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Commensurability effects in one-dimensional Anderson localization: anomalies in eigenfunction statistics

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Abstract.

The one-dimensional (1d) Anderson model (AM), i.e. a tight-binding chain with uncorrelated Gaussian disorder in the on-site energies, has statistical anomalies at any rational point $f = \frac{2a}{\lambda_E}$, where a is the lattice constant and λ_E is the de Broglie wavelength. We develop a regular approach to anomalous eigenfunction statistics at such commensurability points. The approach is based on an exact integral transfer-matrix equation for a generating function $\Phi(u, \phi; r)$ (u and ϕ have a meaning of the squared amplitude and phase of eigenstates, r is the position of the observation point). This generating function can be used to compute local statistics of *normalized* eigenfunctions of 1d AM at *any disorder* and to address the problem of higher-order anomalies at $f = \frac{p}{q}$ with $q > 2$. However, in this paper we concentrate at the principle (center-of-band) anomaly at $E = 0$ ($f = \frac{1}{2}$).

In the leading order in the small disorder we have derived a second-order partial differential equation for the r-independent ("zero-mode") component $\Phi(u, \phi)$ at the $E = 0$ anomaly. This equation is nonseparable in variables u and ϕ. Yet, we show that due to a hidden symmetry, it is integrable and we construct an exact solution for $\Phi(u, \phi)$ explicitly in quadratures. Using this solution we have computed moments $I_m = \langle|\psi|^{2m}\rangle$ ($m > 1$) of the wave function distribution for a very long chain of the length N and found an essential difference between their m-behavior in the center-of-band anomaly and for energies outside this anomaly. Outside the anomaly the function $N\, I_m$ is determined by only one parameter ℓ_0 while more parameters are needed to describe $N\, I_m$ at $E = 0$.

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1. Introduction

Concepts and methods of the localization theory, which counts its origin from the seminal Anderson paper [1], have penetrated almost all branches of modern physics[2, 3] - from the description of transport in disordered media and the Quantum Hall effect to the theory of chaotic systems and turbulence. The one-dimensional (1d) Anderson model [1] (AM) – a tight-binding model with a diagonal disorder – is determined by the Schrödinger equation for the particle wave function ψ_i at the chain site i

$$t[\psi_{i-1} + \psi_{i+1}] + \varepsilon_i \psi_i = E\psi_i \,. \tag{1}$$

Here, the nearest neighbor hopping amplitude t is the same for all bonds (below we put $t = 1$); the on-site energy ε_i is a random variable uncorrelated at different sites and characterized by a zero average $\langle \varepsilon_i \rangle = 0$ and the variance

$$\langle \varepsilon_i \varepsilon_j \rangle = w\delta_{ij} \,, \tag{2}$$

which is a measure of the disorder strength. The variance Eq.(2) is the only quantity that enters our theory at weak disorder $w \ll 1$. For strong disorder or for distributions of ε_i with long tails (e.g. for the Cauchy distribution), we derive equations in terms of the entire distribution of on-site energies (uncorrelated at different sites):

$$\mathcal{F}(\varepsilon) = \langle \delta(\epsilon - \epsilon_i) \rangle. \tag{3}$$

For a finite chain ($i = 1, 2, \cdots, N$), Eq.(1) is supplemented with the definition

$$\psi_0 = 0 = \psi_{N+1} \,. \tag{4}$$

In the absence of the disorder ($\varepsilon \equiv 0$), the normalized wave functions of the chain are $\psi_j = \sqrt{2/N}\sin(kj)$ and the corresponding eigenenergies are

$$E(k) = 2\cos(k) \,, \tag{5}$$

where $k = \pi l/N$ ($l = 1, ..., N$); for an infinite chain ($N \to \infty$) it fills the interval $(0, \pi)$. Due to the band symmetry, it is sufficient to consider only $k \in (0, \pi/2)$.

The most studied is the continuous limit of AM, where the lattice constant $a \to 0$ at ta^2 remaining finite [5, 6, 7, 8]. There was also a great deal of activity [9, 10] aimed at a rigorous mathematical description of 1d AM. However, despite considerable efforts invested, a lot of issues concerning 1d AM still remain unsolved. Among them there are effects of commensurability between the de-Broglie wavelength λ_E (dependent on the energy E) and the lattice constant a, i.e., at rational values of k/π.

The first known manifestation of commensurability effects was found and described quite early [11, 12] for the simplest object - the Lyapunov exponent. The latter is determined in terms of a solution of the Schrödinger equation (1) for a *semi-infinite* chain ($i = 1, 2, \cdots$) supplemented with the definition $\psi_0 = 0$ at only one end. For an arbitrary energy E and a generic boundary condition $\psi_1 \sim 1$, the solution to Eq.(1) is a superposition of two solutions, decreasing and increasing with the increase of i.

The increasing part determines the Lyapunov exponent $\gamma(E)$ and the corresponding localization length $\ell(E)$:

$$\frac{1}{\ell(E)} = \Re\,\gamma(E) = \lim_{N\to\infty}\frac{1}{N}\Re\,\log\left(\psi_N/\psi_1\right) = \lim_{N\to\infty}\frac{1}{N}\Re\sum_{j=2}^{N}\log\left(\psi_j/\psi_{j-1}\right). \qquad (6)$$

In the continuous model with a weak disorder, the Lyapunov exponent $\gamma^0(E)$ and the localization length $\ell_0(E) \propto \sin^2 k/w$ are smooth functions of energy E. However, according to [11, 12], for a discrete chain, the functions $\gamma(E)$ and $\ell(E)$ possess anomalous deviations from $\gamma^0(E)$ and $\ell_0(E)$ in narrow windows of the size $\propto w$ around the points $k = \pi/2$ (i.e., $E = E(k) = 0$) and $k = \pi/3$ (i.e., E= $E(k) = 1$). The Lyapunov exponent sharply *decreases* at $k = \pi/2$ (which is usually associated with an *increase* of the localization length) but may both increase or decrease at $k = \pi/3$ depending on the third moment $\langle\varepsilon_i^3\rangle$ of the on-site energy distribution [12]. It was also conjectured [12] that progressively weakening anomalies (for the weak disorder) may take place at every rational point $k/\pi = m/n$ (with natural m and n) of the band.

