HENSELIAN VALUED QUASILOCAL FIELDS WITH TOTALLY INDIVISIBLE VALUE GROUPS, II

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Abstract. This paper characterizes the quasilocal fields from the class of Henselian valued fields with totally indivisible value groups, over which there exist finite separable extensions of nontrivial defect. We show that every nontrivial divisible subgroup of the quotient group $\mathbb{Q}/\mathbb{Z}$ of the additive group of rational numbers by the subgroup of integers is realizable as a Brauer group of such a quasilocal field.

1. Introduction and statements of the main results

Let K be a field, $K_{\rm sep}$ a separable closure of K, $\mathcal{G}_K = \mathcal{G}(K_{\rm sep}/K)$ the absolute Galois group of K and $\Pi(K)$ the set of those prime numbers for which the Sylow pro-p-subgroups of $\mathcal{G}_K$ are nontrivial. The field K is called primarily quasilocal (abbr, PQL), if every cyclic extension F of K is embeddable as a subalgebra in each central division K-algebra D of Schur index ind(D) divisible by the degree $[F\colon K]$; we say that K is quasilocal, if its finite extensions are PQL-fields. The notion of a quasilocal field extends the one of a local field and defines a class which contains p-adically closed fields and Henselian discrete valued fields with quasifinite residue fields (cf. [23], Ch. XIII, Sect. 3, [21], Theorem 3.1 and Lemma 2.9, and [3], Proposition 6.4). Other examples of quasilocal fields, mostly, of nonarithmetic nature (from the perspective of [3], (1.2), (1.3) and Corollary 5.2), can be found in [5].

The purpose of this paper is to find a satisfactory characterization of the quasilocal property in the class of Henselian valued fields with totally indivisible value groups. When finite separable extensions of the considered Henselian fields are defectless, such a characterization is contained in [1], Theorem 2.1. This, combined with the following result, stated below, solves the considered problem in general. The statement of the result is simplified by the notion of a quasiinertial extension introduced in Section 2.

Theorem 1.1. *Let (K, v) be a Henselian valued field, such that* $\mathrm{char}(\widehat{K}) = q \neq 0$*, and for each $p \in \Pi(K)$, let K_p be the fixed field of some Sylow pro-p-subgroup $G_p \leq \mathcal{G}_K$. Assume that $v(K) \neq pv(K)$, for every $p \in \Pi(K)$, and K possesses at least one finite extension in $K_{\rm sep}$ of nontrivial defect. Then K is quasilocal if and only if it satisfies the following two conditions:*

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(i) $v(K)/qv(K)$ *is of order* q *and* $K_{\rm sep}$ *contains as a subfield a quasiinertial* $\mathbb{Z}_q$*-extension* Y *of* K_q*, such that every finite extension* L_q *of* K_q *in* $K_{\rm sep}$ *with* $L_q \cap Y = K_q$ *is totally ramified;*

(ii) $r(p)_{K_p} \leq 2$*, for each* $p \in \Pi(K) \setminus \{q\}$*.*

For a proof of Theorem 1.1, we refer the reader to [3], Proposition 6.1. Our main objective in this paper is to prove the following:

Theorem 1.2. *Let* (Φ, ω) *be a Henselian discrete valued field, such that* $\widehat{\Phi}$ *is quasifinite of characteristic* $q \neq 0$*, and let* T *be a divisible subgroup of* $\mathbb{Q}/\mathbb{Z}$ *with a nontrivial* q*-component* T_q*. Then there exists a Henselian valued quasilocal field* (K, v) *with the following properties:*

(i) *The Brauer group* $\mathrm{Br}(K)$ *is isomorphic to* T*,* K/Φ *is a field extension of transcendency degree* 1 *and* v *is a prolongation of* ω*;*

(ii) $v(K)$ *is Archimedean and totally indivisible,* $\widehat{K}/\widehat{\Phi}$ *is an algebraic extension, and* K *possesses an immediate* $\mathbb{Z}_q$*-extension* I_∞*.*

Brauer groups of quasilocal fields have influence on a wide spectrum of their algebraic properties (see, e.g., [2], I, Lemma 3.8 and Theorem 8.1, [2], II, Lemmas 2.3 and 3.3, [4], Sects. 3 and 4, and [5], Sects. 1 and 6). Note also that, by [3], Corollary 5.2, $\mathrm{Br}(K)$ is divisible and embeds in $\mathbb{Q}/\mathbb{Z}$ whenever (K, v) is a Henselian valued quasilocal field with $v(K)$ totally indivisible. Therefore, Theorem 1.2 can be viewed as a complement to Theorem 1.1, which clearly shows that the study of the fields dealt with in the present paper does not reduce to the special case singled out by [1], Theorem 2.1.

