SINGULARITIES OF THE ASYMPTOTIC COMPLETION OF DEVELOPABLE MÖBIUS STRIPS

KOSUKE NAOKAWA

ABSTRACT. We prove that the asymptotic completion of a developable Möbius strip in Euclidean three-space must have at least one singular point other than cuspidal edge singularities. Moreover, if the strip contains a closed geodesic, then the number of such singular points is at least three. These lower bounds are both sharp.

1. INTRODUCTION

Let U be an open domain in Euclidean two-space $\mathbb{R}^2$ and $f : U \longrightarrow \mathbb{R}^3$ a C^∞ map. A point $p \in U$ is called a *singular point* of f if the Jacobi matrix of f is of rank less than 2 at p. It is well-known that complete and flat (i.e. zero Gaussian curvature) surfaces immersed in $\mathbb{R}^3$ are cylindrical. This fact implies that the 'asymptotic completion' (see Definition 2.1) of a developable Möbius strip (i.e. a flat ruled Möbius strip) must have singular points. Since the most generic singular points appeared on developable surfaces are cuspidal edge singularities (cf. [4, 5]), we are interesting how often singular points other than cuspidal edge singularities appear on the asymptotic completion of a developable Möbius strip. (Izumiya-Takeuchi [5] is a nice reference for singularities of ruled surfaces or developable surfaces.)

Recently, global properties of flat surfaces with singularities in $\mathbb{R}^3$ were investigated in Murata-Umehara [9]. They defined 'completeness' for flat fronts (cf. [9, Definition 0.2]) and proved that a complete flat front with embedded ends has at least four singular points other than cuspidal edge singularities if the front has singular points. However, we cannot apply this result, since complete flat fronts are all orientable (cf. [9, Theorem A]). Therefore, it is interesting to determine lower bounds on the number of non-cuspidal-edge singular points on developable Möbius strips. We show the following:

Proposition. *The asymptotic completion of a developable Möbius strip has at least one singular point other than cuspidal edge singularities.*

In fact, there are many developable Möbius strips. Chicone-Kalton [2] constructed a developable Möbius strip on each generic closed regular curve in $\mathbb{R}^3$. The topological types of Möbius strips are determined by the isotopy types of their generating curves and Möbius twisting numbers. Røgen [11] showed that there exists a developable Möbius strip of an arbitrarily given topological type.

Date: November 15, 2010.
2010 *Mathematics Subject Classification.* 53A05 · 57R45.
Key words and phrases. cuspidal edge singularity · developable surface · rectifying Moebius strip.

A developable Möbius strip which contains a closed geodesic is called a *rectifying Möbius strip*. Roughly speaking, a rectifying strip can be constructed from an isometric deformation of a rectangular domain on a plane (cf. [8, Proposition 2.14]). We also show the following assertion.

Theorem. *The asymptotic completion of a rectifying Möbius strip has at least three singular points other than cuspidal edge singularities.*

The first explicit construction of a rectifying Möbius strip in $\mathbb{R}^3$ was given by Wunderlich [13]. Recently, Kurono-Umehara [8] proved that there exists a rectifying Möbius strip which is isotopic to any given Möbius strip. See Sabitov [12] for other references and the history.

2. Singularities of developable Möbius strips

First, we define several terminologies. Let $\gamma = \gamma(s) : \mathbb{R} \longrightarrow \mathbb{R}^3$ be a C^∞ map. The map $\gamma(s)$ is called *l-periodic* if $\gamma(s+l) = \gamma(s)$ for $s \in \mathbb{R}$. A C^∞ map $\gamma(s)$ is called *regular* if $\gamma'(s) := d\gamma(s)/ds$ does not vanish on $\mathbb{R}$. We fix such a periodic regular curve γ. An $\mathbb{R}^3$-valued vector field ξ along γ is called *l-odd-periodic* if it satisfies $\xi(s+l) = -\xi(s)$ for $s \in \mathbb{R}$. We also fix such a C^∞ l-odd-periodic vector field ξ. Then, a C^∞ immersion

$$F(s,u) = \gamma(s) + u\xi(s) \qquad (s \in \mathbb{R},\ |u| < \epsilon) \tag{2.1}$$

is called a *ruled Möbius strip* if $\gamma'(s)$ and $\xi(s)$ are linearly independent for each $s \in \mathbb{R}$, where $\epsilon > 0$ is taken to be sufficiently small. In this situation, γ is called the *generating curve* of F and ξ is called the *ruling vector field* of F.