More recently [14, 15] it has been found that also the statistics of conductance in 1d AM are anomalous at the center of band ($E = 0$, $k = \pi/2$). This is not too surprising, because the chain conductance is expressed via the reflection and transition coefficients of an electron wave coming from an ordered region to *one* of the ends of the disordered chain. This problem has a lot of similarities with the problem of calculation of the Lyapunov exponent: solutions to both of them are expressed in terms of a probability distribution function $\mathcal{P}(\phi)$ of a single "angular" variable ϕ. The latter can be interpreted as a "phase" parameter in the representation of the wave function ψ_j in the form $\psi_j = a_j\cos(kj + \phi_j)$ with slowly varying real amplitude and phase. The center of band anomaly corresponds to an emergent non-triviality of the distribution function $\mathcal{P}(\phi)$ as compared to the trivial isotropic angular distribution in the continuous problem.

In a sense, the both problems touch "extrinsic" properties of localization, as the Lyapunov exponent describes only the *tails* of localized wave functions. In the problem of the average logarithm of conductance, the extrinsic character of this quantity is set by the distance $L \gg \ell(E)$ between the ideal leads. In other applications of which most important is the interplay between the localization and non-linearity [4], one is interested in the number of sites with a high amplitude of the wave function. This "intrinsic" picture of localization is better represented by the *inverse participation ratio* (IPR) $I \sim \int dx\ \langle|\psi(x)|^4\rangle$, where $\psi(x)$ is a random *normalized* eigenfunction obeying the Shrödinger equation Eq.(1) with the boundary conditions Eq.(4) at the *both ends* of the chain. The relation between the two descriptions is not clear yet. In particular, it is not known whether the localization length, determined via IPR and sensitive to short-range characteristics of eigenfunctions, coincides at points of anomaly $k/\pi = m/n$ with the length determined via Lyapunov exponent Eq.(6). This question is one of the motivations of the present paper aimed to develop a formalism to tackle a class of "intrinsic" problems connected with statistics of *normalized* eigenfunctions.

Here it is worth noting that computing IPR I and other quantities determined in terms of eigenfuctions is much more difficult than calculation of the Lyapunov exponent, since the latter problem does not require a *stationary* solution to the Shrödinger equation either to obey the boundary conditions at the both ends, or to be normalized. An attack on the "two-end" problem was undertaken in the pioneering paper [11] but with only limited success. The normalization condition imposed on the wave function amplitudes $\{a_j\}$ turned out to be very difficult for an analytical treatment. Only few quantities which are effectively insensitive to the normalization constraint have been calculated: they are the averaged density of states (DOS) and ratios of wave functions at different sites.

The difficulty of treating IPR and other quantities determined by local eigenfunctions is related with the fact that one needs to deal with an unknown joint probability distribution function $\mathcal{P}(a, \phi)$ of two variables, both the phase and the amplitude.

In this paper we develop a regular approach to the phenomenon of wave function statistical anomalies. The approach is based on a transfer-matrix equation (TME) for a generating function of two variables $\Phi(u, \phi)$ (u has a meaning of the squared amplitude of wave functions), which is a universal tool to describe properties of a generic 1d or quasi-1d system. The generating function can be used to compute *any* local statistics of *normalized* eigenfunctions of 1d AM. In principle, it determines a *joint probability distribution* $\mathcal{P}(u, \phi)$ of the (squared) amplitude u and phase ϕ of the random eigenfunction $\psi \sim \sqrt{u}\cos\phi$.

We will concentrate mostly on the study of the principle ("center-of-band") anomaly at $k = \pi/2$ ($E(k) = 0$) at weak disorder and show that the corresponding TME for $\Phi(u, \phi)$ has anomalous terms which make it essentially two-dimensional second-order partial differential equation (PDE). This equation is nonseparable in variables u and phase ϕ. Yet, we show that it is integrable and we construct its exact solution explicitly in quadratures.

In the next section 2 we present an *elementary* derivation of the transfer-matrix equation (TME) which is valid at any strength of disorder. This derivation does not exploit the supersymmetry method [16], used in earlier approaches (see [17, 3, 19]). In section 3 we introduce amplitude-phase variables connected with the representation of eigenfunctions in the form $\psi \sim \sqrt{u}\cos\phi$. In these variables, assuming weak disorder $w \ll 1$, we obtain a partial differential equation for the generating function, section 4. From that moment on we concentrate on the study of the principle, center-of-band anomaly. In section 5 we consider a partial differential equation for the case $k = \pi/2$ ($E(k) = 0$) and show its integrability. Namely, we show that this equation can be factorized in new variables and we can construct its solutions. But it turns out that due to non-Hermitian nature of the differential operator, there is a continuum of possible solutions. This huge redundancy problem is analyzed in section 6 where we show how physical requirements imposed on the generating function allow to find the unique solution. This solution is used in section 7 to compute moments of the wave function distribution. It is shown that these moments cannot be described by only one-parameter. This means invalidity of the one-parameter scaling at the anomalous center-of-band point. The paper is concluded with discussions on analogies and further extensions of the studied phenomenon.

