FAITHFUL ACTIONS OF AUTOMORPHISMS ON THE SPACE OF ORDERINGS OF A GROUP

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Abstract. We study the space of left– and bi–invariant orderings on a torsion–free nilpotent group G. We will show that generally the set of such orderings is equipped with a faithful action of the automorphism group of G. We prove an extension result which allows us to establish the same result when G is assumed to be merely residually torsion–free nilpotent. In particular, we obtain faithful action of mapping class groups of surfaces. We will draw connections between the structure of orderings on residually torsion–free nilpotent, hyperbolic groups and their Gromov boundaries, and we show that in those cases a faithful $\mathrm{Aut}(G)$–action on the boundary is equivalent to a faithful $\mathrm{Aut}(G)$ action on the space of left–invariant orderings.

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Contents

1. Introduction

Let G be a finitely generated group. A fundamental and often quite difficult problem in the combinatorial group theory of G is to describe the space of orderings on G. A **left–invariant ordering** on G is a relation $\leq$ on G which is a total ordering on the elements of G, together with the following left–invariance property: for all triples $a, b, c \in G$, $a \leq b$ implies $ca \leq cb$. An ordering is called **right–invariant** if the analogous right–invariance property holds. An ordering is called **bi–invariant** if it is both

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left– and right–invariant. It is easy to check that an ordering is bi–invariant if and only if it is left–invariant and conjugation–invariant.

Many groups admit no left–invariant orderings at all. For instance, the presence of torsion precludes orderability. Some groups admit finitely many orderings, and these groups have been completely classified by Tararin. See the book of Botto Mura and Rhemtulla [MR] for more details. It is sometimes useful to observe that orderings on a group naturally occur in pairs. For each ordering $\leq$, there is a natural ordering $\leq^{op}$ called the **opposite ordering**, given by $g \leq^{op} h$ if and only if $h \leq g$. On the other hand, many groups admit uncountably many orderings. To organize the set of all orderings of a group, one defines the **space of orderings** on the group, denoted $LO(G)$ in the case of left–invariant orderings and $O(G)$ in the case of bi–invariant orderings. To define this space and equip it with a good topology, we first define the notion of a **positive cone** $\mathcal{P}$ of an ordering. Given an ordering $\leq \in LO(G)$ or $O(G)$, we set

$$\mathcal{P} = \mathcal{P}(\leq) = \{g \in G \text{ such that } 1 < g\}.$$

This gives us a canonical bijective correspondence between orderings and certain subsets of G, since to recover an ordering, we declare $g < h$ if and only if $g^{-1}h \in \mathcal{P}$.

In order for a subset of G to be the positive cone of some left–invariant ordering, it must satisfy some axioms:

(1) $\mathcal{P} \cup \mathcal{P}^{-1} = G \setminus \{1\}$, where $\mathcal{P}^{-1}$ denotes the set of inverses of elements of $\mathcal{P}$.
(2) $\mathcal{P} \cap \mathcal{P}^{-1} = \emptyset$.
(3) $\mathcal{P} \cdot \mathcal{P} \subset \mathcal{P}$.

$\mathcal{P}$ will be the positive cone of some bi–invariant ordering if in addition $\mathcal{P}$ is G–conjugation invariant.

The power set of subsets of G comes with a natural topology which gives it the structure of a Cantor set. This Cantor set will be metrizable whenever G is countable. In particular, the power set of G can be viewed as

$$\{0, 1\}^G,$$

where the two point set has the discrete topology and the product has the product topology. Two points in the power set of G are close in this topology if they agree on a large finite subset.

It is not difficult to show that the conditions for a set to be the positive cone of a left– or bi–invariant ordering are closed conditions in the natural topology on the power set. The details of the proof can be found in chapter 14 of the book [DDRW] by Dehornoy, Dynnikov, Rolfsen and Wiest, which has served as the primary inspiration for the material in this paper. Therefore, $LO(G)$ and $O(G)$ can be viewed as closed subsets of a Cantor set. The topology of this space for various groups has been studied by various authors, such as by Navas for free groups in in [N1], by Navas and Rivas for Thompson's group F in [NR], and by Sikora for finitely generated torsion–free abelian groups in [S].

The groups $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ both have natural actions on $LO(G)$ and $O(G)$ respectively. G acts on $LO(G)$ by conjugation, so $\mathrm{Out}(G)$ also acts on the G–orbits in this space. These actions are given by pulling back and ordering $\leq$ to an ordering $\leq_\phi$ via the automorphism ϕ. Precisely, we define $g \leq_\phi h$ if and only if $\phi(g) \leq \phi(h)$. It is easy to check that the two actions are by homeomorphisms. One sees that this way we get maps

$$\mathrm{Aut}(G) \to \mathrm{Homeo}(LO(G))$$

and

$$\mathrm{Out}(G) \to \mathrm{Homeo}(O(G)).$$

These maps are the primary focus of this paper. Recall that a group G is called **residually torsion–free nilpotent** if every non–identity element of G persists in some torsion–free nilpotent quotient of G. Examples of residually torsion–free nilpotent groups include free groups, surface groups, right-angled Artin groups and pure braid groups. With this terminology, we can state the main result of this paper:

Theorem 1.1. *Let G by a finitely generated, residually torsion–free nilpotent group. Then the map*

$$\mathrm{Aut}(G) \to \mathrm{Homeo}(LO(G))$$

is injective.

