

DERIVED MODULI OF SCHEMES AND SHEAVES

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ABSTRACT. We describe derived moduli functors for a range of problems involving schemes and quasi-coherent sheaves, and give cohomological conditions for them to be representable by derived geometric n -stacks. Examples of problems represented by derived geometric 1-stacks are derived moduli of polarised projective varieties, derived moduli of vector bundles, and derived moduli of polarised abelian varieties.

INTRODUCTION

The driving philosophy behind derived algebraic geometry is that for every moduli problem, there should be an associated derived moduli problem. Thanks to the framework of [TV2] and [Lur1], there is often an obvious candidate for such a functor for derived moduli of schemes:

Given a simplicial algebra A , we first form the ∞ -category $\mathcal{C}(A)$ of derived geometric stacks \mathfrak{X} homotopy-flat over $\mathrm{Spec} A$, with the additional property that

$$\mathfrak{X} \times_{\mathrm{Spec} A}^h \mathrm{Spec} \pi_0 A$$

is a scheme. We can impose additional restrictions (smoothness, properness, dimension) of \mathfrak{X} by requiring that they apply to $\mathfrak{X} \times_{\mathrm{Spec} A}^h \mathrm{Spec} \pi_0 A$. We then consider the ∞ -category $\mathcal{G}(A) \subset \mathcal{C}(A)$ in which the only morphisms are weak equivalences. Finally, we define the derived moduli functor $\mathfrak{M}(A)$ to be the nerve of the ∞ -category $\mathcal{G}(A)$. There are variants of this construction for subschemes and so on.

Similarly, for derived moduli of sheaves or complexes on a scheme X , we can first set $\mathcal{C}(A)$ to be the ∞ -category of quasi-coherent complexes \mathcal{F} (in the sense of [TV2]) on $X \times \mathrm{Spec} A$, such that the complex $\mathcal{F} \otimes_A^L \pi_0 A$ on $X \times \mathrm{Spec}(\pi_0 A)$ satisfies any additional constraints (bounded, fixed Euler characteristic, concentrated in degree 0, free of rank n). We then form $\mathcal{G}(A) \subset \mathcal{C}(A)$ as before, and take the nerve.

We can even combine these approaches to consider moduli of polarised projective schemes. $\mathcal{C}(A)$ then consists of pairs $(\mathfrak{X}, \mathcal{F})$, for \mathfrak{X} a derived geometric stack homotopy-flat over $\mathrm{Spec} A$ and \mathcal{F} a quasi-coherent complex on \mathfrak{X} , with

$$(\mathfrak{X} \times_{\mathrm{Spec} A}^h \mathrm{Spec} \pi_0 A, \mathcal{F} \otimes_A^L \pi_0 A)$$

a projective scheme and an ample line bundle.

Lurie's Representability Theorem provides explicit criteria to check whether such a functor is representable by a geometric derived stack. However, the framework of [TV2] and [Lur1] does not lend itself easily to calculations, making it difficult to verify that these functors satisfy the criteria of the representability theorem. Even to show that the cotangent complex governs first-order deformations of n -geometric stacks, the more concrete approach of Artin hypergroupoids is needed (as in [Pri1] §9.2). Several examples of representable derived moduli functors have already been established, however. Both [TV2] and [Lur1] construct Hom-stacks, while [Lur1] addresses derived moduli of stable curves, as well as derived Picard stacks and derived Hilbert schemes.

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The purpose of this paper is to establish representability more generally for problems involving schemes and quasi-coherent sheaves, building on the explicit characterisations of [Pri1]. In simplistic terms, we show that if the associated underived moduli problem is representable, then the derived moduli problem will be representable provided only that certain cohomology groups satisfy mild finiteness conditions. The main results are Theorem 3.32 for moduli of derived geometric stacks over a fixed base, Theorem 3.35 which has the additional datum of an augmentation, and Theorem 4.12 for moduli of quasi-coherent complexes.

By contrast to the indirect approach of satisfying a representability theorem, [CFK2] and [CFK1] construct explicit derived Hilbert and Quot schemes as dg-schemes with the necessary properties, but there no universal family is given, so the derived moduli spaces lack functorial interpretations. In [Pri2], we will show how to compare these approaches, thereby giving explicit presentations for the derived moduli spaces constructed here.

The structure of the paper is as follows.

In Section 1 we review the framework of simplicial categories, giving several properties of the nerve construction from simplicial categories to simplicial sets. Section 2 then recalls a variant (Theorem 2.10) of Lurie’s Representability Theorem, and shows how the nerve construction transfers good properties from simplicial category-valued functors to simplicial set-valued functors. These results are then applied in the last two sections to a wide range of moduli problems.

The main results of Section 3 are Corollaries 3.10 and 3.15, which deal with derived moduli of affine schemes, and Theorems 3.32 and 3.35, addressing derived moduli of (augmented) geometric n -stacks. These are applied to moduli of finite schemes (Example 3.11), of schemes (Example 3.36), of torsors (Example 3.38), of polarised schemes (Example 3.39), of polarised abelian schemes (Example 3.46) and derived Hom-stacks (Example 3.37).

Section 4 deals, in Theorem 4.12 and Example 4.15, with moduli of quasi-coherent complexes. Lemma 4.16 then shows that the paper’s two possible approaches to derived moduli of vector bundles (via GL_n -torsors and via quasi-coherent complexes) give equivalent results.

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1. SIMPLICIAL CATEGORIES

Let \mathbb{S} denote the category of simplicial sets, and $s\mathbb{S}$ the category of bisimplicial sets.

Definition 1.1. Let $s\text{Cat}$ be the category of small simplicially enriched categories, which we will refer to as simplicial categories. Explicitly, an object $\mathcal{C} \in s\text{Cat}$ consists of a set $\text{Ob}\mathcal{C}$ of objects, together with $\underline{\text{Hom}}_{\mathcal{C}}(x, y) \in \mathbb{S}$ for all $x, y \in \text{Ob}\mathcal{C}$, equipped with an associative composition law and identities.

Definition 1.2. Given a simplicial category \mathcal{C} , the category $\pi_0\mathcal{C}$ is defined to have the same objects as \mathcal{C} , with morphisms

$$\text{Hom}_{\pi_0\mathcal{C}}(x, y) = \pi_0\underline{\text{Hom}}_{\mathcal{C}}(x, y).$$

A morphism in $\underline{\text{Hom}}_{\mathcal{C}}(x, y)_0$ is said to be a homotopy equivalence if its image in $\pi_0\mathcal{C}$ is an isomorphism.

Definition 1.3. Recall from [Ber] Theorem 1.1 that a morphism $f : \mathcal{C} \rightarrow \mathcal{D}$ in $s\text{Cat}$ is said to be a weak equivalence (a.k.a. an ∞ -equivalence) whenever

- (W1) for any objects a_1 and a_2 in \mathcal{C} , the map $\underline{\text{Hom}}_{\mathcal{C}}(a_1, a_2) \rightarrow \underline{\text{Hom}}_{\mathcal{D}}(fa_1, fa_2)$ is a weak equivalence of simplicial sets;
- (W2) the induced functor $\pi_0f : \pi_0\mathcal{C} \rightarrow \pi_0\mathcal{D}$ is an equivalence of categories.

Definition 1.4. Given a simplicial category \mathcal{C} , a morphism in \mathcal{C}_0 is said to be a homotopy equivalence if it becomes an isomorphism in $\pi_0\mathcal{C}$.

Definition 1.5. Recall from [Ber] Theorem 1.1 that a morphism $f : \mathcal{C} \rightarrow \mathcal{D}$ in $s\text{Cat}$ is said to be a fibration whenever

- (F1) for any objects a_1 and a_2 in \mathcal{C} , the map $\underline{\text{Hom}}_{\mathcal{C}}(a_1, a_2) \rightarrow \underline{\text{Hom}}_{\mathcal{D}}(fa_1, fa_2)$ is a fibration of simplicial sets;
- (F2) for any objects $a_1 \in \mathcal{C}$, $b \in \mathcal{D}$, and homotopy equivalence $e : fa_1 \rightarrow b$ in \mathcal{D} , there is an object $a_2 \in \mathcal{C}$ and a homotopy equivalence $d : a_1 \rightarrow a_2$ in \mathcal{C} such that $fd = e$.

Definition 1.6. Given $\mathcal{C} \in s\text{Cat}$, define $\bar{W}\mathcal{C} := \bar{W}B\mathcal{C}$, where $\bar{W} : s\mathbb{S} \rightarrow \mathbb{S}$ is the right adjoint to Illusie's total Dec functor given by $\text{DEC}(X)_{mn} = X_{m+n+1}$, and $B : \text{Cat} \rightarrow \mathbb{S}$ is the nerve. Explicitly

$$(\bar{W}\mathcal{C})_n = \coprod_{\underline{x} \in (\text{Ob}\mathcal{C})^{n+1}} \mathcal{C}_{n-1}(x_{n-1}, x_n) \times \mathcal{C}_{n-2}(x_{n-2}, x_{n-1}) \times \dots \times \mathcal{C}_0(x_0, x_1),$$

with operations

$$\begin{aligned} \partial_i(g_{n-1}, \dots, g_0) &= \begin{cases} (g_{n-2}, \dots, g_0) & i = 0, \\ (\partial_{i-1}g_{n-1}, \dots, \partial_1g_{n-i+1}, (\partial_0g_{n-i})g_{n-i-1}, g_{n-i-2}, \dots, g_0) & 0 < i < n, \\ (\partial_{n-1}g_{n-1}, \dots, \partial_1g_1) & i = n, \end{cases} \\ \sigma_i(g_{n-1}, \dots, g_0) &= (\sigma_{i-1}g_{n-1}, \dots, \sigma_0g_{n-i}, \text{id}_{x_{n-i}}, g_{n-i-1}, \dots, g_0), \end{aligned}$$

Remark 1.7. Another functor from bisimplicial to simplicial sets is the diagonal functor $\text{diag}(Y)_n = Y_{nn}$. In [CR], it is established that the canonical natural transformation

$$\text{diag}Y \rightarrow \bar{W}Y$$

is a weak equivalence for all Y . When $Y = B\mathcal{C}$, this transformation is given by

$$(y_n, h_{n-1}, y_{n-1}, \dots, h_0, y_0) \mapsto (y_n, \partial_0h_{n-1}, \partial_0y_{n-1}, \dots, (\partial_0)^n h_0, (\partial_0)^n y_0).$$

This observation means that throughout this paper, \bar{W} can be replaced with diag , which is simpler to define. Our reasons for preferring \bar{W} are that it produces much smaller objects, as exploited in [DK]. For instance, if Y is a bisimplicial abelian group, then the Dold–Kan normalisation $N\bar{W}Y$ (see Definition 2.1) is just the total complex of the Dold–Kan binormalisation NY .

The following results seem to be folklore, but lacking a suitable reference, we prove them afresh.

Proposition 1.8. *If $f : \mathcal{C} \rightarrow \mathcal{D}$ is a fibration of simplicial categories, with $\pi_0\mathcal{C}$ and $\pi_0\mathcal{D}$ being groupoids, and $\mathcal{B} \rightarrow \mathcal{D}$ is an arbitrary morphism in $s\text{Cat}$, then the homotopy fibre product*

$$(\bar{W}\mathcal{C}) \times_{(\bar{W}\mathcal{D})}^h (\bar{W}\mathcal{B})$$

is weakly equivalent to $\bar{W}(\mathcal{C} \times_{\mathcal{D}} \mathcal{B})$.

Proof. Construct a category $\pi_n\mathcal{C}$ with objects $\text{Ob}\mathcal{C}$ and morphisms $\coprod_{f \in \mathcal{C}_0(x,y)} \pi_n(\mathcal{C}(x,y), f)$ for $x, y \in \text{Ob}\mathcal{C}$. If we form the bisimplicial set BC , then $\pi_n BC = B\pi_n\mathcal{C}$ and we wish to verify the π_* -Kan condition ([GJ] §IV.4) — that

$$B\pi_n\mathcal{C} \rightarrow BC_0$$

is a fibration.

If a functor $F : \mathcal{A} \rightarrow \mathcal{A}'$ of categories is an isomorphism on objects, then $BA \rightarrow BA'$ is a fibration if and only if

- (1) F is full, and
- (2) if a morphism h in \mathcal{A} factors as $F(h) = fg$ in \mathcal{A}' , with either f or g lifting to \tilde{f} or \tilde{g} in \mathcal{A} , then the other lifts uniquely in such a way that $\tilde{f}\tilde{g} = h$.

Since $\pi_n\mathcal{C} \rightarrow \mathcal{C}_0$ has a canonical section (given by $0 \in \pi_n$), it is full. Now, as $\pi_0\mathcal{C}$ is a groupoid, for all $f \in \mathcal{C}_0(x,y)$ the map $f_* : \pi_n \underline{\text{End}}_{\mathcal{C}}(x) \rightarrow \pi_n(\underline{\text{Hom}}_{\mathcal{C}}(x,y), f)$ is an isomorphism. It therefore gives rise to an isomorphism $\text{ad}_f : \pi_n \underline{\text{End}}_{\mathcal{C}}(x) \rightarrow \pi_n \underline{\text{End}}_{\mathcal{C}}(y)$. We can write $\tilde{f} = f_*a$, $\tilde{g} = g_*b$ and $h = f_*g_*c$, so \tilde{f} and \tilde{g} determine each other by the formula $a = \text{ad}_g(cb^{-1})$.

Now, (F2) implies that $B\pi_0\mathcal{C} \rightarrow B\pi_0\mathcal{D}$ is a Kan fibration, while (F1) implies that $BC \rightarrow BD$ is a pointwise fibration. Therefore the Bousfield–Friedlander Theorem ([GJ] Theorem IV.4.9) gives that

$$\text{diag } B(\mathcal{C} \times_{\mathcal{D}} \mathcal{B}) \rightarrow (\text{diag } BC) \times_{(\text{diag } BD)}^h (\text{diag } B\mathcal{B})$$

is a weak equivalence. Finally, [CR], shows that canonical natural transformation $\text{diag } BA \rightarrow \bar{W}A$ is a weak equivalence for all A , which completes the proof. \square

Definition 1.9. Given a category \mathcal{C} , define $c(\mathcal{C})$ to be the set of isomorphism classes in \mathcal{C} .

Corollary 1.10. *If $\mathcal{C} \in s\text{Cat}$ is such that $\pi_0\mathcal{C}$ is a groupoid, then the homotopy fibre of $\bar{W}\mathcal{C} \rightarrow c(\pi_0\mathcal{C})$ over $x \in \text{Ob}\mathcal{C}$ is just*

$$\bar{W}\underline{\text{End}}_{\mathcal{C}}(x).$$

Thus $\pi_n(\bar{W}\mathcal{C}, x) = \pi_{n-1}\underline{\text{End}}_{\mathcal{C}}(x)$, where the basepoint of the $\underline{\text{End}}_{\mathcal{C}}(x)$ is taken to be the identity.

Proof. By using the model structure of [Ber] Theorem 1.1, there is a morphism $g : \mathcal{C} \rightarrow \mathcal{C}'$ over $c(\mathcal{C})$, with $c(\pi_0\mathcal{C}) = c(\pi_0\mathcal{C}')$, $\pi_*\underline{\text{Hom}}_{\mathcal{C}'}(gx, gy) \cong \pi_*\underline{\text{Hom}}_{\mathcal{C}}(x, y)$, and $\mathcal{C}' \rightarrow c(\mathcal{C})$ a fibration. Replacing \mathcal{C} with \mathcal{C}' , we may therefore assume that $\underline{\text{Hom}}_{\mathcal{C}}(x, y)$ is fibrant for all x, y .

Let \mathcal{D} be the groupoid with objects $\text{Ob}\mathcal{C}$, and a unique morphism $x \rightarrow y$ for any pair of objects with the same image in $c(\pi_0\mathcal{C})$. There is therefore a canonical morphism $\mathcal{C} \rightarrow \mathcal{D}$, and this is a fibration. Therefore the proposition gives

$$\bar{W}(\mathcal{C} \times_{\mathcal{D}} \{x\}) \simeq (\bar{W}\mathcal{C}) \times_{(\bar{W}\mathcal{D})}^h \{x\}.$$

Since $\bar{W}\mathcal{D} \simeq c(\pi_0\mathcal{C})$, the right-hand side is just the homotopy fibre of $\bar{W}\mathcal{C} \rightarrow c(\pi_0\mathcal{C})$. Meanwhile, $\mathcal{C} \times_{\mathcal{D}} \{x\}$ is just the simplicial category $\underline{\text{End}}_{\mathcal{C}}(x)$ with unique object x and morphisms $\underline{\text{Hom}}_{\mathcal{C}}(x, x)$, as required. \square

Corollary 1.11. *If \mathcal{C} and \mathcal{D} are simplicial categories for which $\pi_0\mathcal{C}$ and $\pi_0\mathcal{D}$ are groupoids, then a morphism $f : \mathcal{C} \rightarrow \mathcal{D}$ is an ∞ -equivalence if and only if*

$$\bar{W}\mathcal{C} \rightarrow \bar{W}\mathcal{D}$$

is a weak equivalence.

Proof. We have just seen that f is a \bar{W} -equivalence if and only if $c(\pi_0\mathcal{C}) \rightarrow c(\pi_0\mathcal{D})$ is an isomorphism, and $\pi_n \underline{\text{End}}_{\mathcal{C}}(x) \rightarrow \pi_n \underline{\text{End}}_{\mathcal{D}}(fx)$ is an isomorphism for all $n \geq 0$. This is precisely the same as saying that f is an ∞ -equivalence. \square

Corollary 1.12. *If $\mathcal{C} \rightarrow \mathcal{D}$ and $\mathcal{B} \rightarrow \mathcal{D}$ are arbitrary morphisms in $s\text{Cat}$, with $\pi_0\mathcal{C}$ and $\pi_0\mathcal{D}$ groupoids, then*

$$\bar{W}(\mathcal{C} \times_{\mathcal{D}}^h \mathcal{B}) \rightarrow (\bar{W}\mathcal{C}) \times_{(\bar{W}\mathcal{D})}^h (\bar{W}\mathcal{B})$$

is a weak equivalence, where the homotopy fibre product $\mathcal{C} \times_{\mathcal{D}}^h \mathcal{B}$ is taken in the model structure of [Ber].

In particular, if $\pi_0\mathcal{B}$ is also a groupoid, then the fundamental groupoid $\pi_f((\bar{W}\mathcal{C}) \times_{(\bar{W}\mathcal{D})}^h (\bar{W}\mathcal{B}))$ is equivalent to the 2-fibre product

$$(\pi_0\mathcal{C}) \times_{(\pi_0\mathcal{D})}^{(2)} (\pi_0\mathcal{B}),$$

while for any $\alpha := (x, f, y)$, where $x \in \text{Ob}\mathcal{C}$, $y \in \text{Ob}\mathcal{B}$ and $f \in \mathcal{D}_0(\bar{x}, \bar{y})$,

$$\pi_n((\bar{W}\mathcal{C}) \times_{(\bar{W}\mathcal{D})}^h (\bar{W}\mathcal{B}), \alpha) \cong \pi_{n-1}(\underline{\text{End}}_{\mathcal{C}}(x) \times_{f_* \underline{\text{Hom}}_{\mathcal{D}}(\bar{x}, \bar{y}), f^*} \underline{\text{End}}_{\mathcal{B}}(y)).$$

Proof. Factorise $\mathcal{C} \rightarrow \mathcal{D}$ as $\mathcal{C} \rightarrow \mathcal{C}' \rightarrow \mathcal{D}$, a weak equivalence followed by a fibration. By Corollary 1.11,

$$(\bar{W}\mathcal{C}) \times_{(\bar{W}\mathcal{D})}^h (\bar{W}\mathcal{B}) \rightarrow (\bar{W}\mathcal{C}') \times_{(\bar{W}\mathcal{D})}^h (\bar{W}\mathcal{B})$$

is a weak equivalence. By Proposition 1.8, the latter is weakly equivalent to $\bar{W}(\mathcal{C}' \times_{\mathcal{D}} \mathcal{B})$, which is just $\bar{W}(\mathcal{C} \times_{\mathcal{D}}^h \mathcal{B})$, since $s\text{Cat}$ is right proper.

