Two-valued groups, Kummer varieties and integrable billiards

V. M. Buchstaber (*), V. Dragović (**)

(*) Steklov Mathematical Institute, Russian Academy of Sciences
Gubkina Street 8, 119991 Moscow, Russia
e-mail: buchstab@mi.ras.ru

(**)Mathematical Institute SANU
Kneza Mihaila 36, 11000 Belgrade, Serbia
Mathematical Physics Group, University of Lisbon
e-mail: vladad@mi.sanu.ac.rs

Abstract

A natural and important question of study two-valued groups associated with hyperelliptic Jacobians and their relationship with integrable systems is motivated by seminal examples of relationship between algebraic two-valued groups related to elliptic curves and integrable systems such as elliptic billiards and celebrated Kowalevski top. The present paper is devoted to the case of genus 2, to the investigation of algebraic two-valued group structures on Kummer varieties.

One of our approaches is based on the theory of σ-functions. It enables us to study the dependence of parameters of the curves, including rational limits. Following this line, we are introducing a notion of n-groupoid as natural multivalued analogue of the notion of topological groupoid.

Our second approach is geometric. It is based on a geometric approach to addition laws on hyperelliptic Jacobians and on a recent notion of billiard algebra. Especially important is connection with integrable billiard systems within confocal quadrics.

The third approach is based on the realization of the Kummer variety in the framework of moduli of semi-stable bundles, after Narasimhan and Ramanan. This construction of the two-valued structure is remarkably similar to the historically first example of topological formal two-valued group from 1971, with a significant difference: the resulting bundles in the 1971 case were "virtual", while in the present case the resulting bundles are effectively realizable.

MSC2010: 20N20, 14H40, 14H70

Key words: 2-valued groups, Kummer varieties, hyperelliptic Jacobians, integrable billiards, semi-stable bundles

Contents

1 Introduction

The study of structure of multivalued groups has been started in 1971 (see [5]) within the study of characteristic classes of vector bundles, at that time as the structure of formal (local) multivalued groups. In 1990, in [8] has been introduced algebraic two-valued group structure on $\mathbb{C}P^1$ by using addition theorems on elliptic curves. The structure of algebraic multivalued groups has been studied since then extensively (see [9] and references therein) in different contexts.

The deep connection between two-valued groups and integrable systems is well-known, see [7]. However, it has got a new impetus with a recent paper [18], where a connection between two-valued groups on $\mathbb{C}P^1$ and the celebrated Kowalevski top has been discovered. There, it was shown that the "mysterious" Kowalevski change of variables (see, for example, [3] for the terminology) corresponds to the operation in two-valued group W/τ, where W is an elliptic curve, in the standard Weierstrass model of elliptic curve, with τ as the canonical involution. More detailed description of this two-valued group is presented in Section 2.1, see Example 3. This deep connection of Kowalevski integration procedure with a structure of elliptic curve, on the first glance, may be in surprising contrast to a well known fact that the integration of the Kowalevski top is performed on Jacobian of a genus two curve. Moreover, as it has been shown in [18], the Kowalevski integration procedure can be interpreted as a certain deformation of the two-valued group p_2 (see Example 2) toward the structure on W/τ. The structure of p_2 is the rational limit of the one on W/τ. In addition, in [18] it has been proven the equivalence between associativity of the two-valued group on W/τ with the Poncelet theorem of pencils of conics, in a case of triangles.

Higher genera Poncelet type problems, integrable billiards and so-called billiard algebra associated with pencils of quadrics in arbitrary dimensions have been studied in [20] and [19], where the relationship with hyperelliptic curves and their Jacobians has been considered (see also [21]).

From our previous experience and work with elliptic case, as a natural and important question arises the study two-valued groups associated with hyperelliptic Jacobians and their relationship with integrable systems. Especially important is connection with integrable billiards within pencils of quadrics.

The present paper is devoted to the case of genus 2. In this case one comes to the problem of investigation of two-valued group structures on Kummer varieties.

We describe the structure of algebraic two-valued groups on Kummer varieties, its rational limits and relationship with integrable systems. The Kummer variety is a classical and well-studied object of the algebraic geometry. It appears as a variety of orbits of a Jacobian of a curve of genus 2, factorized by the hyperelliptic involutive automorphism. Thus, there are various algebro-geometric and analytical structures on it, related to the moduli of curves of genus 2. We want to understand and present the structure of an algebraic two-valued group associated to different appearances of the Kummer varieties. Let us mention that from the point of view of differential geometry and the theory of abelian functions of genus 2, Kummer varieties have been studied by Baker.

One side of our approach is based on the theory of σ-functions. The choice of σ functions is motivated by our wish to study dependence of parameters of the curves. As we know from the theory of integrable systems, (see [22]), it is important to study not only a single curve and its Jacobian, but rather a whole class of curves and their Jacobians. Following this line, let B denote a set parameterizing nonsingular curves of genus 2 and let U be the analytic bundle over B with a genus 2 curve as a fibre. We denote by $J(U)$ the associated bundle over B with a Jacobian of a genus 2 curve as a fiber. Then, $J(U)$ has a natural structure of a topological groupoid, see [11].

In the next section, after recalling the definition of the n valued group, we are introducing a notion of n-groupoid, which enables us to consider not only a single structure of a two-valued group, but also dependance of the parameters of the curve. Thus the notion of n-groupoid is a natural analogue of the notion of topological groupoid.

Another important reason of using σ-functions, lies in the fact that addition relations for them are well adjusted to two-valued structure, as one can easily see from the genus one case formula:

$$\sigma(u+v)\sigma(u-v) = \sigma(u)^2\sigma(v)^2\big(\wp(v) - \wp(u)\big),$$

The program of construction of σ-functions of higher genera had been proposed by Klein. In genus 2 case, the program got rather advanced development by Baker (see [4]), although in his last review of the program in 1923, Klein noted that program for genus $g > 2$ was still far from being completed. We base our first approach on recent results, see [11] and references therein.

Our second approach is geometric. Following [15], [25], [29] a geometric approach to addition laws on hyperelliptic Jacobians has been developed further in [20] (see also [21]). It has been based on the notion of billiard algebra and it has been connected to integrable billiard systems within confocal quadrics.

Our first approach is developed in Section 6, and the second one in Section 9. The Section 6 is preceded by the three sections of preparation. In Section 3, the derivation of the two-valued group law on W/τ, where W denotes an elliptic curve as above, is based on σ - functions. The rational limit of the last structure is studied in the next Section 4. The study of the rational limit of a Kummer surface and related two-valued group law is performed in Section 5.

In Section 10, the geometric approach from the Section 9, develops further toward integrable systems, through the integrable billiards and the billiard algebra and is motivated by [20] .

The final Section 11 gives yet another construction of the two-valued group law on the Kummer variety, based on the realization of the Kummer variety in the framework of moduli of semi-stable bundles. This framework has been developed by Narasimhan and Ramanan, see [28]. This last construction of the two-valued structure is remarkably similar to the historically first example of topological two-valued group from 1971 (see [5]), with a significant difference.

The resulting bundles in the 1971 case of [5] are "virtual", while in the present case the resulting bundles are effectively realizable.

2 From n-valued groups to n-groupoids.

2.1 Defining notions and basic examples of multivalued groups

Following [9], we give the definition of an n-valued group on X as a map:

$$m:\ X \times X \to (X)^n$$
$$m(x,y) = x * y = [z_1, \ldots, z_n],$$

where $(X)^n$ denotes the symmetric n-th power of X and z_i coordinates therein.

Associativity is the condition of equality of two n^2-sets

$$[x * (y * z)_1, \ldots, x * (y * z)_n]$$
$$[(x * y)_1 * z, \ldots, (x * y)_n * z]$$

for all triplets $(x, y, z) \in X^3$.

An element $e \in X$ is *a unit* if

$$e * x = x * e = [x, \ldots, x],$$

for all $x \in X$.

A map inv : $X \to X$ is *an inverse* if it satisfies

$$e \in \mathrm{inv}(x) * x, \quad e \in x * \mathrm{inv}(x),$$

for all $x \in X$.

Following [9], we say that m defines *an n-valued group structure* (X, m, e, inv) if it is associative, with a unit and an inverse.

An n-valued group X acts on the set Y if there is a mapping

$$\phi:\ X \times Y \to (Y)^n$$
$$\phi(x,y) = x \circ y,$$

such that the two n^2-multisubsets of Y

$$x_1 \circ (x_2 \circ y) \quad (x_1 * x_2) \circ y$$

are equal for all $x_1, x_2 \in X, y \in Y$. It is additionally required that

$$e \circ y = [y, \ldots, y]$$

for all $y \in Y$.

Example 1 *[A two-valued group structure on $\mathbb{Z}_+$, [7]] Let us consider the set of nonnegative integers $\mathbb{Z}_+$ and define a mapping*

$$m: \mathbb{Z}_+ \times \mathbb{Z}_+ \to (\mathbb{Z}_+)^2,$$
$$m(x,y) = [x+y, |x-y|].$$

This mapping provides a structure of a two-valued group on $\mathbb{Z}_+$ with the unit $e = 0$ and the inverse equal to the identity $\mathrm{inv}(x) = x$.

In [7] sequence of two-valued mappings associated with the Poncelet porism was identified as the algebraic representation of this 2-valued group. Moreover, the algebraic action of this group on $\mathbb{C}P^1$ was studied and it was shown that in the irreducible case all such actions are generated by Euler-Chasles correspondences.

Example 2 *[2-valued group on $(\mathbb{C}, +)$] Among the basic examples of multivalued groups, there are n-valued additive group structures on $\mathbb{C}$. For $n = 2$, this is a two-valued group p_2 defined by the relation*

$$\begin{aligned} m_2 &: \mathbb{C} \times \mathbb{C} \to (\mathbb{C})^2 \\ x *_2 y &= [(\sqrt{x} + \sqrt{y})^2, (\sqrt{x} - \sqrt{y})^2] \end{aligned} \tag{1}$$

The product $x *_2 y$ corresponds to the roots in z of the polynomial equation

$$p_2(z,x,y) = 0,$$

where

$$p_2(z,x,y) = (x+y+z)^2 - 4(xy+yz+zx).$$

This two-valued group structure has been connected with degenerations of the Kowalevski top in [18]. Similar integrable systems were studied by Appel'rot, Delone, Mlodzeevskii (see [14], [1], [27], [24].)

As it has been observed in [18], the general Kowalevski case is connected with p_2 together with its deformation on $\mathbb{C}P^1$ as a factor of an elliptic curve, see the next example.

Example 3 *[2-valued group on $S^2 = \hat{\mathbb{C}}$, associated with an elliptic curve] Suppose that a cubic W is given in the standard form*

$$W: t^2 = J(s) = 4s^3 - g_2 s - g_3.$$

Consider the mapping $W \to S^2 = \hat{\mathbb{C}} : (s,t) \mapsto s$, where $\hat{\mathbb{C}}$ represents a complex line extended by ∞.

The curve W as a cubic curve has the group structure. Together with its canonical involutive automorphism $\tau : (s,t) \mapsto (s,-t)$, it defines the standard *two-valued group structure of coset type (see [12], [9]) on S^2 with the unit at infinity in S^2. The product is defined by the formula:*

$$[s_1] *_c [s_2] = \left[\left[-s_1 - s_2 + \left(\frac{t_1 - t_2}{2(s_1 - s_2)} \right)^2 \right], \left[-s_1 - s_2 + \left(\frac{t_1 + t_2}{2(s_1 - s_2)} \right)^2 \right] \right], \tag{2}$$

where $t_i = J(s_i), i = 1, 2$, and

$$[s_i] = \{(s_i, t_i), (s_i, -t_i)\}, \quad s_i = \wp(u_i), t_i = \wp'(u_i),$$

by using addition theorem for the Weierstrass function $\wp(u)$:

$$\wp(u_1 + u_2) = -\wp(u_1) - \wp(u_2) + \left(\frac{\wp'(u_1) - \wp'(u_2)}{2(\wp(u_1) - \wp(u_2))} \right)^2.$$

The Kowalevski integration procedure was explained in [18] as certain deformation of p_2 to (W, τ).

The 2-group structure (W, τ) is also connected with the Poncelet and the Darboux theorem (see [13], [17], [20]).

In this example, g_2, g_3 are parameters of the curve, and they lead to a rational limit, in the standard limit procedure.

Related structure on $\mathbb{C}P^1$ as an algebraic mapping $\mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^2$ (see [12], [9]) is presented in Section 3.

Example 4 *Let X be a topological space. Denote by $W(X)$ the set of all 2-dimensional complex vector bundles $\zeta = \eta + \bar{\eta}$ over X, where η is a linear complex vector bundle over X. Then the formula*

$$\zeta_1 \otimes \zeta_2 = (\eta_1 \otimes \eta_2 + \bar{\eta}_1 \otimes \bar{\eta}_2) + (\bar{\eta}_1 \otimes \eta_2 + \eta_1 \otimes \bar{\eta}_2)$$

gives the structure of 2-valued group on $W(X)$ defined by the 2-valued multiplication

$$\zeta_1 \star \zeta_2 = [(\eta_1 \otimes \eta_2 + \bar{\eta}_1 \otimes \bar{\eta}_2), (\bar{\eta}_1 \otimes \eta_2 + \eta_1 \otimes \bar{\eta}_2)].$$

2.2 Topological n-groupoids

Following [11], let us fix a topological space Y. A space X with a map $p_X : X \to Y$ is called a space over Y, and the map p_X is called anchor. For Y, the anchor p_Y is the identity map.

