Uniqueness on the Class of Odd-Dimensional Starlike Obstacles with Cross Section Data

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Abstract

We determine the uniqueness on starlike obstacles by using the cross section data. We see cross section data as spectral measure in polar coordinate at far field. Cross section scattering data suffice to give the local behavior of the wave trace. These local trace formulas contain the geometric information on the obstacle. Local wave trace behavior is connected to the cross section scattering data by Lax-Phillips' formula. Once the scattering data are identical from two different obstacles, the short time behavior of the localized wave trace is expected to give identical heat/wave invariants.

1 Introduction and the Statement of Main Result

Let H be an embedded hypersurface in $\mathbb{R}^n$ such that

$$\mathbb{R}^n \setminus H = \Omega \cup \mathcal{O}, \text{ with } \overline{\mathcal{O}} \text{ compact and } \overline{\Omega} \text{ connected }, \tag{1.1}$$

where both $\mathcal{O}$ and Ω are open. We call $\mathcal{O}$ an obstacle and Ω as its exterior. Without loss of generality, we assume $\mathcal{O}$ contains the origin.

Mathematically, exterior scattering problem is formulated as follows. Let $u \in \mathcal{C}^\infty(\Omega)$ be the solution to the following exterior problem

$$\begin{cases} \Delta u + \lambda^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } H, \end{cases} \tag{1.2}$$

for Dirichlet condition;

$$\begin{cases} \Delta u + \lambda^2 u = 0 & \text{in } \Omega, \\ \frac{\partial}{\partial \nu} u = 0 & \text{on } H, \end{cases} \tag{1.3}$$

for Neumann condition. Let us call the Laplacian defined above as $\Delta_{\mathcal{O}}$.

Let $u = u(x, \omega, \lambda)$ be the corresponding incoming solution. Let $x := |x|\frac{x}{|x|} = r\theta$. We have the following asymptotic behavior when $r := |x| \to \infty$, for λ near $\mathbb{R}$,

$$u(x, \omega, \lambda) = e^{-i\lambda\omega\cdot x} + v_\omega(x), \tag{1.4}$$

where

$$v_\omega(x) := \frac{e^{i\lambda r}}{r^{(n-1)/2}}(A(\lambda, \theta, \omega) + O(\frac{1}{r})), \text{ as } r = |x| \to \infty. \tag{1.5}$$

The function $A(\lambda, \theta, \omega) \in \mathcal{C}^\infty(\mathbb{R} \setminus \{0\} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ is the scattering amplitude related to obstacle $\mathcal{O}$. Note that, in the sense of distribution on $\mathbb{S}^{n-1}_\theta$,

$$e^{-i\lambda r\theta\cdot\omega} = (\frac{2\pi i}{\lambda})^{\frac{n-1}{2}} r^{-\frac{n-1}{2}}\{e^{-i\lambda r}(\delta_\omega(\theta) + O(\frac{1}{r})) + i^{n-1}e^{i\lambda r}(\delta_{-\omega}(\theta) + O(\frac{1}{r}))\}. \tag{1.6}$$

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We define scattering matrix $S(\lambda)$ as the operator with $\mathcal{C}^\infty$-Schwartz kernel

$$S(\lambda,\omega,\theta) = \delta_\omega(\theta) + c_n\lambda^{(n-1)/2}\overline{A(\lambda,-\theta,\omega)},\ c_n = (2\pi)^{-\frac{n-1}{2}}e^{-\frac{i\pi}{4}(n-1)}. \tag{1.7}$$

A scattering matrix in this form is close to the one in Lax and Phillips [13]. In Melrose [15, p.23], we have the "absolute scattering matrix" defined as

$$\widehat{S}(\lambda,\omega,\theta) = i^{n-1}\delta_{-\omega}(\theta) + \overline{c_n}\lambda^{(n-1)/2}A(-\lambda,\theta,\omega),\ \text{where } A(\lambda,\theta,\omega)\in\mathcal{C}^\infty(\mathbb{R}\setminus\{0\}\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}), \tag{1.8}$$

which is obtained by comparing the coefficient of the $e^{-i\lambda r}$ and the one of $e^{i\lambda r}$ as an operator image in (1.4).

Alternatively, scattering matrix can be derived from Poisson operator $P(\lambda):\mathcal{L}^2(\mathbb{S}^{n-1})\to\mathcal{L}^2(\Omega)$, which has $\mathcal{L}^2$-kernel defined as

$$P(\lambda,x,\omega) := \lambda^{\frac{n-1}{2}}c_n u(x,\omega,\lambda). \tag{1.9}$$

To understand $S(\lambda)$, we begin with the spectral theory of its resolvent. We define

$$\mathcal{P} := \{\lambda\in\mathbb{C}|\Im\lambda > 0\}$$

as the physical plane in this paper. According to Sjöstrand and Zworski [20], the scattering matrices $S(\lambda)$ has the meromorphic extension to $\mathbb{C}$ when n is odd; Λ, logarithmic plane, when n is even. We use

$$(\Delta_{\mathcal{O}} - \lambda^2)^{-1} : \mathcal{L}^2(\Omega)\to\mathcal{H}^2(\Omega) \tag{1.10}$$

as the scattered resolvent, imposed with scatterers described above, which is defined over $\mathcal{P}$ by spectral analysis. As a special case of black-box formalism of Sjöstrand and Zworski [20], $(\Delta_{\mathcal{O}}-\lambda^2)^{-1}:\mathcal{L}^2(\Omega)\to\mathcal{H}^2(\Omega)$ meromorphically extends from $\mathcal{P}$, $\lambda^2\notin Spec_{pp}(\Delta_{\mathcal{O}})$, to $\mathbb{C}$ if n is odd; to Λ, the logarithmic plane, if n is even, as an operator

$$R(\lambda) : \mathcal{L}^2_{\text{comp}}(\Omega)\to\mathcal{H}^2_{\text{loc}}(\Omega) \tag{1.11}$$

In this paper, n is odd. $R(\lambda)$ shares the same spectral structure as the corresponding scattering matrix $S(\lambda)$. The resolvent operator $R(\lambda)$ that we will use in this paper are meromorphically extended. That means they are spatially cut offs. So do the wave groups.