The description of the fundamental groupoid follows from the observation that for $\mathcal{A} \in s\text{Cat}$ and a discrete groupoid Γ , there are canonical isomorphisms

$$\text{Hom}_{\mathbb{S}}(\bar{W}\mathcal{A}, B\Gamma) \cong \text{Hom}_{s\text{Cat}}(\mathcal{A}, \Gamma) \cong \text{Hom}_{\text{Cat}}(\pi_0\mathcal{A}, \Gamma).$$

For $\mathcal{C}' \rightarrow \mathcal{D}$ a fibration, the map

$$\pi_0(\mathcal{C}' \times_{\mathcal{D}} \mathcal{B}) = (\pi_0\mathcal{C}') \times_{(\pi_0\mathcal{D})} (\pi_0\mathcal{B}) \rightarrow (\pi_0\mathcal{C}') \times_{(\pi_0\mathcal{D})}^{(2)} (\pi_0\mathcal{B})$$

is an equivalence, and the latter is equivalent to $(\pi_0\mathcal{C}) \times_{(\pi_0\mathcal{D})}^{(2)} (\pi_0\mathcal{B})$.

Finally, for $\alpha := (x, f, y)$ as above, also write x for the image of x in \mathcal{C}' . Since $\mathcal{C}' \rightarrow \mathcal{D}$ lifts homotopy equivalences, there is a morphism $g : x \rightarrow z$ in \mathcal{C}' mapping to f in \mathcal{D} . Then α defines the same element in $c(\pi_0\mathcal{C}') \times_{(\pi_0\mathcal{D})}^{(2)} (\pi_0\mathcal{B})$ as $\alpha' := (z, \text{id}, y)$, so we need only describe homotopy groups based at α' .

By Corollary 1.10,

$$\pi_n(\bar{W}(\mathcal{C}' \times_{\mathcal{D}} \mathcal{B}), \alpha') = \pi_{n-1}(\underline{\text{End}}_{\mathcal{C}'}(z) \times_{\underline{\text{End}}_{\mathcal{D}}(\bar{y})} \underline{\text{End}}_{\mathcal{B}}(y), \alpha'),$$

and we note that $\underline{\text{End}}_{\mathcal{C}'}(z) \rightarrow \underline{\text{End}}_{\mathcal{D}}(\bar{y})$ is a Kan fibration, since $\mathcal{C}' \rightarrow \mathcal{D}$ is a fibration. Now, since g and f are homotopy equivalences, they induce weak equivalences

$$\begin{aligned} g^* : \underline{\text{End}}_{\mathcal{C}'}(z) &\rightarrow \underline{\text{Hom}}_{\mathcal{C}'}(x, z) \\ g_* : \underline{\text{End}}_{\mathcal{C}'}(x) &\rightarrow \underline{\text{Hom}}_{\mathcal{C}'}(x, z) \\ f^* : \underline{\text{End}}_{\mathcal{D}}(\bar{y}) &\rightarrow \underline{\text{Hom}}_{\mathcal{D}}(\bar{x}, \bar{y}), \end{aligned}$$

giving weak equivalences

$$\underline{\text{End}}_{\mathcal{C}'}(z) \times_{\underline{\text{End}}_{\mathcal{D}}(\bar{y})} \underline{\text{End}}_{\mathcal{B}}(y) \rightarrow \underline{\text{Hom}}_{\mathcal{C}'}(x, z) \times_{\underline{\text{Hom}}_{\mathcal{D}}(\bar{x}, \bar{y})} \underline{\text{End}}_{\mathcal{B}}(y) \leftarrow \underline{\text{End}}_{\mathcal{C}'}(x) \times_{f_* \underline{\text{Hom}}_{\mathcal{D}}(\bar{x}, \bar{y})} \underline{\text{End}}_{\mathcal{B}}(y).$$

Finally, the weak equivalence $\mathcal{C} \rightarrow \mathcal{C}'$ gives a weak equivalence $\underline{\text{End}}_{\mathcal{C}}(x) \rightarrow \underline{\text{End}}_{\mathcal{C}'}(x)$, yielding the expression required. \square

2. DERIVED MODULI FROM SIMPLICIAL CATEGORIES

2.1. Background on representability. Denote the category of simplicial rings by $s\text{Ring}$, the category of simplicial R -algebras by $s\text{Alg}_R$, and the category of simplicial R -modules by $s\text{Mod}_R$. If R is a (dg) \mathbb{Q} -algebra, we let $dg_+\text{Alg}_R$ be the category of differential graded-commutative R -algebras in non-negative chain degrees, and $dg_+\text{Mod}_R$ the category of R -modules in chain complexes in non-negative chain degrees.

Definition 2.1. Given a simplicial abelian group A_\bullet , we denote the associated normalised chain complex by NA . Recall that this is given by $N(A)_n := \bigcap_{i>0} \ker(\partial_i : A_n \rightarrow A_{n-1})$, with differential ∂_0 . Then $H_*(NA) \cong \pi_*(A)$.

Using the Eilenberg–Zilber shuffle product, normalisation N extends to a functor

$$N : s\text{Alg}_R \rightarrow dg_+\text{Alg}_R.$$

When R is a \mathbb{Q} -algebra, this functor is a right Quillen equivalence.

Definition 2.2. Define $dg_+\mathcal{N}_R$ (resp. $s\mathcal{N}_R$) to be the full subcategory of $dg_+\text{Alg}_R$ (resp. $s\text{Alg}_R$) consisting of objects A for which the map $A \rightarrow H_0A$ (resp. $A \rightarrow \pi_0A$) has nilpotent kernel. Define $dg_+\mathcal{N}_R^\flat$ (resp. $s\mathcal{N}_R^\flat$) to be the full subcategory of $dg_+\mathcal{N}_R$ (resp. $s\mathcal{N}_R$) consisting of objects A for which $A_i = 0$ (resp. $N_iA = 0$) for all $i \gg 0$.

From now on, we will write $d\mathcal{N}^\flat$ (resp. $d\text{Alg}_R$, resp. $d\text{Mod}_R$) to mean either $s\mathcal{N}_R^\flat$ (resp. $s\text{Alg}_R$, resp. $s\text{Mod}_R$) or $dg_+\mathcal{N}_R^\flat$ (resp. $dg_+\text{Alg}_R$, resp. $dg_+\text{mod}_R$), noting that we only use chain algebras in characteristic 0.

Definition 2.3. Say that a surjection $A \rightarrow B$ in $dg_+\text{Alg}_R$ (resp. $s\text{Alg}_R$) is a *tiny acyclic extension* if the kernel K satisfies $I_A \cdot K = 0$, and K (resp. NK) is of the form $\text{cone}(M)[-r]$ for some H_0A -module (resp. π_0A -module) M . In particular, $H_*K = 0$.

Definition 2.4. Say that a functor $F : d\mathcal{N}^\flat \rightarrow \mathcal{C}$ to a model category \mathcal{C} is homotopy-preserving if for all tiny acyclic extensions $A \rightarrow B$, the map

$$F(A) \rightarrow F(B)$$

is a weak equivalence.

Definition 2.5. We say that a functor

$$F : d\mathcal{N}^\flat \rightarrow \mathcal{C}$$

to a model category \mathcal{C} is homotopy-homogeneous if for all square-zero extensions $A \rightarrow B$ and all maps $C \rightarrow B$ in $d\mathcal{N}^\flat$, the natural map

$$F(A \times_B C) \rightarrow F(A) \times_{F(B)}^h F(C)$$

to the homotopy fibre product is a weak equivalence.

Definition 2.6. Given a homotopy-preserving homotopy-homogeneous functor $F : d\mathcal{N}^\flat \rightarrow \mathbb{S}$, $A \in d\mathcal{N}^\flat$, and a point $x \in F(A)$, define

$$T_x(F/R) : d\text{Mod}_A \rightarrow \mathbb{S}$$

by

$$T_x(F/R)(M) := F(A \oplus M) \times_{F(A)}^h \{x\}.$$

Definition 2.7. Given F, A, x as above, and $M \in d\text{Mod}_A$, define

$$D_x^{n-i}(F/R, M) := \pi_i(T_x(F/R)(M[-n])),$$

noting that this is well-defined by [Pri3] Lemma 1.12.

Proposition 2.8. *If $F : s\text{Alg}_R \rightarrow \mathbb{S}$ is homotopy-preserving and homotopy-homogeneous, then for any square-zero extension $e : I \rightarrow A \xrightarrow{f} B$ in \mathcal{C} , there is a sequence of sets*

$$\pi_0(FA) \xrightarrow{f_*} \pi_0(FB) \xrightarrow{o_e} \Gamma(FB, D^1(F/R, I)),$$

where $\Gamma(-)$ denotes the global section functor. This is exact in the sense that the fibre of o_e over 0 is the image of f_* . Moreover, there is a group action of $D_x^0(F/R, I)$ on the fibre of $\pi_0(FA) \rightarrow \pi_0(FB)$ over x , whose orbits are precisely the fibres of f_* .

For any $y \in F_0A$, with $x = f_*y$, the fibre of $FA \rightarrow FB$ over x is weakly equivalent to $T_x(F/R, I)$, and the sequence above extends to a long exact sequence

$$\begin{array}{ccccccc} \cdots & \xrightarrow{e_*} & \pi_n(FA, y) & \xrightarrow{f_*} & \pi_n(FB, x) & \xrightarrow{o_e} & D_y^{1-n}(F/R, I) & \xrightarrow{e_*} & \pi_{n-1}(FA, y) & \xrightarrow{f_*} & \cdots \\ & & & & & & & & & & \\ & & & & \cdots & \xrightarrow{f_*} & \pi_1(FB, x) & \xrightarrow{o_e} & D_y^0(F/R, I) & \xrightarrow{-*y} & \pi_0(FA). \end{array}$$

Proof. This is [Pri3] Proposition 1.16 □

Definition 2.9. Given a functor $F : d\mathcal{N}_R^b \rightarrow \mathbb{S}$, define $\pi^0 F : \text{Alg}_{\pi_0 R} \rightarrow \mathbb{S}$ by $\pi^0 F(A) = F(A)$.

The following variant of Lurie's Representability Theorem is [Pri3] Theorem 2.17:

Theorem 2.10. *Let R be a derived G -ring admitting a dualising module (in the sense of [Lur1] Definition 3.6.1) and take a functor $F : d\mathcal{N}_R^b \rightarrow \mathbb{S}$. Then F is the restriction of an almost finitely presented geometric derived n -stack $F' : d\text{Alg}_R \rightarrow \mathbb{S}$ if and only if the following conditions hold*

- (1) F is homotopy-preserving.
- (2) For all discrete rings A , $F(A)$ is n -truncated, i.e. $\pi_i F(A) = 0$ for all $i > n$.
- (3) F is homotopy-homogeneous.
- (4) $\pi^0 F : \text{Alg}_{\pi_0 R} \rightarrow \mathbb{S}$ is a hypersheaf for the étale topology.
- (5) $\pi_0 \pi^0 F : \text{Alg}_{\pi_0 R} \rightarrow \text{Set}$ preserves filtered colimits.
- (6) For all $A \in \text{Alg}_{\pi_0 R}$ and all $x \in F(A)$, the functors $\pi_i(\pi^0 F, x) : \text{Alg}_A \rightarrow \text{Set}$ preserve filtered colimits for all $i > 0$.
- (7) for all finitely generated integral domains $A \in \text{Alg}_{\pi_0 R}$, all $x \in F(A)_0$ and all étale morphisms $f : A \rightarrow A'$, the maps

$$D_x^*(F, A) \otimes_A A' \rightarrow D_{fx}^*(F, A')$$

are isomorphisms.

- (8) for all finitely generated $A \in \text{Alg}_{\pi_0 R}$ and all $x \in F(A)_0$, the functors $D_x^i((F/R), -) : \text{Mod}_A \rightarrow \text{Ab}$ preserve filtered colimits for all $i > 0$.
- (9) for all finitely generated integral domains $A \in \text{Alg}_{\pi_0 R}$ and all $x \in F(A)_0$, the groups $D_x^i(F/R, A)$ are all finitely generated A -modules.
- (10) for all discrete local Noetherian $\pi_0 R$ -algebras A , with maximal ideal \mathfrak{m} , the map

$$\pi^0 F(A) \rightarrow \varprojlim^h_r F(A/\mathfrak{m}^r)$$

is a weak equivalence.

Moreover, F' is uniquely determined by F (up to weak equivalence).

Remark 2.11. The Milnor exact sequence ([GJ] Proposition 2.15) gives a sequence

$$\bullet \rightarrow \varprojlim_r^1 \pi_{i+1} F(A/\mathfrak{m}^r) \rightarrow \pi_i(\varprojlim^h_r F(A/\mathfrak{m}^r)) \rightarrow \varprojlim_r \pi_i F(A/\mathfrak{m}^r) \rightarrow \bullet,$$

which is exact as groups for $i \geq 1$ and as pointed sets for $i = 0$. Thus the final condition above can be rephrased to say that the map

$$f_0 : \pi_0 F(A) \rightarrow \varprojlim_r \pi_0 F(A/\mathfrak{m}^r)$$

is surjective, that for all $x \in F(A)$ the maps

$$f_{i,x} : \pi_i(FA, x) \rightarrow \varprojlim_r \pi_i(F(A/\mathfrak{m}^r), x)$$

are surjective for all $i \geq 1$ and that the resulting maps

$$\ker f_{i,x} \rightarrow \varprojlim_r^1 \pi_{i+1}(F(A/\mathfrak{m}^r), x)$$

are surjective for all $i \geq 0$.

Now, we can say that an inverse system $\{G_r\}_{r \in \mathbb{N}}$ of groups satisfies the Mittag-Leffler condition if for all r , the images $\text{Im}(G_s \rightarrow G_r)_{s \geq r}$ satisfy the descending chain condition. In that case, the usual abelian proof (see e.g. [Wei] Proposition 3.5.7) adapts to show that $\varprojlim^1 \{G_r\}_r = 1$.

Hence, if each system $\{\text{Im}(\pi_i(F(A/\mathfrak{m}^s), x) \rightarrow \pi_i(F(A/\mathfrak{m}^r), x))\}_{s \geq r}$ satisfies the Mittag-Leffler condition for $i \geq 1$, then the final condition of the theorem reduces to requiring that the maps

$$\pi_i F(A) \rightarrow \varprojlim_r \pi_i F(A/\mathfrak{m}^r)$$

be isomorphisms for all i .

Remark 2.12. Note that there are slight differences in terminology between [TV2] and [Lur1]. In the former, only disjoint unions of affine schemes are 0-representable, so arbitrary schemes are 2-geometric stacks, and Artin stacks are 1-geometric stacks if and only if they have affine diagonal. In the latter, algebraic spaces are 0-stacks. A geometric n -stack is called n -truncated in [TV2], and it follows easily that every n -geometric stack in [TV2] is n -truncated. Conversely, every geometric n -stack is $(n+2)$ -geometric.

Theorem 2.10 takes the convention from [Lur1], so “geometric derived n -stack” means “ n -truncated derived geometric stack”.

2.2. Formal quasi-smoothness.

Definition 2.13. Say that a natural transformation $\eta : F \rightarrow G$ of functors $F, G : d\mathcal{N}^b \rightarrow \mathcal{C}$ to a model category \mathcal{C} is homotopic if for all tiny acyclic extensions $A \rightarrow B$, the map

$$F(A) \rightarrow F(B) \times_{G(B)} G(A)$$

is a trivial fibration. Say that F is homotopic if $F \rightarrow \bullet$ is so, where \bullet denotes the final object of \mathcal{C} .

Definition 2.14. Say that a natural transformation $\eta : F \rightarrow G$ of functors $F, G : d\mathcal{N}^b \rightarrow \mathcal{C}$ to a model category \mathcal{C} is formally quasi-presmooth if for all square-zero extensions $A \rightarrow B$, the map

$$F(A) \rightarrow F(B) \times_{G(B)} G(A)$$

is a fibration.

Say that η is formally quasi-smooth if it is formally quasi-presmooth and homotopic.

Definition 2.15. Say that a natural transformation $\eta : F \rightarrow G$ of functors $F, G : d\mathcal{N}^b \rightarrow \mathcal{S}$ is formally presmooth (resp. formally smooth) if it is formally quasi-presmooth (resp. formally quasi-smooth) and for all square-zero extensions $A \rightarrow B$, the map

$$F(A) \rightarrow F(B) \times_{G(B)} G(A)$$

is surjective on π_0 .

Definition 2.16. Say that a natural transformation $\eta : F \rightarrow G$ of functors on $d\mathcal{N}^b$ is formally étale if for all square-zero extensions $A \rightarrow B$, the map

$$F(A) \rightarrow F(B) \times_{G(B)} G(A)$$

is an isomorphism.

2.3. Homogeneity.

Definition 2.17. Say that a natural transformation $F \rightarrow G$ of functors is (relatively) homogeneous if for all square-zero extensions $A \rightarrow B$, the map

$$F(A \times_B C) \rightarrow G(A \times_B C) \times_{[G(A) \times_{G(B)} G(C)]} [F(A) \times_{F(B)} F(C)]$$

is an isomorphism. Say that F is homogeneous if $F \rightarrow \bullet$ is relatively homogeneous.

Proposition 2.18. *Let $\alpha : F \rightarrow G$ be a formally étale morphism of functors $F, G : d\mathcal{N}^p \rightarrow \text{Set}$. If G is homogeneous, then so is F . Conversely, if α is surjective and F is homogeneous, then so is G .*

Proof. Take a square-zero extension $A \rightarrow B$, and a morphism $C \rightarrow B$, noting that $A \times_B C \rightarrow C$ is then another square-zero extension. Since α is formally étale,

$$F(A \times_B C) \cong G(A \times_B C) \times_{GC} FC,$$

and

$$\begin{aligned} FA \times_{FB} FC &\cong [GA \times_{GB} FB] \times_{FB} FC \\ &= GA \times_{GB} FC \\ &= [GA \times_{GB} GC] \times_{GC} FC. \end{aligned}$$

Thus homogeneity of G implies homogeneity of F , and if $\pi_0 FC \rightarrow \pi_0 GC$ is surjective for all C , then homogeneity of F implies homogeneity of G . \square

2.4. Simplicial categories.

Definition 2.19. Given functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{G} \mathcal{C}$ between categories, define the 2-fibre product $\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{C}$ as follows. Objects of $\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{C}$ are triples (a, θ, c) , for $a \in \mathcal{A}, c \in \mathcal{C}$ and $\theta : Fa \rightarrow Gc$ an isomorphism in \mathcal{B} . A morphism in $\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{C}$ from (a, θ, c) to (a', θ', c') is a pair (f, g) , where $f : a \rightarrow a'$ is a morphism in \mathcal{A} and $g : c \rightarrow c'$ a morphism in \mathcal{C} , satisfying the condition that

$$Gg \circ \theta = \theta' \circ Ff.$$

Remark 2.20. This definition has the property that $\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{C}$ is a model for the 2-fibre product in the 2-category of categories. However, we will always use the notation $\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{C}$ to mean the specific model of Definition 2.19, and not merely any equivalent category.

