# Comparison of Dualizing Complexes

Changlong Zhong

**Abstract**

We prove that there is a map from Bloch's cycle complex to Kato's complex of Milnor K-theory, which induces a quasi-isomorphism from étale sheafified cycle complex to the Gersten complex of logarithmic de Rham–Witt sheaves. Next we show that the truncation of Bloch's cycle complex at $-3$ is quasi-isomorphic to Spiess' dualizing complex.

## 1 Introduction

Using Lichtenbaum's weight-two motivic complex, M. Spiess [22] constructed a complex of étale sheaves $\mathcal{K}_X$ on arithmetic surfaces and used it to prove a duality of constructible sheaves. T. Moser [17] also defined a Gersten type complex of logarithmic de Rham–Witt sheaves $\widetilde{\nu}_{r,X}$ and showed that it is a dualizing complex for constructible $\mathbb{Z}/p^r$-sheaves. In more general cases, for instance, schemes over algebraically closed fields, finite fields, local fields and some Dedekind domains, T. Geisser [7] proved that the étale sheafified version of Bloch's cycle complex $\mathbb{Z}^c_X$ of relative dimension 0 [2] is a dualizing complex for constructible sheaves. A natural question arises: are these complexes quasi-isomorphic to each other?

**Theorem 1.1** (Main Theorem 1)**.** *For $X$ a scheme separated and essentially of finite type over a perfect field $k$ of characteristic $p > 0$, there is a map $\hat{\psi} : \mathbb{Z}^c_X/p^r \to \widetilde{\nu}_{r,X}$ which is a quasi-isomorphism.*

**Theorem 1.2** (Main Theorem 2)**.** *Let $X$ be a surface over a field $k$, then $\tau_{\geq -3}\mathbb{Z}^c_X$ is quasi-isomorphic to $\mathcal{K}_X$. If $X$ is smooth, assuming the Beilinson–Soulé Conjecture for smooth surfaces, then $\mathbb{Z}^c_X$ is quasi-isomorphic to $\mathcal{K}_X$.*

To compare Bloch's complex with Moser's complex, first note that the niveau spectral sequence of higher Chow groups induces a canonical map $\phi : \mathbb{Z}^c_X(X) \to$

$C_*^{\mathrm{HC}}(X)$, the latter being the Gersten complex of higher Chow groups. After showing that $C_*^{\mathrm{HC}}(X)$ is isomorphic to Kato's complex of Milnor K-theory $C_*^{\mathrm{M}}(X)$ [14] via the Nesterenko–Suslin isomorphism [19], we obtain a map $\psi : \mathbb{Z}_X^c(X) \to C_*^{\mathrm{M}}(X)$, which is similar to a map defined by Landsburg ([15], p.621). When composing with the isomorphism between Milnor K-theory and global sections of logarithmic de Rham–Witt sheaves of fields (Bloch–Kato, [3], Theorem 2.1), $\psi$ induces a map $\hat{\psi} : \mathbb{Z}_X^c/p^r \to \widetilde{\nu}_{r,X}$. Moreover, it induces an isomorphism of their cohomology groups by a result of Geisser–Levine ([9], Theorem 1.1). Hence we conclude that $\hat{\psi}$ is a quasi-isomorphism.

In this proof, one of the main tools is the Nesterenko–Suslin isomorphism $\chi_F : CH_0(F,n) \to K_n^M(F)$ of a field $F$ ([19], or see Definition 2.5). Another property is that, in Moser's definition of $\widetilde{\nu}_{r,X}$, the differentials are defined so that they are compatible with $C_*^{\mathrm{M}}(X)$ via the Bloch–Kato isomorphism $K_n^M(F)/p^r \cong \nu_{r,F}^n(F)$ ([3], Theorem 2.1). However, one can also define differentials in $\widetilde{\nu}_{r,X}$ from the niveau filtrations of the logarithmic de Rham–Witt sheaf $\nu_{r,X}^d$. According to Gros–Suwa ([10], Lemma 4.11), these two definitions coincide, hence $C_*^{\mathrm{M}}(X)/p^r \cong \widetilde{\nu}_{r,X}(X)$ (see Jannsen-Saito-Sato, [11], Theorem 2.1.1 and Theorem 2.11.3(3) for a more detailed proof).

To prove Main Theorem 2, we define an intermediate complex $\mathcal{C}_X$ and show that it is quasi-isomorphic to $\tau_{\geq -3}\mathbb{Z}_X^c$ and $\mathcal{K}_X$, respectively. Hence, $\tau_{\geq -3}\mathbb{Z}_X^c$ is quasi-isomorphic to $\mathcal{K}_X$. Assuming the Beilinson–Soulé Conjecture for smooth surfaces, we can drop the truncation and conclude that $\mathbb{Z}_X^c$ is quasi-isomorphic to $\mathcal{K}_X$. A key ingredient in this proof is the quasi-isomorphism between a truncation of Bloch's complex and $\mathbb{Z}(F,2)$ of a field $F$, which induces an isomorphism $\theta : CH_0(F,2) \to K_2(F)$ ([4], §7). Here $\mathbb{Z}(F,2)$ is Lichtenbaum's weight-two motivic complex (see [16], Definition 2.1). We prove that $\theta$ agrees with an edge morphism $\theta' : CH_0(F,2) \to K_2(F)$ of the spectral sequence relating motivic cohomology and Quillen K-theory of fields [4]. Therefore, according to [9], Proposition 3.3, $\theta = \theta' = s \circ \chi_F : CH_0(F,2) \xrightarrow{\chi_F} K_2^M(F) \xrightarrow{s} K_2(F)$, where $s$ is the Steinberg symbol.

The paper is organized as follows: in Section 2, we recall the definitions of Bloch's cycle complex and Moser's complex, as well as the duality results of T. Geisser and T. Moser. We also recall the construction of the niveau spectral sequence of higher Chow groups. This spectral sequence, together with the Nesterenko–Suslin isomorphism, induces the map of complexes $\psi : \mathbb{Z}_X^c(X) \to C_*^{\mathrm{M}}(X)$. When composing with the Bloch–Kato isomorphism, $\psi$ induces the quasi-isomorphism $\hat{\psi} : \mathbb{Z}_X^c/p^r \to \widetilde{\nu}_{r,X}$. As another application of this method, we show that, for smooth and projective varieties over finite fields, conjecture A(0) of Geisser (part of Parshin's Conjecture, see [8], Proposition 2.1) is true, if and only if $\psi$ is a quasi-isomorphism.

In Section 3, first we recall the dualizing complex $\mathcal{K}_X$ of sheaves on surfaces defined by M. Spiess and his duality results. Next we construct a new complex $\mathcal{C}_X$ and show that it is quasi-isomorphic to $\tau_{\geq -3}\mathbb{Z}_X^c$, as well as to $\mathcal{K}_X$. Consequently, $\tau_{\geq -3}\mathbb{Z}_X^c$ is quasi-isomorphic to $\mathcal{K}_X$.

**Terminology:** Throughout this paper, the concepts chain complex and cochain complex are used interchangeably. For instance, if $A$ is a chain complex, we think of it as a cochain complex by letting $A^n = A_{-n}$. We use $X$ to denote a scheme separated and of finite type over a perfect field $k$ with characteristic $p \geq 0$, $X_{(n)}$ to denote the set of dimension $n$ points of $X$, and $d = \dim(X)$.

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# 2 Comparison between Bloch's Complex and Moser's Complex

Following the notation in [7], we define $z_0(X, n)$ to be the free abelian group generated by cycles $Z \subset X \times \Delta^n$ that intersect all the faces properly and $k(Z)$ has transcendental degree $n$ over $k$. Then $z_0(\_, n)$ is an étale sheaf. Let $\mathbb{Z}_X^c$ be Bloch's cycle complex of étale sheaves (of relative dimension 0) ([2], [7]):

$$\to z_0(\_, n) \xrightarrow{d} \ldots \to z_0(\_, 1) \to z_0(\_, 0) \to 0,$$

and

$$d(Z) = \sum_i (-1)^i [Z \cap V(t_i)],$$

where $[Z \cap V(t_i)]$ denotes the linear combination of irreducible components of $Z \cap V(t_i)$ with coefficients intersection multiplicities. We put $z_0(\_, n)$ in (homological) degree $n$, and define the motivic Borel–Moore homology to be

$$H_n^c(X, \mathbb{Z}) \stackrel{def}{=} H_n(\mathbb{Z}_X^c(X)) = CH_0(X, n).$$

The complex $\mathbb{Z}^c_X(X)$ is covariant for proper maps and contravariant for quasi-finite, flat maps.