The basic notation, terminology and conventions kept in this paper are standard and essentially the same as in [2], I, [3] and [4]. Throughout, Brauer and value groups are written additively, Galois groups are viewed as profinite with respect to the Krull topology, and by a profinite group homomorphism, we mean a continuous one. We write $\mathbb{P}$ for the set of prime numbers, and for each $p \in \mathbb{P}$, $\mathbb{Z}_p$ is the additive group of p-adic integers and $\mathbb{Z}(p^\infty)$ is the quasicyclic p-group. For any profinite group G, we denote by $\mathrm{cd}(G)$ the cohomological dimension of G, and by $\mathrm{cd}_p(G)$ its cohomological p-dimension, for each $p \in \mathbb{P}$. Given a field E, $\mathrm{Br}(E)_p$ is the p-component of the Brauer group $\mathrm{Br}(E)$, ${}_p\mathrm{Br}(E) = \{\delta \in \mathrm{Br}(E)\colon\ p\delta = 0\}$, for a fixed $p \in \mathbb{P}$, and $P(E) = \{p \in \mathbb{P}\colon\ E(p) \neq E\}$, where $E(p)$ is the maximal p-extension of E in $E_{\rm sep}$. For any $p \in P(E)$, $r(p)_E$ is the rank of $\mathcal{G}(E(p)/E)$ as a pro-p-group; $r(p)_E := 0$ in case $p \notin P(E)$. We write $s(E)$ for the class of finite-dimensional central simple E-algebras, $d(E)$ stands for the class of division algebras $D \in s(E)$, and for each $A \in s(E)$, $[A]$ is the similarity class of A in $\mathrm{Br}(E)$. For any field extension E'/E, $I(E'/E)$ denotes the set of its intermediate fields. The field E is called p-quasilocal, for some $p \in \mathbb{P}$, if $\mathrm{Br}(E)_p \neq \{0\}$, or $p \notin P(E)$, or every extension of E in $E(p)$ of degree p embeds as an E-subalgebra in each $\Delta_p \in d(E)$ of index p. By [2], I, Theorem 4.1, E is PQL if and only if it is p-quasilocal, for each $p \in P(E)$.

2. Preliminaries on Henselian valuations

Let K be a field with a nontrivial (Krull) valuation v, $O_v(K) = \{a \in K\colon\ v(a) \geq 0\}$ the valuation ring of (K, v), $M_v(K) = \{\mu \in K\colon\ v(\mu) > 0\}$

the unique maximal ideal of O_v, $\nabla_0(K) = \{\alpha \in K\colon\ (\alpha - 1) \in M_v(K) \setminus \{0\}\}$, and let $v(K)$ and $\widehat{K}$ be the value group and the residue field of (K, v), respectively. We say that v is Henselian, if it is uniquely, up-to an equivalence, extendable to a valuation v_L on each algebraic field extension L/K. It is known that v is Henselian if and only if (K, v) satisfies the following (Hensel-Rychlik) condition (cf. [11], Sect. 18.1):

(2.2) Given a polynomial $f(X) \in O_v(K)[X]$, and an element $a \in O_v(K)$, such that $2v(f'(a)) < v(f(a))$, where f' is the formal derivative of f, there is a zero $c \in O_v(K)$ of f satisfying the equality $v(c - a) = v(f(a)/f'(a))$.

When v is Henselian and L/K is algebraic, v_L is Henselian and extends uniquely to a valuation v_D on each $D \in d(L)$. Denote by $\widehat{D}$ the residue field of (D, v_D) and put $v(D) = v_D(D)$. By the Ostrowski-Draxl theorem [8], $[D\colon K]$, $[\widehat{D}\colon \widehat{K}]$ and the ramification index $e(D/K)$ are related as follows:

(2.3) $[D\colon K]$ is divisible by $[\widehat{D}\colon \widehat{K}]e(D/K)$ and $[D\colon K]/([\widehat{D}\colon \widehat{K}]e(D/K))$ is not divisible by any $p \in \mathbb{P}$, $p \neq \mathrm{char}(\widehat{K})$.

The K-algebra D is said to be defectless, if $[D\colon K] = [\widehat{D}\colon \widehat{K}]e(D/K)$, and it is called immediate, if $\widehat{D} = \widehat{K}$ and $e(D/K) = 1$. We say that D/K is totally ramified, if $e(D/K) = [D\colon K]$. When v is Henselian with $v(K) \neq pv(K)$, for a given $p \in \mathbb{P}$, (K, v) is subject to the following alternative (see [6], Corollary 6.5):

(2.4) (i) K has a totally ramified proper extension in $K(p)$;

(ii) $\mathrm{char}(K) = 0$, K does not contain a primitive p-th root of unity and the minimal isolated subgroup of $v(K)$ containing $v(p)$ is p-divisible.

Let (K, v) be a Henselian valued field with $\mathrm{char}(\widehat{K}) = p > 0$, and let $M \in I(K(p)/K)$ be a finite extension of K. When $\widehat{M} = \widehat{K}$, $\nabla_0(M)$ equals the pre-image of $\nabla_0(K)$ under the norm map N_K^M, which implies the following:

(2.5) $\varphi(\mu)\mu^{-1} \in \nabla_0(M)$, if $\mu \in M^*$ and φ is a K-automorphism of M.

The extension M/K is called norm-inertial, if $\nabla_0(K)$ is included in the norm group $N(M/K)$. We say that M/K is quasiinertial, if $O_v(M)$ equals the set of those elements $\delta \in M^*$, for which the trace $\mathrm{Tr}_K^M(\delta\mu) \in O_v(K)$, for every $\mu \in O_v(M)$. For each primitive element μ of M/K lying in $O_v(M)$, put $\delta_{M/K}(\mu) = v_M(f'_\mu(\mu))$, where f'_μ is the formal derivative of the minimal (monic) polynomial f_μ of μ over K. It is well-known that $[M\colon K]\delta_{M/K}(\mu) = v(d_\mu)$, where d_μ is the discriminant of f_μ. Our main objective in the rest of this Section is to show that M/K is quasiinertial if and only if any of the following three equivalent conditions is fulfilled:

(2.6) (i) For each $\gamma \in v(K)$, $\gamma > 0$, there exists $\lambda_\gamma \in O_v(K)$, such that $v(\mathrm{Tr}_K^M(\lambda_\gamma)) < \gamma$;

(ii) For each $\gamma' \in v(M)$, $\gamma' > 0$, $O_v(M)$ contains a primitive element $\mu_{\gamma'}$ of M/K satisfying the inequality $\delta_{M/K}(\mu'_{\gamma'}) < \gamma'$;

(iii) There exists $L \in I(M/K)$, such that L/K and M/L are quasiinertial.

