $X^{<\omega}$ carries the following partial order:

$$(x_0, \dots, x_n) \le (y_0, \dots, y_m) \text{ iff } n \le m \text{ and } x_i = y_i \text{ for all } i \le n.$$

The empty sequence $s_\emptyset \in X^0$ is the smallest element (the root) of the tree $X^{<\omega}$. For a finite sequence $s = (x_0, \dots, x_n) \in X^{<\omega}$ and an element $x \in X$ by $s\hat{\ }x = (x_0, \dots, x_n, x)$ we denote the concatenation of s and x. So, $s\hat{\ }x$ is one of $|X|$ many immediate successors of s. The set of all branches of $X^{<\omega}$ can be naturally identified with the countable power X^ω. For each branch $s = (s_n)_{n\in\omega} \in X^\omega$ and $n \in \omega$ by $s|n = (s_0, \dots, s_{n-1})$ we denote the initial interval of length n.

Let $\mathsf{Tr} \subset \mathcal{P}_{X^{<\omega}}$ denote the family of all lower subtrees of the tree $X^{<\omega}$ and $\mathsf{WF} \subset \mathsf{Tr}$ be the subset consisting of all well-founded lower subtrees of $X^{<\omega}$.

In Section 7 we shall exploit some deep facts about the descriptive properties of the sets $\mathsf{WF} \subset \mathsf{Tr} \subset \mathcal{P}_{X^{<\omega}}$ for countable set X. In this case the tree $X^{<\omega}$ is countable and the power-set $\mathcal{P}_{X^{<\omega}}$ carries a natural compact metrizable topology of the Tychonov power $2^{X^{<\omega}}$. So, we can speak about topological properties of the subsets WF and Tr of the compact metrizable space $\mathcal{P}_{X^{<\omega}}$.

We recall that a topological space X is *Polish* if X is homeomorphic to a separable complete metric space. A subset A of a Polish space X is called

- *Borel* if A belongs to the smallest σ-algebra that contains all open subsets of X;
- *analytic* if A is the image of a Polish space P under a continuous map $f : P \to A$;
- *coanalytic* if $X \setminus A$ is analytic.

By Souslin's Theorem 14.11 [3], a subset of a Polish space is Borel if and only if it is both analytic and coanalytic. By Σ^1_1 and Π^1_1 we denote the classes of spaces homeomorphic to analytic and coanalytic subsets of Polish spaces, respectively.

A coanalytic subset X of a compact metric space K is called Π^1_1*-complete* if for each coanalytic subset C is the Cantor cube 2^ω there is a continuous map $f : 2^\omega \to K$ such that $f^{-1}(X) = C$. It follows from the existence of a coanalytic non-Borel set in 2^ω that each Π^1_1-complete set $X \subset K$ is non-Borel.

The following deep theorem is classical and belongs to Lusin, see [3, 32.B,35.23].

Theorem 2.1. *Let X be a countable set.*

1. *The subspace* Tr *is closed (and hence compact) in* $\mathcal{P}_{X^{<\omega}}$.
2. *The set of well-founded trees* WF *is* Π^1_1*-complete in* Tr. *In particular,* WF *is coanalytic but not analytic (and hence not Borel).*

(3) *For each ordinal* $\alpha < \omega_1$ *the subset* $\mathsf{WF}_\alpha = \{T \in \mathsf{WF} : \operatorname{rank}(T) \le \alpha\}$ *is Borel in* Tr.
(4) *Each analytic subspace of* WF *lies in* WF_α *for some ordinal* $\alpha < \omega_1$.

3. Combinatorial characterization of $*$-thin subsets

In this section we prove Theorem 1.1. Let $\mathcal{F} \subset \mathcal{P}_G$ be a left-invariant lower family of subsets of a group G. The theorem 1.1 trivially holds if $\mathcal{F} = \mathcal{P}_G$ (which happens if and only if $G \in \mathcal{F}$). So, it remains to consider the case $G \notin \mathcal{F}$. Let e be the neutral element of G and $G_\circ = G \setminus \{e\}$. We shall work with the tree $G_\circ^{<\omega}$ discussed in the preceding section.

Let A be any subset of G. To each finite sequence $s \in G_\circ^{<\omega}$ assign the set $A_s \subset G$, defined by induction: $A_\emptyset = G$ and $A_{s\hat{\ }x} = A_s \cap xA_s$ and for $s \in G_\circ^{<\omega}$ and $x \in G_\circ$. Repeating the inductive argument of the proof of Proposition 2 [5], we can obtaine the following direct description of the sets A_s:

Claim 3.1. For every sequence $s = (g_0, \dots, g_n) \in G_\circ^{<\omega}$

$$A_s = \bigcap_{k_0,\dots,k_n \in \{0,1\}} g_0^{k_0} \cdots g_n^{k_n} A.$$

The set

$$T_A = \{s \in G_\circ^{<\omega} : A_s \notin \mathcal{F}\}$$

is a subtree of $G_\circ^{<\omega}$ called the *τ-tree* of the set A.

Theorem 3.2. *A set $A \subset G$ belongs to the family $\tau^\alpha(\mathcal{F})$ for some ordinal α if and only if its τ-tree T_A is well-founded and has* $\operatorname{rank}(T_A) \le \alpha + 1$.

Proof. By induction on α. Observe that $A \in \tau^0(\mathcal{F}) = \mathcal{F}$ if and only if $T_A = \{s_\emptyset\}$ if and only if $\operatorname{rank}(T_A) = 1$. So, Theorem holds for $\alpha = 0$.

Assume that for some ordinal $\alpha > 0$ and any ordinal $\beta < \alpha$ we know that a set $A \subset G$ belongs to $\tau^\beta(G)$ if and only if T_A is a well-founded tree of rank $\operatorname{rank}(T_A) \le \beta + 1$. Given a subset $A \subset G$ we should show that that $A \in \tau^\alpha(\mathcal{F})$ if and only if its τ-tree T_A is well-founded and $\operatorname{rank}(T_A) \le \alpha + 1$. This is clear if $A \in \mathcal{F}$. So we assume that $A \notin \mathcal{F}$.

First assume that $A \in \tau^\alpha(\mathcal{F})$. Then for every $x \in G_\circ$ there is an ordinal $\beta_x < \alpha$ such that $A \cap xA \in \tau^{\beta_x}(\mathcal{F})$. By the inductive assumption, the τ-tree $T_{A\cap xA}$ is well-founded and has $\operatorname{rank}(T_{A\cap xA}) \le \beta_x + 1$.