A map $f : X_1 \to X_2$ of two spaces over Y is called a map over Y if $p_{X_2} \circ f = p_{X_1}$. For two spaces X_1, X_2 over Y, their direct product over Y is defined as

$$X_1 \times_Y X_2 = \{(x_1, x_2) \in X_1 \times X_2 \mid p_{X_1}(x_1) = p_{X_2}(x_2)\}$$

with the map $p_{X_1\times_Y X_2}(x_1,x_2) = p_{X_1}(x_1)$. Along this line, one may define a product over Y of n spaces $X_1,\dots X_n$ over Y

$$X_1\times_Y\cdots\times_Y X_n = \{(x_1,\dots,x_n)\in X_1\times X_2\times\cdots\times X_n \mid p_{X_1}(x_1) = p_{X_2}(x_2) = \cdots = p_{X_n}(x_n)\}.$$

In a special case $X_1 = \cdots = X_n = X$, we define *n-th power of X over Y* and denote it as X^n_Y.

For a space X over Y we define its *n-th symmetric power over Y*, denoted as

$$(X)^n_Y$$

as the quotient of X^n_Y by the action of the permutation group.

We define also the diagonal map over Y

$$D: X\to (X)^n_Y,\quad x\mapsto (x,x,\dots,x).$$

Definition 1 *A space X with an anchor $p_X : X\to Y$ and structural maps over Y:*

$$\mu: X\times_Y X\to (X)^n_Y,\quad inv: X\to X,\quad e: Y\to X$$

is called an n-groupoid over Y *if the following conditions are satisfied:*

1 For x_1,x_2,x_3 such that $p_X(x_1) = p_X(x_2) = p_X(x_3)$: if

$$\mu(x_1,x_2) = [z_1,\dots,z_n],\quad \mu(x_2,x_3) = [w_1,\dots,w_n],$$

then

$$[\mu(x_1,w_1),\dots,\mu(x_1,w_n)] = [\mu(z_1,x_3),\dots,\mu(z_n,x_3)].$$

2 For every $x\in X$ and $y = p_X(x)\in Y$:

$$\mu(e(y),x) = \mu(x,e(y)) = D(x).$$

3 For every $x\in X$ and $y = p_X(x)\in Y$:

$$e(y)\in\mu(x,inv(x)),\quad e(y)\in\mu(inv(x),x).$$

Example 5 *Let $X = \mathbb{C}\times\mathbb{C}$, $Y = \mathbb{C}$ and an anchor is defined as projection to the second component*

$$p_X := p_2 : X\to Y,\quad p_X(x,\lambda) := \lambda.$$

We define 2-groupoid over Y starting with an operation over Y

$$\mathcal{A}((x_1,\lambda),(x_2,\lambda)) = (x_1+x_2-\lambda x_1x_2,\lambda).$$

An involutive automorphism I over Y is defined by

$$I: X\to X,\quad (\bar{u},\lambda) = \left(-\frac{u}{1-\lambda u},\lambda\right).$$

If we denote by $u\bar{u} = x$, $v\bar{v} = y$, then the 2-groupoid over Y is defined by

$$\mu((x, \lambda), (y, \lambda)) = [(z_1, \lambda), (z_2, \lambda)]$$

where z_i, $i = 1, 2$ are solutions of the following quadratic equation

$$Z^2 - \left(2(x + y) - \lambda^2 xy\right) Z + (x - y)^2 = 0.$$

Notice that we get p_2 for $\lambda = 0$. Thus the structure of the above 2-groupoid is certain deformation of p_2.

Example 6 *Similarly, we can consider a two-parameter deformation of elementary two-valued group p_2 and we get a structure of 2-groupoid over $Y_1 = \mathbb{C}^2$. We are starting with an operation over Y_1 on $X_1 = \mathbb{C} \times Y_1$, defined by*

$$\mathcal{A}_1 \left((u, \lambda_1, \lambda_2), (v, \lambda_1, \lambda_2)\right) := \left(\frac{u + v - \lambda_1 uv}{1 - \lambda_2 uv}, \lambda_1, \lambda_2\right).$$

The involutive automorphism I_1, over Y_1, is defined by

$$I_1(u, \lambda_1, \lambda_2) = \left(-\frac{u}{1 - \lambda_1 u}, \lambda_1, \lambda_2\right).$$

One can easily deduce corresponding morphism μ_1:

$$\mu_1((x, \lambda_1, \lambda_2), (y, \lambda_1, \lambda_2)) = [(z_1, \lambda_1, \lambda_2), (z_2, \lambda_1, \lambda_2)]$$

where z_i, $i = 1, 2$ are solutions of the following quadratic equation:

$$Z^2 - \frac{B(x, y, \lambda_1, \lambda_2)}{G(x, y, \lambda_1, \lambda_2)} Z + \frac{C(x, y, \lambda_1, \lambda_2)}{G(x, y, \lambda_1, \lambda_2)} = 0.$$

Here we use the notation

$$x = uI_1(u), \quad y = vI_1(v)$$

and

$$B(x, y, \lambda_1, \lambda_2) = x^2 y^2 \lambda_1^2 \lambda_2 (-\lambda_1^2 + \lambda_2) + xy(x + y)\lambda_2(2\lambda_2 - 3\lambda_1) + 2(x + y)$$
$$C(x, y, \lambda_1, \lambda_2) = (x - y)^2$$
$$G(x, y, \lambda_1, \lambda_2) = 1 + xy\lambda_2(\lambda_1^2 - \lambda_2) + xy(x + y)\lambda_1^2 \lambda_2^2 + x^2 y^2 \lambda_2^3 (\lambda_2 - \lambda_1^2)$$

3 Structure of two-valued group on $\mathbb{CP}^1$ and sigma-functions

Let us consider the Weierstrass sigma-function $\sigma(u) = \sigma(u, g_2, g_3)$, associated with a curve

$$V = \{(t, s) \in \mathbb{C}^2 \ : \ t^2 = 4s^3 - g_2 s - g_3\}.$$

On the Jacobian $J_1 = \mathbb{C}^1/\Gamma$ of the curve V, the $\wp$-function $\wp(u) = -\frac{d^2}{du^2} \ln \sigma(u)$ is defined. The function $\sigma(u)$ is odd, thus, the mapping

$$\pi\colon J_1 \longrightarrow \mathbb{C}P^1 \ : \ \pi(u) = (x_1 : x_2),$$

with $x_1 = \sigma(u)^2$ and $x_2 = \sigma(u)^2 \wp(u)$, factorizes as a composition

$$\pi\colon J_1 \longrightarrow J_1/_\pm \xrightarrow{\widehat{\pi}} \mathbb{C}P^1,$$

where $\widehat{\pi}$ is a homeomorphism. Let us note that $x_1(u)$ and $x_2(u)$ are entire functions of u.

The canonical homeomorphism

$$\gamma\colon (\mathbb{C}P^1)^2 \longrightarrow \mathbb{C}P^2 \ : \ \big((x_1 : x_2), (y_1 : y_2)\big) \longrightarrow \big(x_1y_1 : (x_1y_2 + x_2y_1) : x_2y_2\big),$$

corresponds to the mapping

$$\big[(x_1t_1 + x_2t_2), (y_1t_1 + y_2t_2)\big] \longrightarrow x_1y_1t_1^2 + (x_1y_2 + x_2y_1)t_1t_2 + x_2y_2t_2^2.$$

Theorem 1 *Multiplication in the two-valued group*

$$m\colon J_1/_\pm \times J_1/_\pm \longrightarrow (J_1/_\pm)^2$$

is defined by algebraic mapping

$$m\colon \mathbb{C}P^1 \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^2 \ : \ (x_1 : x_2) * (y_1 : y_2) = (z_1 : z_2 : z_3),$$

where $z_1 = (x_1y_2 - x_2y_1)^2$, $z_2 = 2\left[(x_1y_2 + x_2y_1)\big(x_2y_2 - \frac{g_2}{4}x_1y_1\big) - \frac{g_3}{2}x_1^2y_1^2\right]$, $z_3 = \big(x_2y_2 + \frac{g_2}{4}x_1y_1\big)^2 + g_3x_1y_1(x_1y_2 + x_2y_1)$.

Proof. The composition of mappings

$$J_1/_\pm \times J_1/_\pm \xrightarrow{m} (J_1/_\pm)^2 \xrightarrow{(\widehat{\pi})^2} (\mathbb{C}P^1)^2 \xrightarrow{\gamma} \mathbb{C}P^2$$

has the form

$$[u] \times [v] \longrightarrow \big([u+v], [u-v]\big) \longrightarrow \big((x_1 : x_2), (y_1 : y_2)\big) \longrightarrow (z_1 : z_2 : z_3),$$

where $[u] = \{u, -u\}$, $x_1 = \sigma(u+v)^2$, $x_2 = x_1\wp(u+v)$, $y_1 = \sigma(u-v)^2$, $y_2 = y_1\wp(u-v)$.
Thus,

$$z_1 = \sigma(u+v)^2\sigma(u-v)^2, \quad z_2 = z_1\big(\wp(u+v) + \wp(u-v)\big), \quad z_3 = z_1\wp(u+v)\wp(u-v).$$

Our goal is to express z_1, z_2, z_3 as functions of x_1, x_2, y_1, y_2. By using the classical addition theorem for sigma-functions

$$\sigma(u+v)\sigma(u-v) = \sigma(u)^2\sigma(v)^2\big(\wp(v)-\wp(u)\big),$$

we get

$$z_1 = (x_1y_2 - x_2y_1)^2.$$

From

$$\ln\sigma(u+v) + \ln\sigma(u-v) = 2\big(\ln\sigma(u) + \ln\sigma(v)\big) + \ln\big(\wp(v)-\wp(u)\big),$$

applying the operator $\partial = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$, we get

$$2\zeta(u+v) = 2\big(\zeta(u)+\zeta(v)\big) + \frac{\wp'(v)-\wp'(u)}{\wp(v)-\wp(u)}$$

where $\zeta(u) = \frac{\partial}{\partial u}\ln\sigma(u)$. By differentiating previous identity in u, one gets

$$\wp(u+v) = w_1(u,v) - w_2(u,v),$$

where

$$w_1(u,v) = \wp(u) + \frac{1}{2}\frac{\wp''(u)\big(\wp(v)-\wp(u)\big) + \wp'(u)^2}{\big(\wp(v)-\wp(u)\big)^2} \tag{3}$$

$$w_2(u,v) = \frac{1}{2}\frac{\wp'(u)\wp'(v)}{\big(\wp(v)-\wp(u)\big)^2}. \tag{4}$$

Observe, that $w_1(u,-v) = w_1(u,v)$ and $w_2(u,-v) = -w_2(u,v)$. Finally we get,

$$\wp(u+v) + \wp(u-v) = 2w_1(u,v); \quad \wp(u+v)\wp(u-v) = w_1^2 - w_2^2.$$

By using the Weierstrass uniformization of the elliptic curve, we get

$$\wp'(u)^2 = 4\wp(u)^2 - g_2\wp(u) - g_3,$$

and also,

$$\wp''(u) = 6\wp(u)^2 - \frac{g_2}{2}.$$

According to the formula 3 we get

$$\begin{aligned}
&2\big(\wp(v)-\wp(u)\big)^2 w_1(u,v) = \\
&= 2\wp(u)\big(\wp(v)-\wp(u)\big)^2 + \left(6\wp(u)_2 - \frac{g_2}{2}\right)\big(\wp(v)-\wp(u)\big) + 4\wp(u)^3 - g_2\wp(u) - g_3 = \\
&\qquad = 2\wp(u)\wp(v)\big(\wp(v)+\wp(u)\big) - \frac{g_2}{2}\big(\wp(v)+\wp(u)\big) - g_3.
\end{aligned}$$

Thus,

$$\begin{aligned}
2z_1^2 w_1(u,v) &= 2\sigma^4(u)\sigma^4(v)\big(\wp(v)-\wp(u)\big)^2 w_1(u,v) = \\
&= \sigma^4(u)\sigma^4(v)\left[\big(\wp(v)+\wp(u)\big)\left(2\wp(v)\wp(u) - \frac{g_2}{2}\right) - g_3\right].
\end{aligned}$$

As a consequence, we get

$$z_2 = 2\left[(x_1y_2 + x_2y_1)\Big(x_2y_2 - \frac{g_2}{4}\,x_1y_1\Big) - \frac{g_3}{2}\,x_1^2y_1^2\right].$$

Further, we get

$$\begin{aligned}&\big(\wp(v) - \wp(u)\big)^4(w_1^2 - w_2^2) = \\ &= \left[\big(\wp(v)+\wp(u)\big)\Big(2\wp(v)\wp(u) - \frac{g_2}{4}\Big) - \frac{g_3}{2}\right]^2 - \left[2\wp(u)^3 - \frac{g_2}{2}\wp(u) - \frac{g_3}{2}\right]\left[2\wp(v) - \frac{g_2}{2}\wp(v) - \frac{g_3}{2}\right].\end{aligned}$$

From

$$\begin{aligned}z_1z_3 = \sigma^8(u)\sigma^8(v)\big(\wp(v) - \wp(u)\big)^4(w_1^2 - w_2^2) &= \\ \left[(x_1y_2 + x_2y_1)\Big(x_2y_2 - \frac{g_2}{4}\,x_1y_1\Big) - \frac{g_3}{2}\,x_1^2y_1^2\right]^2 &- \\ - x_1y_1\left[2x_2^3 - \frac{g_2}{2}\,x_2x_1^2 - \frac{g_3}{2}\,x_1^3\right]\left[2y_2^3 - \frac{g_2}{2}\,y_2y_1^2 - \frac{g_3}{2}\,y_1^3\right]&,\end{aligned}$$

we get

$$z_1z_3 = \left[\Big(x_2y_2 + \frac{g_2}{4}\,x_1y_1\Big)^2 + g_3x_1y_1(x_1y_2 + x_2y_1)\right].$$