**Theorem 2.1** (Geisser, [7], §5)**.** *Let $k$ be a finite field and $\mathcal{F}$ be a constructible sheaf on $X$, then there are perfect pairings of finite groups*

$$H^i_c(X_{\text{ét}}, \mathcal{F}) \times \operatorname{Ext}^{2-i}_X(\mathcal{F}, \mathbb{Z}^c_X) \to H^2_c(X_{\text{ét}}, \mathbb{Z}^c_X) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

**Theorem 2.2** (Geisser, [7], §5)**.** *Let $k$ be an algebraically closed field and $\mathcal{F}$ be a constructible sheaf on $X$. Then there are perfect pairings of finitely generated groups*

$$H^i_c(X_{\text{ét}}, \mathcal{F}) \times \operatorname{Ext}^{1-i}_X(\mathcal{F}, \mathbb{Z}^c_X) \to H^1_c(X_{\text{ét}}, \mathbb{Z}^c_X) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

If $p > 0$, we define $\widetilde{\nu}_{r,X}$ to be the Gersten complex of logarithmic de Rham–Witt sheaves:

$$0 \to \bigoplus_{X_{(d)}} i_{x*}\nu^d_{r,k(x)} \to ... \to \bigoplus_{X_{(1)}} i_{x*}\nu^1_{r,k(x)} \to \bigoplus_{X_{(0)}} i_{x*}\nu^0_{r,k(x)} \to 0.$$

Here for a scheme $X$ over $k$, $\nu^n_{r,X}$ is the the logarithmic de Rham–Witt sheaf $W_r\Omega^n_{X,\log}$, i.e., the subsheaf of $W_r\Omega^n_X$ generated by $d\log f_1 \wedge ... \wedge d\log f_n$, and $i_{x*}$ is the push-forward map of sheaves defined by $i_x : \operatorname{Spec}(k(x)) \to X$. We put $\bigoplus_{X_{(n)}} i_{x*}\nu^n_{r,k(x)}$ in (homological) degree $n$. The differentials are induced from the niveau filtrations of the complex $\nu^d_{r,X}$.

**Theorem 2.3** (Moser, [17], Theorem 5.6)**.** *Let $k$ be a finite field of characteristic $p$ and $X$ be a $k$-scheme of pure dimension $d$. Then for every $r \geq 1$ and every constructible $\mathbb{Z}/p^r$-sheaf $\mathcal{F}$, there are perfect pairings of finite groups*

$$H^i_c(X_{\text{ét}}, \mathcal{F}) \times \operatorname{Ext}^{2-i}_X(\mathcal{F}, \widetilde{\nu}_{r,X}) \to H^2_c(X_{\text{ét}}, \widetilde{\nu}_{r,X}) \xrightarrow{\sim} \mathbb{Z}/p^r.$$

Let $C^{\mathrm{M}}_*(X)$ be Kato's complex of Milnor K-theory (cf. K. Kato, [14]):

$$0 \to \bigoplus_{X_{(d)}} K^M_d(k(x)) \xrightarrow{d'} ... \to \bigoplus_{X_{(1)}} K^M_1(k(x)) \to \bigoplus_{X_{(0)}} K^M_0(k(x)) \to 0.$$

The differential $d'$ is defined as follows: for any $x \in X_{(n)}$ and any $y \in \overline{\{x\}} \cap X_{(n-1)}$, we take the normalization $\overline{\{x'\}}$ of $\overline{\{x\}}$ with $x'$ its generic point, and define a map

$$\partial_y : K^M_n(k(x)) = K^M_n(k(x')) \xrightarrow{\sum \partial_{y'}} \bigoplus_{y'|y} K^M_{n-1}(k(y')) \xrightarrow{\sum N_{k(y')/k(y)}} K^M_{n-1}(k(y)).$$

Here the notation $y'|y$ means that $y' \in \overline{\{x'\}}_{(n-1)}$ is in the fiber of $y$,

$$N_{k(y')/k(y)} : K^M_{n-1}(k(y')) \to K^M_{n-1}(k(y))$$

is the norm map of Milnor K-theory (see Bass–Tate, [1] and Kato, [13], Section 1.7), and $\partial_{y'}$ is the tame symbol defined by $y'$. Then $d' \overset{def}{=} \sum_{y \in X_{(n-1)} \cap \overline{\{x\}}} \partial_y$. Note that the sum in $\partial_y$ is finite since elements in $K^M_n(k(x'))$ are represented by $n$ elements in $k(x')^*$, and each element in $k(x')^*$ has a finite number of poles and zeros. When applying the tame symbol, only a finite number of terms in the sum are non-zero. We put $\bigoplus_{X_{(n)}} K^M_n(k(x))$ in (homological) degree $n$. The complex $C^{\mathrm{M}}_*(X)$ is covariant for proper maps and contravariant for quasi-finite and flat maps (see Rost, [21], Proposition 4.6(1),(2)).

**Theorem 2.4** (Bloch–Kato, Gros–Suwa, Moser)**.**

$$C^{\mathrm{M}}_*(X)/p^r \cong \widetilde{\nu}_{r,X}(X). \tag{2.0.1}$$

*Proof.* By Bloch–Kato, [3], Theorem 2.1, for any field $F$, there is an isomorphism

$$K^M_n(F)/p^r \cong \nu^n_{r,F}(F)$$

sending $\{f_1, ..., f_n\}$ to $d\log f_1 \wedge ... \wedge d\log f_n$. By [10], Lemma 4.1, this isomorphism respects the differentials in $C^{\mathrm{M}}_*(X)$ and $\widetilde{\nu}_{r,X}(X)$. Hence it induces an isomorphism of complexes $C^{\mathrm{M}}_*(X)/p^r \to \widetilde{\nu}_{r,X}(X)$. $Q.E.D.$

*The Niveau Spectral Sequence:* Let us recall the construction of the niveau spectral sequence of higher Chow groups with $\mathbb{Z}$-coefficients. Let $p$ be the projection $X \times \Delta^n \to X$. Let $F_s(n) \subset z_0(X, n)$ be generated by cycles $Z$ with $\dim p(Z) \leq s$, and $F_s$ be the corresponding subcomplex of $\mathbb{Z}^c_X(X)$. There is a short exact sequence of complexes:

$$0 \to F_{s-1} \to F_s \to F_s/F_{s-1} \to 0,$$

which induces a long exact sequence of abelian groups

$$\to H_{s+t+1}(F_s/F_{s-1}) \to H_{s+t}(F_{s-1}) \to H_{s+t}(F_s) \to H_{s+t}(F_s/F_{s-1}) \to . \tag{2.0.2}$$

Moreover, by the localization property of higher Chow groups, we have

$$H_{s+t}(F_s/F_{s-1}) \cong \bigoplus_{X_{(s)}} H^c_{s+t}(k(x), \mathbb{Z}).$$

Therefore, there is a spectral sequence:

$$E^1_{s,t} = \bigoplus_{X_{(s)}} H^c_{s+t}(k(x), \mathbb{Z}) \Rightarrow H^c_{s+t}(X, \mathbb{Z}). \tag{2.0.3}$$

From the construction we know that

$$H^c_s(X, \mathbb{Z}) = \varinjlim(... \to H_s(F_{s-1}) \to H_s(F_s) \twoheadrightarrow H_s(F_{s+1}) \to ...)$$

which induces a filtration

$$N_s H^c_n(X, \mathbb{Z}) \overset{def}{=} \mathrm{Im}(H_n(F_s) \to H^c_n(X, \mathbb{Z})).$$

This filtration $N.H^c_n(X, \mathbb{Z})$ is called the niveau filtration of higher Chow groups. The complex $C^{\mathrm{HC}}_*(X) \overset{def}{=} E^1_{*,0}$ is the Gersten complex of higher Chow groups and the differentials $d''$ in $E^1_{*,0}$ are induced by the localization property of $\mathbb{Z}^c_X(X)$. Moreover, there is a map of complexes $\phi : \mathbb{Z}^c_X(X) \to C^{\mathrm{HC}}_*(X)$ which induces the edge morphisms of the spectral sequence. More specifically, $\phi$ satisfies the following properties:

i) if $n > d$, $\phi_n = 0$;

ii) if $n \le d$, $\phi_n$ is the composition:

$$z_0(X, n) = F_n(n) \twoheadrightarrow \frac{F_n(n)}{F_{n-1}(n)} \xrightarrow{\sim} \bigoplus_{X_{(n)}} z_0(k(x), n) \twoheadrightarrow \bigoplus_{X_{(n)}} H^c_n(k(x), \mathbb{Z}).$$

For any abelian group $A$, there is the niveau spectral sequence of higher Chow groups with $A$-coefficients as well.