2. Elementary derivation of the transfer-matrix equation (TME)

The quantity

$$I_m(r,E) \equiv \frac{1}{\nu(E)} \sum_\nu \left\langle |\psi_r^{(\nu)}|^{2m} \delta(E-E_\nu) \right\rangle \tag{7}$$

generalizes the concept of the inverse participation ratio (IPR), given by Eq.(7) at $m=2$, to an arbitrary natural m. Here, $\psi_r^{(\nu)}$ is a ν-th eigenfunction of Eqs.(1) and (4) with the eigenenergy E_ν, the summation runs over all states, the angular brackets denote the averaging over the ensemble of random site potentials $\{\varepsilon_j\}$, and the quantity

$$\nu(E) = N^{-1} \sum_\nu \langle \delta(E-E_\nu) \rangle \tag{8}$$

is the averaged density of states (DOS) at a given energy E. For a weak disorder, $\nu(E)$ is a smooth function of energy which only slightly differs (inside the energy band) from the corresponding function $\nu_0(E)$ for the ordered system:

$$\nu(E) \approx \nu_0(E) = \frac{1}{2\pi \sin\left[k(E)\right]} \,. \tag{9}$$

One expects that the dependence of $I_m(r,E)$ on the chain site r disappears when this site is far from the chain ends (on the scale of the localization length).

For $m=1$, the quantity $\sum_r I_{m=1}(r,E) = N$ just due to the normalization of wave functions. This means that far from the ends of a long disordered chain where the wave functions are localized and almost insensitive to the chain length, the local quantity $I_{m=1}(r,E)$ is site independent and $I_{m=1}(r,E)=1$.

For $m \geq 2$, the quantity $I_m(r,E)$ can be expressed in terms of the Green's functions of the chain problem, Eqs.(1) and (4), with the use of a limiting procedure (see, e.g., the review [3]):

$$I_m(r,E) = \lim_{\eta\to+0} \frac{(i\eta)^{m-1}}{2\pi\nu(E)} \langle G_{r,r}^{m-1}(E_+) G_{r,r}(E_-) \rangle \;\; ; \;\; E_\pm = E \pm \frac{i\eta}{2} \,. \tag{10}$$

The Green's functions

$$G_{j,r}(E_\pm) = \sum_\nu \frac{\psi_j^{(\nu)} \psi_r^{(\nu)*}}{E_\pm - E_\nu}$$

(with a source at the site r) obey the equations

$$[G_{j-1,r} + G_{j+1,r}] + \varepsilon_j G_{j,r} + \delta_{j,r} = E_\pm G_{j,r} \tag{11}$$

$$G_{0,r} = 0 = G_{N+1,r} \,. \tag{12}$$

Instead of the supersymmetry approach [16], where Green's functions are represented as functional integrals over the usual complex ("bosonic") and Grassmann ("fermionic") variables, here we will present an elementary derivation. The derivation is close in spirit to the methods used in refs. [11, 12].

For $j \neq r$, dividing Eq.(11) by $G_{j,r}$ (as is justified below, this quantity differs from 0), we obtain the recursive equation:

$$q_j^{\pm} + \frac{1}{q_{j-1}^{\pm}} \equiv \frac{G_{j+1,r}(E_{\pm})}{G_{j,r}(E_{\pm})} + \frac{G_{j-1,r}(E_{\pm})}{G_{j,r}(E_{\pm})} = E_{\pm} - \varepsilon_j \,, \tag{13}$$

supplemented with the definitions (see Eq.(12))

$$1/q_0^{\pm} = 0 = q_N^{\pm} \,, \tag{14}$$

Starting with $j = 1$ and using Eq.(13) to go from $j-1$ to j, one can find all the $q_j^{\pm}$, $0 < j < r$ as functions of $\varepsilon_1, \ldots, \varepsilon_{r-1}$. Similarly, starting with $j = N$ and going from j to $j-1$, one finds all the $q_j^{\pm}$, $r \leq j < N$ as functions of $\varepsilon_{r+1}, \ldots, \varepsilon_N$. Finally, from Eq.(11) at $j = r$ we obtain the quantities of our interest, $G_{r,r}(E_{\pm})$:

$$G_{r,r}(E_{\pm}) = \frac{1}{E_{\pm} - \varepsilon_r - q_r^{\pm} - 1/q_{r-1}^{\pm}} = \mp i \int_0^{\infty} d\lambda \, \exp\left[\pm i\lambda \left(E_{\pm} - \varepsilon_r - q_r^{\pm} - \frac{1}{q_{r-1}^{\pm}}\right)\right]; \tag{15}$$

$$G_{r,r}^{m-1}(E_+) = \frac{(-i)^{m-1}}{(m-2)!} \int_0^{\infty} d\lambda \, \lambda^{m-2} \exp\left[i\lambda \left(E_+ - \varepsilon_r - q_r^+ - \frac{1}{q_{r-1}^+}\right)\right] \;\; ; \;\; m \geq 2 \,. \tag{16}$$

To justify the transition from Eq.(11) to Eq.(13), we should prove that $G_{j,r}(E_{\pm}) \neq 0$. Note that $q_1^+ = E + i\eta/2 - \varepsilon_1$, hence $\Im\, q_1^+ > 0$. Assuming that $\Im\, q_{j-1}^+ > 0$, we obtain (for $j < r$): $\Im\, q_j^+ = -\Im\,(1/q_{j-1}^+) + \eta/2 > 0$, hence $\Im\, q_j^+ > 0$, and by induction this is true for any $j < r$, which means all the corresponding q_j^+ (and $G_{j,r}$) are nonzero. Similarly, for quantities q_j^- we find $\Im\, q_j^- < 0$. And for the case $j > r$, we obtain $\Im\, q_j^+ < 0$ and $\Im\, q_j^- > 0$.

As a by-product of this proof, we have confirmed the expected positiveness (negativeness) of the imaginary part of the denominator of $G_{r,r}(E_+)$ ($G_{r,r}(E_-)$), which justifies the integral representation in Eq.(15).