□

Corollary 1 *The multiplication law of the two-valued group $\mathbb{C}P^1$ is defined in homogeneous coordinates by the formula*

$$\begin{aligned}&(x_1t_1 + x_2t_2) * (y_1t_1 + y_2t_2) = \\ &= (x_1y_2 - x_2y_1)^2t_1^2 + 2\left[(x_1y_2 + x_2y_1)\Big(x_2y_2 - \frac{g_2}{4}\,x_1y_1\Big) - \frac{g_3}{2}\,x_1^2y_1^2\right]t_1t_2 + \\ &\qquad + \left[\Big(x_2y_2 + \frac{g_2}{4}\,x_1y_1\Big)^2 + g_3x_1y_1(x_1y_2 + x_2y_1)\right]t_2^2.\end{aligned}$$

In the rational limit, defined with $g_2 = g_3 = 0$, we get

$$\begin{aligned}&(x_1t_1 + x_2t_2) * (y_1t_1 + y_2t_2) = \\ &\quad = (x_1y_2 - x_2y_1)^2t_1^2 + 2(x_1y_2 + x_2y_1)x_2y_2t_1t_2 + x_2^2y_2^2t_2^2 = \\ &\qquad = [(x_1y_2 + x_2y_1)t_1 + x_2y_2t_2]^2 - 4x_1x_2y_1y_2t_1^2.\end{aligned}$$

4 Elementary two-valued group on $(\mathbb{C}^2, +)$

Let us consider $\mathbb{C}^2$ together with its subgroup $\mathbb{Z}^2$. By the standard coset construction (see [9]),

$$X = \mathbb{C}^2/\pm$$

is equipped by two-valued group structure:

$$\mu:\ X\times X\to (X)^2$$

given in coordinates $u=(u_1,u_2), v=(v_1,v_2)\in\mathbb{C}^2$ by the formula

$$\mu([u,-u],[v,-v])=([u+v,-(u+v)],[(u-v),-(u-v)]).$$

Now, we pass to the embedding

$$\pi:\ X\to\mathbb{C}^3,\quad \pi([(u_1,u_2),-(u_1,u_2)])=(x_1,x_2,x_3)$$

given by the formula

$$x_1=u_1^2,\quad x_2=u_1u_2\quad x_3=u_2^2.$$

By checking, one sees that the inverse image of a point is a two element set: $\pi^{-1}(x_1,x_2,x_3)=[(u_1,u_2),-(u_1,u_2)]$, where

$$u_1=\pm\sqrt{x_1},\quad u_2=\pm\sqrt{x_3}.$$

The image of π is a quadric Q in $\mathbb{C}^3$ defined by the equation

$$Q: x_1x_3=x_2^2.$$

On Q, the multiplication μ can be rewritten according to the formulae

$$\mu([u,-u],[v,-v])=((X_2,X_4,X_6),(Y_2,Y_4,Y_6)),$$

where

$$[u,-u]=\pi^{-1}(x_1,x_2,x_3),\quad [v,-v]=\pi^{-1}(y_1,y_2,y_3).$$

Indices $2,4,6$ for variables X and Y are chosen to fit with a graduation in a sequel. Then

$$X_2=(\sqrt{x_1}+\sqrt{y_1})^2,\quad Y_2=(\sqrt{x_1}-\sqrt{y_1})^2,$$

and X_2,Y_2 are the roots of the equation

$$Z_1^2-2(x_1+y_1)Z_1+(x_1-y_1)^2=0. \tag{5}$$

For X_4 and Y_4 we have

$$X_4=(\sqrt{x_1}+\sqrt{y_1})(\sqrt{x_3}+\sqrt{y_3})$$
$$Y_4=(\sqrt{x_1}-\sqrt{y_1})(\sqrt{x_3}-\sqrt{y_3}).$$

Thus,

$$X_4+Y_4=2(x_2+y_2)$$
$$X_4Y_4=(x_1-y_1)(x_3-y_3),$$

and X_4, Y_4 are the solutions of the equation:

$$Z_2^2 - 2(x_2 + y_2)Z_2 + (x_1 - y_1)(x_3 - y_3) = 0. \qquad (6)$$

The last equation can be written in the form

$$(Z_2 - (x_2 + y_2))^2 = y_1 x_3 + x_1 y_3 + 2x_2 y_2.$$

Finally, for X_6, Y_6 the situation is analogue to X_2, Y_2:

$$X_6 = (\sqrt{x_3} + \sqrt{y_3})^2, \quad Y_6 = (\sqrt{x_3} - \sqrt{y_3})^2,$$

X_6, Y_6 are the roots of the equation

$$Z_3^2 - 2(x_3 + y_3)Z_3 + (x_3 - y_3)^2 = 0. \qquad (7)$$

Three quadratic equations (5, 6, 7) determine four possible pairs of triplets of solutions. But, the constraints

$$X_2 X_4 = X_6^2, \quad Y_2 Y_4 = Y_6^2,$$

select a unique pair of triplets $[(X_2, X_4, X_6), (Y_2, Y_4, Y_6)]$, which is in accordance with the theory of two-valued groups.

5 Rational limit of a Kummer surface and two-valued addition law

We start with a genus two curve V given by affine equation

$$V = \{(x, y) \in \mathbb{C}^2 : y^2 = x^5 + \lambda_4 x^3 + \lambda_6 x^2 + \lambda_8 x + \lambda_{10}\}.$$

Corresponding Jacobian $\mathrm{Jac}(V)$ as two-dimensional complex torus is a factor of $\mathbb{C}^2$ with a lattice Γ. The lattice Γ is determined by the vector $(\lambda_4, \lambda_6, \lambda_8, \lambda_{10})$. Note that the indices $(4, 6, 8, 10)$ of λ are chosen to fit with the graduation in sequel. The Kummer surface K is the factor of the Jacobian by the group of automorphisms of order 2:

$$K = \mathrm{Jac}(V)/\pm.$$

Locally, in a vicinity of $0 = (0, 0)$ the Kummer surface K is isomorphic to $\mathbb{C}^2/\pm$.

Moreover, all constructions allow the rational limit:$\lambda \to 0$.

Starting from the vector $(\lambda_4, \lambda_6, \lambda_8, \lambda_{10})$ and a vector $u = (u_1, u_3)$, a function $\sigma(u, \lambda)$ is constructed as an entire function in u and λ, such that all coefficients $c_{ij}(\lambda)$ in expansion in $u_1^i u_3^j$ are *polynomials* in λ.

5.1 Two-dimensional Weierstrass functions

Following [10], we introduce

$$\zeta_k = \frac{\partial}{\partial u_k} \ln \sigma(u,\lambda), \qquad k = 1,3$$
$$\wp_{kl} = -\frac{\partial^2}{\partial u_k \partial u_l} \ln \sigma(u,\lambda), \qquad k,l = 1,3.$$

The last, *Weierstrass functions* are functions on the Jacobian $\mathrm{Jac}(V)$.

We are going to use **the standard** sigma function, the one defined by *arf-invariants* (ℓ,ℓ') to be equal to

$$\ell = (1,1), \quad \ell' = (2,1).$$

The standard sigma function is odd in u,

$$\sigma(u,\lambda) = -\sigma(-u,\lambda),$$

while the $\wp$- functions are *even*:

$$\wp_{k,l}(u,\lambda) = \wp_{k,l}(-u,\lambda).$$

Thus, the $\wp$- functions $\wp_{k,l}(u,\lambda)$ generate well-defined functions on the Kummer surface K.

There is an embedding with each fixed λ:

$$\pi_\lambda : K \to \mathbb{C}P^3$$
$$[u,-u] \mapsto [\sigma^2(u), \sigma^2(u)\wp_{11}(u), \sigma^2(u)\wp_{13}(u), \sigma^2(u)\wp_{33}(u)].$$

Note that $\sigma^2(u)\wp_{kl}(u)$ are entire functions. The embeddings π_λ serve to describe addition law on K in terms of coordinates on $\mathbb{C}P^3$.

One can easily compute the limit of the functions when λ tends to zero:

$$\lim_{\lambda\to 0} \sigma(u,\lambda) = \sigma_0(u) = u_3 - \frac{1}{3}u_1^3$$
$$\zeta_1^{(0)} = \frac{\partial}{\partial u_1} \ln \sigma_0(u) = -\frac{u_1^2}{\sigma_0(u)}$$
$$\zeta_3^{(0)} = \frac{\partial}{\partial u_3} \ln \sigma_0(u) = -\frac{1}{\sigma_0(u)}$$
$$\wp_{11}^{(0)} = -\frac{\partial^2}{\partial u_1^2} \ln \sigma_0(u) = \frac{2u_1\sigma_0(u) + u_1^4}{\sigma_0^2(u)}$$
$$\wp_{13}^{(0)} = -\frac{\partial}{\partial u_1}\zeta_3^{(0)}(u) == -\frac{u_1^2}{\sigma_0(u)^2}$$
$$\wp_{33}^{(0)} = -\frac{\partial}{\partial u_3}\zeta_3^{(0)}(u) = \frac{1}{\sigma_0^2(u)}.$$

Thus,

$$\sigma_0^2\wp_{13}^{(0)} = -u_1^2, \quad \sigma_0^2\wp_{33}^{(0)} = 1.$$

5.2 Rational limit embedding and two-valued group law

Now, we are going to construct a new two-valued group law on $\mathbb{C}^2/\pm$ associated with an embedding π_K induced by the rational limit of a Kummer surface:

$$\begin{aligned}
&\pi_K : \mathbb{C}^2/\pm \to \mathbb{C}^3\\
&\pi_K([u,-u]) = \left((u_3 - \frac{1}{3}u_1^3)^2, 2u_1u_3 + \frac{1}{3}u_1^4, -u_1^2\right),
\end{aligned}$$

where

$$u = (u_1, u_3).$$

One checks that an inverse image is a two element set if nonempty:

$$\begin{aligned}
x_1 &= \left(u_3 - \frac{1}{3}u_1^3\right)^2\\
x_2 &= 2u_1u_3 + \frac{1}{3}u_1^4\\
x_3 &= -u_1^2.
\end{aligned}$$

The multiplication μ given by the formula

$$\begin{aligned}
\mu([u,-u],[v,-v]) =&[(u_1+v_1), -(u_1+v_1), (u_3+v_3), -(u_3+v_3),\\
&(u_1-v_1), -(u_1-v_1), (u_3-v_3), -(u_3-v_3)]
\end{aligned}$$

after composition with the embedding $(\pi_K)^2$ leads to the formulae

$$\begin{aligned}
\hat{X}_6 &= \left((u_3+v_3) - \frac{1}{3}(u_1+v_1)^3\right)^2\\
\hat{X}_4 &= 2(u_1+v_1)(u_3+v_3) + \frac{1}{3}(u_1+v_1)^4\\
\hat{X}_2 &= -(u_1+v_1)^2
\end{aligned}$$

and

$$\begin{aligned}
\hat{Y}_6 &= \left((u_3-v_3) - \frac{1}{3}(u_1-v_1)^3\right)^2\\
\hat{Y}_4 &= 2(u_1-v_1)(u_3-v_3) + \frac{1}{3}(u_1-v_1)^4\\
\hat{Y}_2 &= -(u_1-v_1)^2.
\end{aligned}$$

The last formulae lead to the following change of variables:

$$\begin{aligned}
\hat{X}_2 &= -X_2\\
\hat{X}_4 &= 2X_4 + \frac{1}{3}X_2^2\\
\hat{X}_6 &= X_6 - \frac{2}{3}X_2X_4 + \frac{1}{9}X_2^3.
\end{aligned}$$

This is an *algebraic* change of variables and the inverse change is given by the formulae:

$$
\begin{aligned}
X_2 &= -\hat{X}_2\\
X_4 &= \frac{1}{2}\left(\hat{X}_4 - \frac{1}{3}\hat{X}_2^2\right)\\
X_6 &= \hat{X}_6 - \frac{1}{3}\hat{X}_2\hat{X}_4 - \frac{2}{9}X_2^3
\end{aligned}
$$

By applying the last algebraic change of variables on the equation of the quadric $Q : X_2X_6 = X_4^2$ we get the equation of the rational limit of the Kummer surface.