To connect $C^{\mathrm{HC}}_*(X)$ with $C^{\mathrm{M}}_*(X)$, we need the Nesterenko–Suslin isomorphism.

**Definition 2.5** (Nesterenko–Suslin, [19], Theorem 4.9)**.** *For any generator $z \in z_0(F, n)$, the Nesterenko–Suslin isomorphism $\chi_F : CH_0(F, n) \to K^M_n(F)$ is defined so that $\chi_F(\bar{z}) = N(\beta_z)$, where $\bar{z}$ is the image of $z$ in $CH_0(F, n)$, $N : K^M_n(k(z)) \to K^M_n(F)$ is the norm map of Milnor K-theory, and*

$$\beta_z = \{\frac{-t_0}{t_n}, ..., \frac{-t_{n-1}}{t_n}\} \in K^M_n(k(z)).$$

*Here $t_i$'s are the coordinates of $z$ in $\Delta^n_F$. Since $z$ intersects all the faces properly, $t_i \in k(z)^*$.*

**Lemma 2.6.** *The Nesterenko–Suslin isomorphism induces an isomorphism of complexes* $\chi : C_*^{HC}(X) \to C_*^{M}(X)$.

*Proof.* The Gersten complex of higher Chow group $C_*^{\mathrm{HC}}(X)$ is the following complex:

$$0 \to \bigoplus_{X_{(d)}} H_d^c(k(x), \mathbb{Z}) \xrightarrow{d''} \dots \to \bigoplus_{X_{(1)}} H_1^c(k(x), \mathbb{Z}) \to \bigoplus_{X_{(0)}} H_0^c(k(x), \mathbb{Z}) \to 0.$$

To prove the lemma, it suffices to show that the following diagram is commutative:

$$\begin{array}{ccc} \bigoplus_{X_{(n)}} H_n^c(k(x), \mathbb{Z}) & \xrightarrow{d''} & \bigoplus_{X_{(n-1)}} H_{n-1}^c(k(x), \mathbb{Z}) \\ \downarrow \chi & & \downarrow \chi \\ \bigoplus_{X_{(n)}} K_n^M(k(x)) & \xrightarrow{d'} & \bigoplus_{X_{(n-1)}} K_{n-1}^M(k(x)). \end{array}$$

Let $X' \to X$ be the normalization of $X$, and $x$ be a codimension 1 point of $X$. Consider the following commutative diagram:

$$\begin{array}{ccc} CH_0(k(X'), n) & \xrightarrow{d''_{X'}} & \bigoplus_{x'|x} CH_0(k(x'), n-1) \\ \Vert & & \downarrow \sum N \\ CH_0(k(X), n) & \xrightarrow{d''_X} & CH_0(k(x), n-1). \end{array}$$

Here $d''_X$ and $d''_{X'}$ are differentials in $C_*^{\mathrm{HC}}(X)$ and $C_*^{\mathrm{HC}}(X')$, respectively, and $N = N_{k(x')/k(x)}$ is the push-forward of higher Chow groups of finite field extensions, namely, the norm map of higher Chow groups. This diagram is commutative by the covariance of Gersten complex. So $d''_X = \sum N \circ d''_{X'}$.

To prove the lemma, consider the following diagram:

$$\begin{array}{ccccc} CH_0(k(X'), n) & \xrightarrow{d''_{X'}} & \bigoplus_{x'|x} CH_0(k(x'), n-1) & \longrightarrow & CH_0(k(x), n-1) \\ \downarrow \chi_{k(X')} & & \downarrow \chi_{k(x')} & & \downarrow \chi_{k(x)} \\ K_n^M(k(X')) & \xrightarrow{\partial_{x'}} & \bigoplus_{x'|x} K_{n-1}^M(k(x')) & \longrightarrow & K_{n-1}^M(k(x)). \end{array}$$

The horizontal maps on the right hand square are norm maps. By definition, the composition of the maps at the bottom is the differential $d'$ in $C_*^{\mathrm{M}}(X)$. By the first part of the proof, the composition of the maps on the top is the differential $d''$ in $C_*^{\mathrm{HC}}(X)$. Hence it suffices to prove commutativity of this diagram. The right hand diagram commutes by [19], Lemma 4.7. On the other hand, Geisser and Levine showed commutativity of the left hand square ([9], Lemma 3.2) (even though their statement is for $\mathbb{Z}/p$-coefficients, their proof is for $\mathbb{Z}$-coefficients). Thus the right hand square commutes. Hence we prove the lemma. $Q.E.D.$

**Definition 2.7.** *We define* $\psi = \chi \circ \phi : \mathbb{Z}_X^c(X) \to C_*^{HC}(X) \overset{\sim}{\to} C_*^{M}(X)$.

Explicitly,

**Definition 2.8.** *Given a generator* $Z \in z_0(X, n)$*, We define* $\psi_n(Z) \in \oplus_{X_{(n)}} K_n^M(k(x))$ *as follows:*

*1) if* $n > d = \dim X$*,* $\psi_n(Z) \overset{def}{=} 0$.

*2) if* $n \leq d$ *and* $\dim p(Z) < n$*,* $\psi_n(Z) \overset{def}{=} 0$.

*3) if* $n \leq d$ *and* $\dim p(Z) = n$*, then* $Z$ *is dominant over some* $x \in X_{(n)}$*. Pulling back* $Z$ *along* $\operatorname{Spec} k(x) \to X$*, we obtain* $Z_x \in z_0(k(x), n)$*, which is sent to* $\overline{Z}_x$ *by the quotient* $z_0(k(x), n) \twoheadrightarrow CH_0(k(x), n)$*. Applying the Nesterenko–Suslin isomorphism* $\chi_{k(x)}$*, we define* $\psi_n(Z) = \chi_{k(x)}(\overline{Z}_x) \in K_n^M(k(x))$*. Since* $Z$ *is dominant over* $x$*,* $Z_x$ *is a closed point in* $\Delta_{k(x)}^n$ *with residue field* $k(Z_x) = k(Z)$*. Therefore, by the definition of* $\chi_{k(x)}$*,*

$$\psi_n(Z) = N_{k(Z)/k(x)}(\beta_Z)$$

*with* $t_i \in k(Z)^*$ *and*

$$\beta_Z = \beta_{Z_x} = \{\frac{-t_0}{t_n}, ..., \frac{-t_{n-1}}{t_n}\}.$$

Note that in Definition 2.8, case 3), $\overline{Z}_x = \phi_n(Z)$.

**Theorem 2.9.** *The map* $\psi$ *defined above is a map of complexes, and it is functorial with respect to pullbacks defined by quasi-finite, flat maps and push-forwards defined by proper maps.*

*Proof.* Since $\psi = \chi \circ \phi$, it is a map of complexes.