Definition 2.1. Let $F(s,u)$ be a ruled Möbius strip as in (2.1). Then a C^∞ map

$$\tilde{F}(s,u) = \gamma(s) + u\xi(s) \qquad (s,u \in \mathbb{R})$$

is called the *asymptotic completion* (or *a-completion*) of F.

Definition 2.2. Let $U_i \subset \mathbb{R}^2$ $(i = 1,2)$ be two open neighborhoods of points $p_i \in \mathbb{R}^2$ and $f_i = f_i(u,v) : U_i \longrightarrow \mathbb{R}^3$ two C^∞ maps. Then f_1 is said to be *right-left equivalent* to f_2 if there exist two diffeomorphisms $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $\Phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $\varphi(p_1) = p_2$, $\Phi \circ f_1(p_1) = f_2(p_2)$ and $\Phi \circ f_1 = f_2 \circ \varphi$ on U_1.

We set

$$f_C(u,v) := \begin{pmatrix} 2u^3 \\ -3u^2 \\ v \end{pmatrix}, \qquad f_S(u,v) := \begin{pmatrix} 3u^4 + u^2 v \\ -4u^3 - 2uv \\ v \end{pmatrix}$$

(See Figures 1 and 2, respectively). A C^∞ map which is right-left equivalent to the map germ f_C (resp. f_S) is called a *cuspidal edge* (resp. a *swallowtail*).

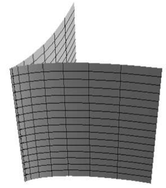

Figure 1. A cuspidal edge

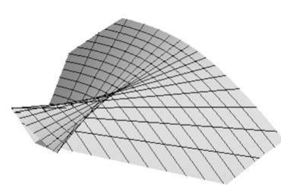

Figure 2. A swallowtail

We recall the criteria for cuspidal edges and swallowtails as in [7]:

Definition 2.3. Let $U \subset \mathbb{R}^2$ be an open domain. A C^∞ map $f : U \longrightarrow \mathbb{R}^3$ is called a *frontal* if there exists a C^∞ map $\nu : U \longrightarrow \mathbb{S}^2$ ($\mathbb{S}^2$ is the unit sphere) such that $\nu(p)$ is perpendicular to $df(T_pU)$ for $p \in U$, where df is the differential of f and T_pU is the tangent space at p to U. Such a map ν is called a *unit normal vector field* of f. Moreover, if the C^∞ map $L := (f, \nu) : U \longrightarrow \mathbb{R}^3 \times \mathbb{S}^2$ is an immersion, f is called a *wave front* (or *front*).

A singular point $p \in U$ of a frontal $f(u, v)$ is *non-degenerate* if the differential $d\lambda$ of $\lambda := \det(f_u, f_v, \nu)$ does not vanish at p, where $f_u := \partial f/\partial u$ and $f_v := \partial f/\partial v$. If a singular point p is non-degenerate, the singular set of f is a regular curve near p on U. This regular curve $c(s)$ is called the *singular curve* of f, and a tangent vector to c is called a *singular direction* of f. Moreover, a nonzero vector $\eta \in T_{c(s)}U$ satisfying $df(\eta) = 0$ is called a *null direction* of f. We can take such $\eta(s)$ as a C^∞ vector field along $c(s)$ near p, and the C^∞ vector field $\eta(s)$ is called a *null vector field* of f.

Fact 2.4 ([7])**.** *Let $f = f(u, v) : U \longrightarrow \mathbb{R}^3$ be a wave front. We denote by $c(s)$ the singular curve near a non-degenerate singular point $p \in U$ of f such that $c(0) = p$, and by $\eta(s)$ a null vector field along $c(s)$. We set $\rho := \det(c', \eta)$. Then*

(*i*) *p is a cuspidal edge if and only if $\rho(0) \neq 0$,*
(*ii*) *p is a swallowtail if and only if $\rho(0) = 0$ and $\rho'(0) \neq 0$.*

Cuspidal edges and swallowtails are wave fronts as C^∞ map germs. It should be remarked that criteria for cuspidal edges and swallowtails of developable surfaces have been given in [5, Theorem 3.7]. One can apply the criteria instead of those in Fact 2.4.

Next, we consider the a-completion of a ruled Möbius strip $F(s, u) = \gamma(s)+u\xi(s)$ with singularities. By a suitable change of parameters, we may assume that s is an arc-length parameter of γ and $\xi(s)$ is a unit vector for $s \in \mathbb{R}$.