Inverse scattering theory asks for the information on the scatterer $\mathcal{O}$ given the knowledge provided by $S(\lambda)$. In particular, let $\mathcal{O}^1$ and $\mathcal{O}^2$ be two obstacles, uniqueness problem asks that if $\mathcal{O}^1=\mathcal{O}^2$ given the information of $S^1(\lambda,\omega,\theta)=S^2(\lambda,\omega,\theta)$ on partial or complete set of $(\lambda,\omega,\theta)\in\mathbb{F}\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$, where $\mathbb{F}=\mathbb{C}$ or Λ. Theoretically, the singularity structure of the scattering matrix may determine the obstacle. We refer to Isakov's papers [7, 8] for an earlier review on the uniqueness and the stability results for obstacle scattering. We refer the inverse scattering problem for convex bodies to [4, Theorem 3.2] in which the case for sound-hard and convex obstacle are discussed. However, there are not too many results on the inverse scattering problem by cross section data. There are some numerical results involved with the determination of the obstacle $\mathcal{O}$ by the corresponding scattering cross section which is defined in this paper as

$$C(\lambda,\theta) := \int_{\mathbb{S}^{n-1}} |A(\lambda,\theta,\theta')|^2 d\theta'. \tag{1.12}$$

As asked by Colton and Sleeman [4], how far can we determine the obstacle $\mathcal{O}$ from the cross section $C(\lambda,\theta)$ provided the obstacle is convex and sound-soft? In [4], the capacity of the obstacle $\mathcal{O}$ and the areas of the shadow projections of $\mathcal{O}$ of all directions can be uniquely determined. In this paper, we will connect the cross section $C(\lambda,\theta)$ to spectral measure and Birman-Krein formula or Hille-Yoshida formula. Therefore, some geometric invariants can be obtained via the asymptotic spectral expansion of heat/wave propagator in short time. Cross section $C(\lambda,\theta)$ can be interpreted as a directional spectral measure propagating along direction θ. How far can we go to tell the geometric difference of these two obstacles by comparing their heat/wave invariants?

Assuming the boundary defining function of obstacle $\mathcal{O}^k$, denoted as x^k, $k=1,2$, is of the form

$$x^k := r^k(\theta),\ \text{where } \theta\in\mathbb{S}^{n-1} \text{ and } r^k\in\mathbb{R}^+, \tag{1.13}$$

we state the main result in this paper as

Theorem 1.1 *Let $\mathcal{O}^k$, $k = 1, 2$, be two starlike obstacles containing the origin in $\mathbb{R}^n$, $n \geq 3$, odd, with smooth boundary imposed either with (1.2) or (1.3). Let $C^k(\lambda, \theta)$ be the cross section data corresponding to obstacle $\mathcal{O}^k$. If, in the neighborhood of one fixed $\lambda_0 \in \mathbb{R} \setminus \{0\}$,*

$$C^1(\lambda, \theta) = C^2(\lambda, \theta), \text{ for all } \theta \in \mathbb{S}^{n-1}, \tag{1.14}$$

then we have

$$r^1(\theta) = r^2(\theta), \forall \theta \in \mathbb{S}^{n-1}. \tag{1.15}$$

In particular,

$$\mathcal{O}^1 = \mathcal{O}^2. \tag{1.16}$$

2 On the Both Sides of Trace Formulas

We recall the following theorem from Lax-Phillips' scattering theory [13].

Theorem 2.1 (Lax and Phillips) *The scattering amplitude satisfies the following relations:*

$$\begin{aligned} & A(\lambda, -\omega, \theta) + (-1)^{\frac{n-1}{2}} \overline{A(\lambda, -\theta, \omega)} \\ = \; & -(\frac{\lambda}{2\pi i})^{\frac{n-1}{2}} \int_{\mathbb{S}^{n-1}} A(\lambda, -\omega, \theta') \overline{A(\lambda, -\theta, \theta')} d\theta' \\ = \; & -(\frac{\lambda}{2\pi i})^{\frac{n-1}{2}} \int_{\mathbb{S}^{n-1}} \overline{A(\lambda, -\theta', \omega)} A(\lambda, -\theta', \theta) d\theta'. \end{aligned} \tag{2.1}$$

Moreover,

$$A(\lambda, \omega, \theta) = A(\lambda, \theta, \omega) \text{ and } A(-\lambda, \omega, \theta) = \overline{A(\lambda, \omega, \theta)}. \tag{2.2}$$

Furthermore,

Lemma 2.2 *In the setting from (1.4) to (1.7), we have*

$$S^{-1}(\lambda, \omega, \theta) = S(-\lambda, \omega, \theta) \tag{2.3}$$

and, formally as coefficient in the spectral expansion (1.4),

$$\widehat{S}^{-1}(\lambda, \omega, \theta) = i^{n-1} \delta_{-\omega}(\theta) + (\frac{\lambda}{2\pi i})^{\frac{n-1}{2}} e^{-2i\lambda r} A(-\lambda, \theta, -\omega). \tag{2.4}$$