For the functoriality, first, we show that $\psi$ is compatible with pull-backs defined by quasi-finite, flat maps $f : X \to Y$. We have to prove that the following diagram is commutative:

$$\begin{array}{ccc} z_0(Y,n) & \xrightarrow{\psi_Y} & \bigoplus_{Y_{(n)}} K_n^M(k(y)) \\ \Big\downarrow{f^*} & & \Big\downarrow{f^{*\prime}} \\ z_0(X,n) & \xrightarrow{\psi_X} & \bigoplus_{X_{(n)}} K_n^M(k(x)). \end{array}$$

Here $f^*$ sends a cycle $Z \in z_0(Y,n)$ to its cycle theoretic pull-back $f^{-1}(Z) \in z_0(X,n)$ and

$$f^{*\prime} : K_n^M(k(y)) \to K_n^M(k(x))$$

is defined by the field extension $k(y) \subset k(x)$ if $x \in X_{(n)}$, $y \in Y_{(n)}$ and $f(x) = y$. Let $p_X : X \times \Delta^n \to X$ and $p_Y : Y \times \Delta^n \to Y$ be the projections. If $\dim p_Y(Z) < n$, then $\dim p_X(f^{-1}(Z)) < n$, so $\psi_X f^*(Z) = 0 = {f^*}'\psi_Y(Z)$.

Suppose that $\dim p_Y(Z) = n$. Without loss of generality, replacing $Y$ by $p_Y(Z)$ and $X$ by $X \times_Y p(Z)$, we can assume that $Y$ is irreducible of dimension $n$ and $Z$ is dominant over $Y$. Since $f$ is quasi-finite and flat, $X$ is of equi-dimension $n$. Let $X = \cup_i X_i$, $X_i$ be the irreducible components of $X$ and $x_i$ be the generic points of $X_i$. Then $\dim X_i = n$. Therefore it suffices to prove commutativity of the following diagram:

$$\begin{array}{ccccc} z_0(Y,n) & \longrightarrow & H_n^c(k(Y),\mathbb{Z}) & \longrightarrow & K_n^M(k(Y)) \\ \Big\downarrow{f^*} & & \Big\downarrow & & \Big\downarrow{f^{*\prime}} \\ z_0(X,n) & \longrightarrow & \bigoplus_i H_n^c(k(x_i),\mathbb{Z}) & \longrightarrow & \bigoplus_i K_n^M(k(x_i)) \end{array} .$$

The square on the left commutes by functoriality of higher Chow groups with respect to flat pull back, the square on the right commutes since the Nesterenko–Suslin isomorphism is covariant with respect to finite field extensions. Therefore $\psi$ commutes with quasi-finite and flat pull-backs.

If $g : X \to Y$ is a proper map, then $\psi$ is also covariant for the push-forward. To see that, it suffices to assume that $X$ is irreducible of dimension $n$ with function field $K$ and prove that the following diagram is commutative:

$$\begin{array}{ccc} z_0(X,n) & \xrightarrow{\psi_X} & K_n^M(K) \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g'_*} \\ z_0(Y,n) & \xrightarrow{\psi_Y} & \bigoplus_{Y_{(n)}} K_n^M(k(y)). \end{array}$$

Here $g_*$ is defined as follows: for any generator $Z \in z_0(X,n)$,

$$g_*(Z) = \begin{cases} 0, & \text{if } \dim g(Z) < n; \\ m_Z \cdot g(Z), & \text{if } \dim g(Z) = n, \end{cases}$$

with $m_Z = [k(Z) : k(g(Z))]$, and

$$g'_* = \begin{cases} 0, & \text{if } \dim g(X) < n; \\ N_{K/k(y)}, & \text{if } X \text{ dominant over } y \in Y_{(n)}. \end{cases}$$

To show that $g'_*\psi_X(Z) = \psi_Y g_*(Z)$, there are three cases:

1) if $\dim g(X) < n$, the $g'_* = 0$. Moreover, for any $Z \in z_0(X,n)$,

$$\dim p_Y(g_*(Z)) = \dim g(p_X(Z)) \leq \dim g(X) < n.$$

Hence $\psi_Y(g_*(Z)) = 0$.

2) if $X$ is dominant over some $y \in Y_{(n)}$ and $\dim p_X(Z) < n$, then $g'_*\psi_X(Z) = g'_*(0) = 0$. $\dim p_X(Z) < n$ also implies $\dim p_Y(g(Z)) < n$, hence $\psi_Y g_*(Z) = \psi_Y(0) = 0$.

3) if $X$ is dominant over some $y \in Y_{(n)}$ and $\dim p_X(Z) = n$, then $\dim p_Y(g(Z)) = \dim g(p_X(Z)) = n$. Therefore, $Z$ is dominant over $X$ and $g(Z)$ is irreducible and dominant over $y$. We have a commutative diagram of field extensions:

$$\begin{array}{ccc} k(Z) & \longleftarrow & K \\ \uparrow & & \uparrow \\ k(g(Z)) & \longleftarrow & k(y) \end{array}$$

Then $\psi_Y g_*(Z) = \psi_Y(m_Z \cdot g(Z)) = N_{k(g(Z))/k(y)}(m_Z \cdot \beta_{g(Z)})$ and $g'_*\psi_X(Z) = N_{K/k(y)}N_{k(Z)/K}(\beta_Z) = N_{k(g(Z))/k(y)}N_{k(Z)/k(g(Z))}(\beta_Z)$. Since $\beta_Z$ is the image of $\beta_{g(Z)}$ under the map $K_n^M(k(g(Z))) \to K_n^M(k(Z))$, $N_{k(Z)/k(g(Z))}(\beta_Z) = m_Z \cdot \beta_{g(Z)}$. Therefore $\psi_Y g_*(Z) = g'_*\psi_X(Z)$.

$Q.E.D.$

**Remark 1.** In [15], Langsburg defined a map from $\mathbb{Z}^c_X(X)$ to $C^{\mathrm{M}}_*(X)$ exactly the same as the one in Definition 2.8, except in case 3), instead of using $\beta_Z$, he used

$$\beta'_Z = \{\frac{t_0}{t_n}, ..., \frac{t_{n-1}}{t_n}\}.$$

By multilinearity of Milnor K-theory, it is easy to see that $\beta_Z = \beta'_Z$ up to a 2-torsion element. Therefore, the map $\psi$ is equal to Langdsburg's map up to a 2-torsion. The advantage of using $\beta_Z$ is that one can use the results in [19]. On the other hand, using the idea in Landsburg's proof of showing that his map is a map of complexes, together with properties of $\chi$ in [19], we can give another proof of showing that $\psi$ is a map of complexes. This proof is lengthy, comparing to the one we give in Theorem 2.9. (There is a small gap in Landsburg's proof, as he only checks compatibility of his map with the differentials in the case of discrete valuation rings).

**Remark 2.** It is easy to see that the map $\psi$ can be generalized to define a map from Bloch's cycle complex (of arbitrary relative dimensions) to the corresponding cycle complex with coefficients in Milnor K-groups or Quillen K-groups defined by M. Rost [21].

**Theorem 2.10.** *For any $X$ separated and essentially of finite type over $k$ of characteristic $p > 0$, the map $\psi$ induces a quasi-isomorphism $\hat{\psi} : \mathbb{Z}^c_X/p^r \to \widetilde{\nu}_{r,X}$.*

*Proof.* Composing $\psi/p^r$ with the isomorphism $C^{\mathrm{M}}_*(X)/p^r \xrightarrow{\sim} \widetilde{\nu}_{r,X}(X)$, we get a map of complexes

$$\hat{\psi} : \mathbb{Z}^c_X/p^r(X) \to \widetilde{\nu}_{r,X}(X).$$

To compare the cohomology of $\mathbb{Z}^c_X/p^r$ and $\widetilde{\nu}_{r,X}$, consider the niveau spectral sequence of higher Chow groups:

$$E^1_{s,t} = \bigoplus_{X_{(s)}} H^c_{s+t}(k(x), \mathbb{Z}/p^r) \Rightarrow H^c_{s+t}(X, \mathbb{Z}/p^r).$$

By [9], Theorem 1.1, this spectral sequence collapses to give edge isomorphisms

$$\Gamma : H^c_s(X, \mathbb{Z}/p^r) \cong H_s(E^1_{*,0}). \qquad (2.0.4)$$

Composing with the isomorphisms $E^1_{*,0} \cong C^{\mathrm{M}}_*(X)/p^r$ (Lemma 2.6) and

$$C^{\mathrm{M}}_*(X)/p^r \cong \widetilde{\nu}_{r,X}(X),$$

we get an isomorphism $\hat{\Gamma} : H^c_s(X, \mathbb{Z}/p^r) \cong H_s(\widetilde{\nu}_{r,X}(X))$. Since $\phi$ induces edge morphisms of the spectral sequence, and $\psi = \chi \circ \phi$, we see that $\hat{\psi}$ induces $\hat{\Gamma}$. Hence $\hat{\psi}$ is a quasi-isomorphism.