With the use of Eq.(15), the expression Eq.(10) can be represented in the form ($m \geq 2$):

$$I_m(r,E) = \frac{1}{2\pi (m-2)!\, \nu(E)} \lim_{\eta \to +0} \eta^{m-1} \int_0^{\infty}\int_0^{\infty} d\lambda_1 d\lambda_2 \lambda_1^{m-2} \, \mathrm{e}^{i(\lambda_1-\lambda_2)E - (\lambda_1+\lambda_2)\eta/2} \langle \mathrm{e}^{-i(\lambda_1-\lambda_2)\varepsilon_r} \rangle \mathcal{R}_{r-1}(\lambda_1,\lambda_2) \tilde{\mathcal{R}}_r(\lambda_1,\lambda_2) \,. \tag{17}$$

Here

$$\mathcal{R}_j(\lambda_1,\lambda_2) \equiv \langle \exp\left[-i\lambda_1/q_j^+ + i\lambda_2/q_j^-\right] \rangle \;\; ; \qquad j < r \,; \tag{18}$$

$$\tilde{\mathcal{R}}_j(\lambda_1,\lambda_2) \equiv \langle \exp\left[-i\lambda_1 q_j^+ + i\lambda_2 q_j^-\right] \rangle \;\; ; \;\; r \leq j \,, \tag{19}$$

where the averaging in Eqs.(18) and (19) is performed over random energies $\varepsilon_1, \ldots, \varepsilon_{r-1}$ and $\varepsilon_{r+1}, \ldots, \varepsilon_N$, respectively. The crucial assumption here is that the random energies ε_r are uncorrelated at different sites. The functions $\mathcal{R}_j$ and $\tilde{\mathcal{R}}_j$ obey recurrent equations. To derive them, we use the following identity for the Bessel function $J_0(x)$:

$$\exp(-z) = -\int_0^{\infty} d\lambda' J_0(2\sqrt{\lambda'}) \frac{\partial}{\partial \lambda'} \exp(-\lambda'/z) \qquad (\Re\, z > 0) \;\; , \tag{20}$$

which allows us to convert in the exponent of Eq.(18) the factor $1/q_j$ to q_j and then to apply Eq.(13). As a result, we obtain:

$$\mathcal{R}_j(\lambda_1,\lambda_2) = \int_0^\infty\int_0^\infty d\lambda_1' d\lambda_2'\, J_0(2\sqrt{\lambda_1\lambda_1'})J_0(2\sqrt{\lambda_2\lambda_2'}) \frac{\partial}{\partial\lambda_1'}\frac{\partial}{\partial\lambda_2'}\left[\left\langle \mathrm{e}^{i(\lambda_1'-\lambda_2')(E-\varepsilon_j)}\right\rangle \mathrm{e}^{-(\lambda_1'+\lambda_2')\eta/2}\mathcal{R}_{j-1}(\lambda_1',\lambda_2')\right] \quad (21)$$

with the initial condition $\mathcal{R}_0(\lambda_1,\lambda_2)=1$. Note that the function of ε_j in the integrand is statistically independent of quantities $\varepsilon_{j-1},\varepsilon_{j-2}$, etc. which determine the function $\mathcal{R}_{j-1}(\lambda_1',\lambda_2')$. In a similar way but proceeding from the site N to the site j, we derive the recursive equation for $\tilde{\mathcal{R}}_j(\lambda_1,\lambda_2)$:

$$\tilde{\mathcal{R}}_j(\lambda_1,\lambda_2) = \int_0^\infty\int_0^\infty d\lambda_1' d\lambda_2'\, J_0(2\sqrt{\lambda_1\lambda_1'})J_0(2\sqrt{\lambda_2\lambda_2'}) \frac{\partial}{\partial\lambda_1'}\frac{\partial}{\partial\lambda_2'}\left[\left\langle \mathrm{e}^{i(\lambda_1'-\lambda_2')(E-\varepsilon_{j+1})}\right\rangle \mathrm{e}^{-(\lambda_1'+\lambda_2')\eta/2}\tilde{\mathcal{R}}_{j+1}(\lambda_1',\lambda_2')\right] \quad (22)$$

with the initial condition $\tilde{\mathcal{R}}_N(\lambda_1,\lambda_2)=1$. For the considered case of site-independent statistics of the local disorder, one can see immediately that the function $\mathcal{R}_{N-j}(\lambda_1,\lambda_2)$ obeys the recursive Eq.(22) and equals unity at $j=N$. Therefore, this function should coincide with the function $\tilde{\mathcal{R}}_j(\lambda_1,\lambda_2)$ and we arrive at an identity:

$$\tilde{\mathcal{R}}_j(\lambda_1,\lambda_2) = \mathcal{R}_{N-j}(\lambda_1,\lambda_2)\,. \quad (23)$$

To perform the limit operation $\eta\to+0$ in Eq.(17) and Eq.(21), we introduce new variables

$$s=\eta(\lambda_1+\lambda_2)/2\ ;\ \ v=\lambda_1-\lambda_2 \quad (24)$$

and a new function:

$$W_j(s,v)\equiv\mathcal{R}_j(s/\eta+v/2,s/\eta-v/2)\ ;\ \tilde{\mathcal{R}}_j(s/\eta+v/2,s/\eta-v/2)=W_{N-j}(s,v)\,. \quad (25)$$