In particular, the conclusions of Theorem 1.1 hold for mapping class groups of surfaces and automorphism and outer automorphism groups of free groups. Theorem 1.1 shows that there are many essentially different positive cones in residually torsion–free nilpotent groups which are not preserved by automorphisms of the group.

The proof of Theorem 1.1 is of a very similar flavor to the proof of asymptotic linearity of the mapping class group, one of the principle results in [K]. Asymptotically faithful actions of mapping class groups have been of recent to various authors, such as Andersen in [A].

As an alternative perspective on Theorem 1.1, we will show that when G is residually torsion–free nilpotent and hyperbolic, $LO(G)$ recovers the boundary ∂G. We will be able to show:

Theorem 1.2. *Suppose that G is residually torsion–free nilpotent and hyperbolic. We have that* $\mathrm{Aut}(G)$ *acts faithfully on ∂G if and only if* $\mathrm{Aut}(G)$ *acts faithfully on $LO(G)$.*

In the case that G is a surface group, Theorem 1.2 can be viewed as a generalization of the classical result of Nielsen, namely that the mapping class group $\mathrm{Mod}_{g,1}$ of a surface of genus $g \geq 2$ with one marked point acts faithfully on the circle. For more details, consult the book of Casson and Bleiler [CB]. It seems that there were few if any connections between orderings on groups and geometric group theory appearing anywhere in literature. It thus appears that Theorem 1.2 gives an example of such a connection.

It is unlikely that one can easily remove the residual condition on G in the statement of Theorem 1.2, since hyperbolic groups can be so diverse. It is not even known whether or not every hyperbolic group is residually finite or virtually torsion–free. For some discussion of virtual properties of hyperbolic groups, the reader might consult the paper [KW] of I. Kapovich and D. Wise.

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3. Abelian groups and extension theorems

In order to prove Theorem 1.1, we will need to understand the conclusion of the theorem for finitely generated torsion–free abelian groups. Our goal is to prove:

Lemma 3.1. *$GL_n(\mathbb{Z})$ acts faithfully on $O(\mathbb{Z}^n)$.*

First, we must understand the structure of $LO(\mathbb{Z}^n) = O(\mathbb{Z}^n)$. When $n = 1$, it is evident that this set has exactly two points. When $n > 1$, Sikora proved in [S] that $O(\mathbb{Z}^n)$ is a Cantor set. To adapt Sikora's Theorem to our setup, we will be quite explicit about a construction of certain orderings on $\mathbb{Z}^n$.

We begin by identifying some useful orderings on $\mathbb{Z}^n$. Let Z denote an rational hyperplane in $\mathbb{R}^n$. Then Z will help determine many positive cones on $\mathbb{Z}^n$ as follows: choose a half of $\mathbb{R}^n$ to be positive. Then choose a hyperplane within Z and declare a half of Z to be positive. Continuing this process, we eventually declare each nonzero integral point in $\mathbb{R}^n$ to be either positive or negative. It is easy to see that we in fact obtain a positive cone on $\mathbb{Z}^n$ this way.

It follows that a flag of rational subspaces of $\mathbb{R}^n$ together with a choice of half–space in each dimension gives rise to an ordering on $\mathbb{Z}^n$. We will call orderings which arise in this fashion **flag orderings**. Note that if Z is an irrational hyperplane in the sense that it contains no rational points other than the origin, Z automatically already determines exactly two orderings: one for each choice of positive half–space.

We have two perspectives on orderings of $\mathbb{Z}^n$. One comes from choosing irrational hyperplanes and rational flags, and the other comes from choosing a positive cone. It is not immediately clear how to reconcile these two descriptions of the orderings on $\mathbb{Z}^n$, even in the case $n = 2$. When $n = 2$, we have a map from $O(\mathbb{Z}^2)$ to $\mathbb{RP}^1$. This map is given by sending an ordering to the line which separates the positive half–plane from the negative half. The fiber over an irrational point in $\mathbb{RP}^1$ consists of two points, one for each choice of positive half–plane. The fiber over a rational point consists of four points, corresponding to the two choices for positive half–plane and the two choices for positive half–line. Thus, one can see that the space of orderings should not be considered with an analytic topology, but rather with a topology which more closely resembles a totally disconnected one.

The rational flag orderings occupy a special place in the study of orderings on $\mathbb{Z}^n$:

Lemma 3.2. *Let $V = \{v_1, \ldots, v_{n+1}\} \subset \mathbb{Z}^n$ be nonzero vectors that do not lie within a single closed rational half space in $\mathbb{R}^n$. Let S denote the semigroup generated by these vectors. Then $0 \in S$.*

Proof. The case where $n = 1$ is trivial. Clearly we may assume that the real span of V is n-dimensional, and indeed that the span of any n vectors in V is n-dimensional. Indeed, otherwise we can take n vectors in V that lie within a proper subspace which is contained in a hyperplane. This hyperplane will divide $\mathbb{R}^n$ into two components. But then V is contained in the closed half space that consists of this hyperplane and the half space which contains the last vector. Consider the line ℓ spanned by v_{n+1} and the set P of positive integral linear combinations of the other n vectors. The positive real span of $\{v_1, \dots, v_n\}$ determines a cone $P_{\mathbb{R}}$ in $\mathbb{R}^n$, and since the span of $\{v_1, \dots, v_n\}$ is n-dimensional and because V is not contained in a closed half-space, it follows that the interior of ℓ intersects the interior of $P_{\mathbb{R}}$.

