Backward uniqueness for the heat equation in cones

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Abstract

It was shown in [5, 13] that a bounded solution of the heat equation in a half-space which becomes zero at some time must be identically zero, even though no assumptions are made on the boundary values of the solutions. In a recent example, Luis Escauriaza showed that this statement fails if the half-space is replaced by cones with opening angle smaller than 90°. Here we show the result remains true for cones with opening angle larger than 110°.

1 Introduction

Consider an open set $\Omega \subset \mathbb{R}^n$. Let u be a bounded solution of the equation

$$u_t - \Delta u + b(x,t)\nabla u + c(x,t)u = 0 \quad \text{in } \Omega \times (0,T), \tag{1}$$

where the coefficients $b = (b_1, \dots, b_n)$, c are measurable and bounded. We say that Ω has the *backward uniqueness property* if the following statement holds:

(BU) *If a bounded* $u\colon \Omega \times (0,T) \to \mathbb{R}$ *satisfies (1) and* $u(\,\cdot\,,T) = 0$, *then* $u \equiv 0$ *in* $\Omega \times (0,T)$.

It is important to emphasize that no assumptions are made about u at the parabolic boundary $\partial\Omega \times (0,T) \cup (\Omega \times \{0\})$. In fact, we can think about the problem in terms of the control theory: we are given some initial data $u_0\colon \Omega \to \mathbb{R}$, and we wish to find a suitable boundary condition g on the lateral boundary $\partial\Omega \times (0,T)$ so that when we solve equation (1) with u_0 as the initial condition and g as the boundary condition, the solution will become exactly zero at time $t = T$. In this interpretation condition

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(BU) means that we can never achieve the exact boundary control of any non-trivial solution.

While the control theory for PDEs seems to be the most natural background for (BU), the problem also appeared in regularity theory of parabolic equations, such as the Navier-Stokes equations, harmonic map heat flows, or semi-linear heat equations, see [5, 11, 14]. The specific unbounded domains which arise in this connection are complements of closed balls (for interior regularity), half-spaces (for boundary regularity at C^1 boundaries), or cones (for boundary regularity in Lipschitz domains).

By classical results we know that in bounded domains we can achieve exact control, and therefore any domain satisfying (BU) has to be unbounded. Classical backward uniqueness results for parabolic equations imply that $\Omega = \mathbb{R}^n$ satisfies (BU). It turns out the the half-space $\Omega = \mathbb{R}^n_+$ also satisfies (BU), although this is harder to prove, see [5]. In general, the smaller the domain, the harder it will be to show that it satisfies (BU). It is immediate that if $\Omega_1 \subset \Omega_2$ and Ω_1 satisfies (BU), then also Ω_2 satisfies (BU).

In this paper we consider the question for cones with opening angle θ. In suitable coordinates

$$\mathcal{O}_\theta = \{x = (x_1, x'), x' \in \mathbb{R}^{n-1}, x_1 > |x| \cos(\theta/2)\}. \tag{2}$$

Luis Escauriaza [1] recently showed that - surprisingly - (BU) fails when $\theta < \pi/2$. We briefly recall the counterexample. Let us denote by Γ the standard heat kernel, i. e. $\Gamma(x,t) = (4\pi t)^{-n/2} \exp(|x|^2/4t)$, and recall Appell's transformation

$$u(x,t) = \Gamma(x,t) v(y,s), \qquad y = \frac{x}{t}, \quad s = \frac{1}{t} \ . \tag{3}$$

This transformation takes the solutions $u(x,t)$ of the heat equation into the solutions $v(y,s)$ of the backward heat equation

$$v_s + \Delta v = 0 \ . \tag{4}$$

By taking $u(x,t) = h(x)$ for a suitable harmonic function h in $\mathcal{O}_\theta$, we can get a counterexample to the backward heat equation form of (BU) for $\theta < \pi/2$. In dimension 2 we can use the real part of the holomorphic function $z \to \exp(-Az^\alpha)$ (for suitable $A > 0$ and a parameter $\alpha > 2$) to obtain an explicit formula:

$$v(y,s) = \operatorname{Re} \frac{1}{s} \ \exp(-A\frac{(y_1 + iy_2)^\alpha}{s^\alpha} + \frac{|y|^2}{4s}) \ . \tag{5}$$

The function $v(y_1, y_2, s)$ is bounded in a sector that $|\arctan \frac{y_2}{y_1}| < \pi/(2\alpha)$, away from the origin. We can shift it to $v(y_1+1, y_2, s)$. The resulted function is bounded in a sector with angle π/α satisfying the backward heat equation $v_s + \triangle v = 0$, and $v(\cdot, \cdot, 0) = 0$.

We note that it is enough to construct a counterexample in dimension $n = 2$. The higher-dimensional example can then be constructed by simply considering the two-dimensional function as a function of n variables, independent of $x_3, \ldots, x_n$.

Escauriaza's example shows that (BU) fails for $\theta < \pi/2$. Since we also know that (BU) is true for $\theta = \pi$, it is easy to see that there exists a borderline angle $\theta_0 \in [\pi/2, \pi]$ such that (BU) is true for $\theta > \theta_0$, and (BU) fails for $\theta < \theta_0$. The borderline case $\theta = \theta_0$ might perhaps present an extra difficulty.

The main result of this paper is the following:

Theorem 1.1. *The cones $\mathcal{O}_\theta$ satisfy (BU) for*

$$\theta > 2\arccos(1/\sqrt{3}) \sim 109.52^\circ .$$

In other words, the critical angle θ_0 introduced above satisfies

$$\theta_0 \leq 2\arccos(1/\sqrt{3}).$$

It is tempting to conjecture that $\theta_0 = \pi/2$. This is supported by the fact that $\theta = \pi/2$ is the borderline case for the above construction of Escauriaza, as can be seen from the Phragmén-Lindölef principle.