Proof The first equality comes from Shenk and Thoe [18]. The proof on the second equality is a straightforward inverse correspondence in (1.4): let $d_n := c_n^{-1}$. We alternatively rewrite (1.4) and (1.5),

$$\begin{aligned} u(x, \omega, \lambda) \; = \; & \lambda^{-\frac{n-1}{2}} d_n r^{-\frac{n-1}{2}} e^{-i\lambda r} \delta_\omega(\theta) + \lambda^{-\frac{n-1}{2}} d_n i^{n-1} r^{-\frac{n-1}{2}} e^{i\lambda r} \delta_{-\omega}(\theta) \\ & + r^{-\frac{n-1}{2}} e^{-i\lambda r} e^{2i\lambda r} A(\lambda, \theta, \omega) + \cdots . \end{aligned} \tag{2.5}$$

Hence,

$$\begin{aligned} u(x, \omega, \lambda) \; = \; & i^{n-1} r^{-\frac{n-1}{2}} \lambda^{-\frac{n-1}{2}} d_n \{ e^{i\lambda r} \delta_{-\omega}(\theta) \\ & + e^{-i\lambda r} [i^{n-1} \delta_\omega(\theta) + i^{n-1} \lambda^{\frac{n-1}{2}} d_n^{-1} e^{2i\lambda r} A(\lambda, \theta, \omega)] \} + \cdots . \end{aligned} \tag{2.6}$$

Observing the correspondence from the coefficient of $e^{i\lambda r}$ to $e^{-i\lambda r}$, we obtain the kernel of $\widehat{S}^{-1}(\lambda)$. □

To connect $C(\lambda, \theta)$ to spectral analysis, we look at the following lemma.

Lemma 2.3 *Under the theorem assumption, we let $R^k(\lambda, x, y)$ be the resolvent kernel corresponding to $\mathcal{O}^k$ with exterior Ω^k, $k = 1, 2$. Then, in a neighborhood of λ_0 in $0i + \mathbb{R}$,*

$$2\lambda \{ R^1(\lambda, x, y) - R^1(-\lambda, x, y) \} = 2\lambda \{ R^2(\lambda, x, y) - R^2(-\lambda, x, y) \}, \forall x, y \in \Omega^1 \cap \Omega^2. \tag{2.7}$$

Proof Starting with

$$[R(\lambda) - R(-\lambda)]d\lambda^2 = (2\pi)^{-1}P(\lambda)P^*(\lambda)d\lambda, \tag{2.8}$$

in which either quantities can serve as the definition of spectral measure. See Reed and Simon [17]. Letting R^k, P^k, u^k, A^k and C^k be the corresponding quantities related to obstacle $\mathcal{O}^k$, we have

$$\begin{aligned}
& 2\lambda\{R^1(\lambda,x,x) - R^1(-\lambda,x,x)\} - 2\lambda\{R^2(\lambda,x,x) - R^2(-\lambda,x,x)\} \\
= & \frac{1}{2\pi}\int_{\mathbb{S}^{n-1}} P^1(\lambda,x,\omega)\overline{P^1(\lambda,\omega,x)}d\omega - \{\text{similar terms from } R^2\} \\
= & \frac{\lambda^{n-1}|c_n|^2}{2\pi}\int_{\mathbb{S}^{n-1}} 1 + \frac{e^{-i\lambda\omega\cdot x}e^{-i\lambda r}}{r^{\frac{n-1}{2}}}\overline{A^1(\lambda,\omega,\theta)} + \frac{e^{i\lambda\omega\cdot x}e^{i\lambda r}}{r^{\frac{n-1}{2}}}A^1(\lambda,\theta,\omega) + \frac{\overline{A^1(\lambda,\omega,\theta)}A^1(\lambda,\omega,\theta)}{r^{n-1}}d\omega \\
& -\{\text{similar terms from } R^2\} + O(\frac{1}{r^n}),\ \text{by}\ \ (1.4),\ (1.5),\ (1.6),\ (1.9), \\
= & \frac{\lambda^{n-1}|c_n|^2}{2\pi}\int_{\mathbb{S}^{n-1}} \frac{-e^{i\lambda\omega\cdot x}e^{-i\lambda r}}{r^{\frac{n-1}{2}}}\overline{A^1(\lambda,\theta,\omega)} + \frac{e^{i\lambda\omega\cdot x}e^{i\lambda r}}{r^{\frac{n-1}{2}}}A^1(\lambda,\theta,\omega) + \frac{\overline{A^1(\lambda,\theta,\omega)}A^1(\lambda,\theta,\omega)}{r^{n-1}}d\omega \\
& -\{\text{similar terms from } R^2\} + O(\frac{1}{r^n}) \\
= & \frac{\lambda^{n-1}|c_n|^2}{2\pi}\frac{e^{-i\lambda r}}{r^{\frac{n-1}{2}}}\int_{\mathbb{S}^{n-1}} e^{-i\lambda\omega\cdot x}\overline{A^1(\lambda,\theta,\omega)}d\omega + \frac{\lambda^{n-1}|c_n|^2}{2\pi}\frac{e^{i\lambda r}}{r^{\frac{n-1}{2}}}\int_{\mathbb{S}^{n-1}} e^{i\lambda\omega\cdot x}A^1(\lambda,\theta,\omega)d\omega \\
& + \frac{(2\pi)^{-1}\lambda^{n-1}|c_n|^2}{r^{n-1}}\int_{\mathbb{S}^{n-1}} \overline{A^1(\lambda,\theta,\omega)}A^1(\lambda,\theta,\omega)d\omega \\
& -\{\text{similar terms from } R^2\} + O(\frac{1}{r^n}).
\end{aligned} \tag{2.9}$$