Each point in P has some distance to ℓ. We may assume that each such distance is nonzero, for otherwise we are done. Consider the $(n-1)$–dimensional real plane spanned by ℓ and $\{v_1, \dots, v_{n-2}\}$. This plane H intersects the interior of $P_{\mathbb{R}}$. H divides $\mathbb{R}^n$ into two halves, and since H intersects the interior of $P_{\mathbb{R}}$, v_{n-1} and v_n cannot lie in the same half space. Positive linear combinations of v_{n-1} and v_n have some distances to H. These distances are either rationally related or not. If they are rationally related, then some positive combination of v_{n-1} and v_n lies in H. If they are not rationally related, the possible distances from positive linear combinations of v_{n-1} and v_n to H accumulate at zero. This observation uses the ergodicity of an irrational rotation on the circle. However, a positive linear combination of v_{n-1} and v_n is always a lattice point. Since H is a rational hyperplane, the set of distances from lattice points to H is a discrete subset of $\mathbb{R}$. It follows that some positive linear combination of v_{n-1} and v_n lies in H.

Let w be a positive combination of v_{n-1} and v_n which lies in H. Then w is linearly independent from $\{v_1, \dots, v_{n-2}\}$. Indeed, this would contradict the assumption that the span of $\{v_1, \dots, v_n\}$ is n-dimensional. By induction, we have the claim. We remark that it is obvious that all the hyperplanes constructed in the proof can by taken to be rational. □

Proposition 3.3. *The set of flag orderings on $\mathbb{Z}^n$ is dense in the space of orderings on $\mathbb{Z}^n$.*

Proof. Let $\{(a_1, b_1), \dots, (a_m, b_m)\}$ be a collection of pairs of distinct lattice points and let $P \in O(\mathbb{Z}^n)$. Suppose that in P, $a_i < b_i$ for all i. We will show that there is a flag ordering in which these relations also hold. This will imply that in any open subset of $O(\mathbb{Z}^n)$ containing P, there is a flag ordering.

By definition, $b_i - a_i \in P$ for each i. By Lemma 3.2, all the $(b_i - a_i)$'s must lie in a closed rational halfspace. If there is a rational hyperplane H such that all the $(b_i - a_i)$'s are in one open half space defined by H, then we are done. Otherwise, we consider the $(b_i - a_i)$'s which lie in H. A repeated application of Lemma 3.2 shows that there is a flag ordering on $\mathbb{Z}^n$ where all the $(b_i - a_i)$'s are positive. □

We are now ready to prove Lemma 3.1. The following proof's strategy was suggested by C. McMullen:

First proof of Lemma 3.1. To each $P \in \mathbb{Z}^n$ we associate a hyperplane H_P in $\mathbb{R}^n$, namely the one which separates $\mathbb{Z}^n$ into positive and negative half spaces. This hyperplane is uniquely determined, as can be seen from Lemma 3.2. Thus the map which associates H_P to the cone P gives a map

$$O(\mathbb{Z}^n) \to \mathbb{P}^{n-1}(\mathbb{R}).$$

Observe that $GL_n(\mathbb{Z})$ acts on $\mathbb{P}^{n-1}(\mathbb{R})$ and that the map above commutes with the action of $GL_n(\mathbb{Z})$ by its very definition. Note that $O(\mathbb{Z}^n)$ is compact and nonempty, and $GL_n(\mathbb{Z})$ has dense orbits in $\mathbb{P}^{n-1}(\mathbb{R})$. It follows that the image of $O(\mathbb{Z}^n)$ is all of $\mathbb{P}^{n-1}(\mathbb{R})$. The faithfulness of the $GL_n(\mathbb{Z})/Z(GL_n(\mathbb{Z}))$ action shows that $GL_n(\mathbb{Z})$ acts faithfully on $O(\mathbb{Z}^n)$, possibly modulo the center. The center of $GL_n(\mathbb{Z})$ consists of integral scalar matrices of determinant one, so that the center is $\pm I$. It is evident that the nontrivial element of the center of $GL_n(\mathbb{Z})$ has no fixed points in $O(\mathbb{Z}^n)$. □

Second proof of Lemma 3.1. We claim that in fact $GL_m(\mathbb{Z})$ acts faithfully on the set of flag orderings of $\mathbb{Z}^n$. Let $A \in GL_n(\mathbb{Z})$. Suppose A is a rational hyperplane preserved by A and v is a nonzero rational vector orthogonal to Z which sits in the positive half–space of some ordering determined by Z. If $A(v) \neq \lambda \cdot v$ for some $\lambda > 0$ then there is clearly an ordering which is not preserved by A. Indeed, either $A(v) = \lambda \cdot v$ for some $\lambda < 0$, in which case any order determined by Z is taken to an order which swaps the signs of the two half–spaces determined by Z, or there is a rational hyperplane in $\mathbb{R}^n$ which contains v and which does not contain $A(v)$. In the latter case, we can choose a flag ordering on $\mathbb{Z}^n$ for which v is positive but $A(v)$ is negative.