Lemma 2.5. *Let $F(s, u) = \gamma(s) + u\xi(s)$ be a ruled Möbius strip where s is arc-length and $\xi(s)$ is a unit vector for $s \in \mathbb{R}$. Then*

$$|F_s \times F_u|^2 = \begin{cases} |\xi'(s)|^2 \left(u + \frac{\gamma'(s)\cdot\xi'(s)}{|\xi'(s)|^2}\right)^2 + \frac{\det(\gamma'(s),\xi(s),\xi'(s))^2}{|\xi'(s)|^2} & (\xi'(s) \neq \mathbf{0}), \\ |\gamma'(s) \times \xi(s)|^2 & (\xi'(s) = \mathbf{0}), \end{cases}$$

where the dot '$\cdot$' is the inner product and the cross '$\times$' is the vector product in $\mathbb{R}^3$.

Proof. Since $F_s = \gamma' + u\xi'$ and $F_u = \xi$, this assertion is obvious when $\xi'(s) = \mathbf{0}$. So we assume $\xi'(s) \neq \mathbf{0}$, and then

$$|F_s \times F_u|^2 = |\xi'|^2 \left(u + \frac{\gamma' \cdot \xi'}{|\xi'|^2}\right)^2 + \frac{|\gamma' \times \xi|^2\,|\xi'|^2 - (\gamma' \cdot \xi')^2}{|\xi'|^2}.$$

Since

$$|(\gamma' \times \xi) \times \xi'|^2 = |\gamma' \times \xi|^2|\xi'|^2 - ((\gamma' \times \xi) \cdot \xi')^2 = |\gamma' \times \xi|^2|\xi'|^2 - \det(\gamma', \xi, \xi')^2$$

and

$$|(\gamma' \times \xi) \times \xi'|^2 = |(\gamma' \cdot \xi')\xi - (\xi \cdot \xi')\gamma'|^2 = (\gamma' \cdot \xi')^2,$$

we get the conclusion. $\square$

The equation $\det(\gamma', \xi, \xi') = 0$ is a necessary and sufficient condition of flatness of ruled Möbius strips. Hence, if F is developable, the C^∞ map

$$\nu(s,u) := \frac{\gamma'(s) \times \xi(s)}{|\gamma'(s) \times \xi(s)|} \tag{2.2}$$

gives a unit normal vector field along $F(s,u)$, so F is a frontal. Since ν does not depend on u when F is developable, we regard as $\nu(s) = \nu(s,u)$ and denote $\nu' = \nu_s$.

Let '$\sim$' be the equivalence relation which regards two points (s,u) and $(s+l, -u)$ as the same point in $\mathbb{R}^2$, where l is the period of the closed curve $\gamma(s)$. We set $M := \mathbb{R}^2/\sim$. Then, F can be regarded as a C^∞ map of M into $\mathbb{R}^3$.

Lemma 2.6. *Suppose that $F(s,u) = \gamma(s) + u\xi(s)$ is a developable Möbius strip, then*

(i) each singular point of F is non-degenerate,
(ii) the singular set $S(F)$ of F is given by

$$S(F) = \left\{ (s,u) \in M\,;\, u = -\frac{|\gamma'(s) \times \xi(s)|^2}{\gamma'(s) \cdot \xi'(s)},\ \xi'(s) \neq \mathbf{0} \right\},$$

(iii) the null vector field of F is given by

$$\frac{\partial}{\partial s} - (\gamma' \cdot \xi)\frac{\partial}{\partial u}.$$

Proof. We set $\lambda := \det(F_s, F_u, \nu)$. Since the singular points of F do not appear on asymptotic lines $u \mapsto (s_1, u)$ for $s_1 \in \mathbb{R}$ satisfying $\xi'(s_1) = \mathbf{0}$, we have $\lambda_u = \gamma' \cdot \xi' / |\gamma' \times \xi| \neq 0$ on $S(F)$. Therefore, we obtain (i). Since $|\gamma' \times \xi|\,|\xi'| = |\gamma' \cdot \xi'|$ by flatness of F and Lemma 2.5, we obtain (ii). Let (s_0, u_0) be a singular point. Since $k := \gamma'(s_0) \cdot \xi(s_0)$ $(\neq 0)$ satisfies $F_s(s_0,u_0) = kF_u(s_0,u_0)$, we have (iii). $\square$

Since ξ is not a constant vector field, there exists a point $s \in \mathbb{R}$ such that $\xi'(s) \neq \mathbf{0}$. Therefore, the singular set $S(F)$ is not empty. The following lemma gives a proof of the proposition in the introduction.