It follows from the inequalities $v_M(y) \leq v(\mathrm{Tr}_K^M(y))$, $y \in O_v(M)$, and the transitivity of traces in towers of finite separable extensions (cf. [17], Ch. VIII, Sect. 5) that if M/K satisfies (2.6) (i) and $M_0 \in I(M/K)$, then M/M_0

and M_0/K satisfy (2.6) (i) as well. When (2.6) (iii) holds, the assertion that M/K is quasiinertial is standardly proved by assuming the opposite, using again trace transitivity. Let $r \in O_v(M)$ be a primitive element of M/K. It is easily obtained (by applying in an obvious manner basic linear algebra, including Vandermonde's determinant) that if $r' \in O_v(M) \setminus \{0\}$ and $\mathrm{Tr}_K^M(r'^{-1}r^{j-1}) \in O_v(K)$, $j = 1, \dots, [M\colon K]$, then $2v_M(r') \leq v(d_r)$. Hence, M/K is quasiinertial when (2.6) (ii) holds. As to (2.6) (i), it is satisfied in case M/K is quasiinertial (because if $a \in M_v(K)\setminus\{0\}$ and $a' \in O_v(M)$, then $\mathrm{Tr}_K^M(a^{-1}a') \in O_v(K)$ if and only if $v(a) \leq v(\mathrm{Tr}_K^M(a')))$. We show that (2.6) (i)$\to$(2.6) (ii). Note first that if the set $v(K)_0 = \{\gamma \in v(K)\colon\ \gamma > 0\}$ contains a minimal element, then the fulfillment of (2.6) (i) or (2.6) (iii) is equivalent to the condition that M/K is inertial (in the sense of [13]). Therefore, it suffices to prove that (2.6) (i)$\to$(2.6) (ii) in case $v(K)_0$ does not contain a minimal element. Assume that (2.6) (i) holds, $[M\colon K] = p^n$ and α is an element of $O_v(M)$, such that $v(\mathrm{Tr}_K^M(\alpha)) < v(p)$. It is easily verified that α is a primitive element of M/K. Put $M' = K(\alpha_1, \dots, \alpha_{p^n})$, where α_u, $u = 1, \dots, [M\colon K]$, are the roots in $K(p)$ of the minimal polynomial f_α. We prove the validity of (2.6) (ii) by showing that $v_{M'}(\alpha_{u'} - \alpha_{u''}) \leq v(\mathrm{Tr}_K^M(\alpha))$, for $1 \leq u' < u'' \leq p^n$. Suppose first that $[M\colon K] = p$ and φ is a generator of $\mathcal{G}(M/K)$. Then $v_M(\varphi^\nu(\alpha) - \alpha) = v_M(\varphi(\alpha) - \alpha)$, for $\nu = 1, \dots, p-1$. As $\alpha \in O_v(M)$ and $v(\mathrm{Tr}_K^M(\alpha)) < v(p)$, this implies the stated inequality. The proof in general is carried out by induction on n, under the inductive hypothesis that $n \geq 2$ and, for a field $K' \in I(M/K)$ of degree $[K'\colon K] = p$, $\mathrm{Tr}_{K'}^M$ is subject to analogous inequalities. Since $\mathrm{Tr}_K^M(\alpha) = \mathrm{Tr}_K^{K'}(\mathrm{Tr}_{K'}^M(\alpha))$, whence $v_{K'}(\mathrm{Tr}_{K'}^M(\alpha)) \leq v(\mathrm{Tr}_K^M(\alpha)) < v(p)$, this yields $v_{M'}(\alpha_{u'} - \alpha_{u''}) \leq v_{K'}(\mathrm{Tr}_{K'}^M(\alpha))$, provided that $u' \neq u''$ and $\alpha_{u'}$ and $\alpha_{u''}$ are conjugate over K'. Now take indices u' and u'' so that $\alpha_{u'}$ and $\alpha_{u''}$ are not conjugate over K'. Then we have $\alpha_{u''} = \psi(\alpha_{u'})$, for some $\psi \in \mathcal{G}(M'/K)$, which induces on K' a generator, say, ψ' of $\mathcal{G}(K'/K)$. Denote by $S_{u'}$ and $S_{u''}$ the sets of roots in M' of the minimal polynomials over K' of $\alpha_{u'}$ and $\alpha_{u''}$, respectively. Using the normality of $\mathcal{G}(M'/K')$ in $\mathcal{G}(M'/K)$, one obtains that if $v_{M'}(\alpha_{u'} - \alpha_{u''}) > v(\mathrm{Tr}_K^M(\alpha))$, then there is a bijection ϵ of $S_{u'}$ on $S_{u''}$, such that $v_{M'}(\alpha_u - \epsilon(\alpha_u)) > v(\mathrm{Tr}_K^M(\alpha))$ whenever $\alpha_u \in S_{u'}$. Our conclusion, however, contradicts the inequality $v_{K'}(\psi'(\mathrm{Tr}_{K'}^M(\alpha)) - \mathrm{Tr}_{K'}^M(\alpha)) \leq v(\mathrm{Tr}_K^M(\alpha))$ and thereby proves that $v_{M'}(\alpha_{u'} - \alpha_{u''}) \leq v(\mathrm{Tr}_K^M(\alpha))$. This completes the proof of the equivalence (2.6) (i)$\leftrightarrow$(2.6) (ii). The obtained results also indicate that if M/K is quasiinertial, then so are M/M_0 and M_0/K, for every $M_0 \in I(M/K)$. Moreover, it becomes clear that each condition in (2.6) is equivalent to the one that M/K is quasiinertial. When M/K is Galois, one concludes in addition that (2.6) (ii) can be restated as follows:

(2.7) For each $\gamma \in v(K)$, $\gamma > 0$, there exists $\beta_\gamma \in O_v(M)$, such that $v_M(\varphi(\beta_M) - \beta_M) < \gamma$, for every $\varphi \in \mathcal{G}(M/K)$, $\varphi \neq 1$.