Since $A \notin \mathcal{F}$, each point $x \in G_\circ = G_\circ^1$ considered as the sequence $(x) \in G^1$ of length 1 belongs to the τ-tree T_A of the set A. So we can consider the upper set $T_A(x) = \{s \in T_A : s \ge x\}$ and observe that the subtree $T_A(x)$ of T_A is isomorphic to the τ-tree $T_{A\cap xA}$ of the set $A \cap xA$ and hence $\operatorname{rank}(T_A(x)) = \operatorname{rank}(T_{A\cap xA}) \le \beta_x + 1 \le \alpha$. It follows that

$$\begin{aligned}\operatorname{rank}(T_A) = \operatorname{rank}_{T_A}(s_\emptyset) + 1 = \big(\sup_{x\in G_\circ}(\operatorname{rank}_{T_A}(x) + 1)\big) + 1 = \\ = (\sup_{x\in G_\circ} \operatorname{rank} T_A(x)) + 1 \le \big(\sup_{x\in G_\circ}(\beta_x + 1)\big) + 1 \le \alpha + 1.\end{aligned}$$

Now assume conversely that the τ-tree T_A of A is well-founded and $\operatorname{rank}(T_A) \le \alpha + 1$. Since $\operatorname{rank}(T_A) = \operatorname{rank}_{T_A}(s_\emptyset) + 1 = \big(\sup_{x\in G_\circ}(\operatorname{rank}_{T_A}(x) + 1)\big) + 1$, we conclude that for each $x \in G_\circ$ we get

$$\operatorname{rank}(T_A(x)) = \operatorname{rank}_{T_A}(x) + 1 = \beta_x + 1$$

for some ordinal $\beta_x < \alpha$. Since the subtree $T_A(x) = T_A \cap {\uparrow}x$ is isomorphic to the τ-tree $T_{A\cap xA}$ of the set $A \cap xA$, we conclude that $T_{A\cap xA}$ is well-founded and has $\mathrm{rank}(T_{xA\cap A}) = \mathrm{rank}(T_A(x)) = \mathrm{rank}_{T_A}(x) + 1 = \beta_x + 1$. Then the inductive assumption guarantees that $A \cap xA \in \tau^{\beta_x}(\mathcal{F}) \subset \tau^{<\alpha}(\mathcal{F})$ and then $A \in \tau^{\alpha}(\mathcal{F})$ by the definition of the family $\tau^{\alpha}(\mathcal{F})$. □

As a corollary of Theorem 3.2, we obtain the following characterization proved in [5]:

Corollary 3.3. *A subset $A \subset G$ belongs to the family $\tau^n(\mathcal{F})$ for some $n \in \omega$ if and only if for each sequence $(g_i)_{i=0}^n \in G_\circ^n$ we get*

$$\bigcap_{k_0,\dots,k_n \in \{0,1\}} g_0^{k_0} \cdots g_n^{k_n} A \in \mathcal{F}.$$

Theorem 3.2 also implies the following explicit description of the family $\tau^*(\mathcal{F})$, which was announced in Theorem 1.1:

Corollary 3.4. *For a subset $A \subset G$ the following conditions are equivalent:*

(1) *$A \in \tau^*(\mathcal{F})$;*
(2) *the τ-tree T_A of A is well-founded;*
(3) *for each sequence $(g_n)_{n\in\omega} \in G_\circ^\omega$ there is $n \in \omega$ such that $(g_0, \dots, g_n) \notin T_A$;*
(4) *for each sequence $(g_n)_{n\in\omega} \in G_\circ^\omega$ there is $n \in \omega$ such that*

$$\bigcap_{k_0,\dots,k_n \in \{0,1\}} g_0^{k_0} \cdots g_n^{k_n} A \in \mathcal{F}.$$

4. Additivity of the families $\tau^{<\alpha}(\mathcal{F})$

In this section we shall prove Theorem 1.2. Let G be an infinite group and e be the neutral element of G.

For a natural number m let 2^m denote the finite cube $\{0,1\}^m$. For vectors $\mathbf{g} = (g_1, \dots, g_m) \in (G \setminus \{e\})^m$ and $\mathbf{x} = (x_1, \dots, x_m) \in 2^m$ let

$$\mathbf{g}^{\mathbf{x}} = g_1^{x_1} \cdots g_m^{x_m} \in G.$$

A function $f : 2^m \to G$ to a group G will be called *cubic* if there is a vector $\mathbf{g} = (g_1, \dots, g_m) \in (G \setminus \{e\})^m$ such that $f(x) = \mathbf{g}^x$ for all $x \in 2^m$.

Lemma 4.1. *If the group G is torsion-free, then for every $n \in \mathbb{N}$, $m > (n-1)^2$, and a cubic function $f : 2^m \to G$ we get $|f(2^m)| > n$.*

Proof. Assume conversely that $|f(2^m)| \le n$. Consider the set $B = \{(k_1, \dots, k_m) \in 2^m : \sum_{i=1}^m k_i = 1\}$ having cardinality $|B| = m > (n-1)^2$. Since $e \notin f(B)$, we conclude that $|f(B)| \le |f(2^m)| - 1 \le n - 1$ and hence $|f^{-1}(y)| \ge n$ for some $y \in f(B)$. Let $B_y = f^{-1}(y)$ and observe that $f(2^m) \supset \{e, y, y^2, \dots, y^{|B_y|}\}$ and thus $|f(2^m)| \ge |B_y| + 1 \ge n + 1$, which contradicts our assumption. □

For every $n \in \mathbb{N}$ let $c(n)$ be the smallest number $m \in \mathbb{N}$ such that for each cubic function $f : 2^m \to G$ we get $|f(2^m)| > n$. It is easy to see that $c(n) \ge n$. On the other hand, Lemma 4.1 implies that $c(n) \le (n-1)^2 + 1$ if G is torsion-free.

For a family $\mathcal{F}$ and a natural number $n \in \mathbb{N}$, let

$$\bigvee_n \mathcal{F} = \{\cup\mathcal{A} : \mathcal{A} \subset \mathcal{F},\ |A| \le n\}.$$

Lemma 4.2. *Let $\mathcal{F} \subset \mathcal{P}_G$ be a left-invariant lower family of subsets in a torsion-free group G. For every $n \in \mathbb{N}$ we get*

$$\bigvee_n \tau(\mathcal{F}) \subset \tau^{c(n)-1}(\bigvee_m \mathcal{F})$$

where $m = n^{2^{c(n)}}$.