Lemma 2.7. *Let $F(s,u) = \gamma(s) + u\xi(s)$ be a developable Möbius strip. The a-completion of F has at least one singular point other than cuspidal edge singularities on each connected component of $S(F)$. In particular, the a-completion of F has at least one singular point other than cuspidal edge singularities.*

Proof. We remark that there exists a point $s \in \mathbb{R}$ such that $\xi'(s) = \mathbf{0}$, since $\gamma' \cdot \xi'$ is an odd-periodic function. Let $\{(s, u(s))\}_{s \in \mathbb{R}}$ be the graph of the singular curve of F in the (s,u)-plane, and let $\{(s,u(s))\}_{s_1 < s < s_2}$ be a connected component of $S(F)$. Then, the two points s_1 and s_2 satisfy $\xi'(s_1) = \xi'(s_2) = \mathbf{0}$ and $\xi'(s) \neq \mathbf{0}$ for $s \in (s_1, s_2)$. Suppose $\gamma'(s) \cdot \xi'(s) > 0$ for $s \in (s_1, s_2)$. By Lemma 2.6 (ii), $u(s)$ satisfies

$$\lim_{s \searrow s_1} u(s) = \lim_{s \nearrow s_2} u(s) = -\infty,$$

where $\searrow$ and $\nearrow$ mean approaching from above and below, respectively. Then, the function

$$P(s) := -u(s) - \int_{s_1}^{s} \gamma'(t) \cdot \xi(t) dt$$

satisfies

$$\lim_{s \searrow s_1} P(s) = \lim_{s \nearrow s_2} P(s) = \infty,$$

since $|\gamma'(s) \cdot \xi(s)| < 1$. This implies that $P(s)$ attains a minimum at a point $s = s_0$. Let $\rho(s)$ be the determinant of the 2×2 matrix consisting of the two vectors for the singular direction and null direction of F. Then, the function $\rho(s) = -u'(s) - \gamma'(s) \cdot \xi(s) = P'(s)$ vanishes at $s = s_0$. By Fact 2.4, the singular point $(s_0, u(s_0))$ is not a cuspidal edge singularity. The case $\gamma' \cdot \xi' < 0$ is similar. □

Remark 2.8. Lemmas 2.6 and 2.7 also imply that the number of non-cuspidal-edge singular points on the a-completion of a developable Möbius strip is greater than or equal to the number of connected components of the zero set of ξ', if these numbers are finite.

We close this section with an example having only one singular point other than cuspidal edge singularities. This implies that the proposition gives the sharpest lower bound.

Example 2.9. We define a 2π-periodic regular curve $\gamma = \gamma(s) : \mathbb{R} \longrightarrow \mathbb{R}^3$ by

$$\gamma(s) := \begin{pmatrix} \sin 2s \\ \cos 2s \\ (1/\sqrt{2}) \sin s \end{pmatrix},$$

whose curvature function $\kappa(s)$ does not vanish. Let $\xi = \xi(s)$ be the 2π-odd-periodic and non-vanishing vector field along γ given by

$$\xi(s) := p(s)\boldsymbol{e}(s) + \cos(s/2)\boldsymbol{n}(s) + \sin(s/2)\boldsymbol{b}(s),$$

where $\boldsymbol{e}$ is the unit tangent vector field, $\boldsymbol{n}$ is the normal vector field and $\boldsymbol{b}$ is the binormal vector field of γ. Moreover, τ is the torsion function of γ and

$$p(s) := \frac{1}{\kappa(s)} \left(\frac{1}{2|\gamma'(s)|} + \tau(s) \right) \Big/ \sin \frac{s}{2}.$$

We remark that $p(s)$ and $\xi(s)$ are both smooth at $s = 0, \pi$. Since $\det(\gamma', \xi, \xi') = 0$, the map $F(s, u) = \gamma(s) + u\xi(s)$ is a developable Möbius strip (See Figure 3).

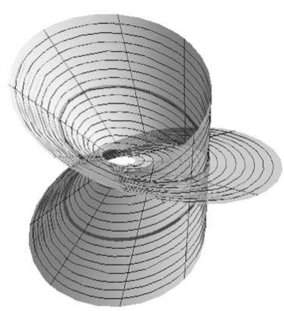

FIGURE 3. The image of $F(s, u)$

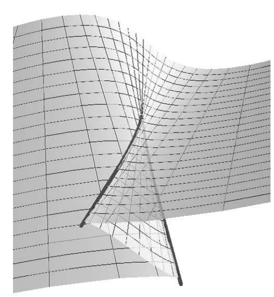

FIGURE 4. An open swallowtail

The generating curve $\gamma(s)$ can be expressed by a rational function; if $x(s) := \tan(s/2)$ for $-\pi < s < \pi$, then we have

$$\gamma(x) = \frac{1}{(1 + x^2)^2} \begin{pmatrix} 4\, x(1 - x^2) \\ (1 - 2x - x^2)(1 + 2x - x^2) \\ \sqrt{2}\, x(1 + x^2) \end{pmatrix}.$$