We compute this term by term. Using (1.6),

$$\begin{aligned}
& \int_{\mathbb{S}^{n-1}} e^{i\lambda\omega\cdot x}A^k(\lambda,\theta,\omega)d\omega \\
\underset{x\to\infty}{\to} & \int_{\mathbb{S}^{n-1}} (\frac{2\pi}{i\lambda})^{\frac{n-1}{2}}r^{-\frac{n-1}{2}}[e^{i\lambda r}\delta_\omega(\theta) + i^{n-1}e^{-i\lambda r}\delta_{-\omega}(\theta)]A^k(\lambda,\theta,\omega)d\omega \\
= & (\frac{2\pi}{i\lambda})^{\frac{n-1}{2}}r^{-\frac{n-1}{2}}e^{i\lambda r}A^k(\lambda,\theta,\theta) + (\frac{2\pi}{i\lambda})^{\frac{n-1}{2}}r^{-\frac{n-1}{2}}i^{n-1}e^{-i\lambda r}A^k(\lambda,\theta,-\theta) + \cdots.
\end{aligned} \tag{2.10}$$

Taking conjugate,

$$\begin{aligned}
& \int_{\mathbb{S}^{n-1}} e^{-i\lambda\omega\cdot x}\overline{A^k(\lambda,\theta,\omega)}d\omega \\
\underset{x\to\infty}{\to} & (\frac{2\pi i}{\lambda})^{\frac{n-1}{2}}r^{-\frac{n-1}{2}}e^{-i\lambda r}\overline{A^k(\lambda,\theta,\theta)} + (\frac{2\pi i}{\lambda})^{\frac{n-1}{2}}r^{-\frac{n-1}{2}}i^{n-1}e^{i\lambda r}\overline{A^k(\lambda,\theta,-\theta)} + \cdots.
\end{aligned} \tag{2.11}$$

However, from the identities in Lemma 2.2,

$$S^k(-\lambda,\omega,\theta) = \delta_\omega(\theta) + (\frac{-\lambda}{2\pi i})^{\frac{n-1}{2}}A^k(\lambda,-\theta,\omega); \tag{2.12}$$

$$\{\widehat{S}^k\}^{-1}(\lambda,\omega,\theta) = i^{n-1}\delta_{-\omega}(\theta) + (\frac{\lambda}{2\pi i})^{\frac{n-1}{2}}e^{-2i\lambda r}A^k(-\lambda,\theta,-\omega). \tag{2.13}$$

Using (1.8) we have

$$\overline{A^k(\lambda,\theta,\omega)} = e^{2i\lambda r}A^k(\lambda,\theta,-\omega). \tag{2.14}$$

Let $\omega = -\theta$. We obtain

$$\overline{A^k(\lambda,\theta,-\theta)} = e^{2i\lambda r}A^k(\lambda,\theta,\theta). \tag{2.15}$$

Therefore, as $|x| \to \infty$,

$$
\begin{aligned}
& \lambda^{n-1}|c_n|^2 \int_{\mathbb{S}^{n-1}} \frac{e^{i\lambda\omega\cdot x}e^{i\lambda r}}{r^{\frac{n-1}{2}}} A^k(\lambda,\theta,\omega) + \frac{e^{-i\lambda\omega\cdot x}e^{-i\lambda r}}{r^{\frac{n-1}{2}}} \overline{A^k(\lambda,\theta,\omega)} d\omega \\
= \; & \lambda^{\frac{n-1}{2}} \overline{c_n} \frac{1}{r^{n-1}} (A^k(\lambda,\theta,-\theta) + (-1)^{\frac{n-1}{2}} \overline{A^k(\lambda,\theta,-\theta)}) \\
& + \lambda^{\frac{n-1}{2}} c_n \frac{1}{r^{n-1}} (e^{2i\lambda r} A^k(\lambda,\theta,\theta) + (-1)^{\frac{n-1}{2}} e^{-2i\lambda r} \overline{A^k(\lambda,\theta,\theta)}) + O(\frac{1}{r^n}), \text{ using (2.1), (2.15),} \\
= \; & -2(\frac{\lambda}{2\pi})^{n-1} r^{-(n-1)} C^k(\lambda,\theta) + O(\frac{1}{r^n}).
\end{aligned} \tag{2.16}
$$

Hence, (2.9) and (2.16) sum up to give

$$
\begin{aligned}
& 2\lambda\{R^1(\lambda,x,x) - R^1(-\lambda,x,x)\} - 2\lambda\{R^2(\lambda,x,x) - R^2(-\lambda,x,x)\} \\
\underset{x\to\infty}{\to} \; & (\frac{-1}{2\pi}) r^{-(n-1)} (\frac{\lambda}{2\pi})^{n-1} \{C^1(\lambda,\theta) - C^2(\lambda,\theta)\} + O(\frac{1}{r^n}).
\end{aligned} \tag{2.17}
$$