Also note that

$$\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{C} = (\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{B}) \times_{\mathcal{B}} \mathcal{C}$$

Definition 2.21. Given functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{G} \mathcal{C}$ between simplicial categories, define the 2-fibre product $\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{C}$ as follows. Objects of $\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{C}$ are triples (a, θ, c) , for $a \in \mathcal{A}, c \in \mathcal{C}$ and $\theta : Fa \rightarrow Gc$ an isomorphism in \mathcal{B}_0 . Morphisms are given by setting

$$\underline{\text{Hom}}_{\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{C}}((a, \theta, c), (a', \theta', c')) := \{(f, g) \in \underline{\text{Hom}}_{\mathcal{A}}(a, a') \times \underline{\text{Hom}}_{\mathcal{C}}(c, c') : Gg \circ \theta = \theta' \circ Ff\}.$$

Definition 2.22. Say that a morphism $F : \mathcal{A} \rightarrow \mathcal{B}$ in $s\text{Cat}$ is a (trivial) 2-fibration if $\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{B} \rightarrow \mathcal{B}$ is a (trivial) fibration in the sense of Definition 1.5. Explicitly, F is a 2-fibration if

- (F1) for any objects a_1 and a_2 in \mathcal{A} , the map $\underline{\text{Hom}}_{\mathcal{A}}(a_1, a_2) \rightarrow \underline{\text{Hom}}_{\mathcal{B}}(Fa_1, Fa_2)$ is a fibration of simplicial sets;
- (F2) for any objects $a_1 \in \mathcal{A}, b \in \mathcal{B}$, and any homotopy equivalence $e : Fa_1 \rightarrow b$ in \mathcal{B} , there is an object $a_2 \in \mathcal{C}$, a homotopy equivalence $d : a_1 \rightarrow a_2$ in \mathcal{C} and an isomorphism $\theta : Fa_2 \rightarrow b$ such that $\theta \circ Fd = e$.

Likewise, F is a trivial 2-fibration if

(TF1) for any objects a_1 and a_2 in \mathcal{A} , the map $\underline{\text{Hom}}_{\mathcal{A}}(a_1, a_2) \rightarrow \underline{\text{Hom}}_{\mathcal{B}}(Fa_1, Fa_2)$ is a trivial fibration of simplicial sets;

(TF2) $F_0 : \mathcal{A}_0 \rightarrow \mathcal{B}_0$ is essentially surjective on objects.

Definition 2.23. Say that a natural transformation $\eta : \mathcal{D} \rightarrow \mathcal{E}$ of functors $\mathcal{D} : d\mathcal{N}^{\flat} \rightarrow s\text{Cat}$ is 2-homotopic if for all tiny acyclic extensions $A \rightarrow B$, the map

$$\mathcal{D}(A) \rightarrow \mathcal{D}(B) \times_{\mathcal{E}(B)}^{(2)} \mathcal{E}(A)$$

is a trivial 2-fibration. Say that \mathcal{D} is 2-homotopic if $\mathcal{D} \rightarrow \bullet$ is so.

Lemma 2.24. *If \mathcal{D} is 2-homotopic, then \mathcal{D} is homotopy-preserving.*

Proof. By [Pri3] Lemma 2.11 and the proof of [Pri3] Proposition 2.7, it suffices to show that \mathcal{D} maps tiny acyclic extensions $A \rightarrow B$ to weak equivalences. Since $\mathcal{D}_0(A) \rightarrow \mathcal{D}_0(B)$ is essentially surjective on objects, so is $\pi_0 \mathcal{D}(A) \rightarrow \pi_0 \mathcal{D}(B)$. On morphisms, we have $\underline{\text{Hom}}_{\mathcal{D}(A)}(x, y) \rightarrow \underline{\text{Hom}}_{\mathcal{D}(B)}(x, y)$ a trivial fibration (and hence a weak equivalence) for all $x, y \in \mathcal{D}(A)$, so $\mathcal{D}(A) \rightarrow \mathcal{D}(B)$ is indeed a weak equivalence. \square

Definition 2.25. Say that a natural transformation $\eta : \mathcal{D} \rightarrow \mathcal{E}$ of functors $\mathcal{D} : d\mathcal{N}^{\flat} \rightarrow s\text{Cat}$ is formally 2-quasi-presmooth if for all square-zero extensions $A \rightarrow B$, the map

$$\mathcal{D}(A) \rightarrow \mathcal{D}(B) \times_{\mathcal{E}(B)}^{(2)} \mathcal{E}(A)$$

is a 2-fibration. If η is also 2-homotopic, then we say that η is formally 2-quasi-smooth.

Say that \mathcal{D} is formally 2-quasi-presmooth (resp. formally 2-quasi-smooth) if $\mathcal{D} \rightarrow \bullet$ is so.

Definition 2.26. Say that a functor $\mathcal{F} : d\mathcal{N}^{\flat} \rightarrow s\text{Cat}$ is 2-homogeneous if for all square-zero extensions $A \rightarrow B$, the map

$$\mathcal{F}(A \times_B C) \rightarrow \mathcal{F}(A) \times_{\mathcal{F}(B)}^{(2)} \mathcal{F}(C)$$

is essentially surjective on objects (for \mathcal{F}_0), and an isomorphism on Hom-spaces.

Lemma 2.27. *If $\mathcal{D} : d\mathcal{N}^{\flat} \rightarrow s\text{Cat}$ is 2-homogeneous and 2-quasi-presmooth, then \mathcal{D} is homotopy-homogeneous.*

Proof. Take a square-zero extension $A \rightarrow B$ and a map $C \rightarrow B$. We first need to show that

$$\theta : \pi_0 \mathcal{D}(A \times_B C) \rightarrow \pi_0(\mathcal{D}(A) \times_{\mathcal{D}(B)}^h \mathcal{D}(C))$$

is essentially surjective on objects.

Any object of the right-hand side is represented by objects $x \in \mathcal{D}(A)$, $y \in \mathcal{D}(B)$, and a homotopy equivalence $f : \bar{x} \rightarrow \bar{y}$ in $\mathcal{D}(B)$. Since \mathcal{D} is 2-quasi-presmooth, we may lift f to some map $\tilde{f} : x \rightarrow z$ in $\mathcal{D}(A)_0$ with $g : \bar{z} \cong \bar{y}$ in $\mathcal{D}(B)_0$. Since \mathcal{D} is 2-homogeneous, there must then be an object $t \in \mathcal{D}(A \times_B C)$ whose image in $\mathcal{D}(A)_0 \times_{\mathcal{D}(B)_0}^{(2)} \mathcal{D}(C)_0$ is isomorphic to (z, g, y) . Since \tilde{f} gives a homotopy equivalence from this to (x, f, y) , we have shown that θ is essentially surjective on objects.

For morphisms, we have

$$\begin{aligned} \underline{\text{Hom}}_{\mathcal{D}(A \times_B C)}(t, t') &\cong \underline{\text{Hom}}_{\mathcal{D}(A)}(t, t') \times_{\underline{\text{Hom}}_{\mathcal{D}(B)}(t, t')} \underline{\text{Hom}}_{\mathcal{D}(C)}(t, t') \\ &\simeq \underline{\text{Hom}}_{\mathcal{D}(A)}(t, t') \times_{\underline{\text{Hom}}_{\mathcal{D}(B)}(t, t')}^h \underline{\text{Hom}}_{\mathcal{D}(C)}(t, t'), \end{aligned}$$

the first isomorphism following from homogeneity, and the second because $\underline{\text{Hom}}_{\mathcal{D}(A)}(t, t') \rightarrow \underline{\text{Hom}}_{\mathcal{D}(B)}(t, t')$ is a fibration, by quasi-smoothness. This completes the proof that

$$\mathcal{D}(A \times_B C) \simeq \mathcal{D}(A) \times_{\mathcal{D}(B)}^h \mathcal{D}(C)$$

in $s\text{Cat}$. \square

Definition 2.28. Say that $\mathcal{D} : d\mathcal{N}^b \rightarrow s\text{Cat}$ has formally quasi-presmooth (resp. formally quasi-smooth) Hom-spaces if $\underline{\text{Hom}}_{\mathcal{D}}(x, y) : A \downarrow d\mathcal{N}^b \rightarrow \mathbb{S}$ is formally quasi-presmooth (resp. formally quasi-smooth) for all $A \in d\mathcal{N}^b$ and $x, y \in \mathcal{D}(A)$.

Definition 2.29. Given a simplicial category \mathcal{C} , define $\mathcal{W}(\mathcal{C})$ to be the full simplicial subcategory in which morphisms are maps whose images in $\pi_0\mathcal{C}$ are invertible. In particular, this means that $\pi_0\mathcal{W}(\mathcal{C})$ is the core (maximal subgroupoid) of $\pi_0\mathcal{C}$.

Proposition 2.30. *If $\mathcal{D} : d\mathcal{N}^b \rightarrow s\text{Cat}$ is 2-homogeneous with formally quasi-smooth Hom-spaces, then $\mathcal{W}(\mathcal{D}) \rightarrow \mathcal{D}$ is formally étale, in the sense that for any square-zero extension $A \rightarrow B$, the map*

$$\mathcal{W}(\mathcal{D})(A) \rightarrow \mathcal{D}(A) \times_{\mathcal{D}(B)} \mathcal{W}(\mathcal{D})(B)$$

is an isomorphism.

Proof. Since $f \in \underline{\text{Hom}}_{\mathcal{C}}(x, y)_n$ lies in $\mathcal{W}(\mathcal{C})$ if and only if $(\partial_0)^n f$ is a homotopy equivalence, it suffices to show that

$$\mathcal{W}(\mathcal{D})_0(A) \rightarrow \mathcal{D}_0(A) \times_{\mathcal{D}_0(B)} \mathcal{W}(\mathcal{D})_0(B)$$

is an isomorphism.

Take a morphism $f \in \underline{\text{Hom}}_{\mathcal{D}(B)}(x, y)_0$, with objects x, y lifting to $x, y \in \mathcal{D}(A)$. If we set $D_f^n(M) := D_f^n(\underline{\text{Hom}}_{\mathcal{D}}(x, y)/A, M)$, then Proposition 2.8 shows that the obstruction $o(f)$ to lifting \tilde{f} to $\underline{\text{Hom}}_{\mathcal{D}(A)}(x, y)_0$ lies in $D_{\tilde{f}}^1(I)$, where $I = \ker(A \rightarrow B)$.

Assume that f is a homotopy equivalence, so there exist $g \in \underline{\text{Hom}}_{\mathcal{D}(B)}(y, x)_0$ and homotopies $h^x \in \underline{\text{Hom}}_{\mathcal{D}(B)}(x, x)_1, h^y \in \underline{\text{Hom}}_{\mathcal{D}(B)}(y, y)_1$ such that

$$\partial_1 h^x = \text{id}_x, \quad \partial_1 h^y = \text{id}_y, \quad \partial_0 h^x = g \circ \tilde{f}, \quad \partial_0 h^y = \tilde{f} \circ g.$$

We now need to study the construction of obstruction maps.

For the acyclic extension $\tilde{B} \rightarrow B$ as in the Proof of Proposition 2.8, we may lift f to \tilde{f} in $\mathcal{D}(\tilde{B})$, and do the same for g . Then the obstruction $o(f \circ g)$ to lifting $f \circ g$ is the class of the image of $\tilde{f} \circ \tilde{g}$ in $\pi_0 \underline{\text{Hom}}_{\mathcal{D}}(B \oplus I[-1])(y, y)$, which is just

$$o(f \circ g) = f \circ g + f_* o(g) + g^* o(f) \in D_{f \circ g}^1(I).$$

Since $\underline{\text{Hom}}_{\mathcal{D}(\tilde{B})}(x, x) \rightarrow \underline{\text{Hom}}_{\mathcal{D}(B)}(x, x)$ is a trivial fibration (by quasi-smoothness), we may also lift the homotopies h^x, h^y to give homotopies $\tilde{h}^x \in \underline{\text{Hom}}_{\mathcal{D}(\tilde{B})}(x, x)_1, \tilde{h}^y \in \underline{\text{Hom}}_{\mathcal{D}(\tilde{B})}(y, y)_1$ making \tilde{f} and \tilde{g} homotopy inverses. Then \tilde{h}^y ensures that $o(f \circ g) = 0$ (since the image of $\tilde{f} \circ \tilde{g}$ is homotopic to id_y), and therefore $f_* o(g) = -g^* o(f) \in D_y^1(I)$. Likewise $g_* o(f) = -f^* o(g) \in D_x^1(I)$. Since $o(g) = (gf)_*(fg)^* o(g)$, this gives $o(g) = -g_* g^* o(f)$

If f lifts to $f' \in \underline{\text{Hom}}_{\mathcal{D}(A)}(x, y)_0$, then $o(f) = 0$, and the formula above ensures that $o(g) = 0$, so g also lifts to g' in $\mathcal{D}(A)$. Now, $f' \circ g'$ is in the fibre of $\underline{\text{End}}_{\mathcal{D}(A)}(y)_0 \rightarrow \underline{\text{End}}_{\mathcal{D}(B)}(y)_0$ over id_y , so $[f' \circ g'] \in \pi_0 \underline{\text{End}}_{\mathcal{D}(A)}(y)$ lies in the image of

$$\pi_0 T_{\text{id}_y}(\underline{\text{End}}_{\mathcal{D}}(y)/A, I) = D_y^0(I).$$

Another choice for g' is of the form $g' + a$ for $a \in D_g^0(I)$, and then $[f' \circ (g' + 1)] = [f' \circ g'] + f_* a$. Since f is a homotopy equivalence, $f_* : D_g^0(I) \rightarrow D_y^0(I)$ is an isomorphism, so we may set $a = -(f_*)^{-1}[f' \circ g']$. Without loss of generality, we may therefore assume that f' and g' are homotopy inverses.

Finally, write $[g' \circ f'] = \text{id}_x + b$, for $b \in D_x^0(I)$. Since $f' \circ g'$ is homotopic to id_y , applying f_* to b gives

$$[f'] + f_* b = [f' \circ g' \circ f'] = [f'],$$

so $f_* b = 0 \in D_f^0(I)$. Thus $b = 0$, as $f_* : D_x^0(I) \rightarrow D_f^0(I)$ is an isomorphism. \square

Corollary 2.31. *If $\mathcal{D} : d\mathcal{N}^b \rightarrow s\text{Cat}$ is formally 2-quasi-smooth and 2-homogeneous, then so is $\mathcal{W}(\mathcal{D})$.*

Proof. For any square-zero extension $A \rightarrow B$ in $d\mathcal{N}^b$, we have

$$\mathcal{W}(\mathcal{D})(A) = \mathcal{D}(A) \times_{\mathcal{D}(B)} \mathcal{W}(\mathcal{D})(B),$$

by Proposition 2.30. Thus

$$\mathcal{W}(\mathcal{D})(A) \times_{\mathcal{W}(\mathcal{D})(B)}^{(2)} \mathcal{W}(\mathcal{D})(B) = \mathcal{D}(A) \times_{\mathcal{D}(B)}^{(2)} \mathcal{W}(\mathcal{D})(B),$$

so $\theta : \mathcal{W}(\mathcal{D})(A) \times_{\mathcal{W}(\mathcal{D})(B)}^{(2)} \mathcal{W}(\mathcal{D})(B) \rightarrow \mathcal{W}(\mathcal{D})(B)$ is the pullback of $\mathcal{D}(A) \times_{\mathcal{D}(B)}^{(2)} \mathcal{D}(B) \rightarrow \mathcal{D}(B)$ along $\mathcal{W}(\mathcal{D})(B) \rightarrow \mathcal{D}(B)$. By 2-quasi-smoothness of \mathcal{D} , this means that θ is a fibration in $s\text{Cat}$, and a trivial fibration whenever $A \rightarrow B$ is a tiny acyclic extension. Thus $\mathcal{W}(\mathcal{D})$ is formally 2-quasi-smooth.

To see that $\mathcal{W}(\mathcal{D})$ is 2-homogeneous, first observe that $\mathcal{D}(A) \times_{\mathcal{D}(B)}^{(2)} \mathcal{D}(C)$ and $\mathcal{W}(\mathcal{D})(A) \times_{\mathcal{W}(\mathcal{D})(B)}^{(2)} \mathcal{W}(\mathcal{D})(C)$ have the same objects. For a square-zero extension $A \rightarrow B$ and a map $C \rightarrow B$, 2-homogeneity of \mathcal{D} then gives that

$$\mathcal{W}(\mathcal{D})_0(A \times_B C) \rightarrow \mathcal{W}(\mathcal{D})_0(A) \times_{\mathcal{W}(\mathcal{D})_0(B)}^{(2)} \mathcal{W}(\mathcal{D})_0(C)$$

is essentially surjective on objects. Proposition 2.30 combines with 2-homogeneity of \mathcal{D} and Lemma 2.18 to give that $\underline{\text{Hom}}_{\mathcal{W}(\mathcal{D})}(x, y)$ is homogeneous, since $\underline{\text{Hom}}_{\mathcal{W}(\mathcal{D})}(x, y) \rightarrow \underline{\text{Hom}}_{\mathcal{D}}(x, y)$ is formally étale. \square

3. DERIVED MODULI OF SCHEMES

3.1. Derived moduli of affine schemes.

Definition 3.1. Given $A \in dg_+\text{Alg}_R$, say that $B \in dg_+\text{Alg}_A$ is quasi-free if the underlying map $A_* \rightarrow B_*$ of graded algebras is freely generated. Say that $B \in dg_+\text{Alg}_A$ is cofibrant if it is a retract of a quasi-free dg algebra.

This is equivalent to satisfying the condition that for all acyclic surjections $C' \rightarrow C$ in $dg_+\text{Alg}_A$, the map $\text{Hom}_{dg_+\text{Alg}_A}(B, C') \rightarrow \text{Hom}_{dg_+\text{Alg}_A}(B, C)$ is surjective.

Definition 3.2. Given $A \in s\text{Alg}_R$, say that $B \in s\text{Alg}_A$ is quasi-free if there are sets $\Sigma_q \subset B_q$ with $B_q = A_q[\Sigma_q]$, such that $\bigcup_q \Sigma_q$ is closed under the degeneracy operations. Say that $B \in dg_+\text{Alg}_A$ is cofibrant if it is a retract of a quasi-free simplicial algebra.

This is equivalent to satisfying the condition that for all acyclic surjections $C' \rightarrow C$ in $dg_+\text{Alg}_A$, the map $\text{Hom}_{s\text{Alg}_A}(B, C') \rightarrow \text{Hom}_{s\text{Alg}_A}(B, C)$ is surjective.

Definition 3.3. Let $\Delta^n \in \mathbb{S}$ be the standard n -simplex, and $\partial\Delta^n \in \mathbb{S}$ its boundary. Given $0 \leq k \leq n$, define the k th horn Λ_k^n of Δ^n to be the simplicial set obtained from Δ^n by removing the interior and the k th face. See [GJ] §I.1 for explicit descriptions.

Definition 3.4. There is a simplicial structure on $s\text{Alg}_A$ given as follows. For $B \in s\text{Alg}_A$ and $K \in \mathbb{S}$, B^K is defined by

$$(B^K)_n := \text{Hom}_{\mathbb{S}}(K \times \Delta^n, B).$$

Spaces $\underline{\text{Hom}}_{s\text{Alg}_A}(C, B) \in \mathbb{S}$ of morphisms are then given by

$$\underline{\text{Hom}}_{s\text{Alg}_A}(C, B)_n := \text{Hom}_{s\text{Alg}_A}(C, B^{\Delta^n}).$$

Definition 3.5. As in [Pri3] Lemma 3.4, put a simplicial structure on $dg_+\text{Alg}_A$ as follows. First set Ω_n to be the cochain algebra

$$\mathbb{Q}[t_0, t_1, \dots, t_n, dt_0, dt_1, \dots, dt_n] / (\sum t_i - 1, \sum dt_i)$$

of rational differential forms on the n -simplex Δ^n . These fit together to form a simplicial complex Ω_\bullet of DG-algebras, and we define B^{Δ^n} as the good truncation $B^{\Delta^n} := \tau_{\geq 0}(B \otimes \Omega_n)$. Spaces $\underline{\mathrm{Hom}}_{dg_+ \mathrm{Alg}_A}(C, B) \in \mathbb{S}$ of morphisms are then given by

$$\underline{\mathrm{Hom}}_{dg_+ \mathrm{Alg}_A}(C, B)_n := \mathrm{Hom}_{dg_+ \mathrm{Alg}_A}(C, B^{\Delta^n}).$$

Definition 3.6. For $A \in dg_+ \mathrm{Alg}_R$ (resp. $A \in s\mathrm{Alg}_R$), define $\mathrm{ALG}(A)$ to be the full simplicial subcategory of the category $dg_+ \mathrm{Alg}_A$ (resp. $s\mathrm{Alg}_A$) on cofibrant objects.