We set

$$\hat{\xi}(x) := \xi(s)/\cos(s/2) = \hat{p}(x)\boldsymbol{e}(x) + \boldsymbol{n}(x) + x\boldsymbol{b}(x).$$

Then $\hat{p}(x)$ is a C^∞ function. Let $\rho(x)$ be the determinant of the 2×2 matrix consisting of the two vectors for the singular direction and null direction of $F(x, v) = \gamma(x) + v\hat{\xi}(x)$. We obtain[1]

$$\rho = \frac{1}{|\hat{\xi} \times \hat{\xi}_x|^2} \frac{\left(b_1 + b_{12}\sqrt{f_2}\right)^2}{\left(a_1 + a_{12}\sqrt{f_2}\right)} \frac{xA(x)}{(1+x^2)^3(f_1)^{7/2}(f_2)^2},$$

where $f_1(x) := 3+5x^2+3x^4$, $f_2(x) := 9+14x^2+9x^4$. Here, a_1, a_{12}, b_1, b_{12} and A are polynomials in x such that they have only even-degree terms and are non-negative. Moreover, the asymptotic line at $x = \infty$ has no singular points, so $\rho(x) = 0$ if and only if $x = 0$ (i.e. $s = 0$). By Fact 2.4, the singular point corresponding to $s = 0$ is not a cuspidal edge singularity. On the other hand, $\nu'(s) = 0$ if and only if $s = 0$, where $\nu(s)$ is defined by (2.2). Therefore, each singular point except at $s = 0$ is a cuspidal edge, again by Fact 2.4.

Remark 2.10. By author's computer graphics, the singularity on the asymptotic line at $s = 0$ looks like an 'open swallowtail' (See Figure 4; cf. [1]).

3. The proof of the theorem in the introduction

Let $F(s, u) = \gamma(s) + u\xi(s)$ be a rectifying Möbius strip (See the introduction). Suppose that s is an arc-length parameter of γ. Then, $\gamma(s)$ and $\xi(s)$ satisfy

$$\gamma''(s) \cdot \xi(s) = 0$$

for $s \in \mathbb{R}$, since $\gamma(s)$ is a geodesic. We normalize the ruling vector $\xi(s)$ for each $s \in \mathbb{R}$ such that the projection of $\xi(s)$ into the rectifying plane at the point $\gamma(s)$ is a unit vector, where the rectifying plane is a plane perpendicular to $\boldsymbol{e}(s) = \gamma'(s)$. Then, ξ can be expressed by

$$D := \frac{\tau}{\kappa}\boldsymbol{e} + \boldsymbol{b} \tag{3.1}$$

when κ is nonzero, where κ is the curvature function, τ is the torsion function and $\{\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{b}\}$ is the Frenet frame of γ. This vector field D is called the *normalized Darboux vector field* of γ. The ratio $\sigma := \tau/\kappa$ is called the *conical curvature* of γ (cf. Heil [3]).

Remark 3.1. In [4], D is called the *modified Darboux vector* along γ. Moreover, the criteria of cuspidal edges and swallowtails on the rectifying developable surfaces associated to γ are given in terms of conical curvature $\sigma = \tau/\kappa$. For example, (s_0, u_0) is a non-cuspidal-edge singularity of $F(s, u) = \gamma(s) + uD(s)$ if and only if $u_0 = -1/\sigma'(s_0)$, $\sigma'(s_0) \neq 0$ and $\sigma''(s_0) = 0$ (see [4, Theorem 2.2]).

We recall the following facts in order to explain properties of the conical curvature of a regular space curve.

Fact 3.2 (cf. [3])**.** *Let $I \subset \mathbb{R}$ be an open interval and $\gamma : I \longrightarrow \mathbb{R}^3$ a regular curve. If the curvature function κ of γ does not vanish, then the geodesic curvature function of the unit tangent vector field $\boldsymbol{e} : I \longrightarrow \mathbb{S}^2$ of γ as a spherical curve is equal to the conical curvature $\sigma = \tau/\kappa$ of γ.*

---

[1] The software Mathematica (Version 7.0.0, Wolfram research) was used for this calculation.