Proposition 1 *The rational limit of the Kummer surface is given by the surface in* $\mathbb{C}^3$ *by the equation*

$$
-9\hat{X}_4^2 - 36\hat{X}_2\hat{X}_6 + 12\hat{X}_2^2\hat{X}_4 + 7\hat{X}_2^4 = 0. \tag{8}
$$

5.3 Rational Kummer two-valued group

In the previous section, we have constructed a two-valued group law in coordinates (X_2, X_4, X_6) - the elementary two-valued group. Now, using the algebraic change of variables, we are going to construct a new two-valued group law, and we will call it *the rational Kummer two-valued group.*

First, we consider $(\hat{X}_2, \hat{Y}_2)$. We have

$$
\begin{aligned}
\hat{X}_2 &= -x_3 - 2u_1v_1 - y_3\\
\hat{Y}_2 &= -x_3 + 2u_1v_1 - y_3
\end{aligned}
$$

therefore

$$
\begin{aligned}
\hat{X}_2 + \hat{Y}_2 &= -2(x_3 + y_3)\\
\hat{X}_2\hat{Y}_2 &= (x_3 - y_3)^2.
\end{aligned}
$$

Thus we see that the pair $(\hat{X}_2, \hat{Y}_2)$ is the solution of the quadratic equation

$$
\mathcal{Z}^2 + 2(x_3 + y_3)\mathcal{Z} + (x_3 - y_3)^2 = 0. \tag{9}
$$

Now, we pass to the pair $(\hat{X}_4, \hat{Y}_4)$. They can be represented in the form

$$
\begin{aligned}
\hat{X}_4 &= \hat{X}_4^+ + \hat{X}_4^-\\
\hat{Y}_4 &= \hat{X}_4^+ - \hat{X}_4^-
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{X}_4^+ &= 2(u_1u_3 + v_1v_3) + \frac{1}{3}(u_1^4 + 6u_1^2v_1^2 + v_1^4)\\
\hat{X}_4^- &= 2(v_1u_3 + u_1v_3) + \frac{4}{3}u_1v_1(u_1^2 + v_1^2).
\end{aligned}
$$

Then, we have

$$\hat{X}_4 + \hat{Y}_4 = 2\hat{X}_4^+$$
$$\hat{X}_4\hat{Y}_4 = (\hat{X}_4^+)^2 - (\hat{X}_4^-)^2.$$

Thus, $(\hat{X}_4, \hat{Y}_4)$ are the roots of the quadratic equation

$$\mathcal{Z}^2 - 2\hat{X}_4^+\mathcal{Z} + (\hat{X}_4^+)^2 - (\hat{X}_4^-)^2 = 0. \tag{10}$$

The last equation is equivalent to

$$(\mathcal{Z} - (\hat{X}_4^+))^2 = (\hat{X}_4^-)^2,$$

where $\hat{X}_4^+, \hat{X}_4^-$ can be rewritten in the form

$$\hat{X}_4^+ = 2(u_1u_3 + v_1v_3) + \frac{1}{3}(u_1^4 + 6u_1^2v_1^2 + v_1^4)$$
$$\hat{X}_4^- = 2(v_1u_3 + u_1v_3) + \frac{4}{3}u_1v_1(u_1^2 + v_1^2).$$

We pass to the last pair $\hat{X}_6, \hat{Y}_6$:

$$\hat{X}_6 = \left((u_3 + v_3) - \frac{1}{3}(u_1 + v_1)^3\right)^2$$
$$\hat{Y}_6 = \left((u_3 - v_3) - \frac{1}{3}(u_1 - v_1)^3\right)^2.$$

One can easily calculate

$$\hat{X}_6 + \hat{Y}_6 = 2[u_3^2 + v_3^2 - \frac{2}{3}(u_1^3u_3 + 3u_1u_3v_1^2 + 3u_1^2v_1v_3 + v_1^4) + \\ + \frac{1}{9}(u_1^6 + 15u_1^4v_1^2 + 15u_1^2v_1^4 + v_1^6)]$$
$$\hat{X}_6\hat{Y}_6 = \left((u_3^2 - v_3^2) + \frac{1}{9}(u_1^2 - v_1^2)^3 - \frac{2}{3}(u_3u_1^3 + 3u_1v_1^2u_3^2 - 3u_1^2v_1v_3 - v_1^3v_3)\right)^2,$$

or, in the old coordinates

$$\hat{X}_6 + \hat{Y}_6 = 2\left(x_3 + y_3 - \frac{2}{3}(x_3x_2 + 3x_2y_1 + 3x_1y_2 + y_1^2) + \frac{1}{9}(x_1^3 + 15x_1^2y_1 + 15x_1y_1^2 + y_1^3)\right)$$
$$\hat{X}_6\hat{Y}_6 = \left((x_3 - y_3) + \frac{1}{9}(x_1 - y_1)^2 - \frac{2}{3}(x_2x_1 + 3x_2y_1 - 3x_1y_2 - y_1y_2)\right)^2.$$

In the new coordinates one may rewrite

$$B_3 := \hat{X}_6 + \hat{Y}_6$$
$$C_3 := \hat{X}_6\hat{Y}_6$$

and to get finally

$$
\begin{aligned}
B_3 =& 2((\hat{x}_6 - \frac{1}{3}\hat{x}_4\hat{x}_2 - \frac{2}{9}\hat{x}_2^3) + (\hat{y}_6 - \frac{1}{3}\hat{y}_4\hat{y}_2 - \frac{2}{9}\hat{y}_2^3) \\
& - \frac{1}{3}(-\hat{x}_2\hat{x}_4 + \frac{1}{3}\hat{x}_2^3 - 3\hat{x}_4\hat{y}_2 + \hat{x}_2^2\hat{y}_2 - 3\hat{x}_2\hat{y}_4 + \hat{x}_2\hat{y}_2^2 + 2\hat{y}_2^2) \\
& + \frac{1}{9}(-\hat{x}_2^3 + 15\hat{x}_2^2\hat{y}_2 + 15\hat{x}_2\hat{y}_2^2 - \hat{y}_2^3))
\end{aligned}
$$

and

$$
\begin{aligned}
C_3 =& [(\hat{x}_6 - \frac{1}{3}\hat{x}_4\hat{x}_2 - \frac{2}{9}\hat{x}_2^3 - \hat{y}_6 + \frac{1}{3}\hat{y}_4\hat{y}_2 + \frac{2}{9}\hat{y}_2^3) + \frac{1}{9}(\hat{y}_2 - \hat{x}_2)^3 \\
& - \frac{1}{9}(\frac{1}{3}\hat{x}_2^2 - \hat{x}_2\hat{x}_4 + \hat{x}_2^2\hat{y}_2 - 3\hat{x}_4\hat{y}_2 + 3\hat{y}_4\hat{x}_2 - \hat{y}_2^2\hat{x}_2 \\
& + \hat{y}_2\hat{y}_4 - \frac{1}{3}\hat{y}_2^3)]^2.
\end{aligned}
$$

Thus, we may conclude that the pair $(\hat{X}_6, \hat{Y}_6)$ is determined as the roots of the quadratic equation

$$\mathcal{Z}^2 - B_3\mathcal{Z} + C_3 = 0, \tag{11}$$

where B_3, C_3 are functions of the coordinates $(\hat{x}_2, \hat{x}_4, \hat{x}_6, \hat{y}_2, \hat{y}_4, \hat{y}_6)$ given above.

6 Two-valued group structures on Kummer varieties and sigma-functions

We start with the sigma-function $\sigma(u) = \sigma(u, \lambda)$,where $u^\top = (u_1, u_3)$, $\lambda^\top = (\lambda_4, \lambda_6, \lambda_8, \lambda_{10})$, associated with a curve

$$V = \{(t, s) \in \mathbb{C}^2 \ : \ t^2 = s^5 + \lambda_4 s^3 + \lambda_6 s^2 + \lambda_8 s + \lambda_{10}\}.$$

We assume that $\lambda \in \mathbb{C}^4$ is a non-discriminant point of the curve V and $u \in \mathbb{C}^2$, where $du_1 = \frac{sds}{t}$, $du_3 = \frac{ds}{t}$. Indexation of the coordinates of the vector of the parameter λ is chosen according to the graduation $\deg s = -2$, $\deg t = -5$, $\deg \lambda_{2i} = -2i$. Moreover, $\deg u_1 = 1$, $\deg u_3 = 3$, and the sigma-function

$$\sigma(u, \lambda) = u_3 - \frac{1}{3}\, u_1^3 + \frac{1}{6}\, \lambda_6 u_3^3 + (u^5)$$

is an entire and homogeneous of degree 3 in u and λ. Recurrent description of the series for $\sigma(u, \lambda)$ is given in [11].

We consider the Jacobian $J_2 = \mathrm{Jac}(V) = \mathbb{C}^2/\Gamma_2$ of the curve V and a vector-function is defined by the formulae

$$\wp(u) = \big(\wp_{33}(u), \wp_{13}(u), \wp_{11}(u)\big),$$

where $\wp_{kl}(u) = -\frac{\partial^2}{\partial u_k \partial u_l} \ln \sigma(u),\ k,l = 1,3$. Since the function $\sigma(u)$ is odd, we get a mapping

$$i\colon J_2 \longrightarrow \mathbb{C}P^3 \ :\ i(u) = (x_0 : x_2 : x_4 : x_6),$$

where $x_0 = \sigma(u)^2\wp_{33}(u),\ x_2 = \sigma(u)^2\wp_{13}(u),\ x_4 = \sigma(u)^2\wp_{11}(u),\ x_6 = \sigma(u)^2$. The last mapping factorizes through the Kummer variety K together with an embedding

$$\widehat{i}\colon K = (J_2/_\pm) \longrightarrow \mathbb{C}P^3.$$

Let us note, that the embedding $\widehat{i}$ is defined with entire homogeneous functions $x_{2k}(u),\ \deg x_{2k} = 2k,\ k = 0, \dots, 3$.

We are going to use a ramified covering

$$\gamma\colon \left(\mathbb{C}P^3\right)^2 \longrightarrow \mathbb{C}P^6,$$

defined by the relation

$$[p(x,t), p(y,t)] \longrightarrow p(x,t)p(y,t),$$

where $p(x,t) = x_0t_3^3 + x_2t_3^2t_1 + x_4t_3t_1^2 + x_6t_1^3$. By putting $\deg t_k = k$, we get $p(x,t)$ as a homogeneous polynomial of degree 9.

Theorem 2 *Multiplication m in the two-valued group on K*

$$m\colon K \times K \longrightarrow (K)^2 \ :\ [u] * [u] = \big([u+v], [u-v]\big)$$

is defined through algebraic mapping

$$\mu\colon \mathbb{C}P^3 \times \mathbb{C}P^3 \longrightarrow \mathbb{C}P^6 \ :$$

and it is uniquely defined by the commuting condition of the following diagram

$$\begin{array}{ccccc} K \times K & \xrightarrow{m} & (K)^2 & \xrightarrow{(\widehat{i})^2} & \left(\mathbb{C}P^3\right)^2 \\ \big\downarrow{\scriptstyle \widehat{i}\times\widehat{i}} & & & & \big\downarrow{\scriptstyle \gamma} \\ \mathbb{C}P^3 \times \mathbb{C}P^3 & & \xrightarrow{\qquad\mu\qquad} & & \mathbb{C}P^6 \end{array}$$

Proof. Set

$$X(u) = \big(\wp_{33}(u), \wp_{31}(u), \wp_{11}(u), 1\big) \qquad \mathcal{X}(u) = \sigma(u)^2 X(u).$$

Consider the canonical projection

$$\pi\colon \mathbb{C}^7 \backslash 0 \longrightarrow \mathbb{C}P^6 \ :\ \pi(z) = (z_0 : z_2 : \ldots : z_{12}) = [z].$$

According to the construction, we have

$$\widehat{i}\,[u] = [\mathcal{X}(u)]\,, \quad \gamma(\,\widehat{i}\,)^2\big([u] * [v]\big) = \gamma\big(\,[\mathcal{X}(u+v), \mathcal{X}(u-v)]\,\big).$$

Thus, we have to show that each coordinate $z_{2k}(u,v),\ k=0,\dots,6$, of the point $\gamma\big([\mathcal{X}(u+v),\mathcal{X}(u-v)]\big)$ is a polynomial of the coordinates $x_{2i},\ y_{2i},\ i=0,\dots,3$ of the points $\mathcal{X}(u)\ \ \mathcal{X}(v)$.

Genus two sigma-function $\sigma(u)$ satisfies the following addition theorem (see [10], [11]):

$$\sigma(u+v)\sigma(u-v)=\mathcal{X}(u)^\top\mathcal{J}\,\mathcal{X}(v).$$

where $\mathcal{J}=\begin{pmatrix}0 & -\mathcal{E}\\ \mathcal{E} & 0\end{pmatrix}$ and $\mathcal{E}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. We have,

$$z_{12}=z_{12}(u,v)=\big(\mathcal{X}(u)^\top\mathcal{J}\,\mathcal{X}(v)\big)^2.$$

Thus, we get that in the mapping $\mu(x,y)=[z]=(z_0:z_2:\ldots:z_{12})\in\mathbb{C}P^6$, the coordinate z_{12} is defined by the formula $z_{12}=(x^\top\mathcal{J}y)^2$, where $(x,y)\in\mathbb{C}P^3\times\mathbb{C}P^3$.

Analogous result for the rest of the coordinates is based on deep facts about algebraic generators of the ring generated by logarithmic derivatives of order 2 and higher of the sigma-function $\sigma(u)$ (see [10]).

Set

$$M(u,v)=X(u)^\top\mathcal{J}\,X(v).$$

We have

$$\ln\sigma(u+v)+\ln\sigma(u-v)=2\big(\ln\sigma(u)+\ln\sigma(v)\big)+\ln M(u,v).$$

Apply the operators $\partial_k=\frac{\partial}{\partial u_k}+\frac{\partial}{\partial v_k}$, for $k=1$ and 3. We get

$$2\zeta_k(u+v)=2\big(\zeta_k(u)+\zeta_k(v)\big)+\frac{1}{M(u,v)}\left(X_k(u)^\top\mathcal{J}\,X(v)+X(u)^\top\mathcal{J}\,X_k(v)\right),$$

where $\zeta_k(u)=\frac{\partial}{\partial u_k}\ln\sigma(u)\ \ X_k(u)=\frac{\partial}{\partial u_k}X(u)$. Now, we apply the operator $\frac{\partial}{\partial u_l}$, and we get

$$\wp_{kl}(u+v)=-\frac{\partial}{\partial u_l}\,\zeta_k(u+v)=\varphi_{kl}(u,v)-\psi_{kl}(u,v),$$

where

$$\varphi_{kl}(u,v)=\wp_{kl}(u)-\frac{1}{2M(u,v)^2}\left\{\big(X_{kl}(u)^\top\mathcal{J}\,X(v)\big)M(u,v)-X_k(u)^\top B(v)X_l(u)\right\},$$

$$\psi_{kl}(u,v)=\frac{1}{2M(u,v)^2}\big(X_l(u)^\top C(u,v)X_k(v)\big)$$

and $B(v)=\mathcal{J}\,X(v)X(v)^\top\mathcal{J}^\top,\quad C(u,v)=\big(M(u,v)-\mathcal{J}\,X(v)X(u)^\top\big)\mathcal{J}$.