With assumptions being as above, let I_∞ be a field from $I(K(p)/K)$. We say that I_∞/K is norm-inertial, if I/K is norm-inertial, for finite extension I of K in I_∞. The extension I_∞/K is called quasiinertial, if finite extensions of K in I_∞ are is quasiinertial. The defined notions are related as follows:

(2.8) (i) I_∞/K is norm-inertial, provided that it is quasiinertial;

(ii) If I_∞/K is norm-inertial and $H \neq pH$ whenever H is a nontrivial isolated subgroup of $v(K)$, then I_∞/K is quasiinertial; when this holds, I_∞/I is quasiinertial, for every $I \in I(I_\infty/K)$.

Statement (2.8) (i) is easily proved by applying (2.2) and (2.6). The latter assertion of (2.8) (ii) is implied by (2.6) and the former one. As in the proof of implication (2.6) (i)→(2.6) (ii), one sees that it suffices to prove the former part of (2.8) (ii) in the special case where $v(K) \leq \mathbb{R}$ and $v(K)$ is noncyclic. Moreover, for each finite extension I of K in I_∞, one obtains that if $\theta \in \nabla_0(I)$, $N_K^I(\theta) = 1 + \theta_0$, $v(\theta_0) < v(p)$ and $v(\theta_0) \notin pv(K)$, then $v_{K(p)}(\theta - \theta') \leq v(\theta_0)$, provided $\theta' \in K(p)$, $\theta' \neq \theta$ and $f_\theta(\theta') = 0$. Since $v(K)$ is dense in $\mathbb{R}$, the obtained result proves the former assertion of (2.8) (ii).

3. Preparation for the proof of Theorem 1.2

Our proof is constructive and relies on the following two lemmas.

Lemma 3.1. *Let (E, v) be a Henselian valued field with* $\mathrm{char}(\widehat{E}) = p \neq 0$. *Assume that $p \in P(E)$, $E(p)/E$ is immediate, $r(p)_E \in \mathbb{N}$, and in the mixed characteristic case, E contains a primitive p-th root of unity. Then:*

(i) *$\widehat{E}$ is perfect, $v(E) = pv(E)$ and* $\mathrm{Br}(E)_p = \{0\}$*;*

(ii) *$\mathcal{G}(E(p)/E)$ is a free pro-p-group; in particular, every cyclic extension L of E in $E(p)$ lies in $I(L_\infty/E)$, for some $\mathbb{Z}_p$-extension L_∞/E, $L \subseteq E(p)$;*

(iii) *If E is perfect and $v(E) \leq \mathbb{R}$, then finite extensions of E in $E(p)$ are quasiinertial, whence every $\mathbb{Z}_p$-extension of E is quasiinertial.*

Proof. The immediacy of $E(p)/E$ ensures that $v(E) = pv(E)$ (cf. [6], Remark 4.2). Hence, by [3], Lemma 3.2, and our assumption on $r(p)_E$, $\widehat{E}$ is perfect. We show that $\mathrm{Br}(E)_p = \{0\}$ and $\mathcal{G}(E(p)/E)$ is a free pro-p-group. When $\mathrm{char}(E) = p$, this is a special case of [14], Proposition 4.4.8, and [22], Ch. II, Proposition 2, respectively. If E contains a primitive p-th root of unity, the two assertions are equivalent (by Galois cohomology, see [24], page 265, [22], Ch. I, 4.2, and [26], page 725), so their validity follows from [9], Proposition 3.4 (or [3], Proposition 2.5). This indicates that $\mathcal{G}(E(p)/E) \cong \mathcal{G}_Y$, for some field Y of characteristic p [16], (4.8) (see also [1], Remark 2.6). The obtained result, combined with Galois theory and Witt's lemma (see [7], Sect. 15), completes the proof of Lemma 3.1 (i) and (ii). Since the class of free pro-p-groups is closed under the formation of open subgroups (cf. [22], Ch. I, 4.2 and Proposition 14), the former assertion of Lemma 3.1 (iii) can be deduced from the latter one. When E contains a primitive p-th root of unity, the latter part of Lemma 3.1 has been proved in [6], Sect. 5, so we assume that $\mathrm{char}(E) = p$ and E is perfect. Let F be an extension of E in $E(p)$ of degree p. Then the Artin-Schreier theorem implies the existence of a sequence $t = \{t_n \in M_v(E) \colon\ n \in \mathbb{N}\}$, such that $t_{n+1}^p = t_n \neq 0$ and the polynomial $X^p - X - t_n^{-1}$ is irreducible over E with a root $\xi_n \in F$, for each index n. Observing that $\xi_n^{-1} = t_n \prod_{j=1}^{p-1}(\xi_n + j)$ and $v_F(\xi_n) = p^{-1}v(t_n^{-1})$, one obtains by direct calculations that $v_F(\xi_n^{-1}) = p^{-1}v(t_n)$ and $v_F(\psi(\xi_n^{-1}) - \xi_n^{-1}) = 2v_F(\xi_n^{-1})$. Therefore, $\nabla_0(F)$ contains the elements $\lambda_n = \xi_n\psi(\xi_n^{-1})$, $n \in \mathbb{N}$, and $v_F(\lambda_n - 1) = p^{-1}v(t_n)$, for every index n. In view of (2.6) and (2.7), this result proves Lemma 3.1 (iii). □