A C^∞ function $g = g(s) : I \longrightarrow \mathbb{R}$ is said to be *strictly increasing* (resp. *monotonically increasing in the wider sense*) if $g'(s) > 0$ (resp. $g'(s) \geq 0$) for $s \in I$. A regular spherical curve $\alpha = \alpha(s) : I \longrightarrow \mathbb{S}^2$ is called an *honestly positive spiral* (resp. *positive spiral*) if the geodesic curvature function of α is strictly increasing (resp. monotonically increasing in the wider sense).

Fact 3.3 ([6, 10])**.** *Let $\alpha = \alpha(s) : I \longrightarrow \mathbb{S}^2$ be a honestly positive spiral* (*resp. positive spiral*)*. We denote by $C(s) \subset \mathbb{S}^2$ the osculating circle of α at $s \in I$ and assign $C(s)$ the orientation compatible with that of $\alpha(s)$ for each $s \in I$. Let $D(s)$ be the left-hand domain of $C(s)$. Then, $s_1 < s_2$ implies $\overline{D(s_2)} \subset D(s_1)$ (resp. $D(s_2) \subset D(s_1)$).*

We return to the initial settings in this section. The Frenet frame of $\gamma(s)$ cannot be defined if $\kappa(s) = 0$. However, we can construct an 'extended' Frenet frame defined on $\mathbb{R}$ by using the ruling vector field $\xi(s)$. We set

$$\hat{\boldsymbol{n}} := -\frac{\boldsymbol{e} \times \xi}{|\boldsymbol{e} \times \xi|}, \quad \hat{\boldsymbol{b}} := \boldsymbol{e} \times \hat{\boldsymbol{n}}, \quad \hat{\kappa} := \boldsymbol{e}' \cdot \hat{\boldsymbol{n}}, \quad \hat{\tau} := -\hat{\boldsymbol{b}}' \cdot \hat{\boldsymbol{n}}, \quad \hat{\sigma} := \boldsymbol{e} \cdot \xi.$$

These vector fields and functions are of class C^∞. Then $\{\boldsymbol{e}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{b}}\}$ satisfies

$$\boldsymbol{e}' = \hat{\kappa}\hat{\boldsymbol{n}}, \qquad \hat{\boldsymbol{n}}' = -\hat{\kappa}\boldsymbol{e} + \hat{\tau}\hat{\boldsymbol{b}}, \qquad \hat{\boldsymbol{b}}' = -\hat{\tau}\hat{\boldsymbol{n}}.$$

Therefore, we have $\kappa = |\hat{\kappa}|$. Moreover, if $\kappa(s) \neq 0$, then

$$\hat{\boldsymbol{n}}(s) = \epsilon \boldsymbol{n}(s), \quad \hat{\boldsymbol{b}}(s) = \epsilon \boldsymbol{b}(s),$$

where $\epsilon := \hat{\kappa}(s)/\kappa(s)$ $(= \pm 1)$. The function $\hat{\tau}(s)$ is exactly equal to $\tau(s)$ if $\kappa(s) \neq 0$. We can express the ruling vector field of F by

$$\xi = \hat{\sigma}\boldsymbol{e} + \hat{\boldsymbol{b}}.$$

Since $\det(\boldsymbol{e}, \xi, \xi') = 0$, we have $\hat{\tau} = \hat{\sigma}\hat{\kappa}$. Therefore, we regard $\hat{\boldsymbol{n}}$, $\hat{\boldsymbol{b}}$, $\hat{\kappa}$, $\hat{\tau}$ and $\hat{\sigma}$ as smooth extensions of $\boldsymbol{n}$, $\boldsymbol{b}$, κ, τ and σ, respectively. We set

$$\hat{K}_+ := \{s \in \mathbb{R}\,;\, \hat{\kappa}(s) > 0\}, \ \hat{K}_0 := \{s \in \mathbb{R}\,;\, \hat{\kappa}(s) = 0\}, \ \hat{K}_- := \{s \in \mathbb{R}\,;\, \hat{\kappa}(s) < 0\}.$$