We may therefore assume that A preserves Z and the positive projective class of v. Since we are now free to order $\mathbb{Z}^n \cap Z$, we may repeat the same argument as above with a hyperplane Z' in Z and an orthogonal vector v'. By induction we may assume that we have a flag of hyperplanes

$$Z \supset Z' \supset \cdots \supset Z^{(n)} = \{0\}$$

and a sequence of nonzero rational vectors $\{v, v', \ldots, v^{(n-1)}\}$ orthogonal to the first $n-1$ of these, together with a nonzero vector $v^{(n)}$ in $Z^{(n-1)}$ such that $\{v, v', \ldots, v^{(n)}\}$ form a basis for $\mathbb{R}^n$ and such that the positive projective class of each of these vectors is preserved by A. It follows that up to conjugacy over $\mathbb{Q}$, A is a diagonal matrix with positive rational entries along the diagonal. Note that each entry of this matrix is an eigenvalue of A, which must therefore be a rational algebraic integer since A is an integral matrix. It follows that the entries are all integral, so that A is in fact the identity. □

Together with Lemma 3.1 concerning orderings on abelian groups, certain extension theorems for orderings on torsion–free nilpotent groups will be very important for the proof of Theorem 1.1. Up to this point in our discussion of orderings on groups, we have been considering positive cones which contain "half" of the nonidentity elements in a group. If we are given a positive cone $\mathcal{P}$ which is partial in the sense that $\mathcal{P} \cup \mathcal{P}^{-1}$ is properly contained in $G \setminus \{1\}$, we call $\mathcal{P}$ a **partial ordering**. A partial ordering $\mathcal{P}$ is bi–invariant if it is conjugation–invariant. We now quote the following two strong theorems, the first due to Rhemtulla in [Rh] and the second due to Mal'cev in [M] (see also [MR]):

Theorem 3.4. *Let N be a finitely generated torsion–free nilpotent group and P a partial ordering on N. Then P extends to a total ordering on N.*

Theorem 3.5. *Let N be a finitely generated torsion–free nilpotent group and P a bi–invariant partial ordering on N. Then P extends to a total bi–invariant ordering on N.*

Mal'cev actually proved that it suffices for N to be locally torsion–free nilpotent.

The two extension theorems above can be restated as follows:

Theorem 3.6. *Let N be a torsion–free nilpotent group and let $N' < N$ be a subgroup. Then the restriction maps*

$$\rho_L : LO(N) \to LO(N')$$

and

$$\rho_B : O(N) \to O(N')$$

are both surjective.

4. Representations of automorphism groups and the boundary of a hyperbolic group

In this section, we prove Theorem 1.1:

Theorem 4.1. *Let G be a residually torsion–free nilpotent group and let $\phi \in \mathrm{Aut}(G)$. Then ϕ acts nontrivially on $LO(G)$.*

Proof. Clearly we may suppose that ϕ acts trivially on $G^{ab} \otimes \mathbb{Q}$, since otherwise we may choose an ordering on G^{ab} which is not preserved by ϕ by Lemma 3.1, and then extend it to all of G by a standard ordering.

Suppose $\phi(g) \neq g$. Then there is a torsion–free nilpotent quotient N for which $\phi(g)g^{-1}$ is central and nontrivial. Choose an ordering on the abelian subgroup of N generated by g and $\phi(g)g^{-1}$ and extend it to an ordering on N using Rhemtulla's Extension Theorem. By Lemma 3.1, we see that there are orderings on the subgroup of N generated by g and $\phi(g)g^{-1}$ which are not preserved by ϕ. Since ϕ acts trivially on the center of N, this subgroup is ϕ–invariant, so that it makes sense to consider the action of ϕ on the orderings on that subgroup. Having chosen such an ordering, we can then extend it to all of G using the standard ordering construction. □

In the remainder of this section we shall develop an alternative viewpoint on Theorem 1.1 which makes the result more transparent, at least in the case of surface groups. Recall that a finitely generated group G is called **hyperbolic**, **Gromov hyperbolic** or **negatively curved** if there is a $\delta \geq 0$ such that whenever $g, h \in G$, any geodesic in G (with respect to the word metric) connecting g and h is contained in a δ–neighborhood of the union of two geodesics connecting the identity to g and h respectively. Being δ–hyperbolic is a quasi–isometry invariant, though the precise value of δ which witnesses δ–hyperbolicity depends on the generating set of G.

For basics on hyperbolic groups, the reader is referred to [G]. The property of hyperbolic groups we will be most interested in presently is the notion of the **Gromov boundary** of G, denoted ∂G. Recall that to define ∂G, we fix a basepoint in G and consider equivalence classes of geodesic rays emanating from the basepoint. Two geodesic rays are equivalent if they remain bounded distance from each other. Using the δ–hyperbolicity of G, it is possible to check that ∂G is independent of the basepoint.

If two geodesic rays agree along long initial segments, then they are close. It is therefore easy to produce a dense set of points in ∂G using the elements of G itself. Indeed, note that each primitive $g \in G$ gives rise to a point $x_g \in \partial G$ given by the geodesic ray determined by powers of g.