Lemma 3.2. *Let (E, w) be a Henselian valued field with* $\mathrm{char}(\widehat{E}) = q > 0$, $w(E) \neq qw(E)$ *and* $\mathrm{Br}(E)_q = \{0\}$. *Assume also that $w(E)$ is Archimedean and $E' \in I(E_{\mathrm{sep}}/E)$ is the root field over E of the binomial $X^q - 1$. Then:*

(i) *$\widehat{E}$ is perfect, $w(E)/qw(E)$ is of order q, $q \in P(E)$ and finite extensions of E in $E(q)$ are totally ramified;*

(ii) *For any cyclic extension Φ of E in $E(q)$, there exists $\Gamma_0 \in I(E'(q)/E)$, such that $E'(q)/\Gamma_0$ is a $\mathbb{Z}_q$-extension and $\Phi \cap \Gamma_0 = E$.*

Proof. The assertion that $q \in P(E)$ is implied by the fact that (E, w) satisfies condition (i) of (2.4). Since, by [15], Theorem 3.16, E is a nonreal field, [27], Theorem 2, indicates that $E(q)$ contains as a subfield a $\mathbb{Z}_q$-extension Γ of E. In particular, $[E(q) \colon E] = \infty$. Let L be a finite extension of E in $E(q)$, and let $[L \colon E] = q^k$. It is clear from [4], Theorem 3.1, and the triviality of $\mathrm{Br}(E)_q$ that $N(L/E) = E^*$. Hence, by the Henselian property of w, $q^k w(L) = w(E)$, which implies in conjunction with (2.3) and the inequality $w(E) \neq qw(E)$ that $\widehat{\Phi} = \widehat{E}$ and $w(E)/q^k w(E)$ is a cyclic group of order q^k. These observations prove Lemma 3.2 (i). For the proof of Lemma 3.2 (ii), it suffices to observe that the set $Y(\Phi) = \{Y \in I(E'(q)/E) \colon\ Y \cap \Phi = E\}$, partially ordered by inclusion, satisfies the conditions of Zorn's lemma, to take as Γ_0 any maximal element of $Y(\Phi)$, and again to apply [27], Theorem 2. □

Remark 3.3. Retaining assumptions and notation as in Lemma 3.2 with its proof, put $\Gamma_* = E(q) \cap \Gamma_0$ and denote by Γ_n the extension of Γ_0 in $E'(q)$ of degree q^n, for each $n \in \mathbb{N}$. Observing that E'/E is cyclic and $[E' \colon E] \mid (q-1)$, one obtains that $E'(q)/E$ is Galois and Γ_0 contains a primitive q-th root of unity unless $\mathrm{char}(E) = q$. Note further that $[\Gamma_* \colon E] = \infty$. Indeed, it follows from Lemma 3.2 (i), [6], Remark 4.2, [4], Remark 2.8, and the triviality of $\mathrm{Br}(E)_q$ that $r(q)_E = \infty$. Hence, by Galois theory and Lemma 3.2 (ii), there are infinitely many extensions of E in Γ_* of degree q, so we have $[\Gamma_* \colon E] = \infty$, as claimed. It is therefore clear from Lemmas 3.1 and 3.2, Galois theory and (2.3) that $E'(q)/\Gamma_0$ is an immediate quasiinertial $\mathbb{Z}_q$-extension.

4. Proof of Theorem 1.2

Fix an algebraic closure $\overline{\Phi}$ of Φ_{sep}, put $S(T) = \{p \in \mathbb{P} \colon\ T_p \neq \{0\}\}$, $S'(T) = \{q\} \cup (\mathbb{P} \setminus S(T))$, and let U be the compositum of the inertial extensions of Φ in Φ_{sep}. Denote by U_0 the maximal extension of Φ in U whose finite subextensions have degrees not divisible by any $p \in S(T) \setminus \{q\}$. The assumptions on Φ, ω and $\widehat{\Phi}$ and the definition of U_0 indicate that U_0/Φ is a Galois extension with $\mathcal{G}(U_0/\Phi)$ isomorphic to the topological group product $\prod_{\pi' \in S'(T)} \mathbb{Z}_{\pi'}$; this implies that $q \notin \Pi(\widehat{U}_0)$, whence $\widehat{U}_0$ is infinite. As Φ is quasilocal, the obtained result proves that $\mathrm{Br}(U_0)_{\pi'} = \{0\}$, for each $\pi' \in S'(T)$. At the same time, it follows from (2.4) and the equality $\omega(U_0) = \omega(\Phi)$ that $\Phi(q) \notin I(U_0/\Phi)$, which ensures that $q \in P(U_0)$. Observing that ω_{U_0} is discrete and Henselian, one obtains from [25], Proposition 2.2, that finite extensions of U_0 in Φ_{sep} are defectless. Since $\widehat{\Phi}$ is perfect, U_0 does not possess inertial proper extensions in $U_0(q)$, and $\mathrm{Br}(U_0')_q = \{0\}$, for every $U_0' \in I(\overline{\Phi}/U_0)$, one also concludes that finite extensions of U_0 in $U_0(q)$ are totally ramified and $\mathcal{G}(U_0(q)/U_0)$ is a free pro-q-group. Note that $r(q)_{U_0} = \infty$; since ω_{U_0} is Henselian and discrete, this follows from [20], (2.7), and the infinity of $\widehat{U}_0$ (as well as from Remark 3.3 and the fact that $\mathrm{Br}(U_0)_q = \{0\}$). The rest of our proof relies on the observation that the set Σ of all $\Theta \in I(\Phi_{\mathrm{sep}}/U_0)$ with $\Theta \cap U = U_0$, and such that the degrees of finite extensions of U_0 in Θ are not divisible by q, satisfies the conditions of Zorn's lemma with respect to the partial ordering by inclusion. Fix a maximal element $\Theta' \in \Sigma$ and put $\omega' = \omega_{\Theta'}$. Then it follows from Galois theory, (2.3) and the noted properties of U_0 that Θ' satisfies the following:

(4.1) (i) $\omega'(\Theta') \neq q\omega'(\Theta')$ and $\omega'(\Theta') = p\omega'(\Theta')$, for each $p \in \mathbb{P} \setminus \{q\}$.