Proof. Fix any $A \in \bigvee_n \tau(\mathcal{F})$ and write it as the union $A = A_1 \cup \dots \cup A_n$ of sets $A_1, \dots, A_n \in \tau(\mathcal{F})$. The inclusion $A \in \tau^{c(n)-1}(\bigvee_m \mathcal{F})$ will follow from Corollary 3.3 as soon as we check that

$$\bigcap_{x \in 2^{c(n)}} \mathbf{g}^x A \in \bigvee_m \mathcal{F}$$

for each vector $\mathbf{g} \in (G \setminus \{e\})^{c(n)}$. De Morgan's law guarantees that

$$\bigcap_{x \in 2^{c(n)}} \mathbf{g}^x \cdot (\bigcup_{i=1}^n A_i) = \bigcup_{f \in n^{2^{c(n)}}} \bigcap_{x \in 2^{c(n)}} \mathbf{g}^x A_{f(x)}.$$

So, the proof will be complete as soon as we check that for every function $f : 2^{c(n)} \to n$ the set $\bigcap_{x \in 2^{c(n)}} \mathbf{g}^x A_{f(x)}$ belongs to $\mathcal{F}$. The vector $\mathbf{g} \in (G \setminus \{e\})^{c(n)}$ induces the cubic function $g : 2^{c(n)} \to G$, $g : x \mapsto \mathbf{g}^x$. The definition of the function $c(n)$ guarantees that $|g(2^{c(n)})| > n$. The function $f : 2^{c(n)} \to n$ can be thought as a coloring of the cube $2^{c(n)}$ into n colors. Since $|g(2^{c(n)})| > n$, there are two points $y, z \in 2^{c(n)}$ colored by the same color such that $g(y) \neq g(z)$. Then $\mathbf{g}^y = g(y) \neq g(z) = \mathbf{g}^z$ but $f(y) = f(z) = k$ for some $k \le n$. Consequently,

$$\bigcap_{x \in 2^{c(n)}} \mathbf{g}^x A_{f(x)} \subset \mathbf{g}^y A_k \cap \mathbf{g}^z A_k \in \mathcal{F}$$

because the set $A_k \in \tau(\mathcal{F})$. □

Now consider the function $c : \mathbb{N} \times \omega \to \omega$ defined recursively as $c(n, 0) = 0$ for all $n \in \mathbb{N}$ and $c(n, k+1) = c(n) - 1 + c(n^{2^{\alpha(n)}}, k)$ for $(n, k) \in \mathbb{N} \times \omega$. Observe that $c(n, 1) = c(n) - 1$ for all $n \in \mathbb{N}$.

Lemma 4.3. *If the group G is torsion-free and $\mathcal{F} \subset \mathcal{P}_G$ is a left-invariant ideal, then*

$$\bigvee_n \tau^k(\mathcal{F}) \subset \tau^{c(n,k)}(\mathcal{F})$$

for all pairs $(n, k) \in \mathbb{N} \times \omega$.

Proof. By induction on k. For $k = 0$ the equality $\bigvee_n \tau^0(\mathcal{F}) = \mathcal{F} = \tau^{c(n,0)}(\mathcal{F})$ holds because $\mathcal{F}$ is additive.

Assume that Lemma is true for some $k \in \omega$. By Lemma 4.2 and by the inductive assumption, for every $n \in \mathbb{N}$ we get

$$\bigvee_n \tau^{k+1}(\mathcal{F}) = \bigvee_n \tau(\tau^k(\mathcal{F})) \subset \tau^{c(n)-1}\big(\bigvee_{n^{2^{c(n)}}} \tau^k(\mathcal{F})\big) \subset$$

$$\tau^{c(n)-1}(\tau^{c(n^{2^{c(n)}},k)}(\mathcal{F})) = \tau^{c(n)-1+\alpha(n^{2^{c(n)}},k)}(\mathcal{F}) = \tau^{c(n,k+1)}(\mathcal{F}).$$

□

Now we are able to present:

Proof of Theorem 1.2. Assume that G is a torsion-free group G and $\mathcal{F} \subset \mathcal{P}_G$ is a left-invariant ideal. By transfinite induction we shall prove that for each limit ordinal α the family $\tau^{<\alpha}(\mathcal{F})$ is additive. For the smallest limit ordinal $\alpha = 0$ the additivity of the family $\tau^0(\mathcal{F}) = \mathcal{F}$ is included into the hypothesis. Assume that for some non-zero limit ordinal α we have proved that the families $\tau^{<\beta}(\mathcal{F})$ are additive for all limit ordinals $\beta < \alpha$. Two cases are possible:

1) $\alpha = \beta + \omega$ for some limit ordinal β. By the inductive assumption, the family $\tau^{<\beta}(\mathcal{F})$ is additive. Then Lemma 4.3 implies that the family $\tau^{<\alpha}(\mathcal{F}) = \tau^{<\omega}(\tau^{<\beta}(\mathcal{F}))$ is additive.

2) $\alpha = \sup B$ for some family $B \not\ni \alpha$ of limit ordinals. By the inductive assumption for each limit ordinal $\beta \in B$ the family $\tau^{<\beta}(\mathcal{F})$ is additive and then the union

$$\tau^{<\alpha}(\mathcal{F}) = \bigcup_{\beta \in B} \tau^{<\beta}(\mathcal{F})$$

is additive too.

This completes the proof of the additivity of the families $\tau^{<\alpha}(\mathcal{F})$ for all limit ordinals α. Since the torsion-free group G is infinite, the ordinal $\alpha = |G|^+$ is limit and hence the family $\tau^*(\mathcal{F}) = \tau^{<\alpha}(\mathcal{F})$ is additive. Being left-invariant and lower, the family $\tau^*(\mathcal{F})$ is a left-invariant ideal in $\mathcal{P}_G$. $\square$

Remark 4.4. Theorem 1.2 is not true for an infinite Boolean group G. In this case Theorem 1(2) of [5] implies that $\tau^*(\mathcal{F}_G) = \tau(\mathcal{F}_G)$. Then for any infinite thin subset $A \subset G$ and any $x \in G \setminus \{e\}$ the union $A \cup xA$ is not thin as $(A \cup xA) \cap x(A \cup xA) = A \cup xA$ is infinite. Consequently, the family $\tau^*(\mathcal{F}_G) = \tau(\mathcal{F}_G)$ is not additive.