Furthermore, we see that $\int_{|x|=s} 2\lambda\{R^1(\lambda,x,y) - R^1(-\lambda,x,y) - R^2(\lambda,x,y) + R^2(-\lambda,x,y)\} dS_x$ is a solution of the exterior problem (1.2) or (1.3) for $|y| \gg d$, $\forall s \in [c,d] \subset \mathbb{R}, c \gg 1$. Therefore, using Jensen's inequality, for some constant C depending only on n and s,

$$
\begin{aligned}
& \int_{|y|=r} \{\int_{|x|=s} 2\lambda\{R^1(\lambda,x,y) - R^1(-\lambda,x,y) - R^2(\lambda,x,y) + R^2(-\lambda,x,y)\} dS_x\}^2 dS_y \\
\leq \; & C \int_{|y|=r} \int_{|x|=s} \{2\lambda\{R^1(\lambda,x,y) - R^1(-\lambda,x,y) - R^2(\lambda,x,y) + R^2(-\lambda,x,y)\}\}^2 dS_x dS_y \\
\leq \; & C \int_{|y|=r} \int_{|x|=r} |2\lambda\{R^1(\lambda,x,y) - R^1(-\lambda,x,y) - R^2(\lambda,x,y) + R^2(-\lambda,x,y)\}|^2 dS_x dS_y \\
\leq \; & C\{\int_{|x|=r} |2\lambda\{R^1(\lambda,x,x) - R^1(-\lambda,x,x) - R^2(\lambda,x,x) + R^2(-\lambda,x,x)\}| dS_x\}^2,
\end{aligned} \tag{2.18}
$$

where the last inequality comes from the fact that Hilbert-Schmidt norm is controlled by trace norm. The theorem assumption $C^1(\lambda,\theta) = C^2(\lambda,\theta)$, (2.17) and (2.18) yield

$$
\begin{aligned}
& \int_{|y|=r} \{\int_{|x|=s} 2\lambda\{R^1(\lambda,x,y) - R^1(-\lambda,x,y) - R^2(\lambda,x,y) + R^2(-\lambda,x,y)\} dS_x\}^2 dS_y \\
\lesssim \; & C\{\int_{|x|=r} |x|^{-n} dS_x\}^2 \\
\leq \; & Cr^{-2}, \text{ where } C's \text{ are constants.}
\end{aligned} \tag{2.19}
$$

Using the Kato's uniqueness theorem as in Shenk and Thoe [19, Lemma 4.4], we have

$$
\int_{|x|=s} 2\lambda\{R^1(\lambda,x,y) - R^1(-\lambda,x,y) - R^2(\lambda,x,y) + R^2(-\lambda,x,y)\} dS_x \equiv 0, \forall y \in \Omega^1 \cap \Omega^2, \forall s \in [c,d]. \tag{2.20}
$$

By Lebesgue integration theory, $\forall y \in \Omega^1 \cap \Omega^2$, $\{R^1(\lambda,x,y) - R^1(-\lambda,x,y) - R^2(\lambda,x,y) + R^2(-\lambda,x,y)\}$ is *a.e.* zero with respect to $|x| \in [c,d]$. Since it is continuous to x, provided by Theorem 5.1 part(3) in [19], it is identically zero with respect to $|x| \in [c,d]$, $\forall y \in \Omega^1 \cap \Omega^2$. Now we apply the unique continuation property of elliptic differential equation with analytic coefficients. See, Bers, John and Schechter [1]. Here, we have Helmholtz equation as a special case.

Again, using the unique continuation property of Helmholtz equation with respect to x, we have

$$
2\lambda\{R^1(\lambda,x,y) - R^1(-\lambda,x,y) - R^2(\lambda,x,y) + R^2(-\lambda,x,y)\} \equiv 0, \forall x, y \in \Omega^1 \cap \Omega^2. \tag{2.21}
$$

□

As a result of continuation outside all possible poles, we have in particular that

Corollary 2.4 *In $0i+\mathbb{R}$, there exist a cutoff function $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n;[0,1])$ which is 1 near $\mathcal{O}^1 \cup \mathcal{O}^2$ such that*

$$(1-\chi)\{R^1(\lambda,\cdot,\cdot)-R^1(-\lambda,\cdot,\cdot)\} \equiv (1-\chi)\{R^2(\lambda,\cdot,\cdot)-R^2(-\lambda,\cdot,\cdot)\}. \tag{2.22}$$

Let the naturally regularized wave trace

$$u(t) := 2\{\cos t\sqrt{\Delta_{\mathcal{O}}} - \cos t\sqrt{\Delta_0}\}, \tag{2.23}$$

where $\Delta_0 := -\Delta_{\mathbb{R}^n} = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$. $\cos t\sqrt{\Delta_{\mathcal{O}}}$ has a kernel satisfying the following Cauchy problem:

$$\begin{cases} (\frac{\partial^2}{\partial t^2}+\Delta_{\mathcal{O}})\cos t\sqrt{\Delta_{\mathcal{O}}}(x,y)=0; \\ \cos t\sqrt{\Delta_{\mathcal{O}}}(x,y)|_{t=0}=\delta(x-y); \\ \frac{\partial \cos t\sqrt{\Delta_{\mathcal{O}}}(x,y)}{\partial t}|_{t=0}=0. \end{cases} \tag{2.24}$$

Furthermore, $u(t)$ has a distributional trace. See Zworski [22]. We recall from Petkov and Stoyanov [16] that, for a non-trapping obstacle,

$$\text{singular support of } Tr\{u(t)\} = \{0\}. \tag{2.25}$$

Moreover, $\cos t\sqrt{\Delta_{\mathcal{O}}}(x,y)$, $t \geq 0$, is interpreted as the data given at $(0,y)$ received at (t,x) along the geodesic. Hence, we see $\cos t\sqrt{\Delta_{\mathcal{O}}}(x,x)$, $t\geq 0$, as the data given at $(-t/2,x)$ received at $(t/2,x)$ along the geodesic.