For the classical heat equation, corresponding to the case $b = 0$ and $c = 0$ in (1), and $\theta = \pi$ (the half-space), the statement (BU) can be proved by a relatively simple application of Fourier transformation and some classical complex analysis results, see [13]. We were not able to find such simple proof of the case $b = 0$, $c = 0$ when $\theta < \pi$.

While completing this paper, we learned about reference [8].[1] Theorem 6 in [8] states that for the classical heat equation (corresponding to $b = 0$, $c = 0$), (BU) holds if and only if $\theta \leq \pi/2$. Unfortunately, it seems the proof is not available in print.

Our proof of Theorem (1.1) relies on two Carleman-type inequalities, along lines similar to [5]. The first inequality, Proposition 2.1, is taken from [5] and is applied in the same way to obtain fast decay rates for the solutions, see Lemma 2.2. We note that Carleman inequalities of this type are can be found already in [2, 6, 3, 4].

[1]We thank Gregory Seregin for pointing out this article.

The second inequality, Proposition 2.3, is the main new tool used in our proof. The heuristics behind this inequality is somewhat similar to the heuristics behind the second Carleman-type inequality in [5] (Proposition 6.2). However, the proof of Proposition 2.3 requires a new idea, as for $\theta < \pi$ certain critical terms appearing in the proofs of the inequalities lose convexity.

In addition to determining the critical angle, another interesting open problem is to optimize the assumptions on the coefficient $b(x,t)$ and $c(x,t)$, in the spirit of [9]. For example, it is conceivable that the result remains true for $b \in L^{n+2}_{x,t}$ and $c \in L^{(n+2)/2}_{x,t}$, but it might be a difficult problem to decide whether this is the case.

In what follows we will work with the inequality

$$|u_t - \Delta u| \leq c_1(|\nabla u| + |u|) \tag{6}$$

rather than (8). It is not hard to see that when assuming the boundedness of b and c, the two formulations are equivalent.

2 Backward uniqueness

Without loss of generality we assume $T = 1$ and work with the backward form of (6).

Recall that

$$\mathcal{O}_\theta = \{x = (x_1, x'), x' \in \mathbb{R}^{n-1}, x_1 > |x| \cos(\theta/2)\}. \tag{7}$$

Suppose that $u(x,t)$ is a solution to the backward heat equation for some $\theta > 2\arccos(1/\sqrt{3})$.

$$|u_t + \Delta u| \leq c_1(|\nabla u| + |u|) \qquad \text{in } \mathcal{O}_\theta \times (0,1), \tag{8}$$

$$u(\cdot, 0) = 0 \qquad \text{in } \mathcal{O}_\theta\,. \tag{9}$$

In addition,

$$|u| < M \qquad \text{in } \mathcal{O}_\theta \times (0,1). \tag{10}$$

Then $u \equiv 0$.

To prove the above statement we firstly need the following Carleman inequality from [5], by which we obtain a decay result for solutions of backward heat equation.

Proposition 2.1 ([5]). *For any function* $u \in C_0^\infty(\mathbb{R}^n \times (0,2); \mathbb{R}^n)$ *and any positive number* a,

$$\int_{\mathbb{R}^n \times (0,2)} h^{-2a}(t) e^{-\frac{|x|^2}{4t}} \left(\frac{a}{t}|u|^2 + |\nabla u|^2\right) dxdt \\ \leq c_0 \int_{\mathbb{R}^n \times (0,2)} h^{-2a}(t) e^{-\frac{|x|^2}{4t}} |\partial_t u + \Delta u|^2 dxdt, \quad (11)$$

where c_0 *is a positive absolute constant and* $h(t) = te^{\frac{1-t}{3}}$.

Lemma 2.2 below immediately implies exponential decay of the solution u. The proof is by using the Carleman inequality in Proposition 2.1. The decay of u enables us to apply the Carleman inequality in Proposition 2.3 and reach the conclusion in Theorem 1.1.

Lemma 2.2. *Let* B_R *denote the ball with radius* R *in* $\mathbb{R}^n$. *Assume that* $R > 2$. *Consider a function* u *satisfying the following conditions, with some positive constants* c_1 *and* M.

$$|u_t + \triangle u| \leq c_1(|\nabla u| + |u|) \qquad \text{in } B_R \times (0,T), \qquad (12)$$

$$u(x,0) = 0 \qquad \text{in } B_R, \qquad (13)$$

$$|u| < M \qquad \text{in } B_R \times (0,T). \qquad (14)$$

Then there exist some constants β, γ, *such that for* $t \in (0,\gamma)$,

$$u(0,t) \leq \frac{c_2}{\min\{1,T\}} M e^{-\beta \frac{R^2}{t}}, \qquad (15)$$

where β *is a small enough absolute constant,* c_2 *depends on* c_1, γ *depends on* c_1 *and* T.

We will give the proof of the lemma in the next section.

The following Carleman inequality in Proposition 2.1 is a key tool used in our proof of the backward uniqueness in cones. We define the set

$$Q_\theta = (\mathcal{O}_\theta \cap \{x_1 > 1\}) \times (0,1).$$

The purpose of "cutting off the corner" is to avoid singularities at the origin.