Using asymptotic of the Bessel function, integrating by parts in Eq.(21), and neglecting infinitely fast oscillating terms, we arrive at the following recursive equation:

$$W_j(s,v)=\frac{\sqrt{s}}{2\pi}\int_{-\infty}^\infty dv'\int_0^\infty\frac{ds'}{(s')^{3/2}}\mathrm{e}^{-s'}\cos\left[\sqrt{ss'}\left(\frac{v}{s}+\frac{v'}{s'}\right)\right]\mathrm{e}^{iv'E}\,\chi(v')\,W_{j-1}(s',v'), \quad (26)$$

where $\chi(v')$ is the characteristic function of the on-site energy distribution $\mathcal{F}(\varepsilon)$, Eq.(3):

$$\chi(v)=\int d\varepsilon\,\mathcal{F}(\varepsilon)\,e^{-i\varepsilon v}\equiv\left\langle \mathrm{e}^{-iv'\varepsilon_j}\right\rangle. \quad (27)$$

In new variables, Eq.(17) takes the form ($m\geq 2$):

$$I_m(r,E)=\frac{1}{2\pi(m-2)!\,\nu(E)}\int_{-\infty}^\infty dv\int_0^\infty ds\,s^{m-2}\,\mathrm{e}^{ivE-s}\,\chi(v)\,W_{r-1}(s,v)W_{N-r}(s,v)\,. \quad (28)$$

Free of any limit operation, Eqs.(26) and (29) are the starting point of our analysis. For the 1d problem of interest, they were obtained in [19], and still earlier in [17, 18] in a study of the localization transition on the Bethe lattice. The presented here elementary derivation of these equations is considerably simpler than the supersymmetry approach used in [19, 17]. Also, it allows to establish a relation between the generating function $W_j(s,v)$ of our interest and the phase distribution function $\mathcal{P}(\phi)$ (see below).

3. The "amplitude-phase" variables z and ϕ

3.1. Exact equations in (s,q) variables

To proceed, we introduce the Fourier-transform of $W_j(s,v)$ in the variable v: $\tilde{W}_j(s,q) = \int dv \exp{(iqv)} W_j(s,v)$. The basic equations (29),(26) in the new variables (s,q) take the following form:

$$I_m(r,E) = \frac{1}{2\pi(m-2)!\,\nu(E)} \int_0^\infty ds\, s^{m-2}\, \mathrm{e}^{-s}$$
$$\int_{-\infty}^{\infty} \frac{dq\, dq'}{2\pi}\, \tilde{W}_{r-1}(s,q)\, \mathcal{F}(E-q-q')\, \tilde{W}_{N-r}(s,q')\,, \tag{29}$$

$$\tilde{W}_j(s,q) = \frac{\mathrm{e}^{-sq^2}}{q^2} \int_{-\infty}^{\infty} dq'\, \mathcal{F}(E-q^{-1}-q')\, \tilde{W}_{j-1}(sq^2,q')\,. \tag{30}$$

Eqs.(29),(30) are *exact* for uncorrelated on-site energies with the arbitrary distribution function $\mathcal{F}(\varepsilon)$. However, the advantage of the (s,q) variables become clear only for weak disorder when $\mathcal{F}(\varepsilon)$ is sharply peaked at $\varepsilon = 0$. Compared to $\mathcal{F}(\varepsilon)$ all other functions can be considered as smoothly varying.

3.2. Weak disorder: expression for moments in the amplitude-phase variables

At weak disorder one can replace $\mathcal{F}(\varepsilon)$ by a δ-function in Eq.(29) and immediately do integration over q'. The physical meaning of the two remaining variables become more transparent if we introduce yet another couple of variables, z and ϕ, determined by:

$$s = z\cos^2{(\phi+k)} \quad ; \quad q = \frac{\cos\phi}{\cos{(\phi+k)}}\,, \tag{31}$$

and a new function:

$$\Phi_j(z,\phi) = \tilde{W}_j(s,q)\frac{\sin k}{2\pi\cos^2{(\phi+k)}}, \tag{32}$$

with the boundary condition

$$W_0(s,v) = \mathcal{R}_0(s/\eta+v/2, s/\eta-v/2) = 1 \;\Rightarrow$$
$$\Rightarrow\; \Phi_0(z,\phi) = 2\pi\delta(q)\frac{\sin k}{2\pi\cos^2{(\phi+k)}} = \delta(\phi-\pi/2). \tag{33}$$

Here $z \in (0,\infty)$; k is determined by the relation $E = 2\cos k$; and the "phase" variable ϕ changes within the interval $(0,\pi)$, where there is one-to-one correspondence between ϕ and $q(\phi)$. Alternatively, we may use so called "extended band" representation, where ϕ is arbitrary, but the function $\Phi_j(z,\phi)$ obeys the periodicity condition

$$\Phi_j(z,\phi+\pi) = \Phi_j(z,\phi+\pi) \;\;;\;\; \forall\phi\,, \tag{34}$$

and integrations over ϕ can be taken over any interval of the length π.

The product $\tilde{W}_{r-1}(s,q)\tilde{W}_{N-r}(s,E-q)$ which arises in Eq.(29) at $\mathcal{F}(\varepsilon) \approx \delta(\varepsilon)$ should be transformed into a product $\Phi_{r-1}(z,\phi)\Phi_{N-r}(z',\phi')$ according to Eq.(31) and the conditions:

$$s(z,\phi) = s = s(z',\phi') \;\;;\;\; q(\phi) = q \;\;;\;\; q(\phi') = E-q\,. \tag{35}$$

The last two conditions and Eqs.(5) and (31) result in

$$\frac{\cos\phi'}{\cos(\phi'+k)} = 2\cos k - \frac{\cos\phi}{\cos(\phi+k)} \Rightarrow -\tan(\phi'+k) = \tan(\phi+k)\,, \quad (36)$$

from where it follows that

$$\phi' = -\phi - 2k\,,\ \operatorname{mod}(\pi)\,. \quad (37)$$

As a consequence, $\cos^2(\phi+k) = \cos^2(\phi'+k)$ and the quantities $z' \equiv s/\cos^2(\phi'+k)$ and z coincide. Then taking into account Eq.(9), one reduces (for weak disorder, $w \ll 1$) Eq.(29) to:

$$I_m(r,E) = \frac{2\pi}{(m-2)!}\int_0^\pi d\phi\, \cos^{2m}(\phi)\int_0^\infty dz\, z^{m-2}\Phi_{r-1}(z,\phi-k)\Phi_{N-r}(z,-\phi-k)\,. \quad (38)$$

This expression is completely determined by the "generating function" $\Phi_j(z,\phi)$.