Proposition 3.7. *The functor $\mathrm{ALG} : d\mathcal{N}^b \rightarrow s\mathrm{Cat}$ defined above is 2-homogeneous and formally 2-quasi-smooth.*

Proof. We first prove that ALG is 2-homogeneous, taking a square-zero extension $A \rightarrow B$ and a morphism $C \rightarrow B$. Homogeneity of $\underline{\mathrm{Hom}}_{\mathrm{ALG}}$ is almost immediate, because $S \in \mathrm{ALG}(A \times_B C)$ is flat over $A \times_B C$ as a graded algebra (resp. levelwise flat over $A \times_B C$), giving the pullback square

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{\mathrm{ALG}(A \times_B C)}(S, S') & \longrightarrow & \underline{\mathrm{Hom}}_{\mathrm{ALG}(A)}(S \otimes_{A \times_B C} A, S' \otimes_{A \times_B C} A) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}_{\mathrm{ALG}(C)}(S \otimes_{A \times_B C} C, S' \otimes_{A \times_B C} C) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathrm{ALG}(B)}(S \otimes_{A \times_B C} B, S' \otimes_{A \times_B C} B). \end{array}$$

Given objects $S_A \in \mathrm{ALG}(A)$ and $S_C \in \mathrm{ALG}(C)$, and an isomorphism $\alpha : S_A \otimes_A B \rightarrow S_C \otimes_C B$, we can define $S \in \mathrm{ALG}(A \times_B C)$ by $S := S_A \times_{\alpha, S_C \otimes_C B} S_C$, noting that this is automatically cofibrant, with $S \otimes_{A \times_B C} A \cong S_A$ and $S \otimes_{A \times_B C} C \cong S_C$. This completes the proof that ALG is 2-homogeneous.

Now, since any $S \in \mathrm{ALG}(A)$ is cofibrant over A , the map $f : \underline{\mathrm{Hom}}_{\mathrm{ALG}(A)}(S, S') \rightarrow \underline{\mathrm{Hom}}_{\mathrm{ALG}(B)}(S \otimes_A B, S' \otimes_A B)$ is a fibration for all square-zero extensions $A \rightarrow B$, because $S' \rightarrow S' \otimes_A B$ is also a square-zero extension. If $A \rightarrow B$ is also a weak equivalence, then so is $S' \rightarrow S' \otimes_A B$ (as S' is levelwise flat over A , so the kernel is $S' \otimes_A I$), and f is therefore a trivial fibration. We have therefore shown that $\underline{\mathrm{Hom}}_{\mathrm{ALG}}(S, S')$ is formally quasi-smooth.

It only remains to show that $\mathrm{ALG}(A) \rightarrow \mathrm{ALG}(B)$ satisfies Condition (F2) from Definition 2.22 for all square-zero extensions $A \rightarrow B$, and satisfies condition (TF2) whenever the extension is acyclic. [Pri1] Proposition 10.6 shows that the obstruction (in the simplicial case) to lifting $S \in \mathrm{ALG}(B)$ to $\mathrm{ALG}(A)$ lies in

$$\mathbb{E}\mathrm{xt}_B^2(\Omega^{S/B}, S \otimes_B I) = \mathrm{D}_{\mathrm{id}_S}^2(\underline{\mathrm{Hom}}_{\mathrm{ALG}}(S, S), I),$$

where $I = \ker(A \rightarrow B)$. The idea is that as an almost simplicial algebra (i.e. discarding ∂_0), S has a unique levelwise flat lift to \tilde{S} over A . We can then choose a lifting δ of ∂_0 with $\partial_i \delta = \delta \partial_{i-1}$ for all $i > 1$, and the obstruction comes from comparing δ^2 and $\delta \partial_1$. The same proof adapts to the dg case, where we lift S as a flat graded algebra, lift d to δ , and get δ^2 as the obstruction. In particular, this ensures that (TF2) is satisfied, since if $H_*(I) = 0$, then $\mathbb{E}\mathrm{xt}_B^2(\Omega^{S/B}, S \otimes_B I) = 0$.

We now establish (F1): take $S \in \mathrm{ALG}(A)$, write $\bar{S} := S \otimes_A B$, and take a homotopy equivalence $\theta : \bar{S} \rightarrow T$. We now proceed with the proof in the dg case — the simplicial case is entirely similar, replacing graded algebras with almost simplicial algebras. As a graded algebra, T has a unique lift \tilde{T} over A , and we can then lift θ to $\tilde{\theta} : S \rightarrow \tilde{T}$ (as a graded morphism), since S is smooth. We also lift the differential d on T to δ on \tilde{T} . The obstruction to these giving a morphism of dg algebras is then

$$(u, v) := (\delta^2, \delta \circ \tilde{\theta} - \tilde{\theta} \circ d) \in \mathrm{HOM}_B^2(\Omega^{T/B}, T \otimes_B I) \times \mathrm{HOM}_B^1(\Omega^{\bar{S}/B}, T \otimes_B I),$$

which satisfies $du = 0, dv = u \circ \theta$. Here, HOM_B^i denotes B -linear graded morphisms of chain degree $-i$. Alternative choices for $(\delta, \tilde{\theta})$ would be of the form $(\delta + a, \tilde{\theta} + b)$, for

$$(a, b) \in \mathrm{HOM}_B^1(\Omega^{T/B}, T \otimes_B I) \times \mathrm{HOM}_B^0(\Omega^{\bar{S}/B}, T \otimes_B I);$$

these send (u, v) to $(u + da, v + db + a \circ \theta)$. Thus the obstruction to lifting θ and T compatibly lies in H^2 of the cone complex

$$\mathrm{HOM}_B^\bullet(\Omega^{T/B}, T \otimes_B I) \xrightarrow{\theta^*} \mathrm{HOM}_B^\bullet(\Omega^{\bar{S}/B}, T \otimes_B I).$$

Since θ is a weak equivalence, θ^* is a quasi-isomorphism, so cohomology of this complex is 0, and suitable liftings thus exist. \square

Definition 3.8. Given a functor $\mathcal{C} : \mathrm{Alg}_{\mathbb{H}_0R} \rightarrow s\mathrm{Cat}$ and a functorial simplicial subcategory $\mathcal{M} \subset \mathcal{C}$, say that \mathcal{M} is an *open* simplicial subcategory of \mathcal{C} if

- (1) \mathcal{M} is a full simplicial subcategory, i.e. for all $X, Y \in \mathcal{M}(A)$, the map $\underline{\mathrm{Hom}}_{\mathcal{M}(A)}(X, Y) \rightarrow \underline{\mathrm{Hom}}_{\mathcal{C}(A)}(X, Y)$ is a weak equivalence;
- (2) the map $\mathcal{M} \rightarrow \mathcal{C}$ is homotopy formally étale, in the sense that for any square-zero extension $A \rightarrow B$, the map

$$\pi_0\mathcal{M}(A) \rightarrow \pi_0\mathcal{C}(A) \times_{\pi_0\mathcal{C}(B)}^{(2)} \pi_0\mathcal{M}(B)$$

is essentially surjective on objects.

Definition 3.9. Given a simplicial category \mathcal{C} for which $\pi_0\mathcal{C}$ is a groupoid, say that \mathcal{C} is n -truncated if

$$\pi_i \underline{\mathrm{Hom}}_{\mathcal{C}(A)}(X, Y) = 0$$

for all $i \geq n$. Thus \mathcal{C} is n -truncated if and only if $\pi_i(\bar{W}\mathcal{C}) = 0$ for all $i > n$.

Corollary 3.10. Take an n -truncated hypersheaf $\mathcal{M} : \mathrm{Alg}_{\mathbb{H}_0R} \rightarrow s\mathrm{Cat}$, open in $\pi^0\mathcal{W}(\mathrm{ALG})$ (in the sense of Definition 3.8), such that

- (1) for all finitely generated $A \in \mathrm{Alg}_{\mathbb{H}_0R}$ and all $X \in \mathcal{M}(A)$, the functors

$$\mathbb{E}\mathrm{xt}_{f(X)}^i(\mathbb{L}^{f(X)/A}, f(X) \otimes_A^{\mathbf{L}} -) : \mathrm{Mod}_A \rightarrow \mathrm{Ab}$$

preserve filtered colimits for all $i > 1$.

- (2) for all finitely generated integral domains $A \in \mathrm{Alg}_{\mathbb{H}_0R}$ and all $X \in \mathcal{M}(A)$, the groups $\mathbb{E}\mathrm{xt}_{f(X)}^i(\mathbb{L}^{f(X)/A}, f(X))$ are all finitely generated A -modules.

- (3) for all discrete local Noetherian \mathbb{H}_0R -algebras A , with maximal ideal \mathfrak{m} , the map

$$\mathcal{M}(A) \rightarrow \varprojlim^h \mathcal{M}(A/\mathfrak{m}^n)$$

is a weak equivalence (see Remark 2.11 for a reformulation).

- (4) The functor $c(\pi_0\mathcal{M}) : \mathrm{Alg}_{\pi_0R} \rightarrow \mathrm{Set}$ of components of the groupoid $\pi_0\mathcal{M}$ preserves filtered colimits.
- (5) For all $A \in \mathrm{Alg}_{\pi_0R}$ and all $X \in \mathcal{M}(A)$, the functors $\pi_i \underline{\mathrm{Hom}}_{\mathcal{M}}(X, X) : \mathrm{Alg}_A \rightarrow \mathrm{Set}$ preserve filtered colimits for all $i \geq 0$.

Let $\tilde{\mathcal{M}} : d\mathcal{N}_R^p \rightarrow s\mathrm{Cat}$ be the full simplicial subcategory of $\mathcal{W}(\mathrm{ALG})(A)$ consisting of objects C for which $C \otimes_A \mathbb{H}_0A$ is weakly equivalent (in $\mathcal{W}(\mathrm{ALG})(\mathbb{H}_0A)$) to an object of $\mathcal{M}(\mathbb{H}_0A)$. Then the nerve $\bar{W}\tilde{\mathcal{M}}$ is a geometric derived n -stack.

Proof. Since $\pi_0\mathcal{W}(\mathrm{ALG})$ is a groupoid-valued functor, and $\mathcal{M} \hookrightarrow \pi^0\mathcal{W}(\mathrm{ALG})$ is open, the map

$$\mathcal{M} \rightarrow \pi^0\mathcal{W}(\mathrm{ALG})$$

is homotopy formally étale. This also implies that $\bar{W}\tilde{\mathcal{M}}$ is a model for

$$A \mapsto \bar{W}\mathcal{M}(\mathbb{H}_0A) \times_{\bar{W}\mathcal{W}(\mathrm{ALG})(\mathbb{H}_0A)}^h \bar{W}\mathcal{W}(\mathrm{ALG})(A).$$

Now observe that this satisfies the conditions of Theorem 2.10, using Lemmas 2.24 and 2.27, Propositions 2.30 and 3.7, and Corollary 2.31. The proof of Proposition 3.7 shows that

$$D_X^i(\bar{W}\tilde{\mathcal{M}}/R, M) = \mathbb{E}\mathrm{xt}_{f(X)}^{i+1}(\mathbb{L}^{f(X)/A}, f(X) \otimes_A^{\mathbf{L}} M).$$

\square

Example 3.11 (Moduli of finite schemes). Define $\mathcal{M} : \text{Alg}_{\mathbb{H}_0 R} \rightarrow s\text{Cat}$ by setting $\mathcal{M}(A) \subset \pi^0 \mathcal{W}(\text{ALG})(A)$ to consist of dg (resp. simplicial) A -algebras S for which $\mathbb{H}_0 S$ is a finite flat A -module, and $\mathbb{H}_i S = 0$ for all $i > 0$. Then we get

$$\tilde{\mathcal{M}} \simeq \mathcal{M} \times_{\pi^0 \text{ALG}}^{(2)} \text{ALG}.$$

Now, \mathbb{H}_0 gives a functor from $\mathcal{M}(A)$ to the groupoid $\mathcal{W}(\text{FALG}(A))$ of finite flat A -algebras, and this is a weak equivalence in $s\text{Cat}$, so

$$\mathcal{M} \simeq \mathcal{W}(\text{FALG}),$$

which is a hypersheaf by [Pri3] Remark 1.29. Thus the corollary above implies that the nerve $\tilde{\mathcal{W}}\tilde{\mathcal{M}}$ is a geometric derived 1-stack, which we can now regard as the derived moduli stack of finite flat algebras.

We now check show that this example is consistent with the flatness criteria used in [TV2] and [Lur1]

Definition 3.12. Given $A \in dg_+ \text{Alg}_R$ (resp. $A \in s\text{Alg}_R$) say that $M \in dg_+ \text{Mod}_A$ (resp. $M \in s\text{Mod}_A$) is homotopy-flat over A if $\mathbb{H}_0 M$ is flat over $\mathbb{H}_0 A$, and the maps

$$\mathbb{H}_n(A) \otimes_{\mathbb{H}_0 A} \mathbb{H}_0 M \rightarrow \mathbb{H}_n M$$

are isomorphisms for all n .

Note that [TV2] Lemma 2.2.2.2 implies that this is the same as flatness in the sense of [TV2] Definition 1.2.4.1, which corresponds to the functor $\otimes_A^{\mathbf{L}} M : \text{Ho}(dg_+ \text{Mod}_A) \rightarrow \text{Ho}(dg_+ \text{Mod}_A)$ (resp. $\otimes_A^{\mathbf{L}} M : \text{Ho}(s\text{Mod}_A) \rightarrow \text{Ho}(s\text{Mod}_A)$) preserving homotopy pullbacks.

Lemma 3.13. *A morphism $f : A \rightarrow B$ in $A \in dg_+ \text{Alg}_R$ (resp. $s\text{Alg}_R$) is homotopy-flat if and only if $B \otimes_A^{\mathbf{L}} \mathbb{H}_0 A$ is weakly equivalent to a discrete flat $\mathbb{H}_0 A$ -algebra.*

Proof. If f is flat, then the proof of [TV2] Lemma 2.2.2.2(2) shows that for any A -module N , we have $\mathbb{H}_*(\mathbf{L}f^* N) \cong \mathbb{H}_0 B \otimes_{\mathbb{H}_0 A} \mathbb{H}_* N$. Setting $N = \mathbb{H}_0 A$, this gives $\mathbf{L}f^* \mathbb{H}_0 A \simeq \mathbb{H}_0 B$. This is flat over $\mathbb{H}_0 A$ by hypothesis, completing the proof of one implication.

For the converse, assume that $\mathbf{L}f^* \mathbb{H}_0 A \simeq \mathbb{H}_0 B$, flat over $\mathbb{H}_0 A$. By considering free simplicial resolutions, this implies that $\mathbf{L}f^* M \simeq \mathbb{H}_0 B \otimes_{\mathbb{H}_0 A} M$ for all $\mathbb{H}_0 A$ -modules M . We have a spectral sequence

$$\mathbb{H}_i(\mathbf{L}f^* \mathbb{H}_j A) \implies \mathbb{H}_{i+j}(B),$$

giving $\mathbb{H}_j B \cong \mathbb{H}_0 B \otimes_{\mathbb{H}_0 A} \mathbb{H}_j A$, which completes the proof. \square

In particular, this means that B is homotopy-flat over A if and only if for all $\mathbb{H}_0 A$ -algebras C and all $i > 0$, we have $\mathbb{H}_i(B \otimes_A^{\mathbf{L}} C) = 0$.

Proposition 3.14. *For $T \in dg_+ \text{Alg}_R$ (resp. $T \in s\text{Alg}_R$) flat as a graded algebra over R , the functor $\text{ALG}_T := \text{ALG}(T \otimes -) : d\mathcal{N}^{\flat} \rightarrow s\text{Cat}$ is 2-homogeneous and formally 2-quasi-smooth.*

Proof. The proof of Proposition 3.7 carries over to this generality, noting that for any (acyclic) square-zero extension $A \rightarrow B$, we get another (acyclic) square-zero extension $T \otimes A \rightarrow T \otimes B$. \square

Corollary 3.15. *Take an n -truncated hypersheaf $\mathcal{M} : \text{Alg}_{\mathbb{H}_0 R} \rightarrow s\text{Cat}$, open in $\pi^0 \mathcal{W}(\text{ALG}_T)$, such that*

- (1) *for all finitely generated $A \in \text{Alg}_{\mathbb{H}_0 R}$ and all $X \in \mathcal{M}(A)$, the functors*

$$\mathbb{E}xt_{f(X)}^i(\mathbb{L}^{f(X)/A \otimes T}, f(X) \otimes_A^{\mathbf{L}} -) : \text{Mod}_A \rightarrow \text{Ab}$$

preserve filtered colimits for all $i > 0$.

- (2) for all finitely generated integral domains $A \in \text{Alg}_{\mathbb{H}_0 R}$ and all $X \in \mathcal{M}(A)$, the groups $\mathbb{E}xt_{f(X)}^i(\mathbb{L}^{f(X)/A \otimes T}, f(X))$ are all finitely generated A -modules.
- (3) for all discrete local Noetherian $\mathbb{H}_0 R$ -algebras A , with maximal ideal \mathfrak{m} , the map

$$\mathcal{M}(A) \rightarrow \varprojlim^h \mathcal{M}(A/\mathfrak{m}^n)$$

is a weak equivalence (see Remark 2.11 for a reformulation).

- (4) The functor $c(\pi_0 \mathcal{M}) : \text{Alg}_{\pi_0 R} \rightarrow \text{Set}$ of components of the groupoid $\pi_0 \mathcal{M}$ preserves filtered colimits.
- (5) For all $A \in \text{Alg}_{\pi_0 R}$ and all $X \in \mathcal{M}(A)$, the functors $\pi_i \underline{\text{Hom}}_{\mathcal{M}}(X, X) : \text{Alg}_A \rightarrow \text{Set}$ preserve filtered colimits for all $i \geq 0$.

Let $\tilde{\mathcal{M}} : d\mathcal{N}_R^b \rightarrow s\text{Cat}$ be the full simplicial subcategory of $\mathcal{W}(\text{ALG}_T)(A)$ consisting of objects C for which $C \otimes_A \mathbb{H}_0 A$ is weakly equivalent (in $\mathcal{W}(\text{ALG}_T)(\mathbb{H}_0 A)$) to an object of $\mathcal{M}(\mathbb{H}_0 A)$. Then the nerve $\bar{W}\tilde{\mathcal{M}}$ is a geometric derived n -stack.

Proof. The proof of Corollary 3.10 carries over. \square

Example 3.16 (Moduli of finite schemes over an affine base). Fix $T \in dg_+ \text{Alg}_R$ (resp. $T \in s\text{Alg}_R$) flat as a graded algebra over R , define $\mathcal{M} : \text{Alg}_{\mathbb{H}_0 R} \rightarrow s\text{Cat}$ by setting $\mathcal{M}(A) \subset \pi^0 \mathcal{W}\text{ALG}(T \otimes_R A)$ to consist of dg (resp. simplicial) $T \otimes_R A$ -algebras S for which $\mathbb{H}_0 S$ is a finite flat A -module, and $\mathbb{H}_i S = 0$ for all $i > 0$. Then we get

$$\tilde{\mathcal{M}} \simeq \mathcal{M} \times_{\pi^0 \text{ALG}_T}^{(2)} \text{ALG}_T.$$

Now, \mathbb{H}_0 gives a functor from $\mathcal{M}(A)$ to the groupoid $\mathcal{W}(\text{FALG}_{\mathbb{H}_0 T}(A))$ of $(\mathbb{H}_0 T \otimes_{\mathbb{H}_0 R} A)$ -algebras, finite and flat over A , and this is a weak equivalence in $s\text{Cat}$, so

$$\mathcal{M} \simeq \text{WFALG}_{\mathbb{H}_0 T},$$

which is a hypersheaf by [Pri3] Remark 1.29. Thus the corollary above implies that the nerve $\bar{W}\tilde{\mathcal{M}}$ is a geometric derived 1-stack, which we can now regard as the derived moduli stack of finite flat algebras equipped with a morphism from T .