Proposition 2.3. *Let* $\phi(x,t) = a\Lambda(t)\varphi(x) + t^2$, *where* $\Lambda(t) = \dfrac{1-t}{t^{\alpha/2}}$, *and* $\varphi(x) = x_1^\alpha - \varepsilon^\alpha r^\alpha$, *where* $r = |x|$, *and* $\varepsilon = \cos(\theta/2)$. *For any* $\varepsilon \in (0, 1/\sqrt{3})$, *that is,* $\theta \in (2\arccos(1/\sqrt{3}), \pi)$, *there exists some* $\alpha = \alpha(\varepsilon) \in (1,2)$ *such* *that the following inequality holds for* $u \in C_0^\infty(Q_\theta)$ *and* $a > a_0$ *for some constant* a_0.

$$\int_{Q_\theta} e^{2\phi(x,t)} \left[a\left(\Lambda(t)+\varphi(x)\right)u^2 + |\nabla u|^2\right] dxdt \\ \leq 4\int_{Q_\theta} e^{2\phi(x,t)}|\partial_t u + \Delta u|^2 dxdt. \tag{16}$$

We apply this Carleman inequality to prove the main result of the paper in the remaining part of this section. The proof of Proposition 2.3 is postponed to the last section.

For $x \in \mathcal{O}_\theta$ we denote by $d_\theta(x)$ the distance between x and the boundary of $\mathcal{O}_\theta$, explicitly given by

$$d_\theta(x) = x_1 \sin(\theta/2) - |x'|\cos(\theta/2). \tag{17}$$

Let ${\mathcal{O}_\theta}^{+2} = \{x \in \mathcal{O}_\theta \mid d_\theta(x) > 2\}$. With any other number c, the set ${\mathcal{O}_\theta}^{+c}$ is defined in the same way.

The next lemma is a consequence of the decay result from Lemma 2.2 and Proposition 2.3 . It implies Theorem 1.1 immediately.

Lemma 2.4. *Assume that for some* $\theta \in (2\arccos(1/\sqrt{3}), \pi)$ *a function* u *satisfies (8) – (10), then there is a number* $\gamma_1(c_1) \in (0, \gamma/2)$ *such that*

$$u(x,t) \equiv 0 \tag{18}$$

in $\mathcal{O}_\theta \times (0, \gamma_1)$.

Proof. Lemma 2.2 implies that

$$|u(x,t)| \leq c_2 M e^{-\beta \frac{d_\theta^2(x)}{t}} \tag{19}$$

for all $(x,t) \in {\mathcal{O}_\theta}^{+2} \times (0,\gamma)$. By local gradient estimates for the heat equation [10] we can assume that

$$|u(x,t)| + |\nabla u(x,t)| \leq c_3 M e^{-\frac{\beta d_\theta^2(x)}{2t}} \tag{20}$$

for all $(x,t) \in {\mathcal{O}_\theta}^{+3} \times (0, \gamma/2]$.

By scaling we define a function v by

$$v(y,s) = u(\lambda y, \lambda s^2 - \gamma_1) \tag{21}$$

for $(y, s) \in \mathcal{O}_\theta \times (0, 1)$ with $\lambda = \sqrt{2\gamma_1}$. This function satisfies the relations

$$|\partial_s v + \Delta v| \le c_1 \lambda (|\nabla v| + |v|) \qquad \text{in } \mathcal{O}_\theta \times (0, 1) \tag{22}$$

$$v(y, s) = 0 \qquad \text{in } \mathcal{O}_\theta \times (0, 1/2), \tag{23}$$

and

$$|v(y, s)| + |\nabla v(y, s)| \le c_3 M e^{-\frac{\beta \lambda^2 d_\theta^2(y)}{2(\lambda^2 s - \gamma_1)}} \le c_3 M e^{-\beta \frac{d_\theta^2(y)}{2s}} \tag{24}$$

for $1/2 < s < 1$ and $y \in \mathcal{O}_\theta{}^{+3/\lambda} = \{y \in \mathcal{O}_\theta \mid d_\theta(y) > 3/\lambda\}$.

To apply Proposition 2.3, we need certain decay of $|v(y, s)|$ when $|y|$ is large. Notice that the preferred decay can be obtained by considering a cone with slightly small opening. Proposition 2.3 holds for angles in $(2 \arccos(1/\sqrt{3}), \pi)$. We thus consider the median of θ and $2 \arccos(1/\sqrt{3})$,

$$\delta = \frac{\theta + 2 \arccos(1/\sqrt{3})}{2}.$$

In the smaller cone $\mathcal{O}_\delta = \{x \in R^n, x_1 > |x| \cos(\delta/2)\}$ we have the estimate $d_\theta(y) \ge |y| \sin\left(\frac{\theta - \delta}{2}\right)$. It follows (24) that

$$|v(y, s)| + |\nabla v(y, s)| \le c_3 M e^{-\beta' \frac{|y|^2}{s}} \tag{25}$$

for $1/2 < s < 1$ and $y \in \mathcal{O}_\delta \cap \mathcal{O}_\theta{}^{+3/\lambda}$, with the constant $\beta' = \beta \sin^2(\frac{\theta - \delta}{2})$. We can further have

$$|v(y, s)| + |\nabla v(y, s)| \le c_3' M e^{-\beta' \frac{|y|^2}{s}} \tag{26}$$

for $1/2 < s < 1$ and $y \in \mathcal{O}_\delta \cap \{y_1 > 3/\lambda\}$ for some other constant c_3'.