3.3. Physical meaning of (z,ϕ) variables.

The integrand in Eq.(38) for the m-th moment of the quantum-mechanical probability density $I_m \sim \langle|\psi^2|^m\rangle$ contains $\cos^{2m}(\phi)\, z^m$ which suggests the physical meaning of ϕ and $\sqrt{z}$ as a phase and the amplitude of the eigenfunction $\psi \propto \sqrt{z}\,\cos(\phi)$. However, one may ask a question what is the meaning of a phase for a wave function in one dimensions which may always be chosen *real*. The answer is that the complete description of eigenfunctions requires two variables defined on a link. To formalize this statement, we follow Kappus and Wegner [11] and introduce $a_j > 0$ and ϕ_j variables defined on the link between the chain sites j $(0 < j < r)$ and $j+1$ in such a way that

$$G_{j,r}(E) = a_j\cos\phi_j \ ; \ G_{j+1,r}(E) = a_j\cos(\phi_j+k) \ \Rightarrow \ q_j = \frac{\cos(\phi_j+k)}{\cos\phi_j} \quad (39)$$

One can see that such an ansatz is compatible with Eq.(13). For the link between the sites 0 and 1 we define $\phi_0 = \pi/2$, so that $1/q_0 \equiv 0$. For brevity, superscripts $\pm$ in Eq.(40) are not indicated.

Near the poles $E \to E_\mu$, where E_μ is an exact eigenvalue, Eq.(40) can be considered as an equation for the eigenfunction $\psi_\mu(r)$ on a link $(j, j+1)$:

$$\psi_\mu(j) = \sqrt{z_j}\cos\phi_j \ ; \ \psi_\mu(j+1) = \sqrt{z_j}\cos(\phi_j+k). \quad (40)$$

3.4. Generating function and the phase distribution function

An exact equation Eq.(30) for the generating function $\Phi_j(z,\phi)$ in the variables z,ϕ reads:

$$\Phi_{j+1}(z,\phi) = \frac{\sin k\, \mathrm{e}^{-z\cos^2\phi}}{\cos^2\phi}\int_0^\pi d\phi' \mathcal{F}\left((\sin k\,(\tan\phi' - \tan\phi)\right)\, \Phi_j\left(\frac{z\cos^2\phi}{\cos^2\phi'}, \phi'-k\right)\,. \quad (41)$$

At vanishing disorder when the on-site energy distribution function $\mathcal{F}(\varepsilon) = \delta(\varepsilon)$, Eq.(41) equation reduces to

$$\Phi_{j+1}(z,\phi) = \mathrm{e}^{-z\cos^2\phi}\Phi_j(z,\phi-k)\,. \quad (42)$$

In particular, at $z = 0$ using the boundary conditions Eq.(33) we obtain:

$$\Phi_j(0, \phi) = \delta(\phi - jk - \frac{\pi}{2}) \,. \tag{43}$$

As the phase ϕ at vanishing disorder is varying like kj with the site number j, Eq.(43) suggests that $\Phi_j(z = 0, \phi)$ is the phase distribution function. Now we prove that this statement is true at an arbitrary disorder.

Indeed, from definitions Eqs.(18) and (25) we obtain

$$W_j(s = 0, v) = \left\langle \exp\left[-i\frac{v}{2}\left(\frac{1}{q_j^+} + \frac{1}{q_j^-} \right) \right] \right\rangle \Rightarrow \tilde{W}_j(0, q) = 2\pi \left\langle \delta\left[q - \frac{\cos(\phi_j)}{\cos(\phi_j + k)} \right] \right\rangle , \tag{44}$$

where it was taken into account the merging of q_j^+ and q_j^- in the limit $\eta \to 0$. Passing to the variables z and ϕ, Eq.(31), we find for the generating function $\Phi_j(z, \phi)$, Eq.(32), at $z = 0$:

$$\Phi_j(z = 0, \phi) = \langle \delta(\phi - \phi_j) \rangle \equiv \mathcal{P}_j(\phi) \,. \tag{45}$$

Thus $\Phi_j(z = 0, \phi)$ is equal to what is essentially the definition of the phase distribution function.

This important relation establishes an exact correspondence between the generating function $\Phi_j(z = 0, \phi)$ and the probability distribution function $\mathcal{P}_j(\phi)$ for the phase variable ϕ_j. This identity will be used later for the proper normalization of the constructed "stationary" (site-independent) solution $\Phi(z, \phi)$. It shows also that the generating function $\Phi_j(z, \phi)$ contains much more information about the system than $\mathcal{P}_j(\phi)$. This is the reason why the problem of our interest - calculation of the generalized IPR, Eqs.(7), (38), determined by the whole $\Phi_j(z, \phi)$, is much more difficult than the calculation of the Lyapunov exponent and similar quantities determined merely by $\mathcal{P}_j(\phi)$. To emphasize the difference between the two problems, note that the generalized IPR, Eq.(38), is not linear but *quadratic* in $\Phi(z, \phi)$. Hence the generating function $\Phi(z, \phi)$ itself *cannot* be considered as a joint probability distribution function of z and ϕ. The problem of finding the joint probability distribution is not simple, and involves, as the first step, the calculation of the moments I_m for integer $m > 0$. This will be our main goal in this paper.