5. h-invariant families of subsets

Let G be a group and $h : H \to K$ be an isomorphism between subgroups of G. A family $\mathcal{F}$ of subsets of G is called *h-invariant* if a subset $A \subset H$ belongs to $\mathcal{F}$ if and only if $h(A) \in \mathcal{F}$.

Example 5.1. The ideal $\mathcal{F}_{\mathbb{Z}}$ of finite subsets of the group $\mathbb{Z}$ is h-invariant for each isomorphism $h_k : \mathbb{Z} \to k\mathbb{Z}$, $h : x \mapsto kx$, where $k \in \mathbb{N}$.

Proposition 5.2. *Let $h : H \to K$ be an isomorphism between subgroups of a group G. For any h-invariant family $\mathcal{F} \subset \mathcal{P}_G$ and any ordinal α the family $\tau^\alpha(\mathcal{F})$ is h-invariant.*

Proof. For $\alpha = 0$ the h-invariance of $\tau^0(\mathcal{F}) = \mathcal{F}$ follows from our assumption. Assume that for some ordinal α we have established that the families $\tau^\beta(\mathcal{F})$ are h-invariant for all ordinals $\beta < \alpha$. Then the union $\tau^{<\alpha}(\mathcal{F}) = \bigcup_{\beta<\alpha} \tau^\beta(\mathcal{F})$ is also h-invariant.

We shall prove that the family $\tau^\alpha(\mathcal{F})$ is h-invariant. Given a set $A \subset H$ we need to prove that $A \in \tau^\alpha(\mathcal{F})$ if and only if $h(A) \in \tau^\alpha(\mathcal{F})$.

Assume first that $A \in \tau^\alpha(\mathcal{F})$. To show that $h(A) \in \tau^\alpha(\mathcal{F})$, take any element $y \in G \setminus \{e\}$. If $y \notin K$, then $h(A) \cap yh(A) = \emptyset \in \tau^{<\alpha}(\mathcal{F})$. If $y \in K$, then $y = h(x)$ for some $x \in H$ and then $h(A) \cap yh(A) = h(A \cap xA) \in \tau^{<\alpha}(\mathcal{F})$ since $A \cap xA \in \tau^{<\alpha}(\mathcal{F})$ and the family $\tau^{<\alpha}(\mathcal{F})$ is h-invariant.

Now assume that $A \notin \tau^\alpha(\mathcal{F})$. Then there is an element $x \in G \setminus \{e\}$ such that $A \cap xA \notin \tau^{<\alpha}(\mathcal{F})$. Since $A \subset H$, the element x must belong to H (otherwise

$A \cap xA = \emptyset \in \tau^{<\alpha}(\mathcal{F})$). Then for the element $y = h(x)$ we get $h(A) \cap yh(A) \notin \tau^{<\alpha}(\mathcal{F})$ by the h-invariance of the family $\tau^{<\alpha}(\mathcal{F})$. Consequently, $h(A) \notin \tau^{\alpha}(\mathcal{F})$. □

Corollary 5.3. *Let $h : H \to K$ be an isomorphism between subgroups of a group G. For any h-invariant family $\mathcal{F} \subset \mathcal{P}_G$ the family $\tau^*(\mathcal{F})$ is h-invariant.*

Definition 5.4. A *left-invariant* family $\mathcal{F} \subset \mathcal{P}_G$ of subsets of a group G is called

- *auto-invariant* if $\mathcal{F}$ is h-invariant for each injective homomorphism $h : G \to G$;
- *sub-invariant* if $\mathcal{F}$ is h-invariant for each isomorphism $h : H \to K$ between subgroups $K \subset H$ of G.
- *strongly invariant* if $\mathcal{F}$ is h-invariant for each isomorphism $h : H \to K$ between subgroups of G.

It is clear that

$$\text{auto-invariant} \Rightarrow \text{sub-invariant} \Rightarrow \text{strongly invariant}$$

Remark 5.5. Each auto-invariant family $\mathcal{F} \subset \mathcal{P}_G$, being left-invariant is also right-invariant.

Proposition 5.2 implies:

Corollary 5.6. *If $\mathcal{F} \subset \mathcal{P}_G$ is an auto-invariant (sub-invariant, strongly invariant) family of subsets of a group G, then so are the families $\tau^*(\mathcal{F})$ and $\tau^{\alpha}(\mathcal{F})$ for all ordinals α.*

It is clear that the famly $\mathcal{F}_G$ of finite subsets of a group G is strongly invariant. Now we present some natural examples of families, which are not strongly invariant. Following [2], we call a subset A of a group G

- *large* if there is a finite subset $F \subset G$ with $G = FA$;
- *small* if for any large set $L \subset G$ the set $L \setminus A$ remains large.

It follows that the family $\mathcal{S}_G$ of small subsets if G is a left-invariant ideal in $\mathcal{P}_G$. According to [2], a subset $A \subset G$ is small if and only if for every finite subset $F \subset G$ the complement $G \setminus FA$ is large. We shall need the following (probably known) fact.

Lemma 5.7. *Let H be a subgroup of finite index in a group G. A subset $A \subset H$ is small in H if and only if A is small in G.*

Proof. First assume that A is small in G. To show that A is small in H, take any large subset $L \subset H$. Since H has finite index in G, the set L is large in G. Since A is small in G, the complement $L \setminus A$ is large in G. Conseqeuntly, there is a finite subset $F \subset G$ such that $F(L \setminus A) = G$. Then for the finite set $F_H = F \cap H$, we get $F_H(L \setminus A) = H$, which means that $L \setminus A$ is large in H.

Now assume that A is small in H. To show that A is small in G, it suffices to show that for every finite subset $F \subset G$ the complement $G \setminus FA$ is large in G. Observe that $(G \setminus FA) \cap H = H \setminus F_H A$ where $F_H = F \cap H$. Since A is small in H, the set $H \setminus F_H A$ is large in H and hence large in G (as H has finite index in G). Then the set $G \setminus FA \supset H \setminus F_H A$ is large in G too. □

Proposition 5.8. *Let G be an infinite abelian group.*

1. *If G is finitely generated, then the ideal $\mathcal{S}_G$ is strongly invariant.*

(2) *If G is infinitely generated free abelian group, then the ideal $\mathcal{S}_G$ is not auto-invariant.*

Proof. 1. Assume that G is a finitely generated abelian group. To show that $\mathcal{S}_G$ is strongly invariant, fix any isomorphism $h : H \to K$ between subgroups of G and let $A \subset H$ be any subset. The groups H, K are isomorphic and hence have the same free rank $r_0(H) = r_0(K)$. If $r_0(H) = r_0(K) < r_0(G)$, then the subgroups H, K have infinite index in G and hence are small. In this case the inclusions $A \in \mathcal{S}_G$ and $h(A) \in \mathcal{S}_G$ hold and so are equivalent.