Lemma 4.2. *The set $\{x_g \mid g \in G\}$ is dense in ∂G.*

Proof. Let γ be an arbitrary geodesic ray. Then any initial segment of γ is an element of G. But then each longer initial segment g gives rise to an x_g which is closer to the equivalence class of γ in ∂G. □

The points $\{x_g\}$ should be thought of as the **rational points** in ∂G. The motivation for this terminology is taken from lattices in $\mathbb{R}^n$. Notions akin to the Gromov boundary can be defined for non–negatively curved metric spaces, such as $\mathbb{R}^n$. From $\mathbb{R}^n$ we obtain a natural boundary which is homeomorphic to S^{n-1}. In this same way, the boundary of $\mathbb{Z}^n$ should be thought of as S^n. Then the rational points on the boundary are obviously given by lines through the origin, all of whose slopes are rational.

When G is a surface group (i.e. the fundamental group of a closed surface of genus g), G is quasi–isometric to hyperbolic space $\mathbb{H}^2$, whose Gromov boundary is homeomorphic to the circle S^1. Some standard results in hyperbolic geometry show that the mapping class group Mod_g acts on the Gromov boundary (see for instance the book of Casson and Bleiler [CB]):

Lemma 4.3. *Let h be a homeomorphism of a closed, orientable surface of genus g and let h' be a lift to $\mathbb{H}^2$. Then h' has a unique continuous extension to the boundary of $\mathbb{H}^2$.*

Lemma 4.4. *Let h_0 and h_1 be two homotopic homeomorphisms of a closed, orientable surface of genus g and let h'_0 be a lift of h_0 to $\mathbb{H}^2$. Then there is a lift h'_1 of h_1 such that $h'_0 = h'_1$ on the boundary of $\mathbb{H}^2$.*

Recall that an element $1 \neq g \in G$ is called **primitive** if it cannot be written in the form $h^n = g$ for some $h \neq g$ and $n \neq \pm 1$. Our main observation in this section is the following:

Lemma 4.5. *Let G be a hyperbolic, residually torsion–free nilpotent, let $g \in G$ be primitive, let $\{\mathcal{P}_\alpha\}$ be the set of positive cones on G which contain g and let $\{\mathcal{P}'_\alpha\}$ be the set of bi–invariant positive cones on G which contain g. Then*

$$\bigcap_\alpha \mathcal{P}_\alpha = \{g^n \mid n > 0\}.$$

We require G to be hyperbolic, since then there are always primitive elements.

Proof. We must first check that this intersection is nonempty. Clearly, g is nontrivial in some torsion–free nilpotent quotient N of G. We may declare g to be positive, thus defining a partial ordering on N. By Rhemtulla's or Mal'cev's Theorems we can extend this partial ordering to all of N, and then to all of G. Therefore there is at least one positive cone which contains g.

Suppose that $h \neq g^n$ for all n. There is a torsion–free nilpotent quotient N of G where both g and h are nontrivial, and we may assume that the subgroup of N which they generate is abelian. If this subgroup is even virtually cyclic then we may find a larger torsion–free nilpotent quotient N' of G where g and h are nontrivial and differ by an element of the center of N'. We may declare h to be negative in N' and g to be positive, thus partially ordering N'. We may then extend this ordering to an ordering on N' and then on G. It follows that for every nonidentity element h which differs from every power of g, there is a positive cone $\mathcal{P}_h$ on G which contains g but not h. □

It follows that the space $LO(G)$ recovers the Gromov boundary of a hyperbolic group.

Proof of Theorem 1.2. Suppose $\text{Aut}(G)$ acts faithfully on ∂G. Then it must act faithfully on the rational points of ∂G. If a nontrivial automorphism ϕ preserves each left–invariant ordering on G, then ϕ preserves each subset of G of the form $\{g^n \mid n > 0\}$ by Lemma 4.5. But then ϕ preserves each rational point in ∂G, a contradiction.

Conversely, suppose $\text{Aut}(G)$ acts faithfully on the set of left–invariant orderings on G. Then for each $\phi \in \text{Aut}(G)$ there is an ordering $\mathcal{P}$ and a $g \in \mathcal{P}$ such that $g \notin \phi(\mathcal{P})$. It follows that g and all of its powers are negative in the ordering $\phi(P)$. It follows that $\text{Aut}(G)$ acts faithfully on the rational points in ∂G and hence on ∂G. □

5. Homology, orderings, residual finiteness and faithful representations

In this section we will make some remarks about homology representations of $\text{Out}(G)$, $O(G)$ and residual finiteness. Recall that a group is **conjugacy separable with respect to a class of groups** $\mathcal{K}$ if whenever g and h are not conjugate in G, there is a quotient $\overline{G} \in \mathcal{K}$ wherein the images of g and h are not conjugate. It is classical that surface groups and free groups are conjugacy separable with respect to the class of p–groups for any prime p. For a proof of this fact, the reader should consult Lyndon and Schupp's book [LS]. It is easy to see then that free and surface groups are conjugacy separable with respect to the class of torsion–free nilpotent groups.

We say that a group has **property C** if whenever an automorphism ϕ of G preserves the conjugacy class of each element of G, then ϕ is inner. The fact that free and surface groups satisfy property C was first demonstrated by E. Grossman in [Gr] in the course of her proof that mapping class groups are residually finite.

It would be nice if we could formulate and prove analogous theorems to Theorems 1.1 and 1.2 for the action of $\text{Out}(G)$ on $O(G)$, but unfortunately we encounter various difficulties. The proofs as they are given for $LO(G)$ will not work for $O(G)$. One difficulty is the following: any residually finite group has a residually finite automorphism group. On the other hand, it is not true that each residually finite group has a residually finite outer automorphism group. In fact, Wise proves in [W] that every finitely generated group embeds in the outer automorphism group of some residually finite group.