Note $B(-v)=B(v)\ \ C(-u,v)=C(u,-v)=C(u,v)$. Thus, $\varphi_{kl}(u,-v)=\varphi_{kl}(u,v)\ \ \psi_{kl}(u,-v)=-\psi_{kl}(u,v)$. We get

$$\mathcal{X}(u+v)=\sigma(u+v)^2(X_1-X_2)\qquad\mathcal{X}(u-v)=\sigma(u-v)^2(X_1+X_2),$$

where $X_1 = (\varphi_{33}, \varphi_{13}, \varphi_{11}, 1)$ $X_2 = (\psi_{33}, \psi_{13}, \psi_{11}, 0)$. We obtain

$$p\big(\mathcal{X}(u+v), t\big) p\big(\mathcal{X}(u-v), t\big) = z_1 \cdot \big(p(X_1, t)^2 - p(X_2, t)^2\big)$$

where

$$\begin{aligned} p(X_1, t) &= \varphi_{33} t_3^3 + \varphi_{13} t_3^2 t_1 + \varphi_{11} t_3 t_1^2 + t_1^3, \\ p(X_2, t) &= t_3(\psi_{33} t_3^2 + \psi_{13} t_3 t_1 + \psi_{11} t_1^2). \end{aligned}$$

From the above formulae for $\varphi_{kl}(u, v)$ $\psi_{kl}(u, v)\psi_{pq}(u, v)$, it immediately follows that these functions are polynomials of $\wp_{ij}(u)$, $\wp_{ijk}(u)\wp_{i'j'k'}(u)$, $\wp_{ijpq}(u)$ $\wp_{ij}(v)$, $\wp_{ijk}(v)\wp_{i'j'k'}(v)$, $\wp_{ijpq}(v)$. The functions $\wp_{ijk}(u)\wp_{i'j'k'}(u)$ $\wp_{ijpq}(u)$, where $i, j, k, i', j', k', p, q$ take values 1 or 3 independently, are polynomials of $\wp_{ij}(u)$ (see [10]). Consequently, using $(x, y)|_{\widehat{i}\times\widehat{i}} = \big(\mathcal{X}(u), \mathcal{X}(v)\big)$, we get coordinates z_{2k} of the vector $\mu(x, y)$ as polynomials of the coordinates of the vectors x and y. □

7 Homomorphism of rings of functions, induced by Abel mapping in genus 2

Let $V = \big\{(s, \mu) \in \mathbb{C}^2 \ : \ \mu^2 = s^5 + \lambda_4 s^3 + \lambda_6 s^2 + \lambda_8 s + \lambda_{10}\big\}$ denotes a hyperelliptic curve.

Theorem 3 *The Abel mapping*

$$\mathcal{A}\colon (V)^2 \longrightarrow \mathrm{Jac}V$$

induces a homomorphism of rings of functions

$$\mathcal{A}^*\colon \mathcal{F}(\mathrm{Jac}V) \longrightarrow \mathcal{F}\big((V)^2\big)$$

such that

$$\wp_{11}(u) = s_1 + s_2, \quad \wp_{13}(u) = -s_1 s_2, \quad \wp_{33}(u) = \frac{F(s_1, s_2) - 2\mu_1\mu_2}{(s_1 - s_2)^2},$$

where

$$F(s_1, s_2) = 2\lambda_{10} + \lambda_8(s_1 + s_2) + s_1 s_2\big(2\lambda_6 + \lambda_4(s_1 + s_2)\big),$$

$$\wp_{111}(u) = 2\frac{\mu_1 - \mu_2}{s_1 - s_2}, \quad \wp_{113}(u) = 2\frac{s_1\mu_2 - s_2\mu_1}{s_1 - s_2}$$

$$\wp_{331}(u) = -2\frac{s_1^2\mu_2 - s_2^2\mu_1}{s_1 - s_2}, \quad \wp_{333}(u) = 2\frac{\psi(s_1, s_2)\mu_2 - \psi(s_2, s_1)\mu_1}{(s_1 - s_2)^3},$$

and

$$\psi(s_1, s_2) = 4\lambda_{10} + \lambda_8(3s_1 + s_2) + 2\lambda_6 s_1(s_1 + s_2) + \lambda_4 s_1^2(s_1 + 3s_2) + s_1^3 s_2(3s_1 + s_2).$$

The proof of the Theorem can be found in [10]. Let us note that we use the indexation here different from [10], in correspondence with the graduation:

$$\deg s = -2\,;\ \deg \mu = -5\,;\ \deg \lambda_{2k} = -2k,\ k = 2,3,4,5\,;\ \deg u_i = i,\ i = 1,3.$$

This provides additional opportunity to check the formulae. Observe that $\deg \wp_{kl}(u) = -(k+l),\ \deg \wp_{klp}(u) = -(k+l+p)$.

Any Abelian function on Jac$V$ represents a linear function of $\wp_{111}(u)$ with coefficients which are rational functions of $\wp_{11}(u)$ and $\wp_{13}(u)$. On the other hand, from the theory of polysymmetric functions (see [23] and [12]), it is known that the field of rational functions on $(\mathbb{C}^2)^2$ in the coordinates $[(s_1,\mu_1),(s_2,\mu_2)]$ is generated by polysymmetric functions

$$e_{10} = s_1+s_2,\ e_{20} = s_1s_2,\ e_{01} = \mu_1+\mu_2,\ e_{02} = \mu_1\mu_2,\ e_{11} = s_1\mu_2+s_2\mu_1,$$

which are related by unique (for $(\mathbb{C}^2)^2$) relation

$$(e_{10}^2-4e_{20})(e_{01}^2-4e_{02}) = (e_{10}e_{01}-2e_{11}).$$

The mapping $\mathcal{A}$ is a birational equivalence, thus $\mathcal{A}^*$ induces isomorphism of the field of Abelian functions $\mathcal{F}(\mathrm{Jac}V)$ on Jac$V$ with the field of rational functions $\mathcal{F}\left((V)^2\right)$ on $(V)^2$. We have:

$$\mathcal{A}^*(\wp_{11}(u)) = e_{10}, \quad \mathcal{A}^*(\wp_{13}(u)) = -e_{20}, \quad \mathcal{A}^*(\wp_{111}(u)) = \frac{e_{10}e_{01}-2e_{11}}{e_{10}^2-4e_{20}}.$$

In this way, the Theorem 3 completely describes the isomorphism $\mathcal{A}^*$ and gives an opportunity to express explicitly, for example, the even functions $\wp_{klp}(u)\wp_{k'l'p'}(u)$ as polynomials of $\wp_{kl}(u)$. The explicit formulae for those polynomials are given in the book [10].

The hyperelliptic involution acts on $(V)^2$ according to the formula

$$[(s_1,\mu_1),(s_2,\mu_2)] \longrightarrow [(s_1,-\mu_1),(s_2,-\mu_2)]\,.$$

The Abel mapping is invariant with respect to this involution on $(V)^2$ and the involution $u \to -u$ on Jac$V$.

From the above formulae one can see that the images of the functions $\wp_{kl}(u)$ and $\wp_{klp}(u)\wp_{k'l'p'}(u)$ are even, while the images of the functions $\wp_{klp}(u)$ are odd. Thus, the mapping

$$\widehat{\mathcal{A}}\colon (V)^2/_{\pm} \longrightarrow K = (\mathrm{Jac}V)/_{\pm},$$

is defined and it induces a homomorphism between rings of functions.

There is an addition law on $(V)^2$, with the Abel mapping $\mathcal{A}$ as a homomorphism. Explicit form of this operation in the coordinates $[(s_1,\mu_1),(s_2,\mu_2)]$ has been described in [11].

Thus, on $(V)^2/_{\pm}$ there is corresponding two-valued addition, such that $\widehat{\mathcal{A}}$ is a homomorphism with respect to the two-valued group structure on the Kummer variety K, defined above.

8 Solutions of the system of equations of S. V. Kowalevski in genus two $\wp$-functions

We follow chapter IV of the Golubev book [24].

Kowalevski introduced variables $s_1,\ s_2$, which satisfy the system of equations (see equations (17) and(18) from [24]:

$$\frac{ds_1}{\sqrt{\Phi(s_1)}} + \frac{ds_2}{\sqrt{\Phi(s_2)}} = 0, \tag{12}$$

$$\frac{s_1 ds_1}{\sqrt{\Phi(s_1)}} + \frac{s_2 ds_2}{\sqrt{\Phi(s_2)}} = \frac{i}{2}\, dt, \tag{13}$$

where $\Phi(s)$ is a polynomial of fifth degree. The Abel mapping

$$\mathcal{A}\colon (V)^2 \longrightarrow \mathrm{Jac}V$$

where $V = \big\{(s,\mu) \in \mathbb{C}^2\ :\ \mu^2 = \Phi(s)\big\}$ $\Phi(s) = s^5 + \lambda_4 s^3 + \lambda_6 s^2 + \lambda_8 s + \lambda_{10}$, is defined by the system of equations

$$\frac{ds_1}{\mu_1} + \frac{ds_2}{\mu_2} = du_3, \tag{14}$$

$$\frac{s_1 ds_1}{\mu_1} + \frac{s_2 ds_2}{\mu_2} = du_1. \tag{15}$$

We are going to describe solutions of the system of equations (12), (13), following the book [10].

Let us consider the sigma-function $\sigma(u) = \sigma(u,\lambda)$, associated with the curve V, and corresponding Abelian functions

$$\wp_{k,l}(u) = -\frac{\partial}{\partial u_k \partial u_l} \ln \sigma(u),\ k = 1, 3.$$

The general solution of the system of equations (14), (15), which is the solution of the Jacobi inversion problem for the Abel mapping, is represented by the point $[(s_1,\mu_1),(s_2,\mu_2)]$, where (s_1,s_2) are the solutions of the equation

$$s^2 - \wp_{11}(u)s - \wp_{13}(u) = 0, \tag{16}$$

and for given s_l we have

$$2\mu_l = \wp_{111}(u)s_l + \wp_{113}(u),\ l = 1, 2. \tag{17}$$

Thus, we get the solution $\big[\big(s_1(t),\mu_1(t)\big),\big(s_2(t),\mu_2(t)\big)\big] \in (V)^2$ of the Kowalevski system of equations (12), (13) in terms of genus 2 $\wp$-functions, where $\big(s_l(t),\mu_l(t)\big) = \big(s_l(t,c_1,c_3),\mu_l(t,c_1,c_3)\big),\ l = 1, 2,$ is the solution of the Jacobi problem for

$u = (u_1, u_3) \in \mathrm{Jac}V$, where $u_1 = u_1(t) = c_1 + it$, $u_3 = c_3$ and c_1, c_3 are constants.

The solution of the system (12), (13) based on the sigma - functions, unlike the classical solutions in terms of the theta-functions, is stable under the limit procedure when $\lambda \to 0$.

Without changing the notation, let us consider corresponding functions, associated with the curve $V_0 = \left\{(s,\mu) \in \mathbb{C}^2 \ : \ \mu^2 = s^5\right\}$. We have

$$\begin{aligned}
\sigma(u) &= u_3 - \frac{1}{3}\, u_1^3, \\
\sigma(u)^2 \wp_{11}(u) &= 2u_1 \left(u_3 + \frac{1}{6}\, u_1^3 \right), \\
\sigma(u)^2 \wp_{13}(u) &= -u_1^2.
\end{aligned}$$

Thus, for $\sigma(u) \neq 0$, the equation (16) with $\lambda \to 0$ is equivalent to

$$\widehat{s}^{\,2} - 2u_1 \left(u_3 + \frac{1}{6}\, u_1^3 \right) \widehat{s} + u_1^2 \sigma(u)^2 = 0, \tag{18}$$

where $\widehat{s} = \sigma(u)^2 s$. Further,

$$\begin{aligned}
\sigma(u)^3 \wp_{111}(u) &= \sigma(u)^3 \frac{\partial}{\partial u_1}\, \wp_{11}(u) = 2 \left[u_3^2 + \frac{7}{3}\, u_1^3 u_3 + \frac{1}{9}\, u_1^6 \right], \\
\sigma(u)^3 \wp_{113}(u) &= \sigma(u)^3 \frac{\partial}{\partial u_1}\, \wp_{13}(u) = -2u_1 \left(u_3 + \frac{2}{3}\, u_1^3 \right).
\end{aligned}$$

In this way, for $\sigma(u) \neq 0$, the formula (17) with $\lambda \to 0$ is equivalent to

$$\widehat{\mu}_l = \left[u_3^2 + \frac{7}{3}\, u_1^3 u_3 + \frac{1}{9}\, u_1^6 \right] \widehat{s}_l - u_1 \left(u_3 + \frac{2}{3}\, u_1^3 \right) \sigma(u)^2, \tag{19}$$

where $\widehat{\mu}_l = \sigma(u)^5 \mu_l$. Observe, that $\widehat{\mu}_l^2 = \widehat{s}^{\,5}$.

Thus, the general solution of the inversion problem $[(s_1,\mu_1),(s_2,\mu_2)] \in (V_0)^2$, where $V_0 = \left\{(s,\mu) \in \mathbb{C}^2 \ : \ \mu^2 = s^5\right\}$, has the form

$$\sigma(u)^2 s_{1,2} = u_1 \left[\left(u_3 + \frac{1}{6}\, u_1^3 \right) \pm \sqrt{u_1^3 \left(u_3 - \frac{1}{12}\, u_1^3 \right)} \right], \tag{20}$$

and $\widehat{\mu}_l = \sigma(u)^5 \mu_l$ are given by the formula (19).

Let us consider important particular cases:

1. Let $u_1 = 0$, then $\sigma(u) = u_3$ and

$$s_1 = s_2 = 0, \quad \mu_1 = \mu_2 = 0.$$

2. Let $u_3 = 0$, then $\sigma(u) = -\frac{1}{3}\, u_1^3$ and

$$\widehat{s}_{1,2} = \frac{1}{6} \left(1 \pm i\sqrt{3} \right) u_1^4, \quad \widehat{\mu}_{1,2} = \frac{1}{18} \left(-1 \pm \frac{i}{\sqrt{3}} \right) u_1^{10}.$$

Check: the identity holds

$$\frac{1}{18^2}\left(-1+\frac{i}{\sqrt{3}}\right)^2=\frac{1}{6^5}\left(1+i\sqrt{3}\right)^5.$$

By taking $u_1 = c_1 + \frac{i}{2}t$ and $u_3 = 0$, we get particular solution of the Kowalevski system in the rational limit.