$Q.E.D.$

$\mathbb{Q}$-*coefficients:* Let $k$ be a finite field with characteristic $p$ and $X$ be smooth and projective over $k$. Let $\mathbb{Q}^c_X = \mathbb{Z}^c_X \otimes \mathbb{Q}$. In [8], Proposition 2.1, conjecture A(0) (part of Parshin's conjecture) is equivalent to that, in the niveau spectral sequence

$$E^1_{s,t} = \bigoplus_{X_{(s)}} H^c_{s+t}(k(x), \mathbb{Q}) \Rightarrow H^c_{s+t}(X, \mathbb{Q}),$$

$E^1_{s,t} = 0$ for $t \neq 0$. In other words, it is equivalent to the existence of the following isomorphism:

$$H_s(E^1_{*,0}) \cong H^c_s(X, \mathbb{Q}).$$

Here $E^1_{*,0}$ is the Gersten complex of higher Chow groups with $\mathbb{Q}$-coefficients:

$$\bigoplus_{X_{(d)}} H^c_d(k(x), \mathbb{Q}) \to ... \to \bigoplus_{X_{(1)}} H^c_1(k(x), \mathbb{Q}) \to \bigoplus_{X_{(0)}} H^c_0(k(x), \mathbb{Q}).$$

Similar to the case as above for the $\mathbb{Z}/p^r$-coefficients, we obtain the following theorem:

**Theorem 2.11.** *For smooth and projective varieties over finite fields, conjecture A(0) is true if and only if $\psi$ induces a quasi-isomorphism from $\mathbb{Q}^c_X(X)$ to the Gersten complex of higher Chow groups in $\mathbb{Q}$-coefficients.*

# 3 Comparison between Bloch's Complex and Spiess' Complex

In this section, we assume $\dim X = 2$. Spiess's dualizing complex of étale sheaves for surfaces uses the weight-two motivic complex defined by S. Lichtenbaum [16]. For regular Noetherian ring $A$, let $W = \operatorname{Spec} A[T]$, $Z = \operatorname{Spec} A[T]/T(T-1)$, and $B = \{b_1, ..., b_n\}$ a finite set of exceptional units of $A$ (i.e. $b_i$ and $1 - b_i$ are units). Let $Y_B = \operatorname{Spec} A[T]/\prod_{i=1}^n (T - b_i)$. Then there is an exact sequence induced from relative K-theory

$$K_3(A) \to K_2(W - Y_B, Z) \xrightarrow{\phi_{A,B}} K_1'(Y_B) \xrightarrow{\omega_{A,B}} K_2(A).$$

Taking the limits among all the $B$'s, we obtain an exact sequence

$$K_3(A) \to C_{2,1}(A) \xrightarrow{\phi_A} C_{2,2}(A) \xrightarrow{\omega_A} K_2(A)$$

with $C_{2,1}(A) = \varinjlim K_2(W - Y_B, Z)$, $C_{2,2}(A) = \varinjlim K_1'(Y_B)$. When $X$ is a regular Noetherian scheme, associating the group $C_{2,i}(A)$ to each regular affine scheme $U = \operatorname{Spec} A$ which is étale over $X$, we get a presheaf on $X_{\text{ét}}$. Denote by $\underline{C}_{2,i}(X)$ the associated étale sheaf. Then Lichtenbaum's weight-two motivic complex $\mathbb{Z}(2, X)$ is defined as the (cochain) complex

$$\underline{C}_{2,1}(X) \overset{\phi_X}{\to} \underline{C}_{2,2}(X)$$

with the terms in degree 1 and 2, respectively. If $A = k$ is a field, there is an exact sequence [16]

$$K_3(k) \to C_{2,1}(k) \xrightarrow{\phi_k} C_{2,2}(k) \xrightarrow{\omega_k} K_2(k) \to 0. \tag{3.0.5}$$

Moreover, $C_{2,2}(k) = \coprod_{x \in k \setminus \{0,1\}} K_1(k)_x$, and $\omega_k(\gamma) = (\gamma, \frac{x-1}{x})$ for $\gamma \in K_1(k)_x$ with $(\_, \_) : K_1(k) \otimes K_1(k) \to K_2(k)$ the product of Quillen K-theory.

**Definition 3.1** (Spiess, [22]). *Define the complex of étale sheaves $\mathcal{K}_X$ as follows:*

$$\bigoplus_{X_{(2)}} i_{x*}\underline{C}_{2,1}(k(x)) \overset{c_3}{\to} \bigoplus_{X_{(2)}} i_{x*}\underline{C}_{2,2}(k(x)) \overset{c_2}{\to} \bigoplus_{X_{(1)}} i_{x*}\mathbb{G}_m \overset{c_1}{\to} \bigoplus_{X_{(0)}} i_{x*}\mathbb{Z}.$$

*Here $c_3 = \phi_{k(x)}$, $c_1$ is the map sending a rational function to its associate divisors, and $c_2$ is the composition*

$$\bigoplus_{X_{(2)}} i_{x*}\underline{C}_{2,2}(k(x)) \xrightarrow{\omega_{k(x)}} \bigoplus_{X_{(2)}} i_{x*}K_2(k(x)) \overset{\partial'}{\to} \bigoplus_{X_{(1)}} i_{x*}\mathbb{G}_m,$$

*with $\partial'$ the map from Gersten resolution of K-theory ([20]). The terms are put in (cohomological) degree -3, -2, -1, 0, respectively.*

Note that the degrees are different from the ones in [22].

**Theorem 3.2** (Spiess, [22], Theorem 2.2.2)**.** *For $X$ an equidimensional surface over $\mathbb{Z}$ satisfying the (NR) condition and every constructible sheaf $\mathcal{F}$ on $X$, there are perfect pairings of finite groups*

$$H^i_c(X_{\text{ét}}, \mathcal{F}) \times \operatorname{Ext}^{2-i}_X(\mathcal{F}, \mathcal{K}_X) \to H^2_c(X_{\text{ét}}, \mathcal{K}_X) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

Here we say that $X$ satisfies the (NR) condition if

(NR) $k(x)$ *is not formally real for every* $x \in X$.

**Theorem 3.3** (Spiess, [22], Proposition 2.3.2)**.** *Let $k$ be an algebraically closed field of characteristic $p$, $X$ be an irreducible surface over $k$ and $\mathcal{F}$ be a $n$-torsion constructible sheaf on $X$, where $(n,p) = 1$. Let $\mathcal{K}_X(n) = RHom_X(\mathbb{Z}/n, \mathcal{K}_X)$. Then there are perfect pairings of finitely generated groups:*

$$H^i_c(X_{\text{ét}}, \mathcal{F}) \times \operatorname{Ext}^{1-i}_X(\mathcal{F}, \mathcal{K}_X(n)) \to H^1_c(X_{\text{ét}}, \mathcal{K}_X(n)) \xrightarrow{\sim} \mathbb{Z}/n.$$

In [5], Deninger defined a dualizing complex of étale sheaves

$$\mathcal{G}_Y : 0 \to \bigoplus_{Y_{(1)}} i_{y*}\mathbb{G}_m \to \bigoplus_{Y_{(0)}} i_{y,*}\mathbb{Z} \to 0$$

for curves, and E. Nart [18] compared it with cycle complex by constructing a map from cycle complex to $\mathcal{G}_Y$ which induces a quasi-isomorphism. In what follows we generalize the method, and define a similar complex for surfaces.