We regard $\boldsymbol{e} = \gamma'$ as a closed curve in $\mathbb{S}^2$. The spherical curve $\boldsymbol{e}$ has singular points at zeros of κ. For each $s \in \mathbb{R}$, $\hat{\boldsymbol{n}}(s)$ and $\hat{\boldsymbol{b}}(s)$ can be regarded as a unit tangent vector and a unit conormal vector of the spherical curve $\boldsymbol{e}(s)$, respectively. In particular, $\{\hat{\boldsymbol{n}}, \hat{\boldsymbol{b}}, \boldsymbol{e}\}$ gives a smooth positive orthonormal frame along $\boldsymbol{e}$. Since $\hat{\sigma}$ is of class C^∞, we can smoothly extend to $\mathbb{R}$ the osculating circle $C(s) \subset \mathbb{S}^2$ of $\boldsymbol{e}(s)$. In fact, the extended osculating circle $\hat{C}(s)$ can be canonically defined by a circle on $\mathbb{S}^2$ which passes $\boldsymbol{e}(s)$ and whose center is

$$\exp_{\boldsymbol{e}(s)}\left(\frac{1}{2}\left(\arctan\frac{2}{\hat{\sigma}}\right)\hat{\boldsymbol{b}}(s)\right),$$

where $\exp_p : T_p\mathbb{S}^2 \longrightarrow \mathbb{S}^2$ is the exponential map at a point $p \in \mathbb{S}^2$. We assign $\hat{C}(s)$ the orientation compatible with the direction of $\hat{\boldsymbol{n}}(s)$. If $s \in \hat{K}_+$ (resp. $\hat{K}_-$), then the orientation of $\hat{C}(s)$ is equal (resp. opposite) to that of $C(s)$. Let $\hat{D}(s)$ be the left-hand domain of $\hat{C}(s)$.

Since $\xi' = \hat{\sigma}'\boldsymbol{e}$, by Remark 2.8, it is sufficient to show that the number of the connected components of the zero point set of $\hat{\sigma}'(s)$ is at least three. We suppose that the number of locally maximal or locally minimal points of the odd-periodic function $\hat{\sigma}$ is only one. We may assume that $s = 0$ is the locally minimal point.

Then $\hat{\sigma}$ is a monotonic increasing function in the wider sense on the closed interval $[0, l]$, where l is the period of $\gamma(s)$. The restriction of the spherical curve $\boldsymbol{e}$ to each connected component of $\hat{K}_+$ (resp. $\hat{K}_-$) is a positive (resp. negative) spiral. If we take two points s_1 and s_2 satisfying $s_1 < s_2$ in each connected component of $\hat{K}_+ \cup \hat{K}_-$, we have $D(s_2) \subset D(s_1)$ by Fact 3.3. On the other hand, if we take two points s_1 and s_2 satisfying $s_1 < s_2$ in each connected component of $\hat{K}_0$, it holds that $\hat{\kappa} = \hat{\tau} = 0$ on the closed interval $[s_1, s_2]$. Therefore $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{b}}$ are constant on $[s_1, s_2]$, so we have $D(s_2) \subset D(s_1)$. Since the domain $D(s)$ depends smoothly on $s \in \mathbb{R}$, we have $D(s_2) \subset D(s_1)$ for s_1 and s_2 satisfying $s_1 < s_2$. In particular, we obtain $D(l) \subset D(0)$. On the other hand, the orientation of $C(l)$ is opposite to that of $C(0)$, since $\hat{\boldsymbol{n}}$ is odd-periodic. Hence, $D(0) \cap D(l)$ is empty. However, since $D(l)$ is not empty, this is a contradiction. Since $\hat{\sigma}$ is odd-periodic, $\hat{\sigma}$ must have at least three locally minimal or locally maximal points. Then, by Remark 2.8, we obtain the theorem in the introduction.

Example 3.4. We set

$$\gamma(s) := \frac{1}{1 + (s + s^3)^2} \begin{pmatrix} (2/5)s + s^3 + s^5 \\ s + s^3 \\ -8/5 \end{pmatrix},$$

which gives a closed regular curve of $\mathbb{S}^1 = \mathbb{R} \cup \{\infty\}$ in $\mathbb{R}^3$. Moreover, γ has only one inflection point at $s = \infty$. We set $\hat{\gamma}(t) := \gamma(1/t)$. Since

$$\hat{\gamma}'(t) \times \hat{\gamma}^{(3)}(t)|_{t=0} \neq \mathbf{0}, \quad \det(\hat{\gamma}'(t), \hat{\gamma}^{(3)}(t), \hat{\gamma}^{(4)}(t))|_{t=0} = 0$$

and [8, Corollary 2.11], the C^∞ map $F(s, u) = \gamma(s) + u\xi(s)$ is a rectifying Möbius strip, where $\xi(s)$ is as in (3.1). The a-completion of F has just three singular points other than cuspidal edges (See Figures 5 and 6).