3. Let $u_3 + \frac{2}{3}\,u_1^3 = 0$, then $\sigma(u) = u_1^3$

$$\widehat{s}_{1,2}=\frac{1}{2}\left(-1\pm i\sqrt{3}\right)u_1^4, \quad \widehat{\mu}_l=-\widehat{s}_l u_1^6.$$

Check: the identity holds $\left(-1\pm i\sqrt{3}\right)^3=8$.

4. Let $u_3 = \frac{1}{12}\,u_1^3$, then $\sigma(u) = -\frac{1}{4}u_1^3$

$$\widehat{s}_{1,2}=\frac{1}{2^2}\,u_1^4, \quad \widehat{\mu}_{1,2}=\frac{1}{2^5}\,u_1^{10}.$$

9 Geometric two-valued group laws and Kummer varieties

9.1 A quadric in $\mathbb{C}P^5$ and a line complex in $\mathbb{C}P^3$

Following classics, let us consider a three-dimensional projective space $\mathbb{C}P^3 = P(\mathbb{C}^4)$ and corresponding Grassmannian $Gr(2,4)$ of all lines in $\mathbb{C}P^3$. By Plücker embedding, the Grassmannian $Gr(2,4)$ can be realized as a quadric G in $\mathbb{C}P^5 = P(\wedge^2 V)$, where $V = \mathbb{C}^4$:

$$Gr(2,4) \hookrightarrow \mathbb{C}P^5$$
$$\ell = \ell < v_1, v_2 > \mapsto v_1 \wedge v_2, \quad v_1, v_2 \in V^4.$$

The quadric G parameterizes all decomposable elements $w = v_1 \wedge v_2$ of $P(\wedge^2 V)$ and the quadric is described by the Plucker quadratic relation:

$$G : w \wedge w = 0.$$

For a given element $x \in G$, denote by ℓ_x the line in $\mathbb{C}P^3$ which maps to x by the above embedding.

We consider so-called Schubert cycles:

$$\begin{aligned}\sigma_1(\ell) &= \{x \in G \mid \ell_x \cap \ell \neq \emptyset\}\\ \sigma_2(p) &= \{x \in G \mid \ell_x \ni p\}\\ \sigma_{1,1}(h) &= \{x \in G \mid \ell_x \subset h\}\\ \sigma_{2,1}(p,h) &= \{x \in G \mid p \in \ell_x \subset h\}\end{aligned}$$

with the intersection table

$$\begin{aligned}\sigma_1 \cdot \sigma_1 &= \sigma_2 + \sigma_{1,1}\\ \sigma_1 \cdot \sigma_2 &= \sigma_1 \cdot \sigma_{1,1} = \sigma_{2,1}\\ \sigma_2 \cdot \sigma_2 &= \sigma_{1,1} \cdot \sigma_{1,1} = \sigma_1 \cdot \sigma_{2,1} = 1\\ \sigma_2 \cdot \sigma_{1,1} &= 0.\end{aligned}$$

One can easily see that every cycle $\sigma_1(\ell)$ is a hyperplane section of the quadric G. If the line $\ell \in \mathbb{C}P^3$ is determined by vectors $v_1, v_2 \in V^4$ then the hyperplane of intersection is of the form $H_{v_1 \wedge v_2} = \{w \mid w \wedge v_1 \wedge v_2 = 0\}$.

Every cycle $\sigma_{2,1}(p,h)$ is a line in $\mathbb{C}P^5$. Every line $L \subset G$ is of the form $L = \sigma_{2,1}(p,h)$.

A line $L \subset G$ is a pencil of lines in $\mathbb{C}P^3$. This is a confocal pencil with the common point, *the focus* $p \in \mathbb{C}P^3$. At the same time this pencil is coplanar with the common plane $h \in \mathbb{C}P^{3*}$.

Every cycle of the form $\sigma_2(p)$ or of the form $\sigma_{1,1}(h)$ is a two-plane in $\mathbb{C}P^5$. Conversely, every two-plane in G is of the form $\sigma_2(p)$ or of the form $\sigma_{1,1}(h)$.

Let us recall some general properties of a quadric Q in $\mathbb{C}P^m$. The rank of the quadric is equal to the rank of any of its symmetric $(m+1) \times (m+1)$ matrices. The quadric is smooth if its rank is maximal, i.e. if rank of Q is $m+1$. If the rank of Q is r then it is a cone over a smooth quadric in $\mathbb{C}P^{r-1}$ with a vertex $\mathbb{C}P^{m-r}$. Quadrics of rank $m+1$ and m are called *general*.

Lemma 1 *Let $Q \in \mathbb{C}P^{2m+1}$ be a general quadric.*

(a) The dimension of a maximal linear subspace of Q is equal to m.

(b) The collection of maximal linear subspaces $C(Q)$ forms an algebraic variety of dimension $m(m+1)/2$.

(c) If the rank of Q is even, then $C(Q)$ has two components. Otherwise, the component is unique. A component is unirational variety.

(d) Let $L \subset Q$ be a linear subspace of dimension $m-1$ which does not contain the vertex of Q. For an irreducible family of maximal subspaces $A \in C(Q)$, there is a unique maximal subspace $M = M(L,A)$ of dimension m which belongs to A and contains L.

(e) Suppose that Q is smooth and let M_1, M_2 be two of its maximal linear subspaces. Then

$$\dim M_1 \cap M_2 \equiv m \,(\mod 2)$$

is equivalent to the fact that M_1, M_2 belong to the same irreducible component of $C(Q)$.

Now, we specialize previous statement for the case of a smooth quadric in $\mathbb{C}P^5$.

Proposition 2 *Let G be a smooth quadric in $\mathbb{C}P^5$. Then:*

(a) There exists a four-dimensional vector space V^4 such that G is the Plucker embedding of Grassmannian of lines in $P(V^4)$.

(b) Maximal linear subspaces of G are two-dimensional and they are of the form $\sigma_2(p)$ or of the form $\sigma_{1,1}(h)$, where $p \in \mathbb{C}P^3 = P(V^4)$ and $h \in \mathbb{C}P^{3*}$.

(c) Variety $C(Q)$ of all two-dimensional subspaces of G is three-dimensional and it has two irreducible components, A and B:

$$A = \{\sigma_2(p) \mid p \in \mathbb{C}P^3 = P(V^4)\}$$
$$B = \{\sigma_{1,1}(h) \mid h \in \mathbb{C}P^{3*}\}.$$

9.2 Smooth intersection of two quadrics and Abelian varieties

Now we consider the intersection X of two quadrics G and F in $\mathbb{C}P^5$. Such a set is classically called quadratic complex of lines, if G is understood as a Grassmannian of lines in some $\mathbb{C}P^3$. Together with two quadrics G and F, one may consider the whole *pencil of quadrics*:

$$F_\lambda := F + \lambda G,$$

and X is the base set for the pencil, the common intersection of quadrics from the pencil.

A pencil of quadrics is *generic* if associated pencil of 6×6 symmetric matrices contains six different singular matrices. For a generic pencil F_λ denote by $\lambda_1, \dots, \lambda_6$ corresponding values of the pencil parameter associated with singular matrices.

The condition that X is smooth is equivalent to the condition that the pencil F_λ is generic. Smoothness of X is also equivalent to the fact that all quadrics F_λ are general and that exactly six of them, $F_{\lambda_1}, \dots, F_{\lambda_6}$ are singular.

Proposition 3 *Suppose X is smooth intersection of two quadrics $X = G \cap F$ in $\mathbb{C}P^5$. Then:*

(a) Maximal linear subspaces of X are one-dimensional.

(b) There is a maximal linear subspace through each point of X.

(c) There are four maximal linear subspaces passing through a generic point of X.

Given a smooth intersection of quadrics $X = G \cap F$ in $\mathbb{C}P^5$, following Narasimhan, Ramanan, Reid and Donagi, let us consider the set of one-dimensional linear subspaces

$$\mathcal{A}(X) = \{L \mid L \in X \cap Gr(2,6)\}.$$

Suppose G is realized as a Grassmannian of lines in some $\mathbb{C}P^3$, and denote as before the two components of $C(G)$ of two-dimensional linear subspaces of G as

$$A = \{\sigma_2(p) \mid p \in \mathbb{C}P^3 = P(V^4)\}$$
$$B = \{\sigma_{1,1}(h) \mid h \in \mathbb{P}^{3*}\}.$$

Lemma 2 *Given $L \in \mathcal{A}(X)$. Then:*

(a) There is a unique two-dimensional linear subspace of G, $\sigma_2(p) \in A$ such that $L \subset \sigma_2(p)$. There is a unique one-dimensional linear subspace of X

$$L_1 \in \mathcal{A}(X),$$

such that

$$\sigma_2(p) \cap F = L \cup L_1.$$

(b) There is a unique two-dimensional linear subspace of G, $\sigma_{1,1}(h) \in A$ such that $L \subset \sigma_{1,1}(h)$. There is a unique one-dimensional linear subspace of X

$$L_2 \in \mathcal{A}(X),$$

such that

$$\sigma_{1,1}(h) \cap F = L \cup L_2.$$

The last Lemma introduces two involutions

$$i_1 : \mathcal{A}(X) \to \mathcal{A}(X)$$
$$i_1 : L \mapsto L_1$$
$$i_2 : \mathcal{A}(X) \to \mathcal{A}(X)$$
$$i_2 : L \mapsto L_2.$$

As pencils of lines in $\mathbb{C}P^3$ the subspaces L and L_1 are confocal, they have the same focus p. In the same manner, the subspaces L and L_2 are coplanar, they have the same plane h.

Moreover, there are two mappings

$$\begin{aligned}
k_1 :& \mathcal{A}(X) \to \mathbb{C}P^3 \\
k_1 :& L \mapsto p \\
k_2 :& \mathcal{A}(X) \to \mathbb{C}P^{3*} \\
k_2 :& L \mapsto h.
\end{aligned}$$

The mapping k_1 maps a pencil L to its focus in $\mathbb{C}P^3$ while k_2 maps a pencil to its plane in $\mathbb{C}P^{3*}$.

Denote by $K \subset \mathbb{C}P^3$ the image of $\mathcal{A}(X)$ by k_1. We see that k_1 is a double covering of $\mathcal{A}(X)$ over K and that the involution i_1 interchanges the leaves of the covering. We are going to call K *the Kummer variety* of $\mathcal{A}(X)$. It is associated to the choice of a quadric G from the pencil and to the choice of a connected component A of $C(G)$.

Similarly, denote by $K^* \subset \mathbb{C}P^{3*}$ the image of $\mathcal{A}(X)$ by k_2. The mapping k_2 is a double covering of $\mathcal{A}(X)$ over K and the involution i_2 interchanges the leaves of the covering. We are going to call K^* *the dual Kummer variety* of $\mathcal{A}(X)$. It is associated to the choice of a quadric G from the pencil and to the choice of a connected component B of $C(G)$.

It can be shown that K^* is dual to K, which means that every plane from K^* is tangent to K. From the previous considerations it can also be seen that the degree of K is equal to 4.

For a general point $p \in \mathbb{C}P^3$ the plane $\sigma_2(p) \subset G$ intersects F along a smooth conic. The Kummer variety can be described as a set of the points in $\mathbb{C}P^3$ for which this intersection is a conic which is not smooth. For general point of K this intersection is a degenerate conic which is a union of two different lines L and $i_1(L)$. But, there is a subset $R \in K$ of sixteen points, for which this intersection is a degenerate conic of double line. These sixteen points from R correspond to the fixed points of the involution i_1 and to the ramification points of the double covering. We also denote $R^* \subset K^*$ the set of points that correspond to the fixed points of the involution i_2.

9.3 Pencils of quadrics, hyperelliptic curves and geometric group laws

With a generic pencil of quadrics F_λ in $\mathbb{C}P^5$ with six singular quadrics $F_{\lambda_1}, \dots, F_{\lambda_6}$ one may associate a genus two curve

$$\Gamma : y^2 = \prod_{i=1}^{6}(x - \lambda_i).$$

As it was shown by Reid and Donagi, this correspondence between a generic pencil of quadrics and a hyperelliptic curve is not just formal. After Donagi, denote by E the family of all connected components of $C(F_\lambda)$ of all quadrics from the pencil together with a projection

$$p : E \to \mathbb{C}P^1$$

which maps a given irreducible family of two-subspaces of a quadric F_μ to the value μ of the pencil parameter. The projection p is obviously double covering. It is ramified over the points $\lambda_i, i = 1, \ldots, 6$ since the singular quadrics are the only one with unique component of maximal linear subspaces. Thus, Donagi showed the isomorphism between E and Γ.

But, there is yet another natural realization of the hyperelliptic curve Γ in the context of the pencil F_λ, as it was demonstrated by Reid.

For $L \in \mathcal{A}(X)$ denote by $\mathcal{A}_L(X)$ the closure of the set $\{L' \in \mathcal{A}(X) \mid L \cap L' \neq \emptyset\}$. There is a natural projection

$$q : \mathcal{A}_L(X) \setminus \{L\} \to \mathbb{C}P^1$$

which maps L' to the parameter μ of a quadric F_μ if the space $< L, L' >$ belongs to $C(F_\mu)$. The mapping q is double covering ramified over the six points $\lambda_i, i = 1, \ldots, 6$ and $\mathcal{A}_L(X)$ is isomorphic to the hyperelliptic curve Γ. The natural involution on $\mathcal{A}_L(X)$ which interchanges the folds of q will be denoted by τ_L.

Moreover, as it was shown by Reid and Donagi, $\mathcal{A}(X)$ is an Abelian variety, isomorphic to the Jacobian of the curve Γ.

We will refer to the curves $\mathcal{A}_L(X)$ and E as *Donagi-Reid-Knörrer curves* (DRK) associated with intersection of quadrics X of a generic pencil.