Next we work on the smaller cone $\mathcal{O}_\delta$ with opening δ, where we have exponential decay (26) and the following properties inherited from $\mathcal{O}_\theta$.

$$|\partial_s v + \Delta v| \le c_1 \lambda (|\nabla v| + |v|) \qquad \text{in } \mathcal{O}_\delta \times (0, 1), \tag{27}$$

$$v(y, s) = 0 \qquad \text{in } \mathcal{O}_\delta \times (0, 1/2). \tag{28}$$

Proposition 2.3 requires support condition for the Carleman inequality. For that purpose, let us fix two smooth cut-off functions such that

$$\psi_1(y_1) = \begin{cases} 0, & y_1 < 3/\lambda + 2, \\ 1, & y_1 > 3/\lambda + 3, \end{cases}$$

$$\psi_2(\tau) = \begin{cases} 0, & \tau < -3/4, \\ 1, & \tau > -1/2. \end{cases}$$

We set (for the definition of ϕ, see Proposition 2.3)

$$\phi_B(y,s) = \frac{1}{a}\phi(y,s) - B = (1-s)\frac{y_1^\alpha - \varepsilon^\alpha |y|^\alpha}{s^{\alpha/2}} + \frac{s^2}{a} - B,$$

where $\varepsilon = \cos(\delta/2)$, $B = \frac{2}{a}\phi(y_\lambda, \frac{1}{2})$, with $y_\lambda = (3/\lambda + 3, 0, \cdots, 0)$ and

$$\eta(y,s) = \psi_1(y_1)\psi_2(\frac{\phi_B}{B}), \qquad w(y,s) = \eta(y,s)v(y,s).$$

The function w is not compact supported in Q_δ (recall that $Q_\delta = (\mathcal{O}_\delta \cap \{x_1 > 1\}) \times (0,1)$). However, it follows from (24) and the special structure of the weight function ϕ in Proposition 2.3 that, with w replacing u in (2.3), integrals on both sides converge. If we multiply w by an additional cut-off function ξ such that

$$\xi(x) = \begin{cases} 1, & \tau < R \\ 0, & \tau > 2R \end{cases}$$

and $|\nabla\xi| < c/R$, $|\nabla^2\xi| < c/R^2$, apply Proposition 2.3 to the compact supported function $w\xi$, then let $R \to \infty$, we finally obtain

$$\int_{Q_\delta} e^{2a\phi_B}\left[a\left(\Lambda(s) + \varphi\right)w^2 + |\nabla w|^2\right] dyds \leq 4\int_{Q_\delta} e^{2a\phi_B}|\partial_s w + \Delta w|^2 dyds. \quad (29)$$

From (27) we have

$$\begin{aligned} |\partial_s w + \triangle w| \quad &\leq \quad c_1\lambda(|\nabla w| + |w|) \\ &\quad + c_4(|\nabla v| + |v|)(|\partial_s \eta + |\nabla\eta| + |\Delta\eta|). \end{aligned} \quad (30)$$

Refer to the definition of ψ_2 we know that $\phi_B/B \geq -3/4$ in the support of w. In other words $a\Lambda(s)\varphi(y) + s^2 > aB/4$. For a large enough, this implies that $a(\Lambda(s) + \varphi(y)) > 1$ by Cauchy-Schwartz inequality. In addition, we take $\gamma_1(c_1)$ small enough such that $16c_1^2\lambda^2 < 1/2$. We then have

$$I \equiv \int_{Q_\delta} e^{2a\phi_B}\left(w^2 + |\nabla w|^2\right])dyds \quad (31)$$

$$\leq 32c_4^2\int_{Q_\delta} e^{2a\phi_B}(|v|^2 + |\nabla v|^2)(|\partial_s\eta| + |\nabla\eta| + |\Delta\eta|)^2 dyds. \quad (32)$$

To estimate the right hand side, we need to look into the detail of derivatives of η. In view of the definitions of ψ_1 and ψ_2, the support of derivatives of $\eta(y,s)$ is the closure of the set

$$\{y_1 > 3/\lambda + 2,\ -3B/4 < \phi_B < -B/2\} \\ \cup \{3/\lambda + 2 < y_1 < 3/\lambda + 3,\ \phi_B > -B/2\}.$$

However, the second set has empty intersection with $\mathcal{O}_\delta \times (1/2, 1)$, where the function v is nonzero. Hence the support of the term $(|\nabla v| + |v|)(|\partial_s \eta + |\nabla \eta| + |\Delta \eta|)$ is closure of the set

$$\omega = \{y_1 > 3/\lambda + 2, 1/2 < s < 1, -3B/4 < \phi_B(y,s) < -B/2\}.$$

We denote by $\chi(y,s)$ the characteristic function of the set ω.

Next we estimate the rate of increasing at infinity of derivatives of $\eta(y,s)$ in the set ω. Recall that $\phi(y,s) = a\Lambda(s)\varphi(y) + s^2$, where $\Lambda(s) = \dfrac{1-s}{s^{\alpha/2}}$ and $\varphi(y) = y_1^\alpha - \varepsilon^\alpha\,|y|^\alpha$. The function $\Lambda(s)$ and the derivative $\Lambda'(s)$ are bounded for $s \in (1/2, 1)$. The function φ and its derivatives up to the second order are bounded by (const. $|y|^\alpha$). The cut-off functions ψ_1 and ψ_2 and derivatives up to the second order are bounded by some absolute constant. The value of B is bounded from below regardless of the value of the parameter a. Thus

$$(|\partial_s \eta| + |\nabla \eta| + |\Delta \eta|)^2 < c_5 |y|^{2\alpha} \tag{33}$$

in the set ω.

We now estimate (32) by using (25) and (33). We see that

$$I \le c_6 M e^{-Ba} \int_{Q_\delta} |y|^{2\alpha} e^{-2\beta' \frac{|y|^2}{s}} \chi(y,s)\, dy ds$$

for some constant c_6. The last integral in bounded. Passing to the limit as $a \to \infty$ we see that $v(y,s) = 0$ for $1/2 < s < 1$ and $\phi_B(y,s) > 0$. Using the property of unique continuation across the spatial boundaries (see Theorem 4.1 in [5]), we show that $v(y,s) = 0$ if $y \in \mathcal{O}_\theta$ and $0 < s < 1$. This proves the lemma. □

3 Proof of Lemma 2.2

The proof of Lemma 2.2 is based on the Carleman inequality in [5], which we quoted in Proposition 2.1.