4. Differential equation for the generating function

An exact *integral* equation Eq.(41) contains all the information about local statistics of eigenfunctions at any uncorrelated on-site disorder. However, only at weak disorder the statistical anomalies we are focusing at in this paper are sharp. The point is that the region in the energy E (or in the parameter k) where statistics is anomalous is proportional to $\Delta E \sim w^2$ [12], and for strong disorder $w \sim 1$ the anomalies are rounded off. That is why in what follows we consider only the case of weak disorder $w \ll 1$.

4.1. Recursive differential operator

For the case of a weak disorder, $w \ll 1$, when the "bare" (i.e. for the continuous model)) localization length $\ell_0 \gg 1$, the "typical" squared amplitude of localized eigenfunctions $z_{typ} \sim 1/\ell_0 << 1$ and the exponential factor in front of the integral in Eq.(41) can be expanded in powers of z. From now we will introduce a re-scaled variable

$$u \equiv \ell_0 \, z = \frac{2 \sin^2(k)}{w} z \,, \tag{46}$$

keeping the notation $\Phi(u, \phi) = \Phi(z, \phi)|_{z=u/\ell_0}$ for the function of this variable. For small w, the function $\mathcal{F}(\varepsilon)$ in Eq.(41) is strongly peaked at $\varepsilon = 0$ and the integration in ϕ' is effectively restricted to a narrow vicinity of ϕ. Expanding the remaining part of the integrand in powers of $(\tan\phi' - \tan\phi)$ and keeping only first order terms in w, we represent Eq.(41) in the form:

$$\Phi_{j+1}(u, \phi) = \left[1 + \frac{1}{\ell_0} \left[\mathcal{L}(u, \phi) - c_1(\phi) \, u\right]\right] \Phi_j(u, \phi - k) \,, \tag{47}$$

where $\mathcal{L}(u, \phi)$ is the second order differential operator

$$\mathcal{L}(u, \phi) = c_2(\phi) \, u^2 \partial_u^2 + c_3(\phi) \, (u\partial_u - 1) + c_4(\phi) \, u\partial_u\partial_\phi + c_5(\phi) \, \partial_\phi + c_6(\phi) \, \partial_\phi^2 \,, \tag{48}$$

The coefficients $c_i(\phi)$ in Eqs.(47) and (48) are all combinations of $\cos(2\phi)$ and $\sin(2\phi)$ and at first glance do not show any nice structure:

$$c_1(\phi) = \frac{1}{2}[1 + \cos(2\phi)] \, ; \; c_2(\phi) = 1 - \cos^2(2\phi) \, ; \; c_3(\phi) = -[1 - \cos(2\phi) - 2\cos^2(2\phi)]$$

$$c_4(\phi) = \sin(2\phi)[1 + \cos(2\phi)] \, ; \; c_5(\phi) = -\frac{3}{2} \sin(2\phi)[1 + \cos(2\phi)] \, ; \; c_6(\phi) = \frac{[1 + \cos(2\phi)]^2}{4} \tag{49}$$

Note that in the leading order in the disorder strength Eq.(47) depends only on the *variance* of the on-site disorder distribution $\mathcal{F}(\varepsilon)$.

From Eq.(47) and the established relation Eq.(45), we can immediately write a recursive equation for the phase distribution function $\mathcal{P}(\phi)$:

$$\mathcal{P}_{j+1}(\phi) = \left[1 + \frac{1}{\ell_0} \, \mathcal{L}(\phi)\right] \mathcal{P}_j(\phi - k) \,, \tag{50}$$

where

$$\mathcal{L}(\phi) = -\frac{\partial}{\partial\phi} \left[\frac{\sin(2\phi)[1 + \cos(2\phi)]}{2} - \frac{[1 + \cos(2\phi)]^2}{4} \frac{\partial}{\partial\phi}\right] . \tag{51}$$

Eqs.(47) and (50) are functional rather than differential equations because of different phase arguments in the left- and right-hand sides.

4.2. Differential equation at rational $k/\pi = m/n \neq 1/2$

In Eqs.(47) and (50) the phase argument experiences a finite jump $-k$ at the transition from the site j to $j+1$. When $k = \pi m/n$ with natural m and n, then after n transitions the shift of the phase argument becomes multiple of π and the functional equations become the differential ones due to the periodicity of $\Phi(u,\phi)$, Eq.(34). Thus, iterating Eq.(47) n times we get a closed equation for $\Phi_j(u,\phi)(=\Phi_j(u,\phi-nk))$:

$$\begin{aligned}\Phi_{j+n}(u,\phi) = &\left[1+\frac{1}{\ell_0}\left[\mathcal{L}(u,\phi)-c_1(\phi)\,u\right]\right]...\\ &\left[1+\frac{1}{\ell_0}\left[\mathcal{L}(u,\phi-(n-1)k)-c_1(\phi-(n-1)k)\,u\right]\right]\Phi_j(u,\phi-nk)\,. \end{aligned}\tag{52}$$

Keeping only first order terms in the disorder strength w, we obtain:

$$\Phi_{j+n}(u,\phi)-\Phi_j(u,\phi) = \frac{1}{\ell_0}\left[\sum_{r=0}^{n-1}\mathcal{L}(\phi-r\,\pi m/n)-u\sum_{r=0}^{m-1}c_1(\phi-r\,\pi m/n)\right]\Phi_j(u,\phi). \tag{53}$$

Here the result of the summation is extremely sensitive to the particular value of $k=\pi m/n$ and this is the formal reason of an emerging anomaly. Indeed, the functions Eq.(49) contain only terms ~ 1, $\mathrm{e}^{\pm 2i\phi}$, and $\mathrm{e}^{\pm 4i\phi}$, for which we have

$$\sum_{r=0}^{n-1}e^{2i(\phi-r\,\pi m/n)}=0, \qquad \sum_{r=0}^{n-1}e^{4i(\phi-r\,\pi m/n)}=\begin{cases}0 & n>2\\ 2\mathrm{e}^{4i\phi} & n=2\,.\end{cases} \tag{54}$$