(ii) Finite extensions of Θ' in $\Theta'(q)$ are totally ramified.

(iii) $\mathcal{G}(\Theta'(q)/\Theta')$ is a free pro-q-group, $r(q)_{\Theta'} = \infty$ and $\mathrm{Br}(\Theta'')_q = \{0\}$, for every $\Theta'' \in I(\overline{\Phi}/\Theta')$.

Galois theory and the former assertion of (4.1) (iii) imply the existence of a $\mathbb{Z}_q$-extension Γ of Θ' in Φ_{sep}. Put $\Gamma_0 = \Theta'$, and for each $n \in \mathbb{N}$, let Γ_n be the extension of Θ' in Γ of degree q^n. It follows from Galois theory and the assumption on $\widehat{\Phi}$ that the compositum $U' = \Theta'\Gamma U$ is a Galois extension of Θ' with $\mathcal{G}(U'/\Theta') \cong \prod_{\pi \in S(T)} \mathbb{Z}_\pi$. In particular, this implies $\mathrm{cd}(\mathcal{G}(U'/\Theta')) = 1$, which means that $\mathcal{G}(U'/\Theta')$ is a projective profinite group (cf. [12], Theorem 1). Note also that the set $\widetilde{\Sigma} = \{\widetilde{\Theta} \in I(\overline{\Phi}/\Theta') \colon\ \widetilde{\Theta} \cap U' = \Theta'\}$, partially ordered by inclusion, satisfies the conditions of Zorn's lemma. Let $\widetilde{K}$ be a maximal element of $\widetilde{\Sigma}$, $\tilde{v} = \omega_{\widetilde{K}}$ and $\tilde{k}$ the residue field of $(\widetilde{K}, \tilde{v})$. It is easily verified that $\widetilde{K}$ and $\tilde{k}$ are perfect fields, and it follows from the projectivity of $\mathcal{G}(U'/\Theta')$ that $\overline{\Phi} = U'\widetilde{K}$. Hence, by Galois theory and the equality $\widetilde{K} \cap U' = \Theta'$, $\mathcal{G}_{\widetilde{K}} \cong \mathcal{G}(U'/\Theta')$. Our argument, combined with the former part of (4.1) (iii), also proves that there exists a $\mathbb{Z}_q$-extension of Θ' in $\widetilde{K}$. Since ω is discrete, this enables one to deduce the former part of the following assertion from (4.1) (i), (ii) and (2.3):

(4.2) $\tilde{v}(\widetilde{K}) = \mathbb{Q}$, $\tilde{k}/\widehat{\Phi}$ is an algebraic extension and $\Gamma\widetilde{K}/\widetilde{K}$ is immediate. Moreover, $\widetilde{K}(q) = \Gamma\widetilde{K}$, $\Gamma\widetilde{K}/\widetilde{K}$ is a $\mathbb{Z}_q$-extension with $[\Gamma_n\widetilde{K} \colon \Gamma_{n-1}\widetilde{K}] = q$, for each $n \in \mathbb{N}$, and $U\widetilde{K}(q)/U\widetilde{K}$ is quasiinertial.

As Γ/Θ' is a $\mathbb{Z}_q$-extension and $\widetilde{K} \cap U' = \Theta'$, the latter part of (4.2) follows at once from the former one, Galois theory and Lemma 3.1 (iii). Fix a positive number $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ and a rational function field $\widetilde{K}(X)$ in one indeterminate. It is easily verified that $\tilde{v}$ is uniquely extendable to a valuation v_γ of $\widetilde{K}(X)$ so that $v_\gamma(X) = \gamma$. In addition, it follows from the choice of γ that $v_\gamma(\widetilde{K}(X))$ is Archimedean and equal to the sum of $\mathbb{Q}$ and $\langle\gamma\rangle$. In addition, it becomes clear that $v_\gamma(\widetilde{K}(X)) \cong \mathbb{Q} \oplus \langle\gamma\rangle$ (as abstract groups) and the residue field of $(\widetilde{K}(X), v_\gamma)$ coincides with $\tilde{k}$. Note also that $\bar{v}_\gamma(\overline{\Phi}(X)) = v_\gamma(\widetilde{K}(X))$, where $\bar{v}_\gamma$ is the valuation of $\overline{\Phi}(X)$ naturally extending $\tilde{v}_{\overline{\Phi}}$ and v_γ. Now fix a Henselization K of $\widetilde{K}(X)$ relative to v_γ, and denote by v the Henselian valuation of K , extending v_γ. The established properties of $v_\gamma(\widetilde{K}(X))$ and the equality $v_\gamma(\widetilde{K}(X)) = v(K)$ indicate that $v(K)/pv(K)$ is of order p and $v(\gamma) \notin pv(K)$, for any $p \in \mathbb{P}$; in particular, $v(K)$ is totally indivisible. We show that K, v and I_∞/K have the properties required by Theorem 1.2, where $I_\infty = \Gamma K$. As a first step towards this, we prove the following:

(4.3) (i) $\widetilde{K}$ is algebraically closed in K and $\overline{\Phi}K/K$ is a Galois extension with $\mathcal{G}(\overline{\Phi}K/K) \cong \mathcal{G}_{\widetilde{K}} \cong \prod_{p\in S(T)} \mathbb{Z}_p$; in addition, $v(\overline{\Phi}K) = v(K)$, $\Gamma K/K$ is an immediate $\mathbb{Z}_q$-extension, $[\Gamma_n K : K] = q^n$, for each $n \in \mathbb{N}$;
(ii) $\Gamma\Omega/\Omega$ is a quasiinertial extension, for every $\Omega \in I(\overline{\Phi}/K)$.