Consider the niveau spectral sequence of higher Chow groups for surfaces

$$E^1_{s,t} = \bigoplus_{X_{(s)}} H^c_{s+t}(k(x), \mathbb{Z}) \Rightarrow H^c_{s+t}(X, \mathbb{Z}).$$

Only $E^1_{0,0}$,$E^1_{1,0}$ and $E^1_{2,t}$($t \geq 0$) are non-vanishing,

$$H^c_{s+2}(X, \mathbb{Z}) \cong \bigoplus_{X_{(2)}} H^c_{s+2}(k(x), \mathbb{Z})$$

for $s > 0$, and in the bottom of the spectral sequence, the only nonvanishing terms in $E^1_{*,0}$ are:

$$\bigoplus_{X_{(2)}} H^c_2(k(x), \mathbb{Z}) \xrightarrow{f} \bigoplus_{X_{(1)}} k(x)^* \xrightarrow{d_1} \bigoplus_{X_{(0)}} \mathbb{Z} \tag{3.0.6}$$

with

$$\operatorname{coker} d_1 \cong H^c_0(X, \mathbb{Z}), \frac{\ker d_1}{\operatorname{Im} f} \cong H^c_1(X, \mathbb{Z}), \ker f \cong H^c_2(X, \mathbb{Z}).$$

We define a cochain complex of étale sheaves $\mathcal{C}_X$:

$$\bigoplus_{X_{(2)}} i_{x*}\frac{z_0(k(x),3)}{I_x} \xrightarrow{d_3} \bigoplus_{X_{(2)}} i_{x*}z_0(k(x),2) \xrightarrow{d_2} \bigoplus_{X_{(1)}} i_{x*}\mathbb{G}_m \xrightarrow{d_1} \bigoplus_{X_{(0)}} i_{x*}\mathbb{Z}.$$

Here $I_x = \operatorname{Im}(d_x : z_0(k(x),4) \to z_0(k(x),3))$, $d_3$ is the map induced by $g_x : z_0(k(x),3) \to z_0(k(x),2)$, $d_1 = c_1$, and $d_2$ is the composition

$$\bigoplus_{X_{(2)}} z_0(k(x),2) \overset{\pi}{\twoheadrightarrow} \bigoplus_{X_{(2)}} H_2^c(k(x),\mathbb{Z}) \xrightarrow{f} \bigoplus_{X_{(1)}} k(x)^*$$

with $\pi$ the projection:

$$z_0(k(x),2) \twoheadrightarrow \frac{z_0(k(x),2)}{\operatorname{Im} g_x} \cong H_2^c(k(x),\mathbb{Z}).$$

The (cohomological) degrees of the terms are -3, -2, -1, 0, respectively.

**Theorem 3.4.** *Let $X$ be a surface, then there is a map $\psi'$ from $\mathbb{Z}_X^c$ to $\mathcal{C}_X$ which induces isomorphisms*

$$H^{-i}(\mathbb{Z}_X^c) \cong H^{-i}(\mathcal{C}_X)$$

*for* $0 \le i \le 3$.

*Proof.* Following the same idea of the definition of $\psi$ in §2, we can define a map from $\tau_{\ge -3}\mathbb{Z}_X^c(X)$ to $\mathcal{C}_X(X)$:

$$\begin{array}{ccccccc}
z_0(X,3)/I' & \xrightarrow{d} & z_0(X,2) & \xrightarrow{d} & z_0(X,1) & \xrightarrow{d} & z_0(X,0). \\
\downarrow{\scriptstyle \psi_3'} & & \downarrow{\scriptstyle \psi_2'} & & \downarrow{\scriptstyle \psi_1'} & & \downarrow{\scriptstyle \psi_0'} \\
\bigoplus_{X_{(2)}} z_0(k(x),3)/I_x & \xrightarrow{d_3} & \bigoplus_{X_{(2)}} z_0(k(x),2) & \xrightarrow{d_2} & \bigoplus_{X_{(1)}} k(x)^* & \xrightarrow{d_1} & \bigoplus_{X_{(0)}} \mathbb{Z}
\end{array}$$

Here $I' = \operatorname{Im}(z_0(X,4) \to z_0(X,3)), \psi_0' \overset{def}{=} \psi_0$, $\psi_1' \overset{def}{=} \psi_1$. For $i = 2,3$, $\psi_i'$ is defined as follows: if $\dim p(Z) \le 1$, $\psi_i'(Z) = 0$; if $\dim p(Z) = 2$, then $Z$ is dominant over $\overline{\{x\}}$ for some $x \in X_{(2)}$, so pulling back $Z$ along $\operatorname{Spec} k(x) \to X$, we get an element $\psi_i'(Z) \in z_0(k(x),i)$.

By similar argument as in §2, we see that the diagram is commutative. Moreover, $\psi_i'$'s induce the corresponding isomorphisms from the degeneration of the niveau spectral sequence:

$$H_0^c(X,\mathbb{Z}) \cong \operatorname{coker} d_1 \cong H_0(\mathcal{C}_X(X)),$$

$$H_1^c(X,\mathbb{Z}) \cong \frac{\ker d_1}{\operatorname{Im} f} = \frac{\ker d_1}{\operatorname{Im} d_2} = H_1(\mathcal{C}_X(X)),$$
$$H_3^c(X,\mathbb{Z}) \cong \bigoplus_{X_{(2)}} H_3^c(k(x),\mathbb{Z}) \cong \ker d_3 \cong H_3(\mathcal{C}_X(X)),$$
$$H_2^c(X,\mathbb{Z}) \cong \ker f.$$

To show that $\ker f \cong H_2(\mathcal{C}_X(X))$, consider the following diagram which defines $d_2$:

$$\begin{array}{ccc}
\bigoplus_{X_{(2)}} z_0(k(x),2) & \xrightarrow{\pi} & \bigoplus_{X_{(2)}} H_2^c(k(x),\mathbb{Z}) \\
 & \searrow^{d_2} & \downarrow f \\
 & & \bigoplus_{X_{(1)}} k(x)^*.
\end{array}$$

There is an exact sequence

$$0 \to \ker \pi \to \ker d_2 \to \ker f \to \operatorname{coker} \pi \to \operatorname{coker} d_2 \to \operatorname{coker} f \to 0.$$

Since $\pi$ is surjective, $\operatorname{coker} \pi = 0$, so $\ker f \cong \ker d_2 / \ker \pi$. But $\ker \pi = \operatorname{Im} d_3$. Hence

$$H_2^c(X,\mathbb{Z}) \cong \ker f \cong \ker d_2 / \operatorname{Im} d_3 \cong H_2(\mathcal{C}_X(X)).$$

$Q.E.D.$

**Remark** In the above proof, $\psi_1$ is similar to the map defined by Nart ([18]). As noted by Nart, there are only two types of generators in $z_0(X,1)$, the vertical ones and the horizontal ones. The vertical ones are those $Z$'s with $\dim p(Z) = 0$, and the horizontal ones are those $Z$'s with $p(Z) = \overline{\{x\}}$ for some $x \in X_{(1)}$. Nart defined a map sending the first type to 0 and the second type to $N(\frac{t_0}{t_1})$ with $N : k(Z)^* \to k(x)^*$ the norm map of fields. In our situation, according to the Nesterenko–Suslin isomorphism, $\psi_1$ sends the first type to 0 and the second type to $N(\frac{-t_0}{t_1})$.