It can easily be shown that for any hyperelliptic curve Γ, there exists a pencil of quadrics with the base set X such that $\mathcal{A}(X)$ is isomorphic to the Jacobian of Γ.

The addition laws on the Abelian varieties $\mathcal{A}(X)$ are such that

$$L_1 + L_2 = M_1 + M_2$$

if there exists μ such that

$$< L_1, L_2 >, < M_1, M_2 > \in C(F_\mu)$$

and if the two two-dimensional spaces $< L_1, L_2 >, < M_1, M_2 >$ belong to the same component of $C(F_\mu)$.

Proposition 4 *Every point $e \in E$ determines its Kummer variety involution i_e and its Kummer variety K_e. The hyperelliptic involution on E, τ interchanges a Kummer variety and its dual:*

$$K_{\tau(e)} = K_e^*.$$

We will fix a point $e_0 \in E$ and a line $L_0 \in \mathcal{A}(X)$ as the origin of a group structure on $\mathcal{A}(X)$ such that

$$L_0 + \hat{L}_0 = 0,$$

whenever

$$< L_0, \hat{L}_0 > \in e_0.$$

As above, denote by τ_{L_0} the natural involution on $\mathcal{A}_{L_0}(X)$. Then we have

Lemma 3 *The involutions τ_{L_0} and i_{e_0} are related according to the formula*

$$i_{e_0}|_{\mathcal{A}_{L_0}(X)} = \tau_{L_0}.$$

Now, we are ready to define a two-valued group structure on the Kummer variety K_{e_0}. We will define a mapping

$$\star : K_{e_0} \times K_{e_0} \longrightarrow (K_{e_0})^2$$

by the following procedure.

Take a pair

$$(p_1, p_2) \in K_{e_0} \times K_{e_0}.$$

Denote by

$$[L_1] = \{L_1, \hat{L}_1\}, \quad L_1, \hat{L}_1 \in \mathcal{A}(X)$$

the class of lines in $\mathcal{A}(X)$ which represents the two confocal pencils of lines in $\mathbb{C}P^3$ with the focal point p_1. Similarly, denote by

$$[L_2] = \{L_2, \hat{L}_2\}, \quad L_2, \hat{L}_2 \in \mathcal{A}(X)$$

the class of lines in $\mathcal{A}(X)$ which represents the two confocal pencils of lines in $\mathbb{C}P^3$ with the focal point p_2.

Assume that L_1, L_2 don't intersect L_0. Denote by N_1, N_2 the two lines in $\mathcal{A}_{L_0}(X)$ of intersection of the space $< L_0, L_1 >$ with X and by $e_1, e_2 \in E$ denote the classes determined by

$$< L_0, N_1 > \in e_1, \quad < L_0, N_2 > \in e_2.$$

Denote also by

$$\mu_1 = p(e_1), \quad \mu_2 = p(e_2),$$

and by

$$N_1', N_1'', N_2', N_2'' \in \mathcal{A}_{L_2}(X)$$

the lines which intersect L_2 and which are uniquely defined by the conditions

$$\begin{array}{ll} < N_1', L_2 > \in e_1, & < N_1'', L_2 > \in \tau(e_1) \\ < N_2', L_2 > \in e_1, & < N_2'', L_2 > \in \tau(e_2). \end{array}$$

In other words N_1', N_1'' belong to the two two-dimensional spaces of the different classes of $C(F_{\mu_1})$ which contain L_2; N_2', N_2'' belong to the two two-dimensional spaces of the different classes of $C(F_{\mu_2})$ which contain L_2.

Let M_1 be the fourth intersection line in the intersection of X with the space generated with L_2, N_1', N_2' and let M_2 be the fourth intersection line in the intersection of X with the space generated with L_2, N_1'', N_2''. The line M_1 represents a pencil of lines in $\mathbb{C}P^3$ with the focal point w_1 and M_2 represents a pencil of lines in $\mathbb{C}P^3$ with the focal point w_2.

If we repeat the above procedure with $\hat{L}_1, \hat{L}_2$ instead of L_1, L_2 we come to the lines $\hat{M}_1$ and $\hat{M}_2$ which are confocal with M_1 and M_2.

One can easily adjust previous construction to the case where L_1, L_2 intersect L_0: denote by M_1 the fourth line of the intersection of X with the space generated with $\hat{L}_1, \hat{L}_2, L_0$; denote by M_2 the fourth line of the intersection of X with the space generated with $L_1, \hat{L}_2, L_0$.

Thus we get

Theorem 4 *The mapping defined by the formulae*

$$\star : K_{e_0} \times K_{e_0} \longrightarrow (K_{e_0})^2$$
$$p_1 \star p_2 = (w_1, \, w_2)$$

defines a structure of two-valued group on the Kummer variety K_{e_0}.

Definition 2 *The mapping $\star$ defined by previous construction defines the geometric two-valued group law on the Kummer variety K_{e_0}.*

10 Integrable billiards and two-valued laws

10.1 Pencils of quadrics and billiards, an overview

We begin this Section by repeating basic definitions related to billiard systems of confocal quadrics from [19], [20].

Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two quadrics. Denote by u the tangent plane to $\mathcal{Q}_1$ at point x and by z the pole of u with respect to $\mathcal{Q}_2$. Suppose lines ℓ_1 and ℓ_2 intersect at x, and the plane containing these two lines meet u along ℓ.

Definition 3 If lines ℓ_1, ℓ_2, xz, ℓ are coplanar and harmonically conjugated, we say that rays ℓ_1 and ℓ_2 *obey the reflection law* at the point x of the quadric $\mathcal{Q}_1$ with respect to the confocal system which contains $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

If we introduce a coordinate system in which quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are confocal in the usual sense, reflection defined in this way is same as the standard one.

Theorem 5 (One Reflection Theorem) *Suppose rays ℓ_1 and ℓ_2 obey the reflection law at x of $\mathcal{Q}_1$ with respect to the confocal system determined by quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Let ℓ_1 intersects $\mathcal{Q}_2$ at y_1' and y_1, u is a tangent plane to $\mathcal{Q}_1$ at x, and z its pole with respect to $\mathcal{Q}_2$. Then lines $y_1'z$ and y_1z respectively contain intersecting points y_2' and y_2 of ray ℓ_2 with $\mathcal{Q}_2$. Converse is also true.*

Corollary 2 *Let rays ℓ_1 and ℓ_2 obey the reflection law of $\mathcal{Q}_1$ with respect to the confocal system determined by quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Then ℓ_1 is tangent to $\mathcal{Q}_2$ if and only if is tangent ℓ_2 to $\mathcal{Q}_2$; ℓ_1 intersects $\mathcal{Q}_2$ at two points if and only if ℓ_2 intersects $\mathcal{Q}_2$ at two points.*

Next assertion is crucial for applications to the billiard dynamics.

Theorem 6 (Double Reflection Theorem) *Suppose that $\mathcal{Q}_1$, $\mathcal{Q}_2$ are given quadrics and ℓ_1 line intersecting $\mathcal{Q}_1$ at the point x_1 and $\mathcal{Q}_2$ at y_1. Let u_1, v_1 be tangent planes to $\mathcal{Q}_1$, $\mathcal{Q}_2$ at points x_1, y_1 respectively, and z_1, w_1 their poles with respect to $\mathcal{Q}_2$ and $\mathcal{Q}_1$. Denote by x_2 second intersecting point of the line w_1x_1 with $\mathcal{Q}_1$, by y_2 intersection of y_1z_1 with $\mathcal{Q}_2$ and by ℓ_2, ℓ_1', ℓ_2' lines x_1y_2, y_1x_2, x_2y_2. Then pairs ℓ_1, ℓ_2; ℓ_1, ℓ_1'; ℓ_2, ℓ_2'; ℓ_1', ℓ_2' obey the reflection law at points x_1 (of $\mathcal{Q}_1$), y_1 (of $\mathcal{Q}_2$), y_2 (of $\mathcal{Q}_2$), x_2 (of $\mathcal{Q}_1$) respectively.*

Corollary 3 *If the line ℓ_1 is tangent to a quadric $\mathcal{Q}$ confocal with $\mathcal{Q}_1$ and $\mathcal{Q}_2$, then rays ℓ_2, ℓ_1', ℓ_2' also touch $\mathcal{Q}$.*

For the conclusion, we recall the notion of generalized Cayley's curve from [19], [20].

Definition 4 The *generalized Cayley curve* $\mathcal{C}_\ell$ is the variety of hyperplanes tangent to quadrics of a given confocal family in $\mathbb{C}P^d$ at the points of a given line ℓ.

This curve is naturally embedded in the dual space $\mathbb{C}P^{d\,*}$.

Proposition 5 *The generalized Cayley curve in $\mathbb{C}P^d$, for $d \geq 3$ is a hyperelliptic curve of genus $g = d - 1$. Its natural realization in $\mathbb{C}P^{d\,*}$ is of degree $2d - 1$.*

The natural involution τ_ℓ on the generalized Cayley's curve $\mathcal{C}_\ell$ maps to each other the tangent planes at the points of intersection of ℓ with any quadric of the confocal family.

It was observed in [19] that this curve is isomorphic to the Veselov-Moser isospectral curve.

Now we are going to mention a connection obtained in [20], between generalized Cayley's curve defined above and the curves (see the previous Section) studied by Knörrer, Donagi, Reid. This connetion traces out the relationship between billiard constructions and the algebraic structure of the corresponding Abelian varieties.

The famous Chasles theorem (see [2]) states that any line in the space $\mathbf{R}^d$ is tangent to exactly $d-1$ quadrics from a given confocal family:

$$\mathcal{Q}_\lambda : \; Q_\lambda(x) = 1. \tag{21}$$

where we denote:

$$Q_\lambda(x) = \frac{x_1^2}{a_1 - \lambda} + \cdots + \frac{x_d^2}{a_d - \lambda}.$$

We assume that the family (21) is generic, i.e. the constants $a_1, \ldots, a_d$ are all distinct.

Suppose a line ℓ is tangent to quadrics $\mathcal{Q}_{\alpha_1}, \ldots, \mathcal{Q}_{\alpha_{d-1}}$ from the given confocal family. Denote by $\mathcal{A}_\ell$ the family of all lines which are tangent to the same $d-1$ quadrics. Note that according to the corollary of the One Reflection Theorem, the set $\mathcal{A}_\ell$ is invariant to the billiard reflection on any of the confocal quadrics.

For what follows, the next simple observation is important.

Lemma 4 *Let the lines ℓ and ℓ' obey the reflection law at the point z of a quadric $\mathcal{Q}$ and suppose they are tangent to a confocal quadric $\mathcal{Q}_1$ at the points z_1 and z_2. Then the intersection of the tangent spaces $T_{z_1}\mathcal{Q}_1 \cap T_{z_2}\mathcal{Q}_1$ is contained in the tangent space $T_z\mathcal{Q}$.*

Following [25], together with $d-1$ affine confocal quadrics $\mathcal{Q}_{\alpha_1}, \ldots, \mathcal{Q}_{\alpha_{d-1}}$, one can consider their projective closures $\mathcal{Q}^p_{\alpha_1}, \ldots, \mathcal{Q}^p_{\alpha_{d-1}}$ and the intersection X of two quadrics in $\mathbb{C}P^{2d-1}$:

$$x_1^2 + \cdots + x_d^2 - y_1^2 - \cdots - y_{d-1}^2 = 0, \tag{22}$$

$$a_1 x_1^2 + \cdots + a_d x_d^2 - \alpha_1 y_1^2 - \cdots - \alpha_{d-1} y_{d-1}^2 = x_0^2. \tag{23}$$

Denote by $\mathcal{A}(X)$ the set of all $(d-2)$-dimensional linear subspaces of X. For a given $L \in \mathcal{A}$, denote by $\mathcal{A}_L(X)$ the closure in $\mathcal{A}(X)$ of the set $\{\, L' \in F \mid \dim L \cap L' = d-3 \,\}$. It was shown in [29] that F_L is a nonsingular hyperelliptic curve of genus $d-1$.

The projection

$$\pi' \;:\; \mathbb{C}P^{2d-1} \setminus \{(x,y) | x = 0\} \to \mathbb{C}P^d, \quad \pi'(x,y) = x,$$

maps $L \in \mathcal{A}(X)$ to a subspace $\pi'(L) \subset \mathbb{CP}^d$ of the codimension 2. $\pi'(L)$ is tangent to the quadrics $\mathcal{Q}^{p*}_{\alpha_1}, \ldots, \mathcal{Q}^{p*}_{\alpha_{d-1}}$ that are dual to $\mathcal{Q}^p_{\alpha_1}, \ldots, \mathcal{Q}^p_{\alpha_{d-1}}$.

Thus, the space dual to $\pi'(L)$, denoted by $\pi^*(L)$, is a line tangent to the quadrics $\mathcal{Q}^p_{\alpha_1}, \ldots, \mathcal{Q}^p_{\alpha_{d-1}}$.

We can reinterpret the generalized Cayley's curve $\mathcal{C}_\ell$, which is a family of tangent hyperplanes, as a set of lines from $\mathcal{A}_\ell$ which intersect ℓ. Namely, for almost every tangent hyperplane there is a unique line ℓ', obtained from ℓ by the billiard reflection. Having this identification in mind, it is easy to prove the following

Corollary 4 *There is a birational morphism between the generalized Cayley's curve $\mathcal{C}_\ell$ and Reid-Donagi-Knörrer's curve F_L, with $L = \pi^{*-1}(\ell)$, defined by*

$$j : \ell' \mapsto L', \quad L' = \pi^{*-1}(\ell'),$$

where ℓ' is a line obtained from ℓ by the billiard reflection on a confocal quadric.

Lemma 4, giving a link between the dynamics of ellipsoidal billiards and algebraic structure of certain Abelian varieties, provides a two way interaction: to apply algebraic methods in the study of the billiard motion, but also vice versa, to use billiard constructions in order to get more effective, more constructive and more observable understanding of the algebraic structure.