Whether or not $\mathrm{Out}(G)$ is residually finite for some group G is a question about the closure of G inside $\mathrm{Aut}(G)$ in the profinite topology. In this paper, we have been restricting the topology on $\mathrm{Aut}(G)$ even more by using the pro–nilpotent topology, thus making it even more difficult for G to be closed in $\mathrm{Aut}(G)$. The reason we care about the residual finiteness of $\mathrm{Out}(G)$ is that given a non–inner automorphism ϕ and $g \in G$ such that $\phi(g)$ is not conjugate to g, we can always find some finite (or even better, nilpotent) quotient of G where the images of g and $\phi(g)$ are not conjugate. Thus, unless we have good control over the algebraic properties of $\mathrm{Out}(G)$, even knowing that G has property C will not help us distinguish inner automorphism from non–inner automorphisms on proper quotients of G.

Forgetting automorphisms for the moment, suppose we simply want to construct a bi–invariant ordering on G where two non–conjugate elements g and h are separated in the sense that one is positive and the other is negative. Suppose that we have a nilpotent quotient N of G where g and h remain non–conjugate. Note that this uses some sort of conjugacy separability of G. Suppose furthermore that when we quotient out N by its center $Z(N)$, g and h become conjugate to each other. In that case we can write $h = h'z$, where h' is conjugate to g and $z \in Z(N)$. We might then try to order the group generated by g and z and maybe its conjugates as well, in such a way so that $g > 1$ and $h < 1$. It is conceivable on the other hand that for some n, $h'z^n$ is conjugate to g, in which case $h'z^n$ is also positive. It is easy to check that then it is impossible to choose an ordering which meets our specifications, using the theory of orderings on abelian groups we developed above.

For certain residually torsion–free nilpotent groups however, it is possible to make $\mathrm{Out}(G)$ act faithfully on $O(G)$ just by exploiting the fact that the homology representation

$$\mathrm{Out}(G) \to \mathrm{Out}(H_1(G, \mathbb{Q})$$

is faithful. Consider $\mathrm{Out}(A_\Gamma)$, where Γ is a finite graph and A_Γ is the associated right-angled Artin group. Recall that this is the free group on the vertices of Γ together with the commutation relations between vertices whenever they are connected by an edge. See Charney's expository article [C] for instance.

Whereas abelian and free groups have large and very complicated automorphism groups, it is often the case that right-angled Artin groups have finite outer automorphism groups. In fact, Charney and Farber have recently proved in [ChFar] that a "generic" right-angled Artin group has a finite outer automorphism group.

Consider the following automorphisms of right-angled Artin groups (see Laurence's article [Lau], also Day's article [Day], for instance):

(1) Automorphisms induced by isomorphisms of Γ.
(2) Inversions of the vertices of Γ.

(3) **Dominated transvections**: we say that a vertex y of Γ dominates a vertex x if the link of x is contained in the star of y. The map which sends x to xy and fixes the rest of the vertices is an automorphism of A_Γ.
(4) **Partial conjugacies**: suppose the star of a vertex x separates Γ into components $\Gamma_1, \ldots, \Gamma_k$. Replacing the vertices in a union of these components by their conjugates by x is an automorphism of A_Γ.

It had been a conjecture of Servatius in [Se] and is a theorem of Laurence that these four types of automorphisms generate all of $\mathrm{Aut}(A_\Gamma)$. Suppose Γ is a finite graph with no separating stars. Then each partial conjugacy is actually inner. If there is no pair of vertices in Γ with the link of one contained in the star of the other, there are no dominated transvections in $\mathrm{Aut}(A_\Gamma)$. An example of a graph Γ with no transvections and no non–inner partial conjugacies is an n–cycle, $n \geq 5$. Note also that inversions of the vertices and automorphisms induced by isomorphisms of Γ all act nontrivially on the homology of A_Γ and are thus non–inner. We thus immediately obtain the following:

Corollary 5.1. *Let A_Γ be a right-angled Artin group admitting no dominated transvections and no non–inner partial conjugacies. Then* $\mathrm{Out}(A_\Gamma)$ *acts faithfully on $H_1(A_\Gamma, \mathbb{Q})$. In particular,* $\mathrm{Out}(A_\Gamma)$ *acts faithfully on the subset of $O(G)$ consisting of standard orderings.*

6. Some remarks on orderings of nilpotent groups

Let N be a finitely generated torsion–free nilpotent group. In general, it is an open problem to describe the set of all bi–invariant orderings on an arbitrary torsion–free nilpotent group (see Example 1.4.1 in Navas' notes [N2]). Theorem 2.2.19 in the same notes shows that every non–cyclic torsion–free nilpotent group has a Cantor set worth of left–invariant orderings. In some simpler cases, one can describe the space of orderings. We will do this here for the standard integral Heisenberg group

$$H = \langle x, y, z \mid [x, y] = z,\ [x, z] = [y, z] = 1 \rangle.$$

First, we shall illustrate a well–known method for producing bi–invariant orderings on torsion–free nilpotent groups, and indeed on residually torsion–free nilpotent groups. This method can be found in the book [DDRW], and the interested reader should consult the references therein. Recall that the **lower central series** of a group G is defined by $\gamma_1(G) = G$, $\gamma_{i+1}(G) = [\gamma_i(G), G]$. A group is nilpotent if $\gamma_n(G) = \{1\}$ for some n, and is residually nilpotent (sometimes called ω–nilpotent) if

$$\bigcap_{i=1}^{\infty} \gamma_i(G) = \{1\}.$$

For more details about the lower central series and nilpotent groups in general, see Raghunathan's book [R] for example.