3.2. Derived moduli of schemes. Recall that we write $d\text{Alg}_R$ for either $dg_+ \text{Alg}_R$ or $s\text{Alg}_R$.

3.2.1. Cosimplicial derived rings.

Definition 3.17. Define Δ to be the ordinal number category. For any category \mathcal{C} , this means that \mathcal{C}^Δ (resp. $\mathcal{C}^{\Delta^{\text{opp}}}$) is the category of cosimplicial (resp. simplicial) diagrams in \mathcal{C} .

Definition 3.18. Define a simplicial structure on $(d\text{Alg}_A)^\Delta$ as follows. For $C \in (d\text{Alg}_A)^\Delta$, first define $h(\Delta^n, C) \in (d\text{Alg}_A)^\Delta$ by $h(\Delta^n, C)^i := (C^i)^{\Delta_i^n}$, and then define the Hom-spaces by

$$\underline{\text{Hom}}_{(d\text{Alg}_A)^\Delta}(B, C)_n = \text{Hom}_{(d\text{Alg}_A)^\Delta}(B, h(\Delta^n, C)).$$

Definition 3.19. Given $A \in (d\text{Alg}_R)^\Delta$, define $L^n A$ to represent the functor $M_n \text{Spec } A$ given by

$$(M_n \text{Spec } A)(B) = \{x \in \prod_{i=0}^n \text{Hom}_{d\text{Alg}_R}(A^{n-1}, B) : (\partial^i)^* x_j = (\partial^{j-1})^* x_i \text{ if } 0 \leq i < j \leq n\}.$$

Definition 3.20. Say that a morphism $A \rightarrow B$ in $(d\text{Alg}_R)^\Delta$ is a Reedy cofibration if the latching maps

$$A^n \otimes_{L^n A} L^n B \rightarrow B^n$$

in $d\text{Alg}_A$ are cofibrations (in the sense of Definitions 3.1 and 3.2) for all $n \geq 0$.

This is equivalent to satisfying the conditions that for all acyclic surjections in $C' \rightarrow C$ in $d\text{Alg}_R$, the map

$$\text{Hom}_{d\text{Alg}_R}(B, C') \rightarrow \text{Hom}_{d\text{Alg}_R}(B, C) \times_{\text{Hom}_{d\text{Alg}_R}(A, C)} \text{Hom}_{d\text{Alg}_R}(A, C')$$

is a trivial fibration in \mathbb{S} , where we set $\text{Hom}_{d\text{Alg}_R}(A, C)_n := \text{Hom}_{d\text{Alg}_R}(A^n, C)$.

Definition 3.21. For $A \in d\text{Alg}_R$, define $c\text{ALG}_T(A)$ to be the full simplicial subcategory of the category $(d\text{Alg}_{(T \otimes_R A)})^\Delta$ on cofibrant objects (over $T \otimes_R A$).

Proposition 3.22. *For T levelwise flat as a graded algebra over R , the functor $c\text{ALG}_T : d\mathcal{N}^\flat \rightarrow s\text{Cat}$ defined above is 2-homogeneous and formally 2-quasi-smooth.*

Proof. This is a fairly direct consequence of the proof of Proposition 3.14. Again the only part which is not immediate is showing that $c\text{ALG}_T(A) \rightarrow c\text{ALG}_T(B)$ satisfies Condition (F2) from Definition 2.22 for all square-zero extensions $A \rightarrow B$, and satisfies condition (TF2) whenever the extension is acyclic.

Given a square-zero extension $A \rightarrow B$ and an object S in $c\text{ALG}_T(B)$, Proposition 3.14 gives criteria for lifting S^0 to $\tilde{S}^0 \in \text{ALG}_{T^0}(A)$. Similar criteria apply for lifting the $L^0 S \otimes_{L^0 T} T^1$ -algebra S^1 to an $L^0 \tilde{S} \otimes_{L^0 T} T^1$ -algebra \tilde{S}^1 , with an augmentation $\tilde{S}^1 \rightarrow M^1(\tilde{S})$ extending the augmentation $S^1 \rightarrow M^1 S$. We then proceed by induction on the levels of the cosimplicial diagram. A similar argument applies to lift morphisms. \square

Definition 3.23. Say that a map $f : A \rightarrow B$ in $d\text{Alg}_R$ is homotopy-flat (resp. homotopy-smooth, resp. homotopy-étale) if $H_0 A \rightarrow H_0 B$ is flat (resp. smooth, resp. étale) and the maps

$$H_n A \otimes_{H_0 A} H_0 B \rightarrow H_n B$$

are isomorphisms for all n .

Say that a homotopy-flat map $A \rightarrow B$ is a cover if $H_0 A \rightarrow H_0 B$ is faithfully flat.

Definition 3.24. Say that a Reedy cofibration $A \rightarrow B$ in $(d\text{Alg}_R)^\Delta$ is a homotopy-étale hypercover if the latching maps

$$A^n \otimes_{L^n A} L^n B \rightarrow B^n$$

in $d\text{Alg}_A$ are homotopy-étale covers for all $n \geq 0$.

Definition 3.25. Define $c\text{ALG}_{T, \text{hyp}} : d\mathcal{N}^\flat \rightarrow s\text{Cat}$ by setting $c\text{ALG}_{\text{hyp}}(A)$ to have the same objects as $c\text{Alg}_T(A)$, and morphisms

$$\underline{\text{Hom}}_{c\text{ALG}_{T, \text{hyp}}(A)}(B, C) = \varinjlim_{C \rightarrow \tilde{C}} \underline{\text{Hom}}_{c\text{Alg}_T(A)}(B, \tilde{C}),$$

where the colimit ranges over the category $\text{hyp}(C)$ of all Reedy cofibrant homotopy-étale hypercovers $C \rightarrow \tilde{C}$, with the only morphisms between them being Reedy cofibrant homotopy-étale hypercovers. The argument of [Pri1] Lemma 5.9 shows that $\text{hyp}(C)$ is a filtered inverse category.

Composition is defined as follows. Given elements of $\underline{\text{Hom}}_{c\text{ALG}_{T, \text{hyp}}(A)}(B, C)$ and $\underline{\text{Hom}}_{c\text{ALG}_{T, \text{hyp}}(A)}(C, D)$, represented by $f : B \rightarrow \tilde{C}$ and $g : C \rightarrow \tilde{D}$, we define $g \circ f$ to be the composition

$$B \rightarrow \tilde{C} \rightarrow \tilde{C} \otimes_C \tilde{D}.$$

Proposition 3.26. *The functor $c\text{ALG}_{T, \text{hyp}} : d\mathcal{N}^\flat \rightarrow s\text{Cat}$ defined above is 2-homogeneous and formally 2-quasi-smooth.*

Proof. This follows immediately from Proposition 3.22, making use of the fact that filtered colimits commute with finite limits. \square

3.2.2. Artin hypergroupoids.

Definition 3.27. Define the category $d\text{Aff}_R$ of derived affine R -schemes to be opposite to $d\text{Alg}_R$ (i.e. $dg_+\text{Alg}_R$ or $s\text{Alg}_R$). Write $sd\text{Aff}_R$ for the category of simplicial objects in $d\text{Aff}_R$; this is opposite to $(d\text{Alg}_R)^\Delta$. Say that a morphism in $d\text{Aff}_R$ is a fibration if the corresponding morphism in $d\text{Alg}_R$ is a cofibration.

The following definition is adapted from [Pri1]. For the definitions of $\partial\Delta^m$ and Λ_k^m , see Definition 3.3.

Definition 3.28. Given $Y_\bullet \in sd\text{Aff}_R$, define a strongly quasi-compact relative derived Artin (resp. Deligne–Mumford) n -hypergroupoid over Y_\bullet to be a morphism $X_\bullet \rightarrow Y_\bullet$ in $sd\text{Aff}_R$, satisfying the following conditions :

- (1) for each $m \geq 0$, the matching map

$$X_m \rightarrow \text{Hom}_{\mathbb{S}}(\partial\Delta^m, X) \times_{\text{Hom}_{\mathbb{S}}(\partial\Delta^m, Y)} Y_m$$

is a fibration (this is equivalent to saying that $X \rightarrow Y$ is opposite to a Reedy cofibration in $(d\text{Alg}_R)^\Delta$);

- (2) for each k, m , the partial matching map

$$X_m \rightarrow \text{Hom}_{\mathbb{S}}(\Lambda_k^m, X) \times_{\text{Hom}_{\mathbb{S}}(\Lambda_k^m, Y)} Y_m$$

is a homotopy-smooth (resp. homotopy-étale) cover (in the sense of Definition 3.23);

- (3) for all $m > n$ and all k , the partial matching maps

$$X_m \rightarrow \text{Hom}_{\mathbb{S}}(\Lambda_k^m, X) \times_{\text{Hom}_{\mathbb{S}}(\Lambda_k^m, Y)} Y_m$$

are weak equivalences.

Proposition 3.29. *The simplicial category of strongly quasi-compact n -geometric derived Artin (resp. Deligne–Mumford) stacks over the hypersheafification $\text{Spec}(T \otimes_R^{\mathbf{L}} A)^\sharp$ is weakly equivalent to the full subcategory $\text{HYP}_T^n(A)$ of $c\text{ALG}_{T, \text{hyp}}(A)$ whose objects are dual to strongly quasi-compact derived Artin (resp. Deligne–Mumford) n -hypergroupoids.*

Proof. This is [Pri1] Theorem 7.21. The functor is given by sending $C \in \text{HYP}^n(A)$ to the hypersheafification of the presheaf $\text{Spec } C : d\text{Aff}_R^{\text{opp}} \rightarrow \mathbb{S}$. \square

Definition 3.30. Given a flat ∞ -geometric Artin stack \mathfrak{Y} over H_0R , and $A \in \text{Alg}_{H_0R}$ let $F\text{Stack}_{\mathfrak{Y}}^n(A)$ be the simplicial category of n -geometric Artin stacks over $\mathfrak{Y} \otimes_{H_0R} A$, flat over A , and set $F\text{Stack}_{\mathfrak{Y}}(A) = \varinjlim_n F\text{Stack}_{\mathfrak{Y}}^n(A)$. Likewise, given an ∞ -geometric derived Artin stack \mathfrak{Y} over R and $A \in d\text{Alg}_R$, let $d\text{Stack}_{\mathfrak{Y}}^n(A)$ be the simplicial category of n -geometric derived Artin stacks over $\mathfrak{Y} \otimes_R^{\mathbf{L}} A$, and set $d\text{Stack}_{\mathfrak{Y}}(A) = \varinjlim_n d\text{Stack}_{\mathfrak{Y}}^n(A)$.

Note that if we set $\mathfrak{Y} := \text{Spec}(T)^\sharp$, then Proposition 3.29 shows that sheafification gives an ∞ -equivalence $\text{HYP}_T^n(A) \rightarrow d\text{Stack}_{\mathfrak{Y}}^n(A)$. Also note that the maps $d\text{Stack}_{\mathfrak{Y}}^n(A) \rightarrow d\text{Stack}_{\mathfrak{Y}}(A)$ are homotopy formally étale, as are the maps $F\text{Stack}_{\mathfrak{Y}}^n(A) \rightarrow d\text{Stack}_{\mathfrak{Y}}^n(A)$, $F\text{Stack}_{\mathfrak{Y}}(A) \rightarrow d\text{Stack}_{\mathfrak{Y}}(A)$ and the embedding $F\text{Sch}_{\mathfrak{Y}}(A) \rightarrow F\text{Stack}_{\mathfrak{Y}}(A)$ of algebraic spaces into ∞ -geometric stacks.

Lemma 3.31. *Given an ∞ -geometric derived Artin stack \mathfrak{Y} over R , the functor $d\text{Stack}_{\mathfrak{Y}} : d\mathcal{N}_R^{\flat} \rightarrow s\text{Cat}$ is homotopy-preserving and homotopy-homogeneous.*

Proof. Choose a representative $T \in c\text{ALG}(R)$ with $(\text{Spec } T)^\sharp \simeq \mathfrak{Y}$ (as in Proposition 3.29). If we write $\text{HYP}_T(A) := \varinjlim_n \text{HYP}_T^n(A)$, then Proposition 3.29 allows us to replace $d\text{Stack}_{\mathfrak{Y}}(A)$ with $\text{HYP}_T(A)$. The map $\text{HYP}_T \rightarrow c\text{ALG}_{T, \text{hyp}}(A)$ is homotopy formally étale, so it suffices to show that $c\text{ALG}_{T, \text{hyp}}(A)$ is homotopy-preserving and homotopy-homogeneous. This follows from combining Lemmas 2.24 and 2.27 with Proposition 3.26. \square

Theorem 3.32. *Take an ∞ -geometric derived Artin stack \mathfrak{Y} , homotopy-flat over R .*

Assume that we have an n -truncated hypersheaf $\mathcal{M} : \text{Alg}_{\mathbb{H}_0 R} \rightarrow \text{sCat}$, open in $\pi^0 \mathcal{W}(F\text{Stack}_{\mathfrak{Y}})$, noting that homotopy-flatness ensures that $Y_A := \mathfrak{Y} \otimes_R^{\mathbf{L}} A$ is a flat ∞ -geometric (underived) Artin stack over A for all $A \in \text{Alg}_{\mathbb{H}_0 R}$.

Also assume that this satisfies the following conditions

- (1) *for all finitely generated $A \in \text{Alg}_{\mathbb{H}_0 R}$ and all $X \in \mathcal{M}(A)$, the functors*

$$\mathbb{E}xt_X^i(\mathbb{L}^{X/Y_A}, \mathcal{O}_X \otimes_A^{\mathbf{L}} -) : \text{Mod}_A \rightarrow \text{Ab}$$

preserve filtered colimits for all $i > 1$.

- (2) *for all finitely generated integral domains $A \in \text{Alg}_{\mathbb{H}_0 R}$ and all $X \in \mathcal{M}(A)$, the groups $\mathbb{E}xt_X^i(\mathbb{L}^{X/Y_A}, \mathcal{O}_X)$ are all finitely generated A -modules.*

- (3) *for all discrete local Noetherian $\mathbb{H}_0 R$ -algebras A , with maximal ideal \mathfrak{m} , the map*

$$\mathcal{M}(A) \rightarrow \varprojlim^h \mathcal{M}(A/\mathfrak{m}^n)$$

is a weak equivalence (see Remark 2.11 for a reformulation).

- (4) *The functor $c(\pi_0 \mathcal{M}) : \text{Alg}_{\pi_0 R} \rightarrow \text{Set}$ of components of the groupoid $\pi_0 \mathcal{M}$ preserves filtered colimits.*

- (5) *For all $A \in \text{Alg}_{\pi_0 R}$ and all $X \in \mathcal{M}(A)$, the functors $\pi_i \underline{\text{Hom}}_{\mathcal{M}}(X, X) : \text{Alg}_A \rightarrow \text{Set}$ preserve filtered colimits for all $i \geq 0$.*

Let $\tilde{\mathcal{M}} : d\mathcal{N}_R^{\flat} \rightarrow \text{sCat}$ be the full simplicial subcategory of $\mathcal{W}(c\text{ALG}_{T,\text{hyp}}(A))$ consisting of objects \mathfrak{X} for which $\mathfrak{X} \otimes_A \mathbb{H}_0 A$ is weakly equivalent in $d\text{Stack}_{\mathfrak{Y}}(\mathbb{H}_0 A)$ to an object of $\mathcal{M}(\mathbb{H}_0 A)$. Then the nerve $\bar{W}\tilde{\mathcal{M}}$ is a geometric derived n -stack.

Proof. As in Lemma 3.31, we may replace $c\text{ALG}_{T,\text{hyp}}(A)$ with $d\text{Stack}_{\mathfrak{Y}}(A)$, so

$$\tilde{\mathcal{M}}(A) \simeq \mathcal{M}(\mathbb{H}_0 A) \times_{\mathcal{W}(d\text{Stack}_{\mathfrak{Y}}(\mathbb{H}_0 A))}^h \mathcal{W}(d\text{Stack}_{\mathfrak{Y}}(A)),$$

because \mathcal{M} is open in $F\text{Stack}_{\mathfrak{Y}}$, which is open in $\pi^0 d\text{Stack}_{\mathfrak{Y}}$.

The proof of Corollary 3.10 now carries over to this context, using Lemma 3.31 in place of Proposition 3.7. The only non-trivial part is the calculation

$$D_X^i(\bar{W}\tilde{\mathcal{M}}/R, M) = \mathbb{E}xt_X^{i+1}(\mathbb{L}^{X/Y_A}, \mathcal{O}_X \otimes_A^{\mathbf{L}} M),$$

but this follows from the calculations of [Pri1] §9.2, for instance. \square

Remark 3.33. Since $\bar{W}\tilde{\mathcal{M}}$ is a geometric derived stack, it extends canonically to a functor

$$\bar{W}\tilde{\mathcal{M}} : d\text{Alg}_R \rightarrow \mathbb{S}$$

where $d\text{Alg}_R = s\text{Alg}_R$ or $dg_+ \text{Alg}_R$. The obvious candidate for this functor is given by adapting the description above; $\tilde{\mathcal{M}}$ should be the full simplicial subcategory of $\mathcal{W}(c\text{ALG}_{T,\text{hyp}})$ (or equivalently of $\mathcal{W}(d\text{Stack}_{\mathfrak{Y}})$) consisting of objects \mathfrak{X} for which $\mathfrak{X} \otimes_A^{\mathbf{L}} \mathbb{H}_0 A$ is weakly equivalent in $d\text{Stack}_{\mathfrak{Y}}(\mathbb{H}_0 A)$ to an object of $\mathcal{M}(\mathbb{H}_0 A)$.

To check that this is indeed the case, it would suffice to verify the conditions of [Pri3] Corollary 1.35 for this functor. These reduce to showing that the functor

$$\bar{W}\mathcal{W}(d\text{Stack}_{\mathfrak{Y}}) : d\text{Alg}_R \rightarrow \mathbb{S}$$

is a homotopy-preserving functor (which is certainly true), and nilcomplete in the sense that for all $A \in d\text{Alg}_R$, the map

$$\bar{W}\mathcal{W}(d\text{Stack}_{\mathfrak{Y}})(A) \rightarrow \varprojlim^h \bar{W}\mathcal{W}(d\text{Stack}_{\mathfrak{Y}})(P_k A)$$

is an equivalence, for $\{P_k A\}$ the Postnikov tower of A . Nilcompleteness certainly seems plausible, as similar statements in [Lur1] §8 are asserted to be automatic. However, broadly similar comparisons in [TV2] involve weighty strictification theorems (such as [TV2] Theorem B.0.7 or [TV1] Theorem 4.2.1).

We now give a generalisation which will be used for moduli of pointed schemes.

Definition 3.34. Given a morphism $f : \mathfrak{Z} \rightarrow \mathfrak{Y}$ of ∞ -geometric derived Artin stacks over R , define $d\text{Stack}_{\mathfrak{Y}}^{\mathfrak{Z}} : d\mathcal{N}_R^{\mathfrak{Z}} \rightarrow s\text{Cat}$ as follows. First set $\mathfrak{Z}_A := \mathfrak{Z} \otimes_R^{\mathbf{L}} A$, $\mathfrak{Y}_A := \mathfrak{Y} \otimes_R^{\mathbf{L}} A$; objects of $d\text{Stack}_{\mathfrak{Y}}^{\mathfrak{Z}}(A)$ are factorisations

$$\mathfrak{Z}_A \xrightarrow{i} X \xrightarrow{p} \mathfrak{Y}_A$$

of f in the category of ∞ -geometric derived Artin stacks over A . Morphism spaces are given by homotopy fibre products

$$\underline{\text{Hom}}_{d\text{Stack}_{\mathfrak{Y}}^{\mathfrak{Z}}(A)}(X, X') := \underline{\text{Hom}}_{\mathfrak{Y}_A}(X, X') \times_{i^*, \underline{\text{Hom}}_{\mathfrak{Y}_A}(\mathfrak{Z}_A, X')}^h \{i'\}.$$

Theorem 3.35. Take a morphism $\mathfrak{Z} \rightarrow \mathfrak{Y}$ of homotopy-flat ∞ -geometric derived Artin stacks over R . Assume that we have an n -truncated hypersheaf $\mathcal{M} : \text{Alg}_{\mathbf{H}_0 R} \rightarrow s\text{Cat}$, open in the functor $A \mapsto \mathcal{W}((\mathfrak{Z} \otimes_R^{\mathbf{L}} A) \downarrow F\text{Stack}_{\mathfrak{Y}})$, noting that $Y_A := \mathfrak{Y} \otimes_R^{\mathbf{L}} A$ and $Z_A := \mathfrak{Z} \otimes_R^{\mathbf{L}} A$ are flat ∞ -geometric (underived) Artin stacks over A for all $A \in \text{Alg}_{\mathbf{H}_0 R}$.