Proof of Lemma 2.2. The proof is similar to the one in [5]. We still include the proof here for the convenience of the reader.

In what follows, we always assume that the function u is extended by zero to negative values of t.

The assumption that $R > 2$ results in no loss, since the conclusion is only useful when R is large. According to the local gradient estimates of the heat equation [10], in the smaller cylinder $(x,t) \in B_{R-1} \times (0, T/2)$, we can assume that

$$|u| + |\nabla u(x,t)| \leq \frac{c_7}{\min\{1,T\}} M \tag{34}$$

with some absolute constant c_7.

We fix $t \in (0, \min\{1,T\}/12)$ and introduce a new function v by the usual parabolic scaling:

$$v(y,s) = u(\lambda y, \lambda^2 s - t/2).$$

The function v is well defined on the set $Q_\rho = B(\rho) \times (0,2)$, where $\rho = (R-1)/\lambda$ and $\lambda = \sqrt{3t} \in (0, \min\{1, \sqrt{T}\}/2)$. We have the following relations for v.

$$|\partial_s v + \Delta v| \leq c_1 \lambda (|\nabla v| + |v|), \tag{35}$$

$$|v(y,s)| + |\nabla v(y,s)| < \frac{c_7}{\min\{1,T\}} M \tag{36}$$

for all $(y,s) \in Q_\rho$,

$$v(y,s) = 0 \tag{37}$$

for $(y,s) \in B(\rho) \times (0, 1/6]$.

By the assumption that $R > 2$ and $\lambda < 1/2$, we have $\rho > 2$. In order to apply Proposition 2.1, we take two smooth cut-off functions in Q_ρ:

$$\psi_\rho(y) = \begin{cases} 0, & |y| > \rho - 1/2, \\ 1, & |y| < \rho - 1, \end{cases}$$

$$\psi_t(s) = \begin{cases} 0, & 7/4 < s < 2, \\ 1, & 0 < s < 3/2. \end{cases}$$

By assumption, these functions take values in $[0,1]$ and are such that $|\nabla^k \psi_\rho| < C_k$, $k = 1,2$, and $|\partial_s \psi_t| < C_0$. We set $\eta(y,s) = \psi_\rho(y)\psi_t(s)$ and

$$w(y,s) = \eta(y,s) v(y,s). \tag{38}$$

It follows from (35) that

$$|\partial_s w + \Delta w| \leq c_1 \lambda (|\nabla w| + |w|) + c_8 \chi (|\nabla v| + |v|), \tag{39}$$

where c_8 is a positive constant depending only on c_1 and C_k, $k = 0,1,2$, $\chi(y,s) = 1$ for $(y,s) \in \omega = \{\rho - 1 < |y| < \rho, 0 < s < 2\} \cup \{|y| < \rho - 1, 3/2 <$

$s < 2\}$, and $\chi(y,s) = 0$ for $(y,s) \notin \omega$. The set ω is where the cut-off function η is not constantly 1 in Q_ρ. Obviously, the function w is compactly supported on $\mathbb{R}^2 \times (0,2)$, we may apply Proposition 2.1 and obtain

$$\int_{Q_\rho} h^{-2a}(s)e^{-\frac{|y|^2}{4s}} \left(\frac{a}{s}|w|^2 + |\nabla w|^2\right) dyds \\ \leq c_0 \int_{Q_\rho} h^{-2a}(s)e^{-\frac{|y|^2}{4s}} |\partial_s w + \Delta w|^2 dyds. \quad (40)$$

Taking $a > 2$, and applying (39) we finally obtain that

$$I \equiv \int_{Q_\rho} h^{-2a}(s)e^{-\frac{|y|^2}{4s}}(|w|^2 + |\nabla w|^2)dyds \leq 4c_0(c_1^2\lambda^2 I + c_8^2 I_1), \quad (41)$$

where

$$I_1 = \int_{Q_\rho} \chi(y,s)h^{-2a}(s)e^{-\frac{|y|^2}{4s}}(|\nabla v|^2 + |v|^2)dyds.$$

Taking a sufficiently small value for $\gamma = \gamma(c_1)$ such that in the range $\lambda \in (0,\gamma)$, we can assume that the inequality $4c_0c_1^2\lambda^2 \leq 1/2$ holds, and therefore (41) implies that

$$I \leq 8c_0c_8^2 I_1. \quad (42)$$

Notice that near the origin $\{y = 0, s = 0\}$, where the parametric function $h^{-2a}(s)e^{-\frac{|y|^2}{4s}}$ is not integrable, our characteristic function χ is 0. By (36) we have

$$\begin{aligned} I_1 &\leq \frac{c_7^2 M^2}{\min\{1,T^2\}} \left\{ \int_{3/2}^2 \int_{|y|<\rho-1} h^{-2a}(s)e^{-\frac{|y|^2}{4s}} dyds \right. \\ &\quad \left. + \int_0^2 \int_{\rho-1<|y|<\rho} h^{-2a}(s)e^{-\frac{|y|^2}{4s}} dyds \right\} &(43) \\ &\leq \frac{c_9 M^2}{\min\{1,T^2\}} \left[h^{-2a}(3/2) + \int_0^2 h^{-2a}(s)e^{-\frac{(\rho-1)^2}{4s}} ds \right], &(44) \end{aligned}$$

where c_9 is an absolute constant.