Furthermore, $u(t)$ has a spectral representation.

$$u(t) \;=\; \int_{\mathbb{R}} e^{-i\lambda t}\{R(\lambda)-R^0(\lambda)-R(-\lambda)+R^0(-\lambda)\}d\lambda^2. \tag{2.26}$$

$\{R(\lambda)-R^0(\lambda)-R(-\lambda)+R^0(-\lambda)\}d\lambda^2$ is the spectral measure, where $R^0(\lambda) := (\Delta_0-\lambda^2)^{-1}$. There is no Neumann or Dirichlet eigenvalue of $\Delta_{\mathcal{O}}$ in obstacle scattering problem. Furthermore, when $n \geq 3$, 0 is neither a resonance nor an eigenvalue of $R(\lambda)$. The continuous spectrum is actually where the scattering phenomenon happens. We consider the Fourier inversion formula of (2.26) over $\overline{\mathcal{P}}$

$$\int_{\mathbb{R}} e^{i\lambda t}u(t)dt \;=\; 2\lambda\{R(\lambda)-R^0(\lambda)-R(-\lambda)+R^0(-\lambda)\}. \tag{2.27}$$

Or, locally,

$$\int_{\mathbb{R}} e^{i\lambda t}fu(t)dt \;=\; 2\lambda f\{R(\lambda)-R^0(\lambda)-R(-\lambda)+R^0(-\lambda)\},\ \text{where } f\in\mathcal{C}_0^\infty(\Omega). \tag{2.28}$$

We will focus at the behavior of the localized cutoffed resolvents on the boundary H.

Using Birman-Krein type of theory, we see that, for $\lambda \in \mathbb{R}$,

$$2\lambda Tr\{R(\lambda)-R^0(\lambda)-R(-\lambda)+R^0(-\lambda)\} = \sigma'(\lambda). \tag{2.29}$$

A general treatment in proving the Birman-Krein theorem in black box formalism setting when n is odd can be found in Zworski [22]. Therefore, we can rewrite (2.27) in a distributional sense as

$$\sigma'(\lambda) = \int_{\mathbb{R}} e^{i\lambda t}Tr\{u(t)\}dt, \lambda \in 0+i\mathbb{R}. \tag{2.30}$$

Locally, we can define

$$\sigma_f{}'(\lambda) := 2\lambda Tr\{f(R(\lambda)-R^0(\lambda)-R(-\lambda)+R^0(-\lambda))\}, \forall f \in \mathcal{C}_0^\infty(\mathbb{R}^n). \tag{2.31}$$

In this notation, we can convert (2.28) to a local formula:

$$\sigma_f{}'(\lambda) = \int_{\mathbb{R}} e^{i\lambda t}Tr\{fu(t)\}dt. \tag{2.32}$$

Let $\sigma_f^{k\prime}(\lambda)$ be the quantity corresponding to $\mathcal{O}^k$. Timing f on the distributional resolvent kernel $R^k(\lambda,x,x)$ and carrying out the trace integration, Lemma 2.3 tells us

$$\sigma_f^{1\prime}(\lambda) = \sigma_f^{2\prime}(\lambda),\ \text{in a neighborhood of } \lambda_0, \forall f \in \mathcal{C}_0^\infty(\Omega^1\cap\Omega^2)). \tag{2.33}$$

Now we prove

Proposition 2.5 *Under the same assumption as in the introduction, the inverse Fourier transform corresponding to* $\mathcal{O}^k$ $\int_{\mathbb{R}} e^{i\lambda t} Tr\{fu^k(t)\}dt$*, which is valid for* $\lambda \in 0i + \mathbb{R}$*, depends only on its short time behavior in the following sense:*

$$\int_{\mathbb{R}} e^{i\lambda t} Tr\{f(u^1(t) - u^2(t))\}dt = \int_{\mathbb{R}} e^{i\lambda t} Tr\{f(u^1(t) - u^2(t))\}\rho_1(t)dt + \text{ rapidly decreasing term}, \quad (2.34)$$

whenever $\lambda \in 0i + \mathbb{R}$ *and where* $\rho_1(t) \in \mathcal{C}_0^\infty(\mathbb{R}; [0,1])$ *is a cutoff function supported at* $t = 0$*. Moreover,* $f \in \mathcal{C}_0^\infty(\mathbb{R}^n; [0,1])$ *is 1 near* $\mathcal{O}^1 \cup \mathcal{O}^2$*.*

Proof We divide the inverse Fourier transform into three time intervals:

$$\begin{aligned}
& \int_{-\infty}^{\infty} e^{i\lambda t} Tr\{f(u^1(t) - u^2(t))\}dt \\
:= & \int_{-\infty}^{\infty} e^{i\lambda t} Tr\{f(u^1(t) - u^2(t))\}\rho_1(t)dt + \int_{-\infty}^{\infty} e^{i\lambda t} Tr\{f(u^1(t) - u^2(t))\}\rho_2(t)dt \\
& + \int_{-\infty}^{\infty} e^{i\lambda t} Tr\{f(u^1(t) - u^2(t))\}\rho_3(t)dt \\
:= & I_1(\lambda) + I_2(\lambda) + I_3(\lambda), \qquad (2.35)
\end{aligned}$$

where $\rho_i \in \mathcal{C}^\infty(\mathbb{R}; [0,1])$, $i = 1,2,3$. Let $\rho_1, \rho_3 \in \mathcal{C}^\infty(\mathbb{R}; [0,1])$ be two cutoff functions such that ρ_1 is 1 with small compact support at $t = 0$ and ρ_3 is 1 near $t = \pm\infty$. We take $\rho_1(t) + \rho_2(t) + \rho_3(t) = 1$. We take β such that $\operatorname{supp}(\rho_3(t)) \subset (-\infty, -\beta) \cup (\beta, \infty)$. β is to be chosen. This is a partition of unity.