Also assume that these satisfy the following conditions

- (1) for all finitely generated $A \in \text{Alg}_{\mathbf{H}_0 R}$ and all $Z_A \xrightarrow{i} X$ in $\mathcal{M}(A)$, the functors

$$\text{Ext}_X^j(\mathbb{L}^{X/Y_A}, (\mathcal{O}_X \rightarrow \mathbf{R}i_* \mathcal{O}_{Z_A}) \otimes_A^{\mathbf{L}} -) : \text{Mod}_A \rightarrow \text{Ab}$$

preserve filtered colimits for all $j > 1$.

- (2) for all finitely generated integral domains $A \in \text{Alg}_{\mathbf{H}_0 R}$ and all $X \in \mathcal{M}(A)$, the groups $\text{Ext}_X^j(\mathbb{L}^{X/Y_A}, \mathcal{O}_X \rightarrow \mathbf{R}i_* \mathcal{O}_{Z_A})$ are all finitely generated A -modules.
(3) for all discrete local Noetherian $\mathbf{H}_0 R$ -algebras A , with maximal ideal \mathfrak{m} , the map

$$\mathcal{M}(A) \rightarrow \varprojlim^h \mathcal{M}(A/\mathfrak{m}^n)$$

is a weak equivalence (see Remark 2.11 for a reformulation).

- (4) The functor $c(\pi_0 \mathcal{M}) : \text{Alg}_{\pi_0 R} \rightarrow \text{Set}$ of components of the groupoid $\pi_0 \mathcal{M}$ preserves filtered colimits.
(5) For all $A \in \text{Alg}_{\pi_0 R}$ and all $X \in \mathcal{M}(A)$, the functors $\pi_j \underline{\text{Hom}}_{\mathcal{M}}(X, X) : \text{Alg}_A \rightarrow \text{Set}$ preserve filtered colimits for all $j \geq 0$.

Let $\tilde{\mathcal{M}} : d\mathcal{N}_R^{\mathfrak{Z}} \rightarrow s\text{Cat}$ be the full simplicial subcategory of $\mathcal{W}(d\text{Stack}_{\mathfrak{Y}}^{\mathfrak{Z}})$ consisting of morphisms $i : Z_A \rightarrow \mathfrak{X}$ for which $\mathfrak{X} \otimes_A \mathbf{H}_0 A$ is weakly equivalent in $d\text{Stack}_{\mathfrak{Y}}^{\mathfrak{Z}}(\mathbf{H}_0 A)$ to an object of $\mathcal{M}(\mathbf{H}_0 A)$. Then the nerve $\bar{W}\tilde{\mathcal{M}}$ is a geometric derived n -stack.

Proof. Lemma 3.31 ensures that $d\text{Stack}_{\mathfrak{Y}}$ is homotopy-preserving and homotopy-homogeneous, so it follows immediately that $d\text{Stack}_{\mathfrak{Y}}^{\mathfrak{Z}}$ inherits these properties. Lemma 2.30 ensures that the same is true of $\mathcal{W}(d\text{Stack}_{\mathfrak{Y}}^{\mathfrak{Z}})$.

In order to verify the conditions of Theorem 2.10, we now just need to establish the isomorphisms

$$\mathbf{D}_{(Z_A \xrightarrow{i} X)}^j(\bar{W}\tilde{\mathcal{M}}/R, M) = \text{Ext}_X^{j+1}(\mathbb{L}^{X/Y_A}, \mathcal{O}_X \otimes_A^{\mathbf{L}} M \rightarrow \mathbf{R}i_* \mathcal{O}_{Z_A} \otimes_R^{\mathbf{L}} M).$$

As in the proof of Theorem 3.32, we know that the loop space of $T_{(Z_A \xrightarrow{i} X)}(\bar{W}\tilde{\mathcal{M}}/R)(M)$ is the homotopy fibre of

$$\mathbf{R}\underline{\text{Hom}}_X(\mathbb{L}^{X/Y_A}, \mathcal{O}_X \otimes_A^{\mathbf{L}} M) \rightarrow \mathbf{R}\underline{\text{Hom}}_{Z_A}(\mathbf{L}i^* \mathbb{L}^{X/Y_A}, \mathcal{O}_{Z_A} \otimes_R^{\mathbf{L}} M)$$

over 0. We can rewrite the second space as $\mathbf{R}\underline{\text{Hom}}_X(\mathbb{L}^{X/Y_A}, \mathbf{R}i_* \mathcal{O}_{Z_A} \otimes_R^{\mathbf{L}} M)$, and the result follows by noting that homotopy fibres correspond to mapping cones of complexes. \square

3.3. Examples. In the examples which follow, we will always choose a functorial full subgroupoid \mathcal{M} of the core $\mathcal{W}(F\text{Stack}_Y^n)$ on objects which are fibred in geometric 0-stacks (i.e. algebraic spaces) over $Y \otimes_{\mathbb{H}_0 R} A$ — this has the effect that \mathcal{M} is a bona fide groupoid, not just an ∞ -groupoid. We get a derived 1-stack by taking the nerve $\bar{W}\tilde{\mathcal{M}}$ of $\tilde{\mathcal{M}}$ as constructed in Theorem 3.32.

Our first application generalises the examples of Hilbert schemes and moduli of curves in [Lur1] §8, but beware that the derived schemes featuring in [Lur1] form a larger category than derived geometric 0-stacks.

Example 3.36 (Moduli of schemes). Let \mathfrak{Y} be a homotopy-flat derived geometric 0-stack over R , so $Y := \mathfrak{Y} \otimes_R^{\mathbf{L}} \mathbb{H}_0 R$ is a flat algebraic space over $\mathbb{H}_0 R$. Choose a stack \mathcal{M} equipped with a functorial fully faithful embedding of \mathcal{M} into the core of $F\text{Sch}_Y$, with the property that \mathcal{M} is closed under infinitesimal deformations. This ensures that $\bar{W}\mathcal{M} \rightarrow \bar{W}F\text{Sch}_Y$ is open.

An example satisfying these conditions is when $\mathfrak{Y} = R$ and \mathcal{M} is some moduli stack schemes with fixed dimension. Another example is to let $\mathcal{M}(A)$ be the set of closed subschemes of $Y \otimes_{\mathbb{H}_0 R} A$.

Then we can define $\tilde{\mathcal{M}}$ as above, and we get a derived 1-stack $\bar{W}\tilde{\mathcal{M}}$, with $\pi^0 \bar{W}\tilde{\mathcal{M}} = B\mathcal{M}$.

If \mathcal{M} is known to be an algebraic stack, then the last two conditions of Theorem 3.32 are automatically satisfied, so for geometricity of $\bar{W}\tilde{\mathcal{M}}$ we need only check properties of the cohomology groups

$$D_f^i(\bar{W}\tilde{\mathcal{M}}/R, M) = \mathbb{E}xt_X^{i+1}(\mathbb{L}^{X/Y \otimes_{\mathbb{H}_0 R} A}, \mathcal{O}_X \otimes_A^{\mathbf{L}} M),$$

for $A \in \text{Alg}_{\mathbb{H}_0 R}$ and $X \in \mathcal{M}(A)$.

Our next application gives a generalisation of [TV2] Theorem 2.2.6.11.

Example 3.37 (Derived Hom-stacks). Another example is to fix n -geometric derived stacks $\mathfrak{X}, \mathfrak{Z}$, homotopy-flat over R . Set $X := \pi^0 \mathfrak{X} = \mathfrak{X} \otimes_R^{\mathbf{L}} \mathbb{H}_0 R$, $Z := \pi^0 \mathfrak{Z}$; these are flat n -geometric stacks over $\mathbb{H}_0 R$.

Now, let $\mathfrak{Y} := \mathfrak{X} \times_{\text{Spec } R} \mathfrak{Z}$ with $Y := \pi^0 \mathfrak{Y}$, and set $\mathcal{M}(A) \subset \text{Sch}_Y(A)$ to consist of maps

$$U \times_{\text{Spec } R} \text{Spec } A \rightarrow X \times_{\text{Spec } R} Z$$

for which the projection $U \rightarrow X$ is an equivalence. Then $\bar{W}\mathcal{M}(A)$ is isomorphic to the simplicial set $\underline{\text{Hom}}(X \times_{\text{Spec } \mathbb{H}_0 R} \text{Spec } A, Z)$. The derived moduli stack $\bar{W}\tilde{\mathcal{M}}(A)$ is weakly equivalent to the nerve of

$$d\text{Stack}_{\mathfrak{Y}}(A) \times_{d\text{Stack}(A)}^h \{\mathfrak{Z}\}$$

— $\bar{W}\tilde{\mathcal{M}}$ is the derived Hom-stack $\mathcal{H}om(\mathfrak{X}, \mathfrak{Z})$.

By considering the arrow category of $d\text{Stack}(A)$, we see that this in turn is equivalent to $\underline{\text{Hom}}(\mathfrak{X} \times_{\text{Spec } R} \text{Spec } A, \mathfrak{Z})$. Thus a far simpler characterisation of this moduli functor is given by choosing $B, C \in \text{HYP}_R^n$ with $\mathfrak{X} = (\text{Spec } B)^\sharp, \mathfrak{Z} = (\text{Spec } C)^\sharp$. Then

$$\tilde{\mathcal{M}}(A) \simeq \underline{\text{Hom}}_{\text{HYP}_R^n(A)}(C \otimes_R A, B \otimes_R A).$$

Of course, if $\mathfrak{X}, \mathfrak{Z}$ are affine schemes, then we can take $n = 0$. If they are semi-separated schemes, we can take $n = 1$.

For $f : X \otimes A \rightarrow Z$ in $\mathcal{M}(A)$, $\mathcal{H}om(\mathfrak{X}, \mathfrak{Z})$ has cohomology groups

$$\begin{aligned} D_f^i(\mathcal{H}om(\mathfrak{X}, \mathfrak{Z})/R, M) &= \mathbb{E}xt_{X \otimes A}^{i+1}(\mathbb{L}^{X \otimes A / (X \times Z) \otimes A}, \mathcal{O}_X \otimes_{\mathbb{H}_0 R} M) \\ &= \mathbb{E}xt_{X \otimes A}^{i+1}(f^* \mathbb{L}^{Z/\mathbb{H}_0 R}[1], \mathcal{O}_X \otimes_{\mathbb{H}_0 R} M) \\ &= \mathbb{E}xt_Z^i(\mathbb{L}^{Z/\mathbb{H}_0 R}, \mathbf{R}f_* \mathcal{O}_X \otimes_{\mathbb{H}_0 R} M). \end{aligned}$$

Theorem 3.32 (or a direct application of Theorem 2.10 to $\underline{\mathrm{Hom}}(\mathfrak{X} \times_{\mathrm{Spec} R} \mathrm{Spec} A, \mathfrak{Z})$) then shows that $\mathcal{H}om(\mathfrak{X}, \mathfrak{Z})$ is a derived n -geometric stack provided X is strongly quasi-compact (i.e. quasi-compact, quasi-separated \dots), that Z is almost of finite presentation, and that the groups $\mathbb{E}xt_X^i(f^*\mathbb{L}^{Z/\mathrm{H}_0R}, \mathcal{O}_X \otimes_{\mathrm{H}_0R} -)$ have suitable finiteness properties (which are satisfied if X is proper).

Example 3.38 (Moduli of torsors). A special case of Example 3.37 is to take a smooth group scheme G over R , and to set \mathfrak{Z} to be the classifying stack BG . Then $\mathcal{H}om(\mathfrak{X}, BG)$ is the derived stack of G -torsors on \mathfrak{X} — when $G = \mathrm{GL}_r$, this gives derived moduli of vector bundles, and when $G = \mathrm{SL}_r$ it gives derived moduli of vector bundles with trivial determinant.

To calculate the cohomology groups $D_f^i(\tilde{M}/R, M)$ for $f : \mathfrak{X} \times \mathrm{Spec} A \rightarrow BG$, write $e : \mathrm{Spec} R \rightarrow G$ for the unit. Then $e^*\Omega(G/R)$ has a G -action, so can be regarded as a quasi-coherent sheaf \mathcal{C} on BG . We then have $\mathbb{L}^{BG/R} \simeq \mathcal{C}[1]$, so

$$D_f^i(\mathcal{H}om(\mathfrak{X}, BG)/R, M) = \mathrm{Ext}_{X \otimes A}^{i+1}(f^*\mathcal{C}, \mathcal{O}_X \otimes_R M).$$

If f corresponds to a G -torsor P , then note that $f^*\mathcal{C} = P \times_G (e^*\Omega(G/R))$.

If $G = \mathrm{GL}(V)$, then $\mathcal{C} \cong \mathrm{End}(V)$ with the adjoint action, and for $f : \mathfrak{X} \times \mathrm{Spec} A \rightarrow B\mathrm{GL}_n$ corresponding to a vector bundle \mathcal{E} , we have $f^*\mathcal{C} \cong \mathcal{E}nd(\mathcal{E})$, so

$$\begin{aligned} D_f^i(\mathcal{H}om(\mathfrak{X}, BG)/R, M) &= \mathrm{Ext}_{X \otimes A}^{i+1}(\mathcal{E}nd(\mathcal{E}), \mathcal{O}_X \otimes_R M) \\ &= \mathrm{Ext}_{X \otimes A}^{i+1}(\mathcal{E}, \mathcal{E} \otimes_R M). \end{aligned}$$

Example 3.39 (Moduli of polarised schemes). For an example combining aspects of all the problems so far, consider moduli of polarised schemes. In Theorem 3.32, take $\mathfrak{Y} = B\mathbb{G}_m$, and let $\mathcal{M}(A)$ be some groupoid of flat schemes X over A , equipped with polarisations $f : X \rightarrow B\mathbb{G}_m$, with morphisms required to respect the polarisations. Then we have $\tilde{\mathcal{M}}$ as defined at the beginning of the section. Again, $\bar{W}\tilde{\mathcal{M}}$ will be a geometric derived 1-stack whenever \mathcal{M} is an algebraic 1-stack and the cohomology groups $D^i(\bar{W}\tilde{\mathcal{M}}/R, -)$ satisfy the requisite finiteness conditions.

Explicitly,

$$D_X^i(\bar{W}\tilde{\mathcal{M}}/R, M) = \mathbb{E}xt_X^{i+1}(\mathbb{L}^{X/B\mathbb{G}_m}, \mathcal{O}_X \otimes_{\mathrm{H}_0R} M).$$

The calculations of the previous example show that $f^*\mathbb{L}^{B\mathbb{G}_m} = \mathcal{O}_X[1]$, so the cotangent complex $\mathbb{L}^{X/B\mathbb{G}_m}$ is an extension of \mathcal{O}_X by $\mathbb{L}^{X/\mathrm{H}_0R}$. This gives a long exact sequence

$$\dots \mathrm{H}^{i+1}(X, \mathcal{O}_X \otimes_{\mathrm{H}_0R} M) \rightarrow D_X^i(\bar{W}\tilde{\mathcal{M}}/R, M) \rightarrow \mathbb{E}xt_X^{i+1}(\mathbb{L}^{X/\mathrm{H}_0R}, \mathcal{O}_X \otimes_{\mathrm{H}_0R} M) \rightarrow \dots,$$

so properness of X gives finiteness.

Remark 3.40. When considering moduli of polarised varieties, it is common only to consider the polarisation up to equivalence, so $(X, \mathcal{L}^m) \sim (X, \mathcal{L}^n)$. The corresponding derived moduli stack can be constructed by taking the filtered colimit

$$\varinjlim_{n \in \mathbb{N}^\times} \bar{W}\tilde{\mathcal{M}},$$

for $\tilde{\mathcal{M}}$ as in Example 3.39, with the transition maps $\tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}$ (corresponding to $m \rightarrow mn$ in \mathbb{N}^\times) coming from the multiplication map $[n] : \mathbb{G}_m \rightarrow \mathbb{G}_m$.

Alternatively, we could simply replace \mathbb{G}_m with the (non-quasi-separated) algebraic group space $\varinjlim_{n \in \mathbb{N}^\times} \mathbb{G}_m = \mathbb{G}_m / \mu_\infty$, where $\mu_\infty = \bigcup_{n \in \mathbb{N}^\times} \mu_n$.

Example 3.41 (Moduli of group schemes). Given a smooth group scheme G over A , the nerve BG is a 1-truncated geometric Artin stack over A . However, beware that $\underline{\mathrm{Hom}}_A(BG, BG')$ is the nerve of the groupoid $[\mathrm{Hom}_{\mathrm{GpSch}}(G, G')/G'(A)]$, where $\mathrm{Hom}_{\mathrm{Gp}}$ denotes group scheme homomorphisms and G' acts by conjugation. If $\mathcal{M}(A)$ is the nerve of some open 2-subcategory of the category of group schemes over A , then B gives a map

$\mathcal{M} \rightarrow F\text{Stack}(A)$, so we may apply Theorem 3.32 to give a derived moduli functor $\tilde{\mathcal{M}}$ whenever \mathcal{M} satisfies the conditions of that theorem.

Since we usually wish to kill the 2-morphisms (given by inner automorphisms), an alternative approach is to apply Theorem 3.35, taking $\mathfrak{J} = \mathfrak{Y} = \text{Spec } R$. Making use of the canonical basepoint $i : \text{Spec } A \rightarrow BG$, we then have

$$\underline{\text{Hom}}_A(BG, BG') \times_{i^*, \underline{\text{Hom}}_A(\text{Spec } A, BG')} \{i'\} \simeq \text{Hom}_{\text{GpSch}}(G, G'),$$

so the nerve functor allows us to regard smooth group schemes over A as an open subcategory of $d\text{Stack}_{\text{Spec } R}^{\text{Spec } R}(A)$.

As in Example 3.38, $\mathbb{L}^{BG/A} \simeq \mathcal{C}[1]$, where $\mathcal{C} = e^*\Omega(G/A)$ with its canonical G -action. Thus

$$D_G^i(\bar{W}\tilde{\mathcal{M}}/R, M) \cong \text{Ext}_{BG}^{i+2}(\mathcal{C}, \mathcal{O}_{BG} \otimes_A^{\mathbf{L}} M)$$

for $i \geq 0$ (or, in the unpointed case, for all i). For $i < -1$, the groups $D_G^i(\bar{W}\tilde{\mathcal{M}}/R, M)$ are 0 in the pointed case, with $i = -1$ giving an exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_{BG}(\mathcal{C}, \mathcal{O}_{BG} \otimes_A^{\mathbf{L}} M) &\rightarrow \text{Hom}_A(e^*\mathcal{C}, M) \rightarrow \\ D_G^{-1}(\bar{W}\tilde{\mathcal{M}}/R, M) &\rightarrow \text{Ext}_{BG}^1(\mathcal{C}, \mathcal{O}_{BG} \otimes_A^{\mathbf{L}} M) \rightarrow 0. \end{aligned}$$

Remark 3.42. Given a flat finitely presented group scheme G over A , [Art] Theorem 6.1 implies that BG is a 1-truncated geometric Artin stack over A . The analysis of the example above then carries over, subject to replacing Ω_G with the complex χ_G defined in [Ill2]. The same substitution will also work in the following example, with [Toë] Theorem 0.1 ensuring that $K(G, n)$ be an n -truncated geometric Artin stack over A .

Example 3.43 (Moduli of commutative group schemes). For smooth commutative group schemes G over A , we can no longer study moduli via BG , since this has non-commutative deformations. However, for all $n \geq 2$, the construction $K(G, n)$ gives an n -truncated geometric Artin stack over A , whose deformations all correspond to commutative group schemes. Rather than fixing one value of n , the most natural approach is to use them all simultaneously.