10.2 Two-valued billiard structure

Thus, we are going to apply this relationship to construct a billiard analogue of the geometric two-valued group structure defined on the Kummer variety K_{e_0} from the previous Section, see Theorem 7 below.

In order to fit together notations from the previous Section and from the last Subsection, we set $d = 3$ and

$$\{\lambda_1, \dots, \lambda_6\} = \{a_1, a_2, a_3, \alpha_1, \alpha_2, \infty\}.$$

We consider the confocal pencil of quadrics Q_λ in $\mathbb{C}P^3$ and the set $\mathcal{A}_\ell$ of lines in $\mathbb{C}P^3$ which are tangent to both quadrics $Q_{\alpha_1}, Q_{\alpha_2}$.

We select a quadric $Q_{p(e_0)}$ together with a family of reflections which corresponds to e_0 and a line $\ell_0 \in \mathcal{A}_\ell$.

We introduce an involution I_{e_0} on $\mathcal{A}_\ell$ induced by $Q_{p(e_0)}$ with e_0, which maps a line $m \in \mathcal{A}_\ell$ to a line $m' \in \mathcal{A}_\ell$ uniquely defined by the condition that m and m' are obtained by billiard reflection of each other from $Q_{p(e_0)}$ from the given family e_0.

We suppose that the restriction of this involution I_{e_0} restricted on C_{ℓ_0} coincides with the natural involution on the generalized Cayley curve C_{ℓ_0}.

Denote by

$$[\ell_1] = \{\ell_1, \hat{\ell}_1\}, \quad \ell_1, \hat{\ell}_1 \in \mathcal{A}_\ell$$

and by

$$[\ell_2] = \{\ell_2, \hat{\ell}_2\}, \quad \ell_2, \hat{\ell}_2 \in \mathcal{A}_\ell$$

two classes of lines in $\mathcal{A}_\ell / I_{e_0}$.

Assume that ℓ_1, ℓ_2 don't intersect ℓ_0. Denote by n_1, n_2 the two lines in $C_{\ell_0}(X)$ which form a double reflection configuration with ℓ_0, ℓ_1. Denote by Q_{μ_1} the quadric of billiard reflection of ℓ_0, n_1 and of billiard reflection of ℓ_1, n_2. Similarly, denote by Q_{μ_2} the quadric of billiard reflections of pairs ℓ_0, n_2 and ℓ_1, n_1.

The quadrics Q_{μ_1}, Q_{μ_2} intersect line ℓ_2. Denote by

$$n_1', n_1'', n_2', n_2'' \in C_{\ell_2}$$

the lines of billiard reflection of the line ℓ_2 from the quadrics Q_{μ_1}, Q_{μ_2}.

The lines n_1', n_1'' are obtained by reflection at Q_{μ_1} and reflection of n_1' belongs to the same family as the reflection of ℓ_0, n_1; the lines n_2', n_2'' are obtained by reflection from Q_{μ_2} and reflection of n_2' belongs to the same family as the reflection ℓ_0, n_2.

Let m_1 be the fourth line of the double reflection configuration determined with ℓ_2, n_1', n_2' and let m_2 be the fourth line of the double reflection configuration generated by ℓ_2, n_1'', n_2''.

If we repeat the above procedure with $\hat{\ell}_1, \hat{\ell}_2$ instead of ℓ_1, ℓ_2 we come to the lines $\hat{m}_1$ and $\hat{m}_2$ which are obtained from m_1 and m_2 by the involution I_{e_0}.

One can easily adjust previous construction to the case where ℓ_1, ℓ_2 intersect ℓ_0: denote by m_1 the fourth line of the double reflection configuration determined with $\hat{\ell}_1, \hat{\ell}_2, \ell_0$; denote by m_2 the fourth line of the double reflection configuration determined with $\ell_1, \hat{\ell}_2$ and ℓ_0.

Thus we get

Theorem 7 *The mapping defined by the formulae*

$$\begin{gathered}\star_b : \mathcal{A}_\ell / I_{e_0} \times \mathcal{A}_\ell / I_{e_0} \longrightarrow (\mathcal{A}_\ell / I_{e_0})^2 \\ [\ell_1] \star_b [\ell_2] = ([m_1],\ [m_2])\end{gathered}$$

defines a structure of two-valued group on the variety $\mathcal{A}_\ell / I_{e_0}$.

Definition 5 *The mapping $\star_b$ defined by previous construction defines the billiard two-valued group law on the variety $\mathcal{A}_\ell / I_{e_0}$.*

11 Moduli of semi-stable bundles and two-valued group structure on Kummer varieties

As we have mentioned before, historically first examples of 2-valued groups appeared in topological context in the study of the characteristic classes of vector bundles, see [5]. There one-dimensional symplectic bundles over $\mathbb{HP}^n$, and together with the canonical projection $\mathbb{C}P^{2n+1} \to \mathbb{H}P^n$, the associated two-dimensional complex vector bundles over $\mathbb{C}P^{2n+1}$ with $c_1 = 0$, were considered.

For a pair of such two-dimensional bundles,

$$\xi_1,\, \xi_2, \quad c_1(\xi_i) = 0,\, i = 1, 2$$

its tensor product

$$\xi_1 \otimes_C \xi_2$$

is a four-dimensional bundle, with the first two Potryagin classes $p_1(\xi_1 \otimes_C \xi_2)$ and $p_2(\xi_1 \otimes_C \xi_2)$. In [5], to the initial pair of bundles, a pair of *virtual* two-bundles, were associated according to the formula

$$Z^2 - p_1(\xi_1 \otimes_C \xi_2)Z + p_2(\xi_1 \otimes_C \xi_2) = 0.$$

The solutions $Z_{1,2}$ of the last quadratic equation play a role of the first Pontryagin classes of two virtual two-bundles. The last quadratic equation defines a two-valued group structure in $x = p_1(\xi_1)$ and $y = p_1(\xi_2)$, since $p_1(\xi_1 \otimes_C \xi_2) = \Theta_1(x, y)$ and $p_2(\xi_1 \otimes_C \xi_2) = \Theta_2(x, y)$, where Θ_i are certain series.

In this Section, we are going to construct the two-valued group structure on the Kummer variety, in the context of two-dimensional varieties, semi-stable in the sense of Mumford and Seshadri. The significance of the present situation, lies in the fact that obtained resulting two-bundles *are not virtual* - they are realized as a pair of two-bundles, semi-stable but not stable.

To get this, yet another interpretation of Kummer varieties, we are going back to [28] Let X be a genus two curve. Following Mumford and Seshadri, the notions of stable and semi-stable vector bundles of rank n and degree d have been introduced respectively. For a holomorphic nonzero vector bundle W on X one introduces a rational-valued function $\mu(W) = \deg W/\mathrm{rank} W$. A vector bundle W is stable if for every proper subbundle V the condition

$$\mu(V) < \mu(W)$$

is satisfied. Similarly, a bundle is semi-stable if in the last inequality the sign $<$ is replaced by $\leq$. Any semi-stable bundle W has a strictly decreasing filtration

$$W = W_0 \supset W_1 \supset \cdots \supset W_n = \{0\}$$

such that W_{i-1}/W_i are stable and $\mu(W_{i-1}/W_i) = \mu(W)$. Denote by $GrW = \oplus W_{i-1}/W_i$. As Seshadri defined, two semi-stable bundles W_1 and W_2 are S-equivalent if $GrW_1 \approx GrW_2$; a normal, projective $(n^2 + 1)$-dimensional variety of S-equivalence classes of semi-stable bundles of degree d and rank n denote $U(n, d)$.

Following [28], for $U(2, 0)$, denote by S its three-dimensional sub-variety of bundles with trivial determinant. The non-stable bundles in S are of the form

$$j \oplus j^{-1},$$

with j a line bundle of degree 0. The Kummer surface K associated to the Jacobian of X is isomorphic to the set of all non-stable bundles in S.

Now, we define a structure of two-valued group on K. It is an important development of the Example 4 from Section 2. Denote $a, b \in K$, where

$$a = j \oplus j^{-1}, \quad b = l \oplus l^{-1},$$

where j, l are line bundles on X of degree 0. Then:

$$a \star_s b := (j \otimes l \oplus j^{-1} \otimes l^{-1}, j \otimes l^{-1} \oplus j^{-1} \otimes l). \tag{24}$$

Proposition 6 *The operation*

$$\star_s : K \times K \longrightarrow (K)^2$$

determined by the relation 24 defines a two-valued group structure on K.

Acknowledgements

The research of one of the authors (V. D.) was partially supported by the Serbian Ministry of Science and Technological Development, Project *Geometry and Topology of Manifolds, Classical Mechanics and Integrable Dynamical Systems* and by the Mathematical Physics Group of the University of Lisbon, Project *Probabilistic approach to finite and infinite dimensional dynamical systems, PTDC/MAT/104173/2008*.

References

[1] G. G. Appel'rot, *Some suplements to the memoir of N. B. Delone*, Tr. otd. fiz. nauk, **6** (1893)

[2] V. Arnol'd, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1978.

[3] M. Audin, *Spinning Tops. An introduction to integrable systems*, Cambridge studies in advanced mathematics **51** 1999

[4] H. F. Baker, *On the hyperelliptic sigma functions*, Amer. Jour. Math. p. 301-384, 1898

[5] V. M. Buchstaber, S. P. Novikov, *Formal groups, power systems and Adams operators*, Mat. Sb. (N. S) **84 (126)** (1971) 81-118 (in Russian)

[6] V. M. Buchstaber, E. G. Rees, *Multivalued groups, their representations and Hopf algebras*, Transform. Groups **2** (1997) 325-249

[7] V. M. Buchstaber, A. P. Veselov, *Integrable correspondences and algebraic representations of multivalued groups*, Internat. Math. Res. Notices, (1996) 381-400

[8] V. M. Buchstaber, *Functional equations that are associated with addition theorems for elliptic functions, and two valued groups*, Uspekhi Math. Nauk 45 (1990) No 3 (273) 185-186 (Russian) English translation: Russian Math. Surveys 45, (1990), No 3, 213-215

[9] V. M. Buchstaber, *n-valued groups: theory and applications*, Moscow Mathematical Journal, **6** (2006) 57-84

[10] V. M. Buchstaber, V. Z. Enolskii, D. V. Leikin, *Kleinian functions, hyperelliptic Jacobians and applications.*, Reviews in Mathematics and Math. Physics, I. M. Krichever, S. P. Novikov Editors, v. 10, part 2, Gordon and Breach, London, 1997, 3–120.

[11] V. M. Buchstaber, D. V. Leykin, *Addition laws on Jacobian varieties of plane algebraic curves.*, Proceedings of the Steklov Math. Inst., v. 251, Issue 4, (2005) 49–120.

[12] V. M. Buchstaber, E. G. Rees, *The Gelfand map and symmetric products*, Selecta Math.,(N.S.), (2002) v.8, N4, p.523-535.

[13] G. Darboux, *Principes de géométrie analytique*, Gauthier-Villars, Paris (1917) 519 p.

[14] N. B. Delone, *Algebraic integrals of motion of a heavy rigid body around a fixed point*, Petersburg, 1892

[15] R. Donagi, *Group Law on the Intersection of Two Quadrics* Ann. Sc. Norm. Sup. Pisa, (1980), 217-239.

[16] V. Dragović, *Poncelet-Darboux curves, their complete decomposition and Marden theorem*, Internat. Math. Res. Notes, 2010, doi:10.1093/imrn/rnq229, arXiv:0812.48290

[17] V. Dragović, *Multi-valued hyperelliptic continous fractions of generalized Halphen type*, Internat. Math. Res. Notices Vol. 2009 no. 10, p. 1891-1932

[18] V. Dragović, *Generalization and Geometrization of the Kowalevski top*, Comm. Math. Phys. 2010, Vol. 298, no. 1, p. 37-64,

[19] V. Dragović, M. Radnović, *Geometry of integrable billiards and pencils of quadrics*, Journal Math. Pures Appl. **85** (2006), 758-790.
arXiv: math-ph/0512049

[20] V. Dragović, M. Radnović, *Hyperelliptic Jacobians as Billiard Algebra of Pencils of Quadrics: Beyond Poncelet Porisms*, Adv. Math., **219** (2008) 1577-1607.
arXiv: math-ph/0710.3656

[21] V. Dragović, M. Radnović, *Integrable Billiards, Quadrics and Higher-dimensional Poncelet Porisms* RCD, Moscow - Izhevsk, 340 p, 2010

[22] B. A. Dubrovin, S. P. Novikov, *A periodic problem for the Korteweg-de Vries and Sturm-Liouville equations. Their connection with algebraic geometry* (Russian) Dokl. Akad. Nauk SSSR 219 (1974), 531534, English translation: Soviet Math. Dokl. 15 (1974), no. 6, 1597–1601 (1975)

[23] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, *Discriminants, resultants and multidimensional deteminants*, Birhauser.-Boston, 1994.

[24] V. V. Golubev, *Lectures on the integration of motion of a heavy rigid body around a fixed point*, Gostechizdat, Moscow, 1953 [in Russian], English translation: Israel program for scintific literature, 1960.

[25] H. Knörrer, *Geodesics on the ellipsoid*, Inventiones math. **59** (1980), 119-143.

[26] S. Kowalevski, *Sur la probleme de la rotation d'un corps solide autour d'un point fixe*, Acta Math. **12** (1889) 177-232

[27] B. K. Mlodzeevskii, *About a case of motion of a heavy rigid body aroud a fixed point*, Mat. Sb. **18** (1895)

[28] M. Narasimhan, Ramanan, *Moduli of vector bundles on compact Riemann surfaces*, Ann. of Math. (2) **89** (1969) p. 14-51

[29] M. Reid, Cambridge Thesis, June 1972, unpublished.