Using Paley-Wiener's theorem for $I_1(\lambda)$,

$$|\int_{-\infty}^{\infty} e^{i\lambda t} Tr\{f(u^1(t) - u^2(t))\}\rho_1(t)dt| \leq C(1 + |\lambda|)^N e^{h(-\Im\lambda)}, \quad (2.36)$$

for some $N \in \mathbb{N}$ and for some constant C. h is the support function of $Tr\{f(u^1(t) - u^2(t))\}\rho_1(t)$. We just keep $I_1(\lambda)$. N will be specified by Ivrii's result [9]. $I_1(\lambda)$ is holomorphic and well-defined as a Fourier-Laplace transform.

We also apply Paley-Wiener's theorem to $I_2(\lambda)$. By (2.25), for each β, $Tr\{f(u^1(t) - u^2(t))\}\rho_2(t)$ is a smooth function with compact support. By construction $\rho_2(t)$ is the union of two cutoff functions. One, denoted as $\rho_2^+(t)$, is supported on $\mathbb{R}^+$ while the other one, denoted as $\rho_2^-(t)$, supported on $\mathbb{R}^-$. For the first one, we choose $\Im\lambda > 0$, the upper half complex plane, for

$$|I_2^+(\lambda)| := |\int_{-\infty}^{\infty} e^{i\lambda t} Tr\{f(u^1(t) - u^2(t))\}\rho_2^+(t)dt| \leq C_N(1 + |\lambda|)^{-N} e^{a^+(-\Im\lambda)}; \quad (2.37)$$

if supported on $\mathbb{R}^-$, we choose $\Im\lambda < 0$, the lower half complex plane, for

$$|I_2^-(\lambda)| := |\int_{-\infty}^{\infty} e^{i\lambda t} Tr\{f(u^1(t) - u^2(t))\}\rho_2^-(t)dt| \leq C_N(1 + |\lambda|)^{-N} e^{a^-(-\Im\lambda)}, \quad (2.38)$$

where $a^\pm$ is the supporting function of $\rho^\pm(t)$. This form of Paley-Wiener's theorem appears in Hörmander's book [6]. In this case,

$$|I_2(\lambda)| \leq C_N(1 + |\lambda|)^{-N}, \forall N \in \mathbb{N}, \text{ whenever } \lambda \in 0i + \mathbb{R}. \quad (2.39)$$

This is a rapidly decreasing term.

For $I_3(\lambda)$, we see $\rho_3(t)$ is also an union of two cutoff functions supported on $(-\infty, -\beta)$ and (β, ∞) respectively. By domain of dependence argument on $u^k(t, x, x)$ along the geodesic toward $0 \in \mathbb{R}^n$ hitting the obstacle boundaries and back to x such that $\omega = -\theta$, we choose β large such that

$$f(u^1(t, r\theta, r\theta) - u^2(t, r\theta, r\theta))\rho_3(t) \equiv 0, \forall\theta \in \mathbb{S}^{n-1}. \quad (2.40)$$

There are infinitely many geodesics starting at x and back to x. We consider only the one carries backscattering information. Under starlike assumption, all such geodesics are transversal reflections.

Henceforth,

$$\mathrm{Tr} f(u^1(t) - u^2(t))\rho_3(t) = \int_0^\infty \int_{\mathbb{S}^{n-1}} f(r\theta)(u^1(t, r\theta, r\theta) - u^2(t, r\theta, r\theta))\rho_3(t) r^{n-1} d\theta dr \equiv 0. \tag{2.41}$$

Therefore,

$$I_3(\lambda) \equiv 0. \tag{2.42}$$

□

Accordingly,

Corollary 2.6 *$I_1(\lambda)$ and $I_2(\lambda)$ are entire functions.*

Proof We see that $\rho_1(t) + \rho_2(t) \in \mathbb{C}_0^\infty(\mathbb{R})$. □

3 Proof of Theorem 1.1

Let us define

$$\Phi_k(t) := \mathcal{F}_{\lambda \to t}[|\lambda|^k], \tag{3.1}$$

where one needs to replace $|\lambda|^k$ by its certain regularization when $k \leq -1$. According to Ivrii [9, 10], when $t \to 0^+$, we have

$$Tr\{f \cos t\sqrt{\Delta_{\mathcal{O}}}\rho_1(t)\} = \sum_{j=0}^{\infty} c_j \Phi_{n-j-1}(t), \tag{3.2}$$

where c_j's are nonzero multiples of heat invariants $a_{j/2}$'s. See Branson and Gilkey [2]. In particular,

$$c_0 = \alpha_0 \int_{\mathcal{O}} f(x)dx, \text{ where the constant } \alpha_0 \neq 0. \tag{3.3}$$

Let

$$D(\lambda) := \int_{-\infty}^{\infty} e^{i\lambda t} Tr\{fu^1(t)\}\rho_1(t)dt - \int_{-\infty}^{\infty} e^{i\lambda t} Tr\{fu^2(t)\}\rho_1(t)dt. \tag{3.4}$$