The loop space functor $(X, x) \mapsto X^{S^1} \times_X \{x\}$ gives a endofunctor Ω of $d\text{Stack}_{\text{Spec } R}^{\text{Spec } R}$. This maps $K(G, n)$ to $K(G, n-1)$ (up to weak equivalence). We define $d\text{SpStack}_R(A)$ to be the homotopy inverse limit of the diagram

$$\dots \xrightarrow{\Omega} d\text{Stack}_{\text{Spec } R}^{\text{Spec } R} \xrightarrow{\Omega} d\text{Stack}_{\text{Spec } R}^{\text{Spec } R}(A);$$

this is the ∞ -category of spectral derived stacks. Defining $F\text{SpStack}_R(A)$ from $F\text{Stack}_R^R(A)$ analogously, we have a functor from the category $SG\text{pSch}(A)$ smooth commutative group schemes G over A to $F\text{SpStack}_R(A)$, given by $G \mapsto \{K(G, n)\}_n$. Explicitly, $K(G, 0) = G$, and we form $K(G, n)$ from the group $n-1$ -stack $K(G, n-1)$ as the nerve $K(G, n) := \bar{W}K(G, n-1)$. Then $SG\text{pSch}$ is an open subcategory of $F\text{SpStack}_R(A)$, each of the projections $K(-, n) : SG\text{pSch} \rightarrow d\text{Stack}_{\text{Spec } R}^{\text{Spec } R}(A)$ also being open for $n \geq 2$.

We can now adapt Theorem 3.35, replacing $d\text{Stack}_{\mathfrak{Y}}^{\mathfrak{J}}$ with $d\text{SpStack}$, noting that the latter is homotopy-preserving and homotopy-homogeneous. In order to determine which open subcategories \mathcal{M} of $SG\text{pSch}$ yield derived geometric stacks

$$\tilde{\mathcal{M}}(A) := \mathcal{M}(\text{H}_0 A) \times_{F\text{SpStack}(\text{H}_0 A)}^h d\text{SpStack}(A),$$

we need to calculate the cohomology groups $D_G^i(\bar{W}\tilde{\mathcal{M}}/R, M)$. The theorem implies that these groups are given by cohomology of

$$\begin{aligned} & \varprojlim_n \mathbf{R}\mathrm{HOM}_{K(G,n)}(\mathbb{L}^{K(G,n)/A}, \mathcal{O}_{K(G,n)} \otimes_A^{\mathbf{L}} M \rightarrow \mathbf{R}i_*M)[1] \\ &= \varprojlim_n \mathbf{R}\mathrm{HOM}_{K(G,n)}(\mathcal{O}_{K(G,n)} \otimes_A e^*\Omega(G/A)[n+1], \mathcal{O}_{K(G,n)} \otimes_A^{\mathbf{L}} M \rightarrow \mathbf{R}i_*M) \\ &= \mathbf{R}\mathrm{HOM}_A(e^*\Omega(G/A), \varprojlim_n \mathbf{R}\tilde{\Gamma}(K(G,n), \mathcal{O}_{K(G,n)} \otimes_A^{\mathbf{L}} M)[n+1]), \end{aligned}$$

where $\mathbf{R}\tilde{\Gamma} = \mathbf{R}\Gamma/H^0$. We can rewrite this as

$$D_G^i(\bar{W}\tilde{\mathcal{M}}/R, M) = \mathbb{E}\mathrm{xt}_{\mathbb{Z}}^{i+1}(\varinjlim_n N\mathbb{Z}K(G, n)[n], \mathrm{Hom}_A(e^*\Omega(G/A), M)),$$

where N denotes Dold–Kan normalisation, and $\mathbb{E}\mathrm{xt}$ is taken in the category of abelian fppf sheaves over $\mathrm{Spec} A$. This ties in with the approach of [Ill1] §VII.4 (as sketched in [Ill2] Theorem 5.3).

Now, [Car] Theorem 9.7 (adapted to abelian sheaves as in [Bre] Theorem 3) shows that

$$\varinjlim_n \mathcal{H}_n(N\mathbb{Z}K(G, n)) = G,$$

and that for fixed q , the groups $\mathcal{H}_{n+q}(N\mathbb{Z}K(G, n))$ are constant for $n > q$, and can be expressed in terms of G/pG and ${}_pG$ for primes p . [Explicit descriptions up to $q=5$ are given at the end of [Bre] §1.] In particular, $\mathcal{H}_{n+1}(\varinjlim_n N\mathbb{Z}K(G, n)) = 0$ for $n > 1$, so

$$D_G^i(\bar{W}\tilde{\mathcal{M}}/R, M) = \mathrm{Ext}_{\mathbb{Z}}^{i+1}(G, \mathrm{Hom}_A(e^*\Omega(G/A), M))$$

for all $i \leq 0$.

Remark 3.44. This approach means that we can think of (derived) commutative group schemes as functors from simplicial (or dg) algebras to spectra. By contrast, derived stacks are functors from simplicial algebras to simplicial sets. This might help to explain why Shimura varieties are one of the rare examples of moduli spaces which extend naturally to brave new stacks (i.e. functors from ring spectra to simplicial sets), as observed in [Lur2].

Example 3.45 (Moduli of abelian schemes). A special case of the previous example is when \mathcal{M} is an open subcategory of the category of abelian schemes. The description of the cohomology groups in this case simplifies greatly. The key observation is that

$$\mathbf{R}\Gamma(G, \mathcal{O}_G) \simeq \mathrm{Sym}_A(H^1(G, \mathcal{O}_G)[-1]),$$

with $H^1(G, \mathcal{O}_G)$ projective over A .

If we have $H^1(G, \mathcal{O}_G) \cong A^g$, then the equivalence becomes

$$\mathbf{R}\Gamma(G, \mathcal{O}_G) \simeq C^\bullet(K(\mathbb{Z}^g, 1), \mathbb{Z}) \otimes_{\mathbb{Z}}^{\mathbf{L}} A.$$

Thus

$$\begin{aligned} \varprojlim_n \mathbf{R}\Gamma(K(G, n), \mathcal{O}_{K(G,n)})[n] &\simeq \varprojlim_n C^\bullet(K(\mathbb{Z}^g, n+1), \mathbb{Z})[n] \otimes_{\mathbb{Z}}^{\mathbf{L}} A \\ &= C_{\mathrm{st}}^{\bullet-1}(H\mathbb{Z}^g, \mathbb{Z}) \otimes_{\mathbb{Z}}^{\mathbf{L}} A \\ &= C_{\mathrm{st}}^{\bullet-1}(H\mathbb{Z}, \mathbb{Z}) \otimes_{\mathbb{Z}}^{\mathbf{L}} A^g, \end{aligned}$$

the complex of stable cochains of the integral Eilenberg–Mac Lane spectrum $H\mathbb{Z}$.

In general, $\varprojlim_n \mathbf{R}\Gamma(K(G, n), \mathcal{O}_{K(G,n)})[n]$ can be rewritten A -linearly in terms of the cocommutative A -coalgebra $\mathbf{R}\Gamma(G, \mathcal{O}_G)$ (a model for which is the normalised Čech complex with Eilenberg–Zilber comultiplication). This gives us a morphism

$$C_{\mathrm{st}}^\bullet(H\mathbb{Z}, \mathbb{Z}) \otimes_{\mathbb{Z}}^{\mathbf{L}} \varprojlim_n \mathbf{R}\Gamma(K(G, n), \mathcal{O}_{K(G,n)})[n] \rightarrow \varprojlim_n \mathbf{R}\Gamma(K(G, n), \mathcal{O}_{K(G,n)})[n],$$

inducing a map

$$\mathbf{C}_{\text{st}}^{\bullet-1}(H\mathbb{Z}, \mathbb{Z}) \otimes_{\mathbb{Z}}^{\mathbf{L}} \mathbf{H}^1(G, \mathcal{O}_G) \rightarrow \varprojlim_n \mathbf{R}\Gamma(K(G, n), \mathcal{O}_{K(G, n)})[n].$$

We know that this is an isomorphism locally on A (as $\mathbf{H}^1(G, \mathcal{O}_G)$ is locally free), so it is an isomorphism in general.

Thus

$$D_G^i(\bar{W}\tilde{\mathcal{M}}/R, M) \cong \text{Hom}_A(e^*\Omega(G/A), \mathbf{H}^1(G, \mathcal{O}_G)) \otimes_A \mathbf{H}_{\text{st}}^i(H\mathbb{Z}, M).$$

These stable cohomology groups can be calculated by applying the universal coefficient theorem to [Car] Theorem 7; alternatively we could apply the universal coefficient theorem for Tor to the integral Steenrod algebra $\mathbf{H}_{\text{st}}^i(H\mathbb{Z}, \mathbb{Z})$. The important features are that $\mathbf{H}_{\text{st}}^{<0}(H\mathbb{Z}, M) = 0$, while $\mathbf{H}_{\text{st}}^0(H\mathbb{Z}, M) = M$ and $\mathbf{H}_{\text{st}}^{>0}(H\mathbb{Z}, M)$ is an invariant of the torsion and cotorsion in M . In particular, if M is a \mathbb{Q} -module, then $\mathbf{H}_{\text{st}}^{>0}(H\mathbb{Z}, M) = 0$. We always have $\mathbf{H}_{\text{st}}^1(H\mathbb{Z}, M) = 0$, which is why moduli of abelian varieties are unobstructed.

The example above only gives a derived Artin stack when a suitable moduli stack of abelian varieties exists. This is seldom the case, so we now consider polarisations.

Example 3.46 (Moduli of polarised abelian schemes). A polarisation on an abelian scheme G yields a line bundle \mathcal{L} for which

$$\mu^* \mathcal{L} \cong \text{pr}_1^* \mathcal{L} \otimes \text{pr}_2^* \mathcal{L},$$

where $\mu : G \times G \rightarrow G$ is the multiplication map. This is equivalent to a morphism $f : G \rightarrow B\mathbb{G}_m$ with $f \circ \mu = B\mu \circ (f, f)$. Since $K(G, n) = \bar{W}K(G, n-1)$, this gives us compatible maps

$$K(G, n) \rightarrow K(\mathbb{G}_m, n+1)$$

for all $n \geq 0$.

Therefore the Eilenberg–Mac Lane spectrum $G \mapsto HG$ gives a functor from polarised abelian schemes over A to $F\text{SpStack}_R(A) \downarrow \Omega^{-1}H(\mathbb{G}_m \otimes A)$, where $\Omega^{-1}H\pi = \{K(\pi, n+1)\}_n$. A pair (G, \mathcal{L}) as above is a polarised (resp. principally polarised) abelian scheme if the induced map $G \rightarrow G^\vee$ is an isogeny (resp. an isomorphism). Since isogenies and isomorphisms are open in the space of group scheme homomorphisms, we can take $\mathcal{M}(A)$ to be the category of polarised (or principally polarised) abelian schemes over A .

In order to proceed further, we need to compute the cotangent complexes $\mathbb{L}^{K(G, n)/K(\mathbb{G}_m, n+1)}$. The calculations of Example 3.39 adapt to show that this is an extension of $\mathcal{O}_G[n]$ by $e^*\Omega(G/A) \otimes_A \mathcal{O}_G[n]$. Now, there is a spectral sequence

$$\mathbf{H}^i(K(G, n)_j, \mathcal{O}_{K(G, n)_j}) \implies \mathbf{H}^{i+j}(K(G, n), \mathcal{O}_{K(G, n)});$$

since $K(G, n)_j = \text{Spec } A$ for all $j < n$, this means that $\mathbf{H}^j(K(G, n), \mathcal{O}_{K(G, n)}) = 0$ for all $0 < j < n$. In particular, we have $\text{Ext}_{K(G, n)}^1(e^*\Omega(G/A) \otimes_A \mathcal{O}_G, \mathcal{O}_G) = 0$ for all $n \geq 2$, giving a decomposition

$$\mathbb{L}^{K(G, n)/K(\mathbb{G}_m, n+1)} \cong (e^*\Omega(G/A) \oplus A) \otimes_A \mathcal{O}_G[n].$$

Therefore

$$D_G^i(\bar{W}\tilde{\mathcal{M}}/R, M) \cong \text{Hom}_A(e^*\Omega(G/A) \oplus A, \mathbf{H}^1(G, \mathcal{O}_G)) \otimes_A \mathbf{H}_{\text{st}}^i(H\mathbb{Z}, M),$$

where

$$\tilde{\mathcal{M}}(A) = \mathcal{M}(H_0A) \times_{(F\text{SpStack}(H_0A) \downarrow \Omega^{-1}H(\mathbb{G}_m \otimes H_0A))}^h (d\text{SpStack}(A) \downarrow \Omega^{-1}H(\mathbb{G}_m \otimes A)).$$

Since these groups satisfy the requisite finiteness conditions, and as \mathcal{M} is representable, this means that $\tilde{\mathcal{M}}$ satisfies Theorem 2.10, so is a 1-truncated derived geometric Deligne–Mumford stack.

Remark 3.47. If M is a \mathbb{Q} -module, then $H_{\text{st}}^i(H\mathbb{Z}, M) = 0$ for all $i > 0$, so in Example 3.46, we have

$$D_G^i(\bar{W}\tilde{\mathcal{M}}/R, M) \cong \begin{cases} \text{Hom}_A(e^*\Omega(G/A) \oplus A, H^1(G, \mathcal{O}_G)) \otimes_A M & i = 0 \\ 0 & i \neq 0. \end{cases}$$

Since \mathcal{M} is a smooth scheme, the map $\mathcal{M} \rightarrow \tilde{\mathcal{M}}$ is therefore an equivalence of derived stacks (since it is an isomorphism on π^0 and on the groups D^i). In other words, the derived moduli stack of (principally) polarised abelian schemes is just the ordinary moduli stack in characteristic zero.

However, even in this case, Example 3.46 provides us with new information: it provides us with the notion of a derived abelian scheme \mathcal{G} over a dg algebra A . Explicitly, \mathcal{G} is a spectral derived stack over $H\mathbb{G}_m \times \text{Spec } A$ for which the derived pullback $\mathcal{G} \otimes_A^{\mathbf{L}} H_0A$ is equivalent to the Eilenberg–Mac Lane spectrum HG , for some polarised abelian scheme G over H_0A .

4. DERIVED MODULI OF QUASI-COHERENT SHEAVES

4.1. Derived quasi-coherent sheaves. Given a simplicial ring A_\bullet , the simplicial normalisation functor N^s of Definition 2.1 induces an equivalence between the categories $s\text{Mod}(A)$ of simplicial A_\bullet -modules, and $dg_+\text{Mod}(N^sA)$ of N^sA -modules in non-negatively graded chain complexes, where the Eilenberg–Zilber shuffle product makes N^sA into a graded-commutative algebra. As observed in [TV2] §2.2.1, this extends to give a weak equivalence between the ∞ -category of stable A -modules, and the ∞ -category $dg\text{Mod}(N^sA)$ of N^sA -modules in \mathbb{Z} -graded chain complexes, and hence an equivalence between the corresponding homotopy categories. More precisely, the ∞ -structure on $dg\text{Mod}(N^sA)$ comes from a model structure:

Definition 4.1. In $dg\text{Mod}(N^sA)$, fibrations are surjections, weak equivalences are quasi-isomorphisms, and cofibrations satisfy LLP with respect to trivial fibrations. Explicitly, $f : L \rightarrow M$ is a cofibration if it is a retract of a transfinite composition of monomorphisms $f' : L' \rightarrow M'$ with $\text{coker } f' \cong (N^sA)[n]$ for some n .

In particular, if A is discrete and f is a monomorphism with $\text{coker } f$ bounded below and levelwise projective, then f is a cofibration.

Definition 4.2. Given $L, M \in dg\text{Mod}(N^sA)$, we can then define $\text{HOM}_A(L, M)$ to be the chain complex given in level n by A -linear maps $L \rightarrow M[n]$ of graded objects. The differential on $\text{HOM}_A(L, M)$ is given by $d(f) = d \circ f \pm f \circ d$. The simplicial structure on $dg\text{Mod}(N^sA)$ is then given by setting

$$\underline{\text{Hom}}_A(L, M) := (N^s)^{-1}(\tau_{\geq 0}\text{HOM}_A(L, M)),$$

where $\tau_{\geq 0}$ is good truncation.

The associated ∞ -category is given by the full simplicial subcategory of $dg\text{Mod}(N^sA)$ on cofibrant objects. Equivalently, we have derived function complexes on $dg\text{Mod}(N^sA)$ given by $\mathbf{R}\underline{\text{Hom}}(L, M) := \underline{\text{Hom}}(\tilde{L}, M)$, for $\tilde{L} \rightarrow L$ a cofibrant replacement.

Definition 4.3. An object X_\bullet in $sd\text{Aff}_R$ is said to be a strongly quasi-compact homotopy derived Artin n -hypergroupoid if it admits a map $X_\bullet \rightarrow \tilde{X}_\bullet$ to a strongly quasi-compact derived Artin n -hypergroupoid with the property that the level maps $X_n \rightarrow \tilde{X}_n$ are all weak equivalences in $d\text{Aff}_R$.

Definition 4.4. If X is a strongly quasi-compact homotopy derived Artin l -hypergroupoid, define the simplicial cosimplicial algebra $O(X)_\bullet$ by $O(X)_m^n =$

$\Gamma(X_n^m, \mathcal{O}_{X_n^m})$. Define the category $dg\text{Mod}(X)$ to have objects M consisting of complexes $M_\bullet^n \in dg\text{Mod}(N^s\mathcal{O}(X)_\bullet^n)$ for all n , together with morphisms

$$\begin{aligned} \partial^i &: M_\bullet^n \otimes_{N^s\mathcal{O}(X)_\bullet^n, \partial_i^*} N^s\mathcal{O}(X)_\bullet^{n+1} \rightarrow M_\bullet^{n+1} \\ \sigma^i &: M_\bullet^n \otimes_{N^s\mathcal{O}(X)_\bullet^n, \sigma_i^*} N^s\mathcal{O}(X)_\bullet^{n-1} \rightarrow M_\bullet^{n-1}, \end{aligned}$$

satisfying the usual cosimplicial identities.

Definition 4.5. A morphism $f : M \rightarrow N$ in $dg\text{Mod}(X)$ is said to be a weak equivalence each map $f^n : M^n \rightarrow N^n$ is a weak equivalence in $dg\text{Mod}(N^s\mathcal{O}(X)_\bullet^n)$. Say that f is a fibration if each map $f^n : M^n \rightarrow N^n$ is surjective.

Finally, f is said to be a cofibration if it satisfies the LLP with respect to trivial fibrations. For an explicit characterisation, first form the n th latching object $L^n M \in dg\text{Mod}(N^s\mathcal{O}(X)_\bullet^n)$ as the cokernel of

$$(\alpha - \beta) : \bigoplus_{i=0}^n \bigoplus_{j=0}^{i-1} \partial_i^* \partial_j^* M^{n-2} \rightarrow \bigoplus_{i=0}^{n-1} \partial_i^* M^{n-1},$$

where for $x \in \partial_i^* \partial_j^* M^{n-2} = \partial_j^* \partial_{i-1}^* M^{n-2}$, we define $\alpha(x) = \partial_i^*(\partial^j)x \in \partial_i^* M^{n-1}$, and $\beta(x) = \partial_j^*(\partial^{i-1})x \in \partial_j^* M^{n-1}$. Then f is a cofibration if and only if the maps

$$L^n N \oplus_{L^n M} M^n \rightarrow N^n$$

are cofibrations in $dg\text{Mod}(N^s\mathcal{O}(X)_\bullet^n)$ for all n .

We can then form the simplicial structure by analogy with $dg\text{Mod}(N^s A)$: define the chain complex

$$\text{HOM}_X(M, N)_n := \text{Hom}_{\mathcal{O}(X)}(M, N[n]),$$

where Hom respects cosimplicial, but not chain, structures. Then $\underline{\text{Hom}}(M, N) := (N^s)^{-1}(\tau_{\geq 0} \text{HOM}_X(M, N))$, with $\mathbf{R}\underline{\text{Hom}}$ given by taking a cofibrant replacement for M .