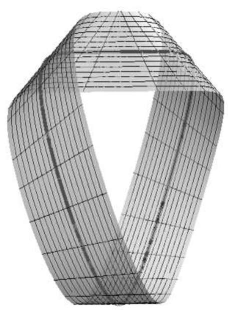

FIGURE 5. The image of $F(s, u)$

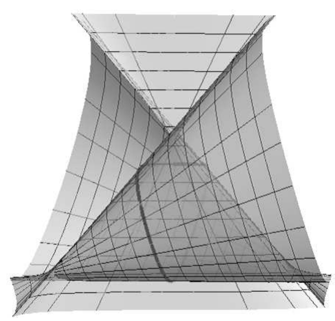

FIGURE 6. The a-completion of F

By Lemma 2.6, the singular curve of F is given by

$$s \longmapsto \left(s, u(s) := -\frac{|\gamma'(s)|}{\hat{\sigma}'(s)} \right).$$

Moreover, the null vector field of F is $\partial/\partial s$. We denote by $\rho(s)$ the determinant of the 2×2 matrix consisting of the two vectors for the singular direction and null direction of F. Then, we can calculate[2]

$$\rho(s) := u'(s) = \frac{(1+s^2+2s^4+s^6)a(s)^{3/2}Q(s)}{s^2 b(s)^2},$$

where $a(s)$, $b(s)$, $Q(s)$ are certain polynomials which have only even-dimensional terms and $a(s), b(s) > 0$. It can be rigorously checked that the polynomial $Q(s)$ has just two roots by Sturm's theorem. Moreover, we have $\rho(s) \to 0$ $(s \to \infty)$, so $\rho(s)$ has three zeros including $s = \infty$. On the other hand, since $\nu(s,u) := \hat{\boldsymbol{n}}(s)$ is a unit normal vector field along F, the C^∞ map $L = (F, \nu)$ is not immersed only at $(s,u) = (0, u(0))$. Then, F has exactly three non-cuspidal-edge singular points by Fact 2.4. We remark that the singularity at $(s,u) = (0, u(0))$ is a shape like an open swallowtail (See Figure 4). The other two non-cuspidal-edge singularities are both swallowtails.

Acknowledgements. The author is grateful to Masaaki Umehara for encouragement and suggestions. The author also thanks to Wayne Rossman for carefully reading and commenting to this paper. Moreover, the author thanks to Yoshio Agaoka and Kentaro Saji for helpful information.

References

[1] V. I. Arnol'd: *Lagrangian manifolds with singularities, asymptotic rays and the open swallowtail*, Funct. Anal. Appl. **15** (1981), 235–246.

[2] C. Chicone and N. J. Kalton: *Flat embeddings of the Möbius strip in* $\mathbb{R}^3$, Commun. Appl. Nonlinear Anal. **9** (2002), 31–50.

[3] E. Heil: *A four-vertex theorem for space curves*, Math. Pannon. **10** (1999), 123–132.

[4] S. Izumiya, H. Katsumi and T. Yamasaki: *The rectifying developable and the spherical Darboux image of a space curve*, Banach Center Publications **50** (1999), 137–149.

[5] S. Izumiya and N. Takeuchi: *Geometry of ruled surfaces*, Applicable Mathematics in the Golden Age (edited by J. C. Misra), Narosa Publishing House, New Delhi, India (2003), 305–338

[6] A. Kneser: *Bermerkungen über die Anzahl der Extreme der Krümmung auf geschlossenen Kurven und über verwandte Fragen in einer nichteuklidischen Geometrie*, Festschrift zum 70, Geburtstag von H. Weber 1912, 170–180.

[7] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada: *Singularities of flat fronts in hyperbolic 3-space*, Pacific J. Math. **221** (2005), 303–351.

[8] Y. Kurono and M. Umehara: *Flat Möbius strips of given isotopy type in* R^3 *whose centerlines are geodesics or lines of curvature*, Geom. Dedicata **134** (2008), 109–130.

[9] S. Murata and M. Umehara: *Flat surfaces with singularities in Euclidean 3-space*, J. Diff. Geom. **82** (2009), 279–316.

[10] U. Pinkall: *On the four-vertex theorem*, Aequationes Math. **34** (1987), 221–230.

[11] P. Røgen: *Embedding and knotting of flat compact surfaces in 3-space*, Comment. Math. Helv. **76** (2001), 589–606.

[12] I. K. Sabitov: Isometric immersions and embeddings of locally Euclidean metrics, Cambridge Scientific Publishers, 2009.

[13] W. Wunderlich: *Über ein abwickelbares Möbiusband*, Monatsh. Math. **66** (1962), 276–289.

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

E-mail address: `k-naokawa@cr.math.sci.osaka-u.ac.jp`

[2] The software Mathematica (Version 7.0.0, Wolfram research) was used for this calculation.