Using (44) we obtain the estimate

$$\begin{aligned} D &\equiv \int_{B(1)} \int_{1/2}^{1} |w|^2 dyds = \int_{B(1)} \int_{1/2}^{1} |v|^2 dyds \\ &\leq c_{10} \int_{Q_\rho} h^{-2a}(s) e^{-\frac{|y|^2}{4s}} (|w|^2 + |\nabla w|^2) dyds \\ &\leq \frac{c_{11} M^2}{\min\{1, T^2\}} \left[h^{-2a}(3/2) + \int_0^2 h^{-2a}(s) e^{-\frac{\rho^2}{16s}} \, ds \right] \\ &= \frac{c_{11} M^2}{\min\{1, T^2\}} e^{-\beta\rho^2} \left[h^{-2a}(3/2) e^{\beta\rho^2} + \int_0^2 h^{-2a}(s) e^{\beta\rho^2 - \frac{\rho^2}{16s}} \, ds \right]. \end{aligned}$$

We take $\beta < 1/64$ and then let

$$a = \beta\rho^2 / (2 \log h(3/2)).$$

This choice of a leads to the estimate

$$D \leq c_{11} e^{-\beta\rho^2} \left[1 + \int_0^2 g(s) ds \right],$$

where $g(s) = h^{-2a}(s) e^{-\frac{\rho^2}{32s}}$. By simple calculation we have that

$$g'(s) = h^{-2a}(s) e^{-\frac{\rho^2}{32s}} \left[-\frac{\beta\rho^2}{\log h(3/2)} \left(\frac{1}{s} - \frac{1}{3}\right) + \frac{\rho^2}{32s^2} \right].$$

One can readily verify that $g(2) < 1$ and $g'(s) \geq 0$ for any $s \in (0,2)$ if $\beta < \frac{1}{64} \log h(3/2)$. Therefore,

$$D \leq 3 \frac{c_{11} M^2}{\min\{1, T^2\}} e^{-\beta\rho^2} = 3 \frac{c_{11} M^2}{\min\{1, T^2\}} e^{-\beta \frac{R^2}{12t}}.$$

On the other hand, the regularity theory implies that

$$|u(0,t)|^2 = |v(0, 1/2)|^2 \leq c_{12} D.$$

Finally we obtain that

$$|u(0,t)| \leq \frac{c_2}{\min\{1, T\}} M e^{-\beta \frac{R^2}{24t}}.$$

Taking another constant equals $\beta/24$, and still denotes by β, we reach the conclusion of the lemma. □

4 Proof of the Carleman inequality

One of the difficulties in the proof of the Carleman inequality in Proposition 2.3 is that – by comparison with the case $\theta = \pi$ – some loss of the convexity of the weight φ cannot be avoided. Therefore we have to investigate in more detail some of the terms which can be neglected when $\theta_0 \geq \pi$.

Proof of Proposition 2.3. We denote $\phi = a\Lambda(t)\varphi(x) + t^2$. Let u be an arbitrary function in $C_0^\infty(Q_\theta)$ and $v = e^\phi u$. Then

$$Lv \equiv e^\phi(\partial_t u + \Delta u) = \Delta v + |\nabla\phi|^2 v - \partial_t\phi v + \partial_t v - 2\nabla\phi\nabla v - \Delta\phi v. \quad (45)$$

We decompose L into symmetric and skew-symmetric parts

$$L = S + A,$$

where

$$Sv = \Delta v + |\nabla\phi|^2 v - \partial_t\phi v \quad (46)$$

and

$$Av = \partial_t v - 2\nabla\phi\nabla v - \Delta\phi v. \quad (47)$$

The right hand side of the inequality (16) is

$$\int |Lv|^2 dxdt = \int |Sv|^2 dxdt + \int |Av|^2 dxdt + \int ([S, A]v)v dxdt, \quad (48)$$

where $[S, A] = SA - AS$ is the commutator of S and A. By simple calculations we have that

$$\begin{aligned} ([S, A]v, v) &= \int 4\phi_{,kl} v_{,k} v_{,l} dxdt & (49)\\ &+ \int \left(2\nabla\phi\nabla|\nabla\phi|^2 - \Delta^2\phi + \partial_t^2\phi - 2\partial_t|\nabla\phi|^2\right)|v|^2 dxdt. & (50)\end{aligned}$$

The Hessian of the function $\phi = a\Lambda(t)(x_1^\alpha - \varepsilon^\alpha r^\alpha)$ is not positive-definite. To compensate the term $\int 4\phi_{,kl} v_{,k} v_{,l} dxdt$ in (49) we introduce a function $F(x, t)$ to be determined.

$$\begin{aligned} (Sv, Fv) &= \int \Delta v F v + (|\nabla\phi|^2 - \partial_t\phi) F v^2 dx \\ &= \int -F|\nabla v|^2 + (\frac{1}{2}\triangle F + |\nabla\phi|^2 F - \partial_t\phi F) v^2 dxdt. \end{aligned}$$

Cauchy-Schwartz inequality implies that

$$\begin{aligned}(Sv,Sv) &\geq -(Sv,Fv) - \frac{1}{4}\int F^2 v^2 dxdt \\ &\geq \int F|\nabla v|^2 - (\frac{1}{2}\triangle F + |\nabla\phi|^2 F - \partial_t\phi F + \frac{1}{4}F^2)v^2 dxdt. \quad (51)\end{aligned}$$

Combining (49) and (51) we have

$$([S,A]v,v) + (Sv,Sv) \geq \int 4\phi_{,kl}v_{,k}v_{,l} + F|\nabla v|^2 dxdt \quad (52)$$

$$+\int \left(2\nabla\phi\nabla|\nabla\phi|^2 - \Delta^2\phi + \partial_t^2\phi - 2\partial_t|\nabla\phi|^2\right) v^2 dxdt \quad (53)$$

$$+\int -(\frac{1}{2}\triangle F + |\nabla\phi|^2 F - \partial_t\phi F + \frac{1}{4}F^2)v^2 dxdt. \quad (54)$$