If the ranks $r_0(H) = r_0(K)$ and $r_0(G)$ coincide, then H and K are subgroups of finite index in the finitely generated group G. By Lemma 5.7, a subset $A \subset H$ is small in G if and only if A is small in H if and only if $h(A)$ is small in the group $h(H) = K$ if and only if $h(A)$ is small in G.

2. Now assume that G is an infinitely generated free abelian group. Then G is isomorphic to the direct sum $\oplus^\kappa\mathbb{Z}$ of $\kappa = |G| \ge \aleph_0$ many copies of the infinite cyclic group $\mathbb{Z}$. Take any subset $\lambda \subset \kappa$ with infinite complement $\kappa \setminus \lambda$ and cardinality $|\lambda| = |\kappa|$ and fix an isomorphism $h : G \to H$ of the group $G = \oplus^\kappa\mathbb{Z}$ onto its subgroup $H = \oplus^\lambda\mathbb{Z}$. The subgroup H has infinite index in G and hence is small in G. Yet $h^{-1}(H) = G$ is not small in G, witnessing that the ideal $\mathcal{S}_G$ of small subsets of G is not auto-invariant. □

6. Thin-completeness of the families $\tau^\alpha(\mathcal{F})$

In this section we shall prove that in general the families $\tau^\alpha(\mathcal{F})$ are not thin-complete. Let us recall that a family $\mathcal{F} \subset \mathcal{P}_G$ is called an *ideal* if $G \notin \mathcal{F}$ and $\mathcal{F}$ is additive and lower. Our principal result is the following theorem that implies Theorem 1.3 announced in the Introduction.

Theorem 6.1. *Let G be a group containing a free abelian subgroup H of cardinality $|H| = |G|$. If $\mathcal{F}$ is a sub-invariant ideal of subsets of G such that $\tau(\mathcal{F}) \cap \mathcal{P}_H \not\subset \mathcal{F}$, then $\tau^*(\mathcal{F}) \ne \tau^\alpha(\mathcal{F}) \ne \tau^{<\alpha}(\mathcal{F})$ for all ordinals $\alpha < |G|^+$.*

We divide the proof of this theorem in a series of lemmas.

Lemma 6.2. *Let $h : H \to K$ be an isomorphism between subgroups of a group G, $\mathcal{F}$ be an h-invariant left-invariant monotone family of subsets of G. If a subset $A \subset H$ does not belong to $\tau^\alpha(\mathcal{F})$ for some ordinal α, then for every point $z \in G\setminus\{e\}$ the set $h(A) \cup zh(A) \notin \tau^{\alpha+1}(\mathcal{F})$.*

Proof. Proposition 5.2 implies that $h(A) \notin \tau^\alpha(\mathcal{F})$. Since

$$(h(A) \cup zh(A)) \cap z^{-1}(h(A) \cup zh(A)) \supset h(A) \notin \tau^n(\mathcal{F}),$$

the set $h(A) \cup zh(A) \notin \tau^{\alpha+1}(\mathcal{F})$ by the definition of $\tau^{\alpha+1}(\mathcal{F})$. □

In the following lemma for a subgroup K of a group H by

$$Z_H(K) = \{z \in H : \forall x \in K \;\; zx = xz\}$$

we denote the centralizer of K in H.

Lemma 6.3. *Let $h : H \to K$ be an isomorphism between subgroups $K \subset H$ of a group G such that there is a point $z \in Z_H(K)$ with $z^2 \notin K$. Let $\mathcal{F} \subset \mathcal{P}_G$ be an h-invariant left-invariant ideal. If a subset $A \subset H$ belongs to the family $\tau^\alpha(\mathcal{F})$ for some ordinal α, then $h(A) \cup zh(A) \in \tau^{\alpha+1}(\mathcal{F})$.*

Proof. By induction on α. For $\alpha = 0$ and $A \in \mathcal{F}$ the inclusion $h(A) \cup zh(A) \in \mathcal{F} \subset \tau(\mathcal{F})$ follows from the h-invariance and the additivity of $\mathcal{F}$.

Now assume that for some ordinal α we have proved that for every $\beta < \alpha$ and $A \in \mathcal{P}_H \cap \tau^\beta(\mathcal{F})$ the set $h(A) \cup zh(A)$ belongs to $\tau^{\beta+1}(\mathcal{F})$. Given any set $A \in \mathcal{P}_H \cap \tau^\alpha(\mathcal{F})$, we need to prove that $h(A) \cup zh(A) \in \tau^{\alpha+1}(\mathcal{F})$. This will follow as soon as we check that $(h(A) \cup zh(A)) \cap y(h(A) \cup zh(A) \in \tau^\alpha(\mathcal{F})$ for every $y \in G \setminus \{e\}$.

If $y \notin K \cup zK \cup z^{-1}K$, then

$$(h(A) \cup zh(A)) \cap y(h(A) \cup zh(A)) \subset (K \cup zK) \cap y(K \cup zK) = \emptyset \in \tau^{\alpha+1}(\mathcal{F}).$$

So, it remains to consider the case $y \in K \cup zK \cup z^{-1}K$. If $y \in K$, then

$$(h(A) \cup zh(A)) \cap y(h(A) \cup zh(A)) = (h(A) \cap yh(A)) \cup z(h(A) \cap y\,h(A)).$$

Since $y \in K$, there is an element $x \in H$ with $y = h(x)$. Since $A \in \tau^\alpha(\mathcal{F})$, $A \cap xA \in \tau^\beta(\mathcal{F})$ for some $\beta < \alpha$ and then

$$(h(A) \cup zh(A)) \cap y(h(A) \cup zh(A)) = h(A \cap xA) \cup zh(A \cap xA) \in \tau^{\beta+1}(\mathcal{F}) \subset \tau^\alpha(\mathcal{F})$$

by the inductive assumption. If $y \in zK$, then $z^2 \notin K$ implies that

$$(h(A) \cup zh(A)) \cap y(h(A) \cup zh(A)) = zh(A) \cap yh(A) \subset zh(A) \in \tau^\alpha(\mathcal{F})$$

by the h-invariance and the left-invariance of the family $\tau^\alpha(\mathcal{F})$, see Proposition 5.2.