As $v(K) \le \mathbb{R}$, K is $\widetilde{K}(X)$-isomorphic to the algebraic closure of $\widetilde{K}(X)$ in $\widetilde{K}(X)_{v_\gamma}$. At the same time, it follows from the definition of v_γ that an element $\rho \in \overline{\Phi}$ lies in $\widetilde{K}(X)_{v_\gamma}$ if and only if $\rho \in \widetilde{K}_{\tilde{v}}$. Observing finally that $\widetilde{K}$ is algebraically closed in $\widetilde{K}_{\tilde{v}}$ (because $\widetilde{K}$ is perfect and $\tilde{v}$ is Henselian), one concludes that $\widetilde{K}$ is algebraically closed in K. In view of Galois theory, this means that $\overline{\Phi}K/K$ is a Galois extension with $\mathcal{G}(\overline{\Phi}K/K) \cong \mathcal{G}_{\widetilde{K}}$. Using the equalities $\bar{v}_\gamma(\overline{\Phi}(X)) = v_\gamma(\widetilde{K}(X)) = v(K)$, and replacing $\widetilde{K}$ by any of its finite extensions in $\overline{\Phi}$, one obtains further that $v(\overline{\Phi}K) = v(K)$. As $\mathrm{cd}_{p'}(\mathcal{G}_{\tilde{k}}) = 0$, for every $p' \in \mathbb{P} \setminus S(T)$, this result implies in conjunction with (2.3) and (4.2) that $\Gamma K/K$ is immediate and $\Gamma \cap K = \Theta'$, so (4.3) (i) is proved. Note that $\Gamma\widetilde{K}/\widetilde{K}$ is quasiinertial; this follows from the concluding assertion of (4.2), trace transitivity in towers of finite separable extensions, and the fact that q does not divide the degree of any finite extension of $\widetilde{K}$ in $U\widetilde{K}$. Since $v(K) \le \mathbb{R}$, v prolongs $\tilde{v}$ upon K, and $\widetilde{K}$ is algebraically closed in K, this enables one to deduce (4.3) (ii) from Galois theory and (2.6).

Our next objective is to show that $\mathrm{Br}(K)_p \ne \{0\}$ if and only if $p \in S(T)$. Suppose first that $p \notin S(T)$. Then $p \dagger [\widetilde{M} : \widetilde{K}]$, for any finite extension $\widetilde{M}$ of $\widetilde{K}$, which ensures that $\mathrm{Br}(K)_p \cap \mathrm{Br}(\overline{\Phi}K/K) = \{0\}$ (cf. [19], Sect. 13.4). On the other hand, $\overline{\Phi}K/\overline{\Phi}$ is a field extension of transcendency degree 1, so it follows from Tsen's theorem (see [19], Sect. 19.4) that $\mathrm{Br}(\overline{\Phi}K) = \{0\}$. It is therefore easy to see that $\mathrm{Br}(K) = \mathrm{Br}(\overline{\Phi}K/K)$ and $\mathrm{Br}(K)_p = \{0\}$. Assume now that $p \in S(T)$. Then it follows from Galois theory and (4.3) that $I(\overline{\Phi}K/K)$ contains a cyclic extension Y_p of K of degree p. Moreover, (4.3) (i) ensures that $v(Y_p) = v(K)$, whence the uniqueness of v_{Y_p} implies $N(Y_p/K) \subseteq \{\lambda \in K^* :\ v(\lambda) \in pv(K)\}$. Since $v(K) \ne pv(K)$, this means that $\mathrm{Br}(Y_p/K) \ne \{0\} \ne \mathrm{Br}(K)_p$. In order to complete the proof of Theorem 1.2 it remains to be shown that $\mathrm{Br}(K_p) \cong \mathbb{Z}(p^\infty)$ and K is quasilocal (see [2], I, Lemma 8.3, and [3], Lemma 3.3 (i)). Let G_p be a Sylow pro-p-subgroup of $\mathcal{G}_K$ and K_p the fixed field of G_p. The equality $v(K) = v_\gamma(\widetilde{K}(X))$ and the isomorphism $v(K_p)/pv(K_p) \cong v(K)/pv(K)$ guarantee that $v(K_p)/pv(K_p)$ is of order p. When $p \ne q$, this enables one to deduce from (4.3) and [10], Lemma 1.2, that K_p^*/K_p^{*p} is a group of order p^2. As K_p contains a primitive p-th root of unity and $\mathrm{Br}(K)_p \cap \mathrm{Br}(K_p/K) = \{0\}$, the obtained results and Galois cohomology (see [26], Lemma 7, [18], (11.5), and [22], Ch. I, 4.2) prove that G_p is a Demushkin group, $r(p)_{K_p} = 2$ and $\mathrm{Br}(K_p) \cong \mathbb{Z}(p^\infty)$. Hence, by [2], I, Lemma 3.8, K_p is p-quasilocal. To conclude with, we show that K_q is q-quasilocal and $\mathrm{Br}(K_q) \cong \mathbb{Z}(q^\infty)$. As $\tilde{k}$ is perfect, $\mathrm{cd}_q(\mathcal{G}_{\tilde{k}}) = 0$ and $\widehat{K} = \tilde{k}$, $\widehat{K}_q$ is an algebraic closure of $\tilde{k}$, so we have $\widehat{Z} = \widehat{K}_q$, for each $Z \in I(K_{\mathrm{sep}}/K_q)$. In addition, it follows from Tsen's theorem that $\mathrm{Br}(K_q) =$