Definition 4.6. Define $dg\text{Mod}(X)_{\text{cart}} \subset dg\text{Mod}(X)$ to be the full subcategory consisting of those M for which the morphisms ∂^i are all weak equivalences. Let $\text{Ho}(dg\text{Mod}(X))_{\text{cart}}$ be the category obtained by localising at weak equivalences. Note that $dg\text{Mod}(X)_{\text{cart}}$ also inherits a simplicial structure from $dg\text{Mod}(X)$. Objects of $dg\text{Mod}(X)_{\text{cart}}$ are called derived quasi-coherent sheaves on X .

Proposition 4.7. *Fix a strongly quasi-compact homotopy derived Artin n -hypergroupoid X . The simplicial category of derived quasi-coherent sheaves on X is equivalent to the ∞ -category $dg\text{Mod}_{\text{cart}}(X^\sharp)$ of homotopy-Cartesian modules on X^\sharp (as in [TV2] Definition 1.2.12.1) or equivalently, of quasi-coherent complexes on X^\sharp (in the sense of [Lur1] §5.2).*

Proof. This is [Pri1] Proposition 4.7. □

In particular, this means that $dg\text{Mod}(X)_{\text{cart}}$ is essentially independent of the atlas X chosen for X^\sharp .

Definition 4.8. Define a strongly quasi-compact Artin (resp. Deligne–Mumford) n -hypergroupoid over a ring R to be an object $X_\bullet \in s\text{Aff}_R$, such that the partial matching maps

$$X_m \rightarrow M_{\Lambda_k^n} X$$

are smooth (resp. étale) surjections for all k, m , and isomorphisms for all $m > n$ and all k .

In particular, note that every strongly quasi-compact Artin (resp. Deligne–Mumford) n -hypergroupoid is a strongly quasi-compact homotopy derived Artin (resp. Deligne–Mumford) n -hypergroupoid. Specialising to underived n -stacks, Proposition 3.29 shows that every strongly quasi-compact n -geometric Artin (resp. Deligne–Mumford) stack

is of the form X^\sharp , for some strongly quasi-compact Artin (resp. Deligne–Mumford) n -hypergroupoid X ([Pri1] Theorem 5.16).

For our purposes, the main applications of derived quasi-coherent sheaves stem from the following:

Proposition 4.9. (1) *Take a strongly quasi-compact Artin n -hypergroupoid X . Then the category of quasi-coherent sheaves on the geometric stack X^\sharp is equivalent to the full subcategory of $\mathrm{Ho}(\mathrm{dgMod}(X)_{\mathrm{cart}})$ consisting of complexes concentrated in chain degree 0.*

(2) *If Y is a semi-separated quasi-compact scheme, take an open affine cover $\{U_i\}_{i=1}^n$. Let $U = \coprod_i U_i$, and set X to be the strongly quasi-compact Deligne–Mumford 1-hypergroupoid given by*

$$X_n = \overbrace{U \times_Y U \times_Y \dots \times_Y U}^{n+1}.$$

Then $\mathrm{Ho}(\mathrm{dgMod}(X)_{\mathrm{cart}})$ is equivalent to the derived category of complexes of quasi-coherent sheaves on Y .

Proof. The first statement is [Pri1] Corollary 6.7. The second is [Hüt] Theorem 4.5.1, once we note that $X^\sharp = Y$. \square

4.2. Moduli.

Definition 4.10. Take $A \in \mathrm{dg}_+ \mathrm{Alg}_R$ (resp. $A \in \mathrm{sAlg}_R$), and a strongly quasi-compact homotopy derived Artin n -hypergroupoid X over R , chosen so that each X_n is levelwise flat over R (which holds for instance if X is a derived Artin n -hypergroupoid). Then observe that $X \otimes_R A$ is a strongly quasi-compact homotopy derived Artin n -hypergroupoid over A , and define

$$d\mathrm{CART}_X(A)$$

to be the full simplicial subcategory of the category of $\mathrm{dgMod}(X \otimes_R A)_{\mathrm{cart}}$ on cofibrant objects.

Proposition 4.11. *The functor $d\mathrm{CART}_X : d\mathcal{N}^\flat \rightarrow \mathrm{sCat}$ defined above is 2-homogeneous and formally 2-quasi-smooth.*

Proof. The proof of Proposition 3.22 carries over to this context, with cofibrancy once again ensuring the existence of suitable lifts. \square

Theorem 4.12. *Take an m -geometric derived Artin stack \mathfrak{Y} over R , and choose a strongly quasi-compact homotopy derived Artin n -hypergroupoid \mathfrak{Y} with $\mathfrak{Y}^\sharp = \mathfrak{Y}$ and each \mathfrak{Y}_n flat over R .*

Assume that we have an n -truncated hypersheaf $\mathcal{M} : \mathrm{Alg}_{\mathrm{H}_0 R} \rightarrow \mathrm{sCat}$, open in the functor $A \mapsto \mathcal{W}(\mathrm{dgMod}_{\mathrm{cart}}(\mathfrak{Y}) \otimes_R^{\mathbf{L}} A)$.

Also assume that this satisfies the following conditions:

(1) *for all finitely generated $A \in \mathrm{Alg}_{\mathrm{H}_0 R}$ and all $\mathcal{E} \in \mathcal{M}(A)$, the functors*

$$\mathbb{E}\mathrm{xt}_{\mathfrak{Y} \otimes_R^{\mathbf{L}} A}^i(\mathcal{E}, \mathcal{E} \otimes_A^{\mathbf{L}} -) : \mathrm{Mod}_A \rightarrow \mathrm{Ab}$$

preserve filtered colimits for all $i \neq 1$.

(2) *for all finitely generated integral domains $A \in \mathrm{Alg}_{\mathrm{H}_0 R}$ and all $\mathcal{E} \in \mathcal{M}(A)$, the groups $\mathbb{E}\mathrm{xt}_{\mathfrak{Y} \otimes_R^{\mathbf{L}} A}^i(\mathcal{E}, \mathcal{E})$ are all finitely generated A -modules.*

(3) *The functor $c(\pi_0 \mathcal{M}) : \mathrm{Alg}_{\pi_0 R} \rightarrow \mathrm{Set}$ of components of the groupoid $\pi_0 \mathcal{M}$ preserves filtered colimits.*

(4) for all discrete local Noetherian H_0R -algebras A , with maximal ideal \mathfrak{m} , the map

$$c(\pi_0\mathcal{M}(A)) \rightarrow \varprojlim_r c(\pi_0\mathcal{M}(A/\mathfrak{m}^r))$$

is an isomorphism, as are the maps

$$\begin{aligned} \mathrm{Aut}_{\pi_0\mathcal{M}(A)}(\mathcal{E}) &\rightarrow \varprojlim_r \mathrm{Aut}_{\pi_0\mathcal{M}(A/\mathfrak{m}^r)}(\mathcal{E}/\mathfrak{m}^r) \\ \mathrm{Ext}_{\mathfrak{Y} \otimes_R^{\mathbf{L}} A}^i(\mathcal{E}, \mathcal{E}) &\rightarrow \varprojlim_r \mathrm{Ext}_{\mathfrak{Y} \otimes_R^{\mathbf{L}} A}^i(\mathcal{E}, \mathcal{E}/\mathfrak{m}^r) \end{aligned}$$

for all $\mathcal{E} \in \mathcal{M}(A)$ and all $i < 0$.

Let $\tilde{\mathcal{M}} : d\mathcal{N}_R^b \rightarrow s\mathrm{Cat}$ be the full simplicial subcategory of $\mathcal{W}(d\mathrm{CART}_{\mathfrak{Y}}(A))$ consisting of objects \mathcal{F} for which $\mathcal{F} \otimes_A H_0A$ is weakly equivalent in $dg\mathrm{Mod}_{\mathrm{cart}}(\mathfrak{Y} \otimes_R^{\mathbf{L}} H_0A)$ to an object of $\mathcal{M}(H_0A)$. Then the nerve $\bar{W}\mathcal{M}$ is an geometric derived n -stack.

Proof. First note that Proposition 4.7 allows us to replace $d\mathrm{CART}_{\mathfrak{Y}}(A)$ with $dg\mathrm{Mod}_{\mathrm{cart}}(\mathfrak{Y} \otimes_R^{\mathbf{L}} A)$, so

$$\tilde{\mathcal{M}}(A) \simeq \mathcal{M}(H_0A) \times_{\mathcal{W}(dg\mathrm{Mod}_{\mathrm{cart}}(\mathfrak{Y} \otimes_R^{\mathbf{L}} H_0A))}^h \mathcal{W}(dg\mathrm{Mod}_{\mathrm{cart}}(\mathfrak{Y} \otimes_R^{\mathbf{L}} A)),$$

because \mathcal{M} is open in $\pi^0\mathcal{W}(dg\mathrm{Mod}_{\mathrm{cart}}(\mathfrak{Y} \otimes_R^{\mathbf{L}} -))$.

Now, we have

$$D_{\mathcal{E}}^i(\bar{W}\tilde{\mathcal{M}}/R, M) = \mathrm{Ext}_{\mathfrak{Y} \otimes_R^{\mathbf{L}} A}^{i+1}(\mathcal{E}, \mathcal{E} \otimes_A^{\mathbf{L}} M),$$

for instance from the calculations of [Pri1] §9, where Ext groups are also characterised as

$$\mathrm{Ext}_{\mathfrak{Y} \otimes_R^{\mathbf{L}} A}^i(\mathcal{E}, \mathcal{F}) \cong \mathrm{H}_{-i}\mathbf{R}\mathrm{HOM}_{X \otimes_R A}(\mathcal{E}, \mathcal{F}),$$

provided \mathcal{E} is chosen to be cofibrant.

Note that $\pi_i \underline{\mathrm{Hom}}_{\mathcal{M}(B)}(\mathcal{E}, \mathcal{E}) = \mathrm{Ext}^{-i}(\mathcal{E}, \mathcal{E} \otimes_A^{\mathbf{L}} B)$ for $i \geq 1$, and that $\pi_0 \underline{\mathrm{Hom}}_{\mathcal{M}(B)}(\mathcal{E}, \mathcal{E})$ consists of invertible elements in $\mathrm{Ext}^0(\mathcal{E}, \mathcal{E} \otimes_A^{\mathbf{L}} B)$. Thus Condition (1) ensures that for all $A \in \mathrm{Alg}_{\pi_0 R}$ and all $\mathcal{E} \in \mathcal{M}(A)$, the functors $\pi_i \underline{\mathrm{Hom}}_{\mathcal{M}}(\mathcal{E}, \mathcal{E}) : \mathrm{Alg}_A \rightarrow \mathrm{Set}$ preserve filtered colimits for all $i \geq 0$.

Next, we need to establish that for all discrete local Noetherian H_0R -algebras A , with maximal ideal \mathfrak{m} , the map

$$\mathcal{M}(A) \rightarrow \varprojlim^h \mathcal{M}(A/\mathfrak{m}^n)$$

is a weak equivalence. Since

$$\begin{aligned} \pi_1(\bar{W}\mathcal{M}(A), \mathcal{E}) &= \mathrm{Aut}_{\pi_0\mathcal{M}(A)}(\mathcal{E}) \\ \pi_i(\bar{W}\mathcal{M}(A), \mathcal{E}) &= \mathrm{Ext}_{\mathfrak{Y} \otimes_R^{\mathbf{L}} A}^{1-i}(\mathcal{E}, \mathcal{E}) \end{aligned}$$

for $i \geq 2$, by Remark 2.11 it suffices to show that each system

$$\{\mathrm{Im}(\pi_i(\bar{W}\mathcal{M}(A/\mathfrak{m}^s), x = \mathcal{E}) \rightarrow \pi_i(\bar{W}\mathcal{M}(A/\mathfrak{m}^r), \mathcal{E}))\}_{s \geq r}$$

satisfies the Mittag-Leffler condition for $i \geq 1$ and $\mathcal{E} \in \mathcal{M}(A)$, and that Condition (4) holds.

Since $\mathrm{Ext}_{\mathfrak{Y} \otimes_R^{\mathbf{L}} A}^{1-i}(\mathcal{E}, \mathcal{E}/\mathfrak{m}^r)$ is a finitely generated A/\mathfrak{m}^r -module, it is Artinian, so the Mittag-Leffler condition is automatically satisfied for $i \geq 2$. For $i = 1$, we can study the obstruction maps of Proposition 2.8 to deduce that

$$\mathrm{Aut}_{\pi_0\mathcal{M}(A/\mathfrak{m}^r)}(\mathcal{E}) = \mathrm{Ext}_{\mathfrak{Y} \otimes_R^{\mathbf{L}} A}^0(\mathcal{E}, \mathcal{E}/\mathfrak{m}^r) \times_{\mathrm{Ext}_{\mathfrak{Y} \otimes_R^{\mathbf{L}} A}^0(\mathcal{E}, \mathcal{E}/\mathfrak{m})} \mathrm{Aut}_{\pi_0\mathcal{M}(A/\mathfrak{m})}(\mathcal{E}/\mathfrak{m}).$$

Since $\mathrm{Ext}_{\mathfrak{Y} \otimes_R^{\mathbf{L}} A}^0(\mathcal{E}, \mathcal{E}/\mathfrak{m}^r)$ is Artinian, we deduce that the Mittag-Leffler condition holds for $i = 1$, so we have satisfied the conditions of Remark 2.11.

The proof of Corollary 3.10 now carries over to this context, using Proposition 4.11 in place of Proposition 3.7.

□

Remarks 4.13. If $R = H_0R$, we can let $\mathfrak{Y} = Y$ be a semi-separated quasi-compact scheme, then form the Čech nerve \tilde{Y} as in part 2 of Proposition 4.9, giving an explicit description of $\tilde{\mathcal{M}}$.

In order to satisfy the finiteness hypotheses for $\mathbb{E}xt^*$, a likely candidate for a derived geometric n -stack satisfying the hypotheses would be to let $\mathcal{M}(A)$ consist of perfect complexes \mathcal{E}_\bullet of $\mathcal{O}_Y \otimes_R^{\mathbf{L}} A$ -modules of Tor-dimension $(n-1)$ over A , with $\mathcal{H}_i(\mathcal{E})$ quasi-coherent of proper support on Y .

Definition 4.14. Given an m -geometric (underived) Artin stack Y over H_0R , and $A \in \text{Alg}_{H_0R}$, let $F\text{Mod}_Y(A)$ be the category of quasi-coherent sheaves (not complexes) on $Y \otimes_{H_0R} A$, flat over A .

Example 4.15. For \mathfrak{Y} as in Theorem 4.12, set $Y := \pi^0\mathfrak{Y}$ (an m -geometric (underived) Artin stack over H_0R). Of course, if $R = H_0R$, then we will have $Y = \mathfrak{Y}$. Choose a stack \mathcal{M} equipped with functorial fully faithful embeddings of \mathcal{M} into the core of $F\text{Mod}_Y$, with the property that \mathcal{M} is closed under infinitesimal deformations. This ensures that $\bar{W}\mathcal{M} \rightarrow \bar{W}F\text{Mod}_Y$ is open.

Note that flatness ensures that for any morphism $A \rightarrow B$ in Alg_{H_0R} and $\mathcal{F} \in F\text{Mod}_Y(A)$, the complex $\mathcal{F} \otimes_A^{\mathbf{L}} B$ is weakly equivalent to $\mathcal{F} \otimes_A B$, so $F\text{Mod}_Y(A)$ embeds into $\text{Mod}_{\text{cart}}(\mathfrak{Y} \otimes_R^{\mathbf{L}} A)$. In fact, we can characterise $F\text{Mod}_Y(A)$ as being weakly equivalent to the subcategory of $\text{Mod}_{\text{cart}}(\mathfrak{Y} \otimes_R^{\mathbf{L}} A)$ consisting of complexes \mathcal{F} for which the homology sheaves $\mathcal{H}_i(\mathcal{F} \otimes_A^{\mathbf{L}} B) = 0$ for all $i \neq 0$ and all $B \in \text{Alg}_A$. Thus $F\text{Mod} \rightarrow \pi^0\text{Mod}_{\text{cart}}(\mathfrak{Y} \otimes_R^{\mathbf{L}} -)$ is an open embedding.

Provided that the objects of \mathcal{M} satisfy the conditions of Theorem 4.12, this gives us a geometric derived 1-stack of quasi-coherent sheaves of \mathfrak{Y} . An example for which the conditions are satisfied arises when Y is a projective scheme over R and $\mathcal{M}(A)$ consists of coherent sheaves on $Y \otimes_R A$, flat over A .

4.2.1. Comparisons. If we wish to study derived moduli of rank r vector bundles on a homotopy-flat derived geometric ∞ -stack \mathfrak{X} over R , we now have two possible approaches, which we now show are consistent. Let $X = \mathfrak{X} \otimes_R H_0R$, and define $\mathcal{M} : \text{Alg}_{H_0R} \rightarrow \text{Gpd}$ by letting $\mathcal{M}(A)$ be the groupoid of rank r vector bundles on $X \otimes_{H_0R} A$.

Example 3.38 gives one extension $\text{Hom}(\mathfrak{X}, BGL_r) : d\mathcal{N}_R^b \rightarrow \mathbb{S}$ of $B\mathcal{M}$. Theorem 4.12 gives another candidate $\bar{W}\tilde{\mathcal{M}}$ for an extension, where the objects of $\tilde{\mathcal{M}} : d\mathcal{N}_R^b \rightarrow s\text{Cat}$ are quasi-coherent complexes \mathcal{E} on $\mathfrak{X} \otimes_R^{\mathbf{L}} A$ for which $\mathcal{E} \otimes_A^{\mathbf{L}} H_0A$ is quasi-isomorphic to a rank r vector bundle on $\mathfrak{X} \otimes_R^{\mathbf{L}} H_0A$.

Lemma 4.16. *The derived geometric 1-stacks $\text{Hom}(\mathfrak{X}, BGL_r)$ and $\bar{W}\tilde{\mathcal{M}}$ defined above are weakly equivalent.*

Proof. The argument of Example 3.38 gives a morphism from $\text{Hom}(\mathfrak{X}, BGL_n)$ to $\bar{W}\mathcal{W}(\mathcal{T})$, where $\mathcal{T}(A)$ is the full simplicial subcategory of $d\text{Stack}_{\mathfrak{X} \times BGL_r}(A)$ on objects \mathfrak{Z} for which the projection $f_1 : \mathfrak{Z} \rightarrow \mathfrak{X} \otimes_R^{\mathbf{L}}(A)$ is a weak equivalence. We can then define a morphism $\mathcal{W}(\mathcal{T}) \rightarrow \tilde{\mathcal{M}}$ as follows: let \mathcal{U} be the universal rank r vector bundle on BGL_r , and send

$$(\mathfrak{Z} \xrightarrow{(f_1, f_2)} (\mathfrak{X} \times BGL_r) \otimes_R^{\mathbf{L}}(A))$$

in $\mathcal{T}(A)$ to the quasi-coherent complex $\mathbf{R}f_{1*}f^*\mathcal{U}$ on \mathfrak{X} .

Thus we have constructed a map $\text{Hom}(\mathfrak{X}, BGL_n) \rightarrow \bar{W}\tilde{\mathcal{M}}$. It is immediate that this gives an equivalence on π^0 , and straightforward to check that it gives isomorphisms on cohomology groups D^i , so it is a weak equivalence, by [Pri3] Proposition 1.37. □

Remark 4.17. We could adapt Theorem 4.12 to moduli of polarised varieties, by considering pairs (T, M) , for $T \in c\text{ALG}(A)$ and $M \in d\text{CART}_T(A)$. The resulting derived stack

would parametrise pairs $(\mathfrak{X}, \mathcal{E})$, for \mathfrak{X} a derived geometric m -stack and \mathcal{E} a quasi-coherent complex on \mathfrak{X} . The argument of Lemma 4.16 then adapts to show that an open substack of the resulting moduli stack is equivalent to the derived stack from Example 3.39.

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