If $y \in z^{-1}K$, then by the same reason,

$$(h(A) \cup zh(A)) \cap y(h(A) \cup zh(A)) = h(A) \cap yzh(A) \subset h(A) \in \tau^\alpha(\mathcal{F}).$$

□

Given an isomorphism $h : H \to K$ between subgroups $K \subset H$ of a group G, for every $n \in \mathbb{N}$ define the iteration $h^n : H \to K$ of the isomorphism h letting $h^1 = h : H \to K$ and $h^{n+1} = h \circ h^n$ for $n \geq 1$.

The isomorphism $h : H \to K$ will be called *expanding* if $\bigcap_{n\in\mathbb{N}} h^n(H) = \{e\}$.

Example 6.4. For every integer $k \geq 2$ the isomorphism

$$h_k : \mathbb{Z} \to k\mathbb{Z}, \quad h_k : x \mapsto kx,$$

is expanding.

Lemma 6.5. *Let $h : H \to K$ be an expanding isomorphism between torsion-free subgroups $K \subset H$ of a group G and $\mathcal{F}$ be an h-invariant left-invariant ideal of subsets of G. For any limit ordinal α and family $\{A_n\}_{n\in\omega} \subset \tau^{<\alpha}(\mathcal{F})$ of subsets of the group H, the union $A = \bigcup_{n\in\omega} h^n(A_n)$ belongs to the family $\tau^\alpha(\mathcal{F})$.*

Proof. First observe that $\{h^n(A_n)\}_{n\in\omega} \subset \tau^{<\alpha}(\mathcal{F})$ by Proposition 5.2. To show that $A = \bigcup_{n\in\omega} h^n(A_n) \in \tau^\alpha(\mathcal{F})$ we need to check that $A \cap xA \in \tau^{<\alpha}(\mathcal{F})$ for all $x \in G \setminus \{e\}$. This is trivially true if $x \notin H$ as $A \subset H$. So, we assume that $x \in H$. By the expanding property of the isomorphism h, there is a number $m \in \omega$ such that $x \notin h^m(H)$. Put $B = \bigcup_{n=0}^{m-1} h^n(A_n)$ and observe that $A \cap xA \subset B \cup xB \in \tau^{<\alpha}(\mathcal{F})$ as $\tau^{<\alpha}(\mathcal{F})$ is additive according to Theorem 1.2. □

Lemma 6.6. *Assume that a left-invariant ideal $\mathcal{F}$ on a group G is h-invariant for some expanding isomorphism $h : H \to K$ between torsion-free subgroups $K \subset H$ of G such that $Z_K(H) \not\subset K$. If $\tau(\mathcal{F}) \cap \mathcal{P}_H \not\subset \mathcal{F}$, then $\tau^\alpha(\mathcal{F}) \neq \tau^{<\alpha}(\mathcal{F})$ for all ordinals $\alpha < \omega_1$.*

Proof. Fix any point $z \in Z_K(H) \setminus K$. Since H is torsion-free, $z^2 \neq e$. Since the isomorphism h is expanding, $z^2 \notin h^m(H)$ for some $m \in \mathbb{N}$. Replacing the isomorphism h by its iterate h^m, we lose no generality assuming that $z^2 \notin h(H) = K$.

By induction on $\alpha < \omega_1$ we shall prove that $\tau^\alpha(\mathcal{F}) \cap \mathcal{P}_H \neq \tau^{<\alpha}(\mathcal{F}) \cap \mathcal{P}_H$.

For $\alpha = 1$ the non-equality $\tau(\mathcal{F}) \cap \mathcal{P}_H \neq \tau^0(\mathcal{F}) \cap \mathcal{P}_H$ is included into the hypothesis. Assume that for some ordinal $\alpha < \omega_1$ we proved that $\tau^\beta(\mathcal{F}) \cap \mathcal{P}_H \neq \tau^{<\beta}(\mathcal{F}) \cap \mathcal{P}_H$ for all ordinals $\beta < \alpha$.

If $\alpha = \beta + 1$ is a successor ordinal, then by the inductive assumption we can find a set $A \in \tau^\beta(\mathcal{F}) \setminus \tau^{<\beta}(\mathcal{F})$ in the subgroup H. By Lemmas 6.2 and 6.3, $A \cup zA \in \tau^{\beta+1}(\mathcal{F}) \setminus \tau^\beta(\mathcal{F}) = \tau^\alpha(\mathcal{F}) \setminus \tau^{<\alpha}(\mathcal{F})$ and we are done.

If α is a limit ordinal, then we can find an increasing sequence of ordinals $(\alpha_n)_{n\in\omega}$ with $\alpha = \sup_{n\in\omega} \alpha_n$. By the inductive assumption, for every $n \in \omega$ there is a subset $A_n \subset H$ with $A_n \in \tau^{\alpha_n+1}(\mathcal{F}) \setminus \tau^{\alpha_n}(\mathcal{F})$. Then we can put $A = \bigcup_{n\in\omega} h^n(A_n)$. By Proposition 5.2, for every $n \in \omega$, we get

$$h^n(A_n) \in \tau^{\alpha_n+1}(\mathcal{F}) \setminus \tau^{\alpha_n}(\mathcal{F})$$

and thus $A \notin \tau^{\alpha_n}(\mathcal{F})$ for all $n \in \omega$, which implies that $A \notin \tau^{<\alpha}(\mathcal{F})$. On the other hand, Lemma 6.5 guarantees that $A \in \tau^\alpha(\mathcal{F})$. □

Lemma 6.7. *Assume that a left-invariant ideal $\mathcal{F}$ on a group G is h-invariant for some isomorphism $h : H \to K$ between torsion-free subgroups $K \subset H$ of G such that $z^2 \notin K$ for some $z \in Z_K(H)$. Assume that for an infinite cardinal κ there are isomorphisms $h_n : H \to H_n$, $n \in \kappa$, onto subgroups $H_n \subset H$ such that $\mathcal{F}$ is h_n-invariant and $H_n \cdot H_m \cap H_k \cdot H_l = \{e\}$ for all indices $n, m, k, l \in \kappa$ with $\{n, m\} \cap \{k, l\} = \emptyset$.*

If $\tau(\mathcal{F}) \cap \mathcal{P}_H \not\subset \mathcal{F}$, then $\tau^\alpha(\mathcal{F}) \neq \tau^{<\alpha}(\mathcal{F})$ for all ordinals $\alpha < \kappa^+$.