**Theorem 3.5.** *For a surface $X$, $\mathcal{C}_X \cong \mathcal{K}_X$ in the derived category of étale sheaves.*

*Proof.* Let us first define a map $\mathcal{C}_X \to \mathcal{K}_X$ in the derived category. For any $x \in X_{(2)}$, there is a quasi-isomorphism from the complex $z_0(k(x),3)/I_x \to z_0(k(x),2)$ to the complex $C_{2,1}(k(x)) \to C_{2,2}(k(x))$ ([4], Theorem 7.2). In particular, it induces an isomorphism between the cokernels of the two complexes

$H_2^c(k(x),\mathbb{Z}) \xrightarrow[\sim]{\theta_x} K_2(k(x))$ (see (3.0.8) below). Recall that the complex $\mathcal{C}_X$ (resp., $\mathcal{K}_X$) is defined by connecting

$$\bigoplus_{X_{(2)}}(z_0(k(x),3)/I_x) \xrightarrow{d_3} \bigoplus_{X_{(2)}} z_0(k(x),2)$$

$$(\text{resp., } \bigoplus_{X_{(2)}} C_{2,1}(k(x)) \xrightarrow{c_3} \bigoplus_{X_{(2)}} C_{2,2}(k(x)))$$

with

$$\bigoplus_{X_{(1)}} k(x)^* \xrightarrow{d_1} \bigoplus_{X_{(0)}} \mathbb{Z}.$$

via $\oplus_{X_{(2)}} H_2^c(k(x),\mathbb{Z})$ (resp., $\oplus_{X_{(2)}} K_2(k(x))$). Since $c_1 = d_1$, it suffices to show that there is a (anti-)commutative diagram of abelian groups:

$$\begin{CD}
\bigoplus_{X_{(2)}} H_2(k(x),\mathbb{Z}^c) @>{f}>> \bigoplus_{X_{(1)}} k(x)^* \\
@V{\theta_x}VV @| \\
\bigoplus_{X_{(2)}} K_2(k(x)) @>{\partial'}>> \bigoplus_{X_{(1)}} k(x)^*.
\end{CD}$$

Let $\eta \in X_{(2)}$ and $K = k(\eta)$, consider the following diagram:

$$\begin{array}{ccccc}
H_2^c(K,\mathbb{Z}) & & \xrightarrow{\quad f \quad} & & \\
 & \searrow^{\chi} & & \searrow & \\
\downarrow{\theta} & & K_2^M(K) & \xrightarrow{\partial} & \bigoplus_{X_{(1)}} k(x)^* \\
 & \swarrow_{s} & & \nearrow_{\partial'} & \\
K_2(K) & & & & .
\end{array}$$

Here $s$ is the Steinberg symbol, $\partial$ is the tame symbol, $\chi$ is the Nesterenko–Suslin isomorphism. We have to show that the outside triangle is (anti-)commutative, so it suffices to show that the three small triangles are (anti-)commutative. The triangle on the top is commutative by Lemma 2.6. For the triangle on the left, note that there is an isomorphism $\theta' : H_2^c(K,\mathbb{Z}) \xrightarrow{\sim} K_2(K)$ induced by an edge morphism of the spectral sequence

$$E_2^{p,q} = H^{p-q}(\operatorname{Spec} K, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(K) \tag{3.0.7}$$

proved in [4]. Here $\mathbb{Z}(-q)$ is the motivic complex defined by Bloch's cycle complex. By Lemma 3.6 below, $\theta = \theta'$. Then by [9], Proposition 3.3, which claims that $s \circ \chi = \theta'$, we conclude that the triangle on the left is commutative.

For the triangle on the bottom, we have to show commutativity of the following diagram:

$$\begin{array}{ccc} K_2^M(K) & \xrightarrow{\partial} & k^* \\ \downarrow{\scriptstyle s} & \nearrow{\scriptstyle \partial'} & \\ K_2(K) & & \end{array} .$$

Here $k$ is the residue field of a valuation $v$ of $K$, with $\pi$ a prime element. From [21], Definition 1.1 R3e, $\partial'$ has the following property: for any $\rho, u \in K^*$ with $v(u) = 0$, $\partial'(\{u\}\cdot\rho) = -\{\bar{u}\}\cdot v(\rho)$, i.e. $\partial' \circ s(\{u,\rho\}) = \{\bar{u}\}^{-v(\rho)}$. Here $\bar{u}$ is the residue class of $u$ in $k^*$. But

$$\partial(\{u,\rho\}) = (-1)^{v(u)v(\rho)}\overline{\{\frac{u^{v(\rho)}}{\rho^{v(u)}}\}} = \{\bar{u}\}^{v(\rho)}.$$

By multilinearity of Milnor K-theory, any element $\{u_1\pi^n, u_2\pi^m\}$ with $v(u_i) = 0$ can be decomposed into a product of elements of the form $\{u,\pi\}$ and $\{\pi,\pi\} = \{-1,\pi\}$ with $v(u) = 0$. Therefore, the diagram commutes up to sign. So the triangle outside is commutative. In conclusion, there is an morphism from $\mathcal{C}_X$ to $\mathcal{K}_X$ in the derived category of étale sheaves.

Now let us compare the (co)homology groups. It is clear that $H_0(\mathcal{C}_X(X)) \cong H_0(\mathcal{K}_X(X))$. From [4], Theorem 7.2, $H_3(\mathcal{C}_X(X)) \cong H_3(\mathcal{K}_X(X))$. To compare $c_2$ and $d_2$, consider the diagrams defining them

$$\begin{array}{ccc} \bigoplus_{X_{(2)}} z_0(k(x),2) & \xrightarrow{\pi} & \bigoplus_{X_{(2)}} H_2^c(k(x),\mathbb{Z}) \\ & \searrow{\scriptstyle d_2} & \downarrow{\scriptstyle f} \\ & & \bigoplus_{X_{(1)}} k(x)^* \end{array} \qquad \begin{array}{ccc} \bigoplus_{X_{(2)}} C_{2,2}(k(x)) & \xrightarrow{\omega_{k(x)}} & \bigoplus_{X_{(2)}} K_2(k(x)) \\ & \searrow{\scriptstyle c_2} & \downarrow{\scriptstyle \partial'} \\ & & \bigoplus_{X_{(1)}} k(x)^*. \end{array}$$

Here $\pi$ and $\omega_{k(x)}$ are both surjective onto groups that are isomorphic, and $\partial'\theta = f$, so $\operatorname{Im} d_2 = \operatorname{Im} f = \operatorname{Im} \partial' = \operatorname{Im} c_2$. Therefore,

$$H_1(\mathcal{C}_X(X)) \cong \ker d_1 / \operatorname{Im} d_2 \cong \ker c_1 / \operatorname{Im} c_2 \cong H_1(\mathcal{K}_X(X)).$$

Similar to the proof of Theorem 3.4, we have that

$$H_2(\mathcal{C}_X(X)) \cong \ker f \cong \ker \partial' \cong H_2(\mathcal{K}_X(X)).$$

To finish the proof of Theorem 3.5, we have to prove the following Lemma 3.6. Let us recall the notations from [4]. Let $F$ be a field, and $\Delta^p = \Delta^p_F$. A closed subvariety $\sigma : t_{i1} = t_{i2} = ... = t_{iq} = 0$ is called a codimension $q$ face. A closed subvariety $V \subset \Delta^p$ is said to be in good position if $V \cap \sigma$ has codimension $\geq q$ in $V$ for any codimension $q$ face $\sigma$. Let $\mathcal{V}^n = \mathcal{V}^n(\Delta^p) \subset \Delta^p$ denote the union of all codimension $n$ closed subvarieties of $\Delta^p$ in good position. Given a scheme $X$, we write $K(X)$ for some space, functorial in $X$, whose homotopy groups calculate the Quillen K-theory of $X$. For $Y \subset X$ a closed subset, we write $K(X, Y)$ for the homotopy fibre of $K(X) \to K(Y)$. This construction can be iterated. Given $Y_1, ..., Y_n \subset X$, we define the multi-relative K-space

$$K(X; Y_1, ..., Y_n) \overset{def}{=} \text{homotopy fibre of}$$

$$(K(X; Y_1, ..., Y_{n-1}) \to K(Y_n; Y_1 \cap Y_n, ..., Y_{n-1} \cap Y_n)).$$

Let

$$K(\Delta^p, \partial) = K(\Delta^p; \partial_0, ..., \partial_p)$$

and

$$K(\Delta^p, \sum) = K(\Delta^p; \partial_0, ..., \partial_{p-1})$$

where $\partial_i = \partial_i(\Delta^{p-1})$. Let $\Psi = \partial$ or $\sum$. Define

$$K_V(\Delta^p, \Psi) \overset{def}{=} \text{homotopy fibre of } (K(\Delta^p, \Psi) \to K(\Delta^p - V, \Psi - V)).$$