For the remainder of the construction, we will assume that G is residually nilpotent and that the quotients $\gamma_i(G)/\gamma_{i+1}(G)$ are all torsion–free. If G is residually torsion–free nilpotent then the quotients $\gamma_i(G)/\gamma_{i+1}(G)$ may not be torsion–free, but one can modify the construction with little difficulty to accommodate that case.

Choose an arbitrary ordering on each $A_i = \gamma_i(G)/\gamma_{i+1}(G)$. If $g, h \in G$, we consider $g^{-1}h$. Consider the minimal i so that $g^{-1}h \in \gamma_i(G) \setminus \gamma_{i+1}(G)$. We set $g \leq h$ if and only if the image of $g^{-1}h$ in A_i is positive. Since each A_i is equipped with a trivial conjugation action by G, it is immediately clear that all these orderings are bi–invariant.

An ordering on a residually torsion–free nilpotent group which arises in this way shall be called **standard**. All other orderings will be called **exotic**. Standard orderings are not good for the study of the action of $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ on orderings, since if ϕ is an automorphism which acts trivially on $H_1(G, \mathbb{Z})$, ϕ preserves all standard orderings. This results from a standard result on the lower central series of a group (see the paper [BL] of Bass and Lubotzky for instance):

Lemma 6.1. *Let $\phi \in \mathrm{Aut}(G)$ and suppose that ϕ acts trivially on $H_1(G, \mathbb{Z}) = G^{ab}$. Then ϕ acts trivially on $\gamma_i(G)/\gamma_{i+1}(G)$ for all i.*

Proof. The proof is by induction on commutator depth. Suppose that $a \in \gamma_{i-1}(G)$ and $t \in G$ is arbitrary. We may assume that $\phi(t) = t \cdot c$ where $c \in [G, G]$ and $\phi(a) = a \cdot c'$, where $c' \in \gamma_i(G)$. The commutator $[t, a]$ lies in $\gamma_i(G)$, and it suffices to show that $\phi([t, a])$ differs from $[t, a]$ by an element of $\gamma_{i+1}(G)$. We have

$$\phi([t, a]) = \phi(t)^{-1}\phi(a)^{-1}\phi(t)\phi(a) = c^{-1}t^{-1}c'^{-1}a^{-1}tcac'.$$

Switching t^{-1} with c'^{-1} perturbs c'^{-1} by an element of $\gamma_{i+1}(G)$, since $c' \in \gamma_i(G)$. So, there is a $c'' \in \gamma_{i+1}(G)$ such that

$$\phi([t, a]) = c''c^{-1}t^{-1}a^{-1}tcac'.$$

Since $c \in [G, G]$ already, conjugating $t^{-1}a^{-1}t$ by c perturbs it by an element of $\gamma_{i+1}(G)$. The claim follows. □

Standard orderings will allow us to extend orderings on nilpotent quotients of a group G to G in the following manner: assume we are given a (bi–invariant) ordering on $G/\gamma_i(G)$. Choosing an arbitrary ordering on $\gamma_j(G)/\gamma_{j+1}(G)$ for $j \geq i$ will give rise to a (bi–invariant) ordering on G.

Proposition 6.2. *Every bi–invariant ordering on the integral Heisenberg group H is standard.*

Before we produce a proof of Proposition 6.2, we will need the notion of an **isolated ordering**. It is important not to confound an isolated ordering in the context here with an isolated ordering in the topology of $LO(G)$ or $O(G)$, which is to say simply an isolated point. An ordering $\mathcal{P}$ is isolated if for each $g \in G$, $g^n \in \mathcal{P}$ for $n > 0$ implies $g \in \mathcal{P}$. It is easy to see that orderings on $\mathbb{Z}^n$ which are induced either by rational flags or irrational hyperplanes are all isolated.

Proof of Proposition 6.2. We use the generators x, y and z above for H. There are two obvious choices for maximal abelian subgroups of H, which we write $M_x = \langle x, z\rangle$ and $M_y = \langle y, z\rangle$. Note that since $[x, y] = z$, both of these subgroups are normal. Suppose that $\mathcal{P} \in O(H)$ is a bi–invariant ordering. Clearly $\mathcal{P}$ restricts to an ordering on M_x and M_y. Furthermore, if $\mathcal{P}$ contains x, it also contains xz^n for all $n \in \mathbb{Z}$, by bi–invariance. More generally, if $x^a y^b \in \mathcal{P}$, so is $x^a y^b z^c$, where c is any integral linear combination of a and b. If $\gcd(a,b) = \pm 1$ then c can be any integer. If $\gcd(a,b) = d \neq \pm 1$ then we can use the fact that any ordering on $\langle x^a y^b, z\rangle$ is isolated to conclude that if $x^a y^b \in \mathcal{P}$ then so is $x^{a/d} y^{b/d}$.