By Lemma 2.3 and Proposition 2.5, as a result of analytic continuation,

$$D(\lambda) \text{ is rapidly decreasing on } 0i + \mathbb{R}. \tag{3.5}$$

Using (3.2), on the other hand, as $\lambda \to 0i \pm \infty$,

$$D(\lambda) \to \alpha_0(a_0(f, \mathcal{O}^1) - a_0(f, \mathcal{O}^2))|\lambda|^{n-1} + \alpha_{0.5}(a_{0.5}(f, \mathcal{O}^1) - a_{0.5}(f, \mathcal{O}^2))|\lambda|^{n-2} + \cdots. \tag{3.6}$$

Combing (3.5) and (3.6), we obtain

$$a_j(f, \mathcal{O}^1) = a_j(f, \mathcal{O}^2), \forall j \geq 0, \text{ where } f \in \mathcal{C}_0^\infty(\Omega^1 \cap \Omega^2). \tag{3.7}$$

In particular, we have identical localized relative volume

$$a_0(f, \mathcal{O}^1) = a_0(f, \mathcal{O}^2), \tag{3.8}$$

where we choose that $f = f(x) = f(r\omega) = f(\omega)$, where $\omega \in \mathbb{S}^{n-1}$. It suffices to show the obstacle can be shaped by angular argument. By our starlike assumption, we have

$$r^k(\omega) := \sup\{v | v\omega \in \mathcal{O}^k\} \tag{3.9}$$

as the radial function of $\mathcal{O}^k$ in the direction of $\omega \in \mathbb{S}^{n-1}$. According to [5] and [12, equation(2.4)], we have for starlike sets

$$\text{Volume}(\mathcal{O}^k) = \int_{\mathbb{S}^{n-1}} (r^k)^n(\omega)d\omega. \tag{3.10}$$

Hence, (3.3) and (3.8) becomes

$$\int_{\mathbb{S}^{n-1}} (r^1)^n(\omega) f(\omega)d\omega = \int_{\mathbb{S}^{n-1}} (r^2)^n(\omega) f(\omega)d\omega, \forall f \in \mathcal{C}_0^\infty(\mathbb{S}^{n-1}). \tag{3.11}$$

Therefore, we have $r_1(\omega) = r_2(\omega)$. Theorem is proved.

References

[1] L. Bers, F. John and M. Schechter, Partial differential equation, Lectures in applied mathematics, V.3A, 1964, Johan Wiley and sons.

[2] T.P. Branson and P.B. Gilkey, The asymptotics of the Laplacian on a manifold with boundary, Comm. Partial Differential Equations, 15(1990), no. 2, 245-272.

[3] E. Balslev, Absence of positive eigenvalues of Schrödinger operators, Archive for rational mechanics and analysis, V.59, Number 4, 343-357(1975).

[4] D. Colton and B.D. Sleeman, Uniqueness theorems for the inverse problem of acoustic scattering, IMA Jounal of Applied Mathematics, 31(1983), 253-259.

[5] H. Groemer, Geometric applications of Fourier series and spherical harmonics, Encyclopedia of mathematics and its applications, v. 61, Cambridge University Press, New York, 1996.

[6] L. Hörmander, The analysis of linear partial differential operators I, Springer-Verlag, Berlin-Heidelberg, 1990.

[7] V. Isakov, New stability results for soft obstacles in inverse scattering, Inverse Problem, 9(1993), 535-543.

[8] V. Isakov, Uniqueness and stability in multi-dimenstional inverse problem, Inverse Problems, 9(1993), 579-621.

[9] V. Ivrii, Precise spectral asymptotics for elliptic operators acting in fiberings over manifolds with boundary, Lecture notes in mathematics, V.1100, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1984.

[10] V. Ivrii, Second term of the spectral asymptotic expansion of the Laplace-Beltrami operator on manifolds with boundary, Funktsioal'nyi Analiz i Ego Prilozheniya, 14(1980), No.2, 25-34.

[11] V. Ivrii, Exact spectral asymptotics for the Laplace-Beltrami operator in the case of general elleptic boundary conditions, Funksional'nyi Analiz i Ego Prilozheniya, V 15, No.1, 74-75(1981).

[12] A. Koldobsky, Fourier analysis in convex geometry, Mathematical Surveys and Monographs, 116, American Mathematical Society, Providence, RI, 2005.

[13] P.D. Lax and R.S. Phillips, Scattering theory, New York, Acdemic press, 1989.

[14] P.D. Lax and R.S. Phillips, A logarithmic bound on the location of the poles of the scattering matrix, Arch. Rational Mech. Anal, 40, 268-280(1971).

[15] R.B Melrose, Geometric scattering theory, Cambridge university press, 1995.

[16] V. Petkov and G. Popov, Asymptotic behavior of the scattering phase for non-trapping obstacles, Ann. Inst. Fourier, Grenoble, 32(3), 111-149(1982).

[17] Reed and Simon, Methods of Modern Mathematical Physics, v.1 and v.2 , Academic press, new York, 1975.

[18] N. Shenk and D. Thoe, Resonant states and poles of the scattering matrix for perturbations of $-\Delta$, Journal of mathematical analysis and applications, 37, 467-491(1972).

[19] N. Shenk and D. Thoe, Outgoing solutions of $(-\Delta + q - k^2)u = f$ in an exterior domain, Journal of mathematical analysis and applicaitons, 31, 81-116(1970).

[20] J. Sjöstrand and M. Zworski, Complex scaling method and the distribution of scattering poles, J. Amer. Math. Soc. 4(1991), 729-769.

[21] E.C. Titchmarsh, "The Theory of Functions", Oxford University Press, 2nd ed.

[22] M. Zworski, Poisson formula for resonances in even dimension, Asian J. Math, 2(3)(1998), 615-624.