If $W \subset V$ is inclusion of closed subvarieties, then there is a canonical map

$$K_W(\Delta^p, \Psi) \to K_V(\Delta^p, \Psi).$$

Hence we can define

$$K_{\mathcal{V}^n}(\Delta^p, \Psi) = \varinjlim_{V \subset \mathcal{V}^n} K_V(\Delta^p, \Psi)$$

and

$$K_{\mathcal{V}^n-\mathcal{V}^{n+1}}(\Delta^p,\Psi)=\varinjlim_{V\subset\mathcal{V}^n}\varinjlim_{W\subset\mathcal{V}^{n+1}}K_{V-W}(\Delta^p-W,\Psi-W).$$

There are two maps from $CH_0(F,2)$ to $K_2(F)$ in [4]: the first one $\theta'$ is an edge morphism of the spectral sequence in (3.0.7), the second one $\theta$ is induced from the quasi-isomorphism between $T_F$ and $\mathbb{Z}(2,F)$. Here $T_F$ is the following truncation of cycle complex of fields:

$$z_0(F,3)/I_F\to z_0(F,2)$$

with $I_F\overset{def}{=}\operatorname{Im}(z_0(F,4)\to z_0(F,3))$. Hence $\operatorname{coker}T_F\cong CH_0(F,2)$. From (3.0.5), we know that this quasi-isomorphism induces an isomorphism:

$$\theta: CH_0(F,2)\to K_2(F). \tag{3.0.8}$$

**Lemma 3.6.** *For any field $F$, $\theta=\theta': CH_0(F,2)\to K_2(F)$.*

*Proof.* By their definitions and the following identifications:

$$K_0(\Delta^2,\partial)\xleftarrow{\cong}K_1(\Delta^1,\partial)\xleftarrow{\cong}K_2(\Delta^0),$$

we get that the two maps are induced by the two paths from $CH_0(F,2)$ to $K_1(\Delta^1,\partial)$ and to $K_0(\Delta^2,\partial)$ shown in the following diagram:

$$\begin{array}{ccccc}
K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^2,\sum) & \twoheadrightarrow & K_{0,\mathcal{V}^2}(\Delta^2,\sum) & \xleftarrow[\cong]{} & K_{0,\mathcal{V}^2}(\Delta^2,\partial)\\
\downarrow & & \downarrow\cong & & \downarrow\\
K_{1,\mathcal{V}^1}(\Delta^1,\partial) & & z_0(F,2) & & K_{0,\mathcal{V}^1}(\Delta^2,\partial)\\
\downarrow & & \downarrow & & \downarrow\cong\\
K_1(\Delta^1,\partial) & & CH^2(F,2) & & K_0(\Delta^2,\partial).
\end{array}$$

The map $\theta'$ is the path from $z_0(F,2)$ to $K_{0,\mathcal{V}^2}(\Delta^2,\partial)$ and then to $K_0(\Delta^2,\partial)$, while $\theta$ is the path from $z_0(F,2)$ to $K_1(\Delta^1,\partial)$ via $K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^2,\sum)$. In order to show that $\theta=\theta'$, it suffices to show commutativity of the following diagram: (starting from $K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^2,\sum)$ and end at $K_0(\Delta^2,\partial)$)

$$\begin{array}{ccccccc}
K_{0,\mathcal{V}^2}(\Delta^2,\sum) & \xleftarrow[\cong]{3} & K_{0,\mathcal{V}^2}(\Delta^2,\partial) & \xrightarrow{4} & K_{0,\mathcal{V}^1}(\Delta^2,\partial) & \twoheadrightarrow & K_0(\Delta^2,\partial)\\
{\scriptstyle 1}\uparrow & & & & {\scriptstyle 5}\uparrow & & \cong\uparrow\\
K_{1,\mathcal{V}^1-\mathcal{V}^2}(\Delta^2,\sum) & & \xrightarrow{\quad\quad 2\quad\quad} & & K_{1,\mathcal{V}^1}(\Delta^1,\partial) & \twoheadrightarrow & K_1(\Delta^1,\partial)
\end{array}$$

Here the right hand square is induced from the embedding $(\Delta^1, \partial_0, \partial_1) \subset (\Delta^2, \partial_0, \partial_1)$, so it is commutative. Therefore we have to show that the square on the left commutes, or more precisely, $5 \circ 2 = 4 \circ 3^{-1} \circ 1$.

Recall the long exact sequence of relative K-groups:

$$K_1(Y) \xrightarrow{i} K_0(X, Y) \xrightarrow{j} K_0(X) \xrightarrow{k} K_0(Y).$$

That is the only type of long exact sequence involved in the diagrams above. In particular,

$$1 \sim i, 2 \sim k, 3 \sim j, 4 \sim k, 5 \sim i.$$

Here $1$ is for the embedding $(\Delta^2 - \mathcal{V}^1, \sum) \subset (\Delta^2 - \mathcal{V}^2, \sum)$, 4 is for the embedding $(\Delta^2 - \mathcal{V}^1, \partial) \subset (\Delta^2 - \mathcal{V}^2, \partial)$, and 2, 3 and 5 are for the embedding $(\Delta^1, \partial) \subset (\Delta^2, \sum)$ (embedding of the last face). By functoriality of relative sequence of K-theory, we obtain the commutativity. $Q.E.D.$

**Corollary 3.7.** *a) Suppose $X$ is a two-dimensional scheme over $k$ of $p \geq 0$. For any torsion sheaf $\mathcal{F}$, there is an isomorphism*

$$RHom_X(\mathcal{F}, \mathbb{Z}_X^c) \to RHom_X(\mathcal{F}, \mathcal{K}_X).$$

*b) Let $X$ be a smooth surface. Assuming the Beilinson–Soulé Conjecture for smooth surfaces, then $\mathbb{Z}_X^c \cong \mathcal{C}_X \cong \mathcal{K}_X$ in the derived category.*

*Proof.* a) From Theorem 3.8 and 3.9, we know that there is a map $\mathbb{Z}_X^c \to \mathcal{K}_X$ in the derived category. Let $\mathcal{F}$ be a $\mathbb{Z}/p^r$-sheaf. By the niveau spectral sequence of higher Chow groups with $\mathbb{Z}/p^r$ coefficients, we have $H_t^c(X, \mathbb{Z}/p^r) \cong \oplus_{X_{(2)}} H_t(k(x), \mathbb{Z}^c/p^r)$ for $t \geq 3$. By [9], Theorem 1.1, we get that $H_t^c(X, \mathbb{Z}/p^r) = 0$. Therefore $\mathbb{Z}_X^c/p^r \cong \tau_{\geq -3}\mathbb{Z}_X/p^r \cong \mathcal{K}_X/p^r$ in the derived category. In particular, for such $\mathcal{F}$,

$$RHom_X(\mathcal{F}, \mathbb{Z}_X^c) \to RHom_X(\mathcal{F}, \mathcal{K}_X).$$

Let $\mathcal{F}$ be a $\mathbb{Z}/n$-sheaf with $n$ coprime to $p$. Since both $\mathbb{Z}_X^c$ and $\mathcal{K}_X$ satisfy the Kummer sequence:

$$\to \mathbb{Z}_X^c \xrightarrow{n} \mathbb{Z}_X^c \to \mathbb{Z}/n(2) \to,$$

$$\to \mathcal{K}_X \xrightarrow{n} \mathcal{K}_X \to \mathbb{Z}/n(2) \to,$$

We get that $\mathbb{Z}_X^c/n \cong \mathcal{K}_X/n$ in the derived category. Hence

$$RHom_X(\mathcal{F}, \mathbb{Z}_X^c) \cong RHom_X(\mathcal{F}, \mathcal{K}_X).$$

b) The Beilinson–Soulé Conjecture ([12], Conjecture 5) asserts that for a smooth scheme $X$ and $n > 0$, $H^i(X, \mathbb{Z}(n)) = 0$ for $i \leq 0$ (here $\mathbb{Z}(n)$ is Voevodsky's motivic complex). When $X$ is a smooth surface, $H_n^c(X, \mathbb{Z}) = H^{4-n}(X, \mathbb{Z}(2)) = 0$ for $n \geq 4$ . Hence we get that $\mathbb{Z}_X^c \cong \mathcal{C}_X \cong \mathcal{K}_X$. $Q.E.D.$

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