$\mathrm{Br}(\Gamma K_q/K_q)$. Applying (4.3), (2.8) and (2.6), one also sees that $\nabla_0(\Gamma_1) \subseteq N(\Gamma_n K_q/\Gamma_1 K_q)$, for each $n \in \mathbb{N}$. As $\Gamma_1 K_q/K_q$ is immediate, this enables one to deduce from (2.5) and Hilbert's Theorem 90 that an element $\theta \in K_q^*$ lies in $N(\Gamma_\nu K_q/\Gamma_1 K_q)$, for a given index ν, if and only if $\theta^q \in N(\Gamma_\nu K_q/K_q)$. Since $\mathrm{Br}(\Gamma K_q/K_q) = \cup_{n=1}^{\infty} \mathrm{Br}(\Gamma_n K_q/K_q)$, these observations and the canonical isomorphisms $\mathrm{Br}(\Gamma_n K_q/K_q) \cong K_q^*/N(\Gamma_n K_q/K_q)$, $n \in \mathbb{N}$ (cf. [19], Sect. 15.1, Proposition b), prove that ${}_q\mathrm{Br}(K_q) = \mathrm{Br}(\Gamma_1 K_q/K_q)$. The obtained result, combined with the fact that $\widehat{K}_q$ is algebraically closed and $v(K_q)/qv(K_q)$ is of order q, proves that $N(\Gamma_1 K_q/K_q) = \{\mu \in K_q^*\colon\ v(\mu) \in qv(K_q)\}$, ${}_q\mathrm{Br}(K_q)$ is of order q and $\mathrm{Br}(K_q) \cong \mathbb{Z}(q^\infty)$. Let now Λ be an extension of K_q in K_{sep}, such that $[\Lambda\colon K_q] = q$ and $\Lambda \neq \Gamma_1 K_q$, and let $V_q(\Lambda) = \{\lambda \in \Lambda\colon\ v_\Lambda(\lambda) \in qv(\Lambda)\}$. Applying (4.3) and (2.5), and arguing as in the proof of the isomorphism $\mathrm{Br}(K_q) \cong \mathbb{Z}(q^\infty)$, one obtains consecutively the following results:

(4.4) (i) $V_q(\Lambda) \subseteq N(\Gamma_1\Lambda/\Lambda)$; $\tau(\lambda')\lambda'^{-1} \in N(\Gamma_1\Lambda/\Lambda)$, for each $\lambda' \in \Lambda^*$ and every generator τ of $\mathcal{G}(\Lambda/K_q)$;

(ii) $\mathrm{Br}(\Gamma_1\Lambda/\Lambda) = {}_q\mathrm{Br}(\Lambda) \neq \{0\}$; hence $N(\Gamma_1\Lambda/\Lambda) \neq \Lambda^*$.

Since $\widehat{\Lambda}$ is algebraically closed and $v(\Lambda)/qv(\Lambda)$ is of order q, one also proves the following:

(4.5) (i) $N(\Gamma_1\Lambda/\Lambda) = V_q(\Lambda)$ and $\Gamma_1\Lambda/\Lambda$ is immediate.

(ii) $K^* \subseteq N(\Gamma_1\Lambda/\Lambda)$, provided that Λ is totally ramified over K_q; when this holds, $\mathrm{Br}(\Gamma_1/K_q) \subseteq \mathrm{Br}(\Lambda/K_q) = {}_q\mathrm{Br}(K_q)$.

In view of (4.4) (ii) and (4.5) (ii), it suffices, for the proof of the q-quasilocal property of K_q, to show that Λ/K_q is totally ramified. Assuming the opposite, one gets from (2.3) and the equality $\widehat{\Lambda} = \widehat{K}_q$ that Λ/K_q is immediate. Fix a generator τ of $\mathcal{G}(\Lambda/K_q)$, denote by τ' the Γ_1-automorphism of $\Gamma_1\Lambda$ extending τ, and put $D_\rho = (\Lambda/K_q, \tau, \rho)$, $\Delta_\rho = (\Gamma_1\Lambda/\Gamma_1, \tau', \rho)$, for some $\rho \in K_q^*$. Clearly, $\Delta_\rho \cong D_\rho \otimes_{K_q} \Gamma_1$ over Γ_1. Hence, the equality $\mathrm{Br}(\Gamma_1/K_q) = {}_q\mathrm{Br}(K_q)$ requires that $[\Delta_\rho] = 0$ in $\mathrm{Br}(\Gamma_1)$. On the other hand, (4.5) (i) and the assumption on Λ/K_q imply $\Gamma_1\Lambda/\Gamma_1$ is immediate. This shows that if $v(\rho) \notin qv(K_q)$, then $D_\rho \in d(K_q)$ and $\Delta_\rho \in d(\Gamma_1)$, whence $[\Delta_\rho] \neq 0$. The observed contradiction proves that Λ/K_q is totally ramified, so K_q is q- quasilocal (as seen, with $\mathrm{Br}(K_q) \cong \mathbb{Z}(q^\infty)$). As $S(T) = \{p \in \mathbb{P}\colon\ \mathrm{Br}(K)_p \neq \{0\}\}$, this result, [3], Lemma 3.3 (i), and isomorphisms $\mathrm{Br}(K_p) \cong \mathbb{Z}(p^\infty)$, $p \in S(T)$, yield $\mathrm{Br}(K) \cong T$. Theorem 1.2 is proved.

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