Proof. By induction on $\alpha < \kappa^+$ we shall prove that $\tau^\alpha(\mathcal{F}) \cap \mathcal{P}_H \neq \tau^{<\alpha}(\mathcal{F}) \cap \mathcal{P}_H$.

For $\alpha = 1$ the non-equality $\tau^1(\mathcal{F}) \cap \mathcal{P}_H \neq \tau^0(\mathcal{F}) \cap \mathcal{P}_H$ is included into the hypothesis. Assume that for some ordinal $\alpha < \kappa^+$ we proved that $\tau^\beta(\mathcal{F}) \cap \mathcal{P}_H \neq \tau^{<\beta}(\mathcal{F}) \cap \mathcal{P}_H$ for all ordinals $\beta < \alpha$.

If $\alpha = \beta + 1$ is a successor ordinal, then by the inductive assumption we can find a set $A \in \tau^\beta(\mathcal{F}) \setminus \tau^{<\beta}(\mathcal{F})$ in the subgroup H. By Lemmas 6.2 and 6.3, $A \cup zA \in \tau^{\beta+1}(\mathcal{F}) \setminus \tau^\beta(\mathcal{F})$ and we are done.

If α is a limit ordinal, then we can fix a family of ordinals $(\alpha_n)_{n\in\kappa}$ with $\alpha = \sup_{n\in\kappa}(\alpha_n + 1)$. By the inductive assumption, for every $n \in \kappa$ there is a subset $A_n \subset H$ such that $A_n \in \tau^{\alpha_n+1}(\mathcal{F}) \setminus \tau^{\alpha_n}(\mathcal{F})$. After a suitable shift, we can assume that $e \notin A_n$. Since the ideal $\mathcal{F}$ is h_n-invariant, $h_n(A_n) \in \tau^{\alpha_n+1}(\mathcal{F}) \setminus \tau^{\alpha_n}(\mathcal{F})$ according to Lemma 5.2.

Then the set $A = \bigcup_{n\in\omega} h_n(A_n)$ does not belong to $\tau^{<\alpha}(\mathcal{F})$. The inclusion $A \in \tau^\alpha(\mathcal{F})$ will follow as soon as we check that $A \cap xA \in \tau^{<\alpha}(\mathcal{F})$ for all $x \in G \setminus \{e\}$. This is clear if $A \cap xA$ is empty. If $A \cap xA$ is not elmpty, then $x \in h_n(A_n)h_m(A_m)^{-1} \subset H_nH_m$ for some $n, m \in \kappa$. Taking into account that $H_nH_m \cap H_kH_l = \{e\}$ for all $k, l \in \kappa \setminus \{n, m\}$ and $e \notin A$, we conclude that

$$A \cap xA \subset h_n(A_n) \cup h_m(A_m) \cup xh_n(A_n) \cup xh_m(A_m) \in \tau^{<\alpha}(\mathcal{F})$$

as $\tau^{<\alpha}(\mathcal{F})$ is additive according to Theorem 1.2. □

Let us recall that a family $\mathcal{F}$ of subsets of a group G is called *auto-invariant* if for any injective homomorphism $h : G \to G$ a subset $A \subset G$ belongs to $\mathcal{F}$ if and only if $h(A) \in \mathcal{F}$.

Lemma 6.8. *Let G be a free abelian group G and $\mathcal{F}$ be an auto-invariant ideal of subsets of G. If $\mathcal{F}$ is not thin-complete, then for each ordinal $\alpha < |G|^+$ the family $\tau^\alpha(\mathcal{F})$ is not thin-complete.*

Proof. Being free abelian, the group G is generated by some linearly independent subset $B \subset G$. Consider the isomorphism $h : G \to 3G$ of G onto the subgroup $3G = \{g^3 : g \in G\}$ and observe that h is expanding and for each $z \in B$ we get $z^2 \notin 3G$. The ideal $\mathcal{F}$ being auto-invariant, is h-invariant. Applying Lemma 6.6, we conclude that $\tau^\alpha(\mathcal{F}) \neq \tau^{<\alpha}(\mathcal{F})$ for all ordinals $\alpha < \omega_1$. If the group G is countable, then this is exactly what we need.

Now consider the case of uncountable $\kappa = |G|$. Being free abelian, the group G is isomorphic to the direct sum $\oplus^\kappa \mathbb{Z}$ of κ-many copies of the infinite cyclic group $\mathbb{Z}$. Write the cardinal κ as the disjoint union $\kappa = \bigcup_{\alpha\in\kappa} \kappa_\alpha$ of κ many subsets $\kappa_\alpha \subset \kappa$ of cardinality $|\kappa_\alpha| = \kappa$. For every $\alpha \in \kappa$ consider the free abelian subgroup $G_\alpha = \oplus^{\kappa_\alpha}\mathbb{Z}$ of G and fix any isomorphism $h_\alpha : G \to G_\alpha$. It is clear that $G_\alpha \oplus G_\beta \cap G_\gamma \oplus G_\delta = \{0\}$ for all ordinals $\alpha, \beta, \gamma, \delta \in \kappa$ with $\{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset$.

Being auto-invariant, the ideal $\mathcal{F}$ is h_α-invariant for every $\alpha \in \kappa$. Now it is legal to apply Lemma 6.7 to conclude that $\tau^\alpha(\mathcal{F}) \neq \tau^{<\alpha}(\mathcal{F})$ for all ordinals $\alpha < \kappa^+$. □

Proof of Theorem 6.1. Let $\mathcal{F}$ be a sub-invariant ideal of subsets of a group G and let $H \subset G$ be a free abelian subgroup of cardinality $|H| = |G|$. Assume that $\tau(\mathcal{F}) \cap \mathcal{P}_H \not\subset \mathcal{F}$.

Consider the ideal $\mathcal{F}_H = \mathcal{P}_H \cap \mathcal{F}$ of subsets of the group H. By transfinite induction it can be shown that $\tau^\alpha(\mathcal{F}_H) = \mathcal{P}_H \cap \tau^\alpha(\mathcal{F})$ for all ordianls α.