By calculation the Hessian of φ is

$$\begin{aligned}D^2\varphi(x) &= \alpha(\alpha-1)\begin{pmatrix} x_1^{\alpha-2} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} - \alpha\,\varepsilon^\alpha\, r^{\alpha-2} E_n \quad (55) \\ &\quad +\alpha(2-\alpha)\,\varepsilon^\alpha\, r^{\alpha-4} x^T x, \quad (56)\end{aligned}$$

where E_n reprensent the n dimensional identity matrix, $x = (x_1, \ldots, x_n)$ is the row vector and x^T denotes the transpose of x. It is easy to see that

$$D^2\varphi(x) + \alpha\,\varepsilon^\alpha\, r^{\alpha-2} E_n \geq 0.$$

We thus let $f(x) = \alpha\,\varepsilon^\alpha\, r^{\alpha-2}$ and

$$F(x,t) = 4a\Lambda(t)f(x) + 1. \quad (57)$$

With this choice of $F(x,t)$, the right hand side of line (52) is positive and

$$\int 4\phi_{,kl}v_{,k}v_{,l} + F|\nabla v|^2 dxdt \geq \int |\nabla v|^2 dxdt.$$

Grouping the remaining terms according to the orders of the parameter a, with A_3 denoting the terms with a^3 and etc, we have that

$$\begin{aligned}([S,A]v,v) + (Sv,Sv) &\geq \int |\nabla v|^2 dxdt \\ &\quad + \int (A_3 + A_2 + A_1 + A_0)v^2 dxdt \quad (58)\end{aligned}$$

where

$$
\begin{aligned}
A_3 &= 4a^3\Lambda^3\varphi_{,kl}\varphi_{,k}\varphi_{,l} - 4a^3\Lambda^3 f|\nabla\varphi|^2, && (59)\\
A_2 &= -4a^2\Lambda\Lambda'|\nabla\varphi|^2 + 4a^2\Lambda\Lambda'\varphi f - 4a^2\Lambda^2(t)f^2 - a^2\Lambda^2|\nabla\varphi|^2, && (60)\\
A_1 &= -a\Lambda\Delta^2\varphi + a\Lambda''\varphi - 2a\Lambda(t)\Delta f + a\Lambda'\varphi - 2a\Lambda(t)f + 8at\Lambda f, && (61)\\
A_0 &= 7/4 + 2t. && (62)
\end{aligned}
$$

We analyze A_3 first. With a^3 as a coefficient A_3 must be non-negative in the set Q_θ. By letting $x_1/r \to \varepsilon$, we see easily that $\varepsilon < 1/\sqrt{3}$ is a necessary condition for $A_3 \geq 0$. This could be seen by letting . Next we show that under the condition $\varepsilon < 1/\sqrt{3}$, we indeed have that $A_3 \geq 0$. Denoting by $\nabla\varphi^T$ the transpose of the row vector $\nabla\varphi$, we notice that

$$
A_3 = 4a^3\Lambda^3(t)\nabla\varphi\left(D^2\varphi(x) - \alpha\,\varepsilon^\alpha\, r^{\alpha-2}\right)\nabla\varphi^T. \tag{63}
$$

It is easy to see that

$$
\begin{aligned}
&D^2\varphi(x) - \alpha\,\varepsilon^\alpha\, r^{\alpha-2}\\
&\geq \alpha(\alpha-1)\begin{pmatrix} x_1^{\alpha-2} & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 \end{pmatrix} - 2\alpha\,\varepsilon^\alpha\, r^{\alpha-2}E_n
\end{aligned} \tag{64}
$$

Apply the fact that $x_1^{\alpha-2} > r^{\alpha-2}$, we have

$$
D^2\varphi(x) - \alpha\,\varepsilon^\alpha\, r^{\alpha-2} \geq \alpha r^{\alpha-2}\begin{pmatrix} \alpha - 1 - 2\,\varepsilon^\alpha & 0\\ 0 & -2\,\varepsilon^\alpha\, E_{n-1} \end{pmatrix}, \tag{65}
$$

where E_{n-1} is the $n-1$ dimensional identity matrix.

The first derivatives of ϕ are as follows.

$$
\varphi_{,1}(x,t) = \alpha x_1^{\alpha-1} - \alpha\,\varepsilon^\alpha\, r^{\alpha-2}x_1, \quad \varphi_{,k}(x,t) = -\alpha\,\varepsilon^\alpha\, r^{\alpha-2}x_k, \quad k = 2,\ldots,n.
$$

We notice that $\varphi_{,1} \geq \alpha(1-\varepsilon^\alpha)x_1^{\alpha-1}$. Thus

$$
A_3 \geq 4a^3\Lambda^3(t)\alpha^3 r^{\alpha-2}\left[(\alpha - 1 - 2\,\varepsilon^\alpha)(1-\varepsilon^\alpha)^2 x_1^{2\alpha-2} - 2\,\varepsilon^{3\alpha}\, r^{2\alpha-4}|x'|^2\right].
$$

Taking into account that $x_1/r > \varepsilon$ and $|x'|^2/r^2 < 1 - \varepsilon^2$,

$$
A_3 \geq 4a^3\Lambda^3(t)\alpha^3 r^{3\alpha-4}\,\varepsilon^{2\alpha-2}\left[(\alpha - 1 - 2\,\varepsilon^\alpha)(1-\varepsilon^\alpha)^2 - 2\,\varepsilon^{\alpha+2}(1-\varepsilon^2)\right].
$$

Let us denote the quantity in the bracket above by $m(\alpha,\varepsilon)$.

$$
m(\alpha,\varepsilon) = (\alpha - 1 - 2\,\varepsilon^\alpha)(1-\varepsilon^\alpha)^2 - 2\,\varepsilon^{\alpha+2}(1-\varepsilon^2).
$$

A_3 is non-negative if $m(\alpha, \varepsilon)$ is. In the set $\alpha \in (1,2)$ and $\varepsilon \in (0, 1/\sqrt{3})$, the function $m(\alpha, \varepsilon)$ is monotone increasing with respect to α and is monotone decreasing with respect to ε. Notice that

$$m(2, 1/\sqrt{3}) = 0.$$

Hence for any $\varepsilon < 1/\sqrt{3}$, there exists a corresponding $\alpha(\varepsilon) < 2$ such that for $\alpha \in (\alpha(\varepsilon), 2)$, $m(\alpha, \varepsilon) \geq 0$, and in turn $A_3 \geq 0$.