Let $Z = \langle z \rangle$. It follows that $\mathcal{P}Z$ is a positive cone on H/Z and hence $\mathcal{P}$ descends to an ordering on H/Z. The proposition follows. □

There are often more orderings on torsion–free nilpotent groups than just the standard ones. Indeed, the following will be an immediate consequence of the proof of Theorem 1.1:

Corollary 6.3. *There exist finitely generated torison–free nilpotent groups with nonstandard bi–invariant orderings.*

We remark that the set of bi–invariant orderings is properly contained in the set of left–invariant orderings by a result of Darnel, Glass and Rhemtulla in [DGR].

7. Some final examples

It is not true in general that $\mathrm{Aut}(G)$ acts faithfully on the left orderings $LO(G)$, nor is it true that $\mathrm{Out}(G)$ acts faithfully on the conjugacy classes in $LO(G)$ or on $O(G)$. Consider, for instance, the fundamental group K of the Klein bottle. We have the presentation

$$K = \langle x, y \mid x^{-1} y x = y^{-1} \rangle.$$

If P is an ordering on K then P is certainly not bi-invariant. Indeed, either $y \in P$ or $y \in P^{-1}$, but conjugation by x takes y to y^{-1}.

It is known that K admits exactly four left-invariant orderings. $\mathrm{Aut}(K)$ is infinite, so it cannot act faithfully on the space of left-invariant orderings $LO(K)$. On the other hand, it is obvious that there are no more than two conjugacy classes of orderings on K by the remarks above. However, $|\mathrm{Out}(K)| = 4$, so that $\mathrm{Out}(K)$ cannot act faithfully on conjugacy classes in $LO(K)$. We can compute the outer automorphism group of K in two different ways. The first illustrates that $\mathrm{Out}(K)$ can be computed directly from the presentation for K we have above, which is generally a rarity for groups:

Proposition 7.1. $\mathrm{Out}(K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Proof. We first note that $y^{-1} x y = x y^2$, and this allows us to completely understand inner automorphisms of K. Let α be an automorphism of K. Then, $\alpha : x \mapsto w_x$ and $\alpha : y \mapsto w_y$. We evidently must have

$$w_x^{-1} w_y w_x = w_y^{-1}.$$

Given any element $w \in K$, there is a well-defined (which is to say conjugation-invariant) notion of the x-exponent sum of w. It follows that the x-exponent sum of w_x must be odd. Furthermore, there can be no occurrences of x in w_y. It follows that the x-exponent sum of w_x must be ± 1. It follows that there are integers m, n, k with $k \neq 0$ such that under the action of α, $x \mapsto y^m x^{\pm 1} y^n$, and $y \mapsto y^k$.

We immediately obtain three non-inner automorphisms of K, namely $\alpha_1 : x \mapsto xy$, $\alpha_2 : x \mapsto yx$, and $\alpha_3 : x \mapsto x^{-1}$, where these are extended to K by letting them fix y in the first two cases and $\alpha_3 : y \mapsto y^{-1}$. Clearly α_3 is distinct from both α_1 or α_2 in $\mathrm{Out}(K)$, but α_1 and α_2 differ by an inner automorphism. Both α_3 and α_1 have order 2 in $\mathrm{Out}(K)$, which already shows that $\mathrm{Out}(K)$ cannot act faithfully on conjugacy classes of orderings on K. The definition of α_3 is such that $\alpha_1 \circ \alpha_3 : x \mapsto y^{-1}x^{-1}$ and $y \mapsto y^{-1}$. On the other hand, $\alpha_3 \circ \alpha_1 : x \mapsto x^{-1}y^{-1}$ and $y \mapsto y^{-1}$, so that α_1 and α_3 commute up to an inner automorphism.

It suffices to show that $k = \pm 1$. Let $\alpha(y) = y^k$ and $\alpha(x) = y^m x y^n$. We must be able to write

$$y = y^{a_1 k}(y^m x y^n)^{b_1} \cdots y^{a_i k}(y^m x y^n)^{b_i}$$

for some integers $a_1, \ldots, a_i$ and $b_1, \ldots b_i$. Clearly,

$$\sum_{j=1}^{i} b_j = 0.$$

It follows that there are powers of y nested between x, x^{-1}-pairs. The group law in K allows us to dispose of these pairs by considering innermost x, x^{-1} pairs, i.e. ones that only have powers of y separating them in the expansion of y above. An easy computation shows that between an innermost x, x^{-1} pair there is a word of the form $y^n y^{a_j k} y^{-n}$. The conjugation law allows us to delete the x, x^{-1} pair and replace $y^n y^{a_j k} y^{-n}$ by $y^{-a_j k}$. We then have a y^m, y^{-m} pair surrounding $y^{-a_j k}$ which cancels.

An easy induction shows that we can reduce the expansion of y above to a power of y whose exponent is divisible by k, the desired conclusion. It follows that the automorphisms α_1 and α_3 generate $\mathrm{Out}(K)$. □

Another way to compute $\mathrm{Out}(K)$ relies on the observation that K contains a copy of $\mathbb{Z}^2$ with index 2. We thus have a homomorphism $K \to \mathbb{Z}/2\mathbb{Z}$ with kernel $\mathbb{Z}^2$. One can argue that any automorphism of K must commute with the deck group action on $\mathbb{Z}^2$ and conclude that $\mathrm{Out}(K)$ can be identified with the Klein 4–group, which is embedded in $GL_2(\mathbb{Z})$ via diagonal matrices with ± 1 along the diagonals.

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