The sub-invariance of $\mathcal{F}$ implies the sub-invariance (and hence auto-invariance) of $\mathcal{F}_H$. By Lemma 6.8, we get $\tau^\alpha(\mathcal{F}_H) \neq \tau^{<\alpha}(\mathcal{F}_H)$ for each $\alpha < |H|^+ = |G|^+$. Then also $\tau^*(\mathcal{F}) \neq \tau^\alpha(\mathcal{F}) \neq \tau^{<\alpha}(\mathcal{F})$ for all $\alpha < |G|^+$. □

7. The descriptive complexity of the family $\tau^*(\mathcal{F})$

In this section given a countable group G and a left-invariant monotone subfamily $\mathcal{F} \subset \mathcal{P}_G$ we study the descriptive complexity of the family $\tau^*(\mathcal{F})$, considered as a subspace of the power-set $\mathcal{P}_G$ endowed with the compact metrizable topology of the Tychonov product 2^G (we identify $\mathcal{P}_G$ with 2^G by identifying each subset $A \subset G$ with its characteristic function $\chi_A : G \to 2 = \{0, 1\}$).

Theorem 7.1. *Let G be a countable group and $\mathcal{F} \subset \mathcal{P}_G$ be a Borel left-invariant lower family of subsets of G.*

1. *For every ordinal $\alpha < \omega_1$ the family $\tau^\alpha(\mathcal{F})$ is Borel in $\mathcal{P}_G$.*
2. *The family $\tau^*(\mathcal{F}) = \tau^{<\omega_1}(\mathcal{F})$ is coanalytic.*
3. *If $\tau^*(\mathcal{F}) \neq \tau^\alpha(\mathcal{F})$ for all $\alpha < \omega_1$, then $\tau^*(\mathcal{F})$ is not Borel in $\mathcal{P}_G$.*

Proof. Let us recall that $G_\circ = G \setminus \{e\}$.

In Section 3 to each subset $A \subset G$ we assigned the τ-tree

$$T_A = \{s \in G_\circ^{<\omega} : A_s \notin \mathcal{F}\},$$

where for a finite sequence $s = (g_0, \dots, g_{n-1}) \in G_\circ^n \subset G_\circ^{<\omega}$ we put

$$A_s = \bigcap_{x_0,\dots,x_{n-1}\in 2^n} g_0^{x_0} \cdots g_{n-1}^{x_{n-1}} A.$$

Consider the subspaces $\mathsf{WF} \subset \mathsf{Tr}$ of $\mathcal{P}_{G_\circ^{<\omega}}$, consisting of all (well-founded) lower subtrees of the tree $G_\circ^{<\omega}$.

Claim 7.2. The function

$$T_* : \mathcal{P}_G \to \mathsf{Tr}, \;\; T_* : A \mapsto T_A$$

is Borel measurable.

Proof. The Borel measurability of T_* means that for each open subset $\mathcal{U} \subset \mathsf{Tr}$ the preimage $T_*^{-1}(\mathcal{U})$ is a Borel subset of $\mathcal{P}_G$. Let us observe that the topology of the space Tr is generated by the sub-base consisting of the sets

$$\langle s\rangle^+ = \{T \in \mathsf{Tr} : s \in T\} \;\text{ and }\; \langle s\rangle^- = \{T \in \mathsf{Tr} : s \notin T\} \text{ where } s \in G_\circ^{<\omega}.$$

Since $\langle s\rangle^- = \mathsf{Tr} \setminus \langle s\rangle^+$, the Borel masurability of T_* will follow as soon as we check that for every $s \in G_\circ^{<\omega}$ the preimage $T_*^{-1}(\langle s\rangle^+) = \{A \in \mathcal{P}_G : s \in T_A\}$ is Borel.

For this observe that the function

$$f : \mathcal{P}_G \times G_\circ^{<\omega} \to \mathcal{P}_G, \;\; f : (A, s) \mapsto A_s,$$

is continuous. Here the tree $G_\circ^{<\omega}$ is endowed with the discrete topology.

Since $\mathcal{F}$ is Borel in $\mathcal{P}_G$, the preimage $\mathcal{E} = f^{-1}(\mathcal{P}_G \setminus \mathcal{F})$ is Borel in $\mathcal{P}_G \times G_\circ^{<\omega}$. Now observe that for every $s \in G_\circ^{<\omega}$ the set

$$T_*^{-1}(\langle s\rangle^+) = \{A \in \mathcal{P}_G : s \in T_A\} = \{A \in \mathcal{P}_G : (A, s) \in \mathcal{E}\}$$

is Borel. □

By Theorem 3.2, $\tau^*(\mathcal{F}) = T_*^{-1}(\mathsf{WF})$ and $\tau^\alpha(\mathcal{F}) = T_*^{-1}(\mathsf{WF}_{\alpha+1})$ for $\alpha < \omega_1$. Now Theorem 2.1 and the Borel measurablity of the function T_* imply that the preimage $\tau^*(\mathcal{F}) = T_*^{-1}(\mathsf{WF})$ is coanalytic while $\tau^\alpha(\mathcal{F}) = T_*^{-1}(\mathsf{WF}_{\alpha+1})$ is Borel for every $\alpha < \omega_1$, see [3, 14.4].

Now assuming that $\tau^{\alpha+1}(\mathcal{F}) \neq \tau^\alpha(\mathcal{F})$ for all $\alpha < \omega_1$, we shall show that $\tau^*(\mathcal{F})$ is not Borel. In the opposite case, $\tau^*(\mathcal{F})$ is analytic and then its image $T_*(\tau^*(\mathcal{F})) \subset \mathsf{WF}$ under the Borel function T_* is an analytic subspace of WF, see [3, 14.4]. By Theorem 2.1(4), $T_*(\tau^*(\mathcal{F})) \subset \mathsf{WF}_{\alpha+1}$ for some $\alpha < \omega_1$ and thus $\tau^*(\mathcal{F}) = T_*^{-1}(\mathsf{WF}_{\alpha+1}) = \tau^\alpha(\mathcal{F})$, which is a contradiction. □

Theorems 6.1 and 7.1 imply:

Corollary 7.3. *For any countable non-torsion group G the ideal $\tau^*(\mathcal{F}_G) \subset \mathcal{P}_G$ is coanalytic but not analytic.*

By [3, 26.4], the Σ^1_1-Determinacy (i.e., the assumption of the determinacy of all analytic games) implies that each coanalytic non-analytic space is Π^1_1-complete. By [6], the Σ^1_1-Determinacy follows from the existence of a measurable cardinal. So, the existence of a measurable cardinal implies that for each countable non-torsion group G the subspace $\tau^*(\mathcal{F}_G) \subset \mathcal{P}_G$, being coanalytic and non-analytic, is Π^1_1-complete.

Question 7.4. Is the space $\tau^*(\mathcal{F}_{\mathbb{Z}})$ Π^1_1-complete in ZFC?

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