For the estimate of A_2 we notice that $\nabla v = \nabla \phi v + e^{\phi} \nabla u$, as $v = e^{\phi} u$. To bound $|e^{\phi} \nabla u|$, we want to apply the inequality $|e^{\phi} \nabla u|^2/2 \leq |\nabla v|^2 + |\nabla \phi|^2 v^2$. We thus look at the inequality (58) in the following way.

$$\begin{aligned} ([S, A]v, v) + (Sv, Sv) \quad \geq \quad & \int (|\nabla v|^2 + |\nabla \phi|^2 v^2) dx dt \\ & + \int [A_3 + (A_2 - |\nabla \phi|^2) + A_1 + A_0] v^2 dx dt. \end{aligned}$$

Next we estimate $A_2 - |\nabla \phi|^2$.

$$A_2 - |\nabla \phi|^2 = -4a^2 \Lambda(t) \Lambda'(t) \left(\left(1 + \frac{\Lambda(t)}{2\Lambda'(t)} \right) |\nabla \varphi|^2 - \varphi f + \frac{\Lambda(t)}{\Lambda'(t)} f^2 \right). \quad (66)$$

$\Lambda(t) = \frac{1-t}{t^{\alpha/2}}$ and $\Lambda'(t) = -\frac{\alpha/2 + (1-\alpha/2)t}{t^{\alpha/2+1}}$. $|\Lambda(t)/\Lambda'(t)| < \frac{1}{2\alpha}$.

$$A_2 - |\nabla \phi|^2 \geq -4a^2 \Lambda(t) \Lambda'(t) \left(\left(1 - \frac{1}{4\alpha} \right) |\nabla \varphi|^2 - \varphi f - \frac{1}{2\alpha} f^2 \right). \quad (67)$$

Notice that $|\nabla \varphi|^2 \geq |\varphi_{,1}|^2 \geq \alpha^2 (1 - \varepsilon^\alpha)^2 x_1^{2\alpha - 2}$ and $\varphi f \leq \alpha \, \varepsilon^\alpha (1 - \varepsilon^\alpha) x_1^{2\alpha - 2}$. From the expression of $m(\alpha, \varepsilon)$ in the estimation of A_3 above, we know that $\varepsilon \leq (\alpha - 1)/2$. Taking into account that $r \geq x_1 > 1$,

$$A_2 - |\nabla \phi|^2 \geq -4a^2 \Lambda(t) \Lambda'(t) \left(\frac{3}{8} x_1^{2\alpha - 2} - \frac{1}{4} r^{2\alpha - 4} \right) \geq -\frac{a^2}{2} \Lambda(t) \Lambda'(t) x_1^{2\alpha - 2}. \quad (68)$$

Finally, we estimate A_1. Recall that

$$A_1 = -a\Lambda \Delta^2 \varphi + a\Lambda'' \varphi - 2a\Lambda(t) \Delta f + a\Lambda' \varphi - 2a\Lambda(t) f + 8at\Lambda f.$$

An simple observation is that

$$A_1 \geq a(\Lambda'' \varphi + a\Lambda') \varphi - a\Lambda(\Delta^2 \varphi + 2\Delta f + 2f), \quad (69)$$

and $\Lambda''(t) + \Lambda'(t) > \alpha - 1 > 0$. The terms in the second parenthesis are of homogeneity less than $\alpha - 2$, thus under the control of A_2. Consequently, there exists some constant a_0 depending on φ such that for $a > a_0$,

$$A_2 + A_1 \geq -\frac{a^2}{4}\Lambda(t)\Lambda'(t)x_1^{2\alpha-2} + a(\alpha-1)\varphi. \tag{70}$$

In addition $|\Lambda'(t)| \geq 1$ and $x_1 > 1$. We thus have

$$\begin{aligned}([S,A]v,v) + (Sv,Sv) \quad &\geq \quad \int (|\nabla v|^2 + |\nabla \phi|^2 v^2) dxdt \\ &\quad + \int \frac{a^2}{4}\Lambda(t) v^2 dxdt + \int a(\alpha-1)\varphi v^2 dxdt\end{aligned}$$

In turn

$$\begin{aligned}\int_{Q_\theta} e^{2a\Lambda(t)\varphi(x)+2t^2} &\left[\left(\frac{a^2}{4}\Lambda(t) + a(\alpha-1)\varphi\right)u^2 + \frac{1}{2}|\nabla u|^2\right] dxdt \\ &\leq \int_{Q_\theta} e^{2a\Lambda(t)\varphi(x)+2t^2} |\partial_t u + \Delta u|^2 dxdt.\end{aligned} \tag{71}$$

To simplify, we can assume $\alpha > 3/2$, and $a > 2$, it follows that

$$\begin{aligned}\int_{Q_\theta} e^{2a\Lambda(t)\varphi(x)+2t^2} &\left[a\left(\Lambda + \varphi\right)u^2 + |\nabla u|^2\right] dxdt \\ &\leq 4\int_{Q_\theta} e^{2a\Lambda(t)\varphi(x)+2t^2} |\partial_t u + \Delta u|^2 dxdt.\end{aligned} \tag{72}$$

□

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