Thus, for $k\neq\pi/2$ (i.e., $E(k)\neq 0$), only ϕ-independent parts of the coefficients Eq.(49) survive in Eq.(53). Assuming $n\ll\ell_0$, expanding the L.H.S. of Eq.(53), and introducing the "continuous" dimensionless coordinate $x=j/(2\ell_0)$ along the chain, we obtain:

$$\partial_x\Phi(u,\phi)=\left[u^2\partial_u^2-u+\frac{3}{4}\partial_\phi^2\right]\Phi\,. \tag{55}$$

The variables u and ϕ are separated and one can immediately find a "stationary" (i.e. independent of x) solution

$$\Phi(u,\phi)=\Phi(u)=\frac{2}{\pi}\sqrt{u}\,K_1(2\sqrt{u})\,. \tag{56}$$

This *zero mode* solution describes the limit of a long chain with the length $L\gg\ell_0$, it is the only one which survives at distances $x\gg\ell_0$. This solution has been earlier obtained [8] in the continuous limit ($n\gg 1$). It also arises in the theory of a multi-channel disordered wire [16, 3]. As follows from Eq.(55), non-zero modes decay at distances $x=j/(2\ell_0)\sim 1$ providing so called "phase randomization": the zero-mode solution corresponds to the absolutely isotropic distribution of the phase ϕ.

The corresponding moments I_m ($m=1,2,...$) are found from Eq.(38) and are equal to:

$$I_m^{norm}=\frac{(m-1)!}{(4\ell_0)^{m-1}} \tag{57}$$

The solution Eq.(56) corresponds to the following probability distribution of squared wave functions $|\psi|^2$ ($=z\cos^2(\phi)$) in a long *strictly* one-dimensional system (amazingly, this result was not known before):

$$\mathcal{P}(|\psi|^2)\,d\,|\psi|^2=\frac{4\ell_0}{L}\,\frac{\exp\left(-4|\psi|^2\ell_0\right)}{|\psi|^2}\,d\,|\psi|^2, \quad (|\psi|^2\ell_0\gg e^{-L/(4\ell_0)}). \tag{58}$$

where L ($=N$ for the unit spacing a) is the chain length. Note that the distribution Eq.(58) is not normalizable, as the normalization integral is logarithmically divergent.

This divergency is an artefact of the zero-mode approximation and is typical to exponentially localized wavefunctions. The point is that the eigenfunction statistics changes drastically for very small values of the amplitude $|\psi|^2\ell_0 \ll e^{-L/(4\ell_0)}$, where the zero-mode approximation no longer applies. This is related with the fact that the *envelope* (i.e. $|\psi|^2$ averaged over oscillations) of the typical localized wave function cannot be significantly smaller than $|\psi|^2 \sim e^{-L/4\ell_0}$. The smaller values of the amplitude $|\psi|^2$ are due to oscillations and nodes of wave functions which probability is different from that of the envelope. Thus in the exact, L-dependent distribution function the logarithmic divergency of the normalization integral is cut at $|\psi|^2\ell_0 < e^{-L/(4\ell_0)}$.

5. Center-of-band anomaly, $k = \pi/2$

As is seen from Eq.(54) for $k = \pi m/n = \pi/2$, terms $\sim \mathrm{e}^{\pm 4i\phi}$ survive in Eq.(53). This leads to a drastic modification of the phase-isotropic equation Eq.(55):

$$\partial_x\Phi \equiv \left[\hat{L} - u\right]\Phi = \Big[[1-\cos(4\phi)]\,u^2\partial_u^2 + \sin(4\phi)\,u\partial_u\partial_\phi \tag{59}$$

$$+\frac{3+\cos(4\phi)}{4}\partial_\phi^2 + 2\cos(4\phi)\,u\partial_u - \frac{3}{2}\sin(4\phi)\partial_\phi - 2\cos(4\phi) - u\Big]\Phi\,, \tag{60}$$

where the differential operator $\hat{L}$ depends explicitly on ϕ. The variables u and ϕ are not separable anymore, which results in an emergent center-of-band ($k = \pi/2 \Rightarrow E = 0$) *anomaly*: the generating function and the phase distribution function become non-isotropic in ϕ. The variables u and ϕ cannot be separated even for the stationary variant of Eq.(59) describing the zero mode:

$$[\hat{L} - u]\Phi = 0\,, \tag{61}$$

Yet, due to a hidden symmetry of Eq.(59), a proper choice of coordinates allows to separate variables in the stationary (zero mode) equation Eq.(61). This will be done in the next subsection.

For completeness, we conclude the present part by derivation of an exact expression for the stationary distribution function of phase $\mathcal{P}(\phi) = \Phi(u=0,\phi)$ (see Eq.(45)). Taking the limit $u \to 0$ in the stationary form of Eq.(59), we obtain the ordinary differential equation:

$$0 = \left[\frac{3+\cos(4\phi)}{4}\partial_\phi^2 - \frac{3}{2}\sin(4\phi)\partial_\phi - 2\cos(4\phi)\right]\mathcal{P}(\phi) =$$

$$\partial_\phi\left[\frac{3+\cos(4\phi)}{4}\partial_\phi - \frac{1}{2}\sin(4\phi)\right]\mathcal{P}(\phi)\,. \tag{62}$$

The only periodic solution to this equation has the form [12]:

$$\mathcal{P}^{(an)}(\phi) = \frac{4\sqrt{\pi}}{\Gamma^2(\frac{1}{4})}\,\frac{1}{\sqrt{3+\cos(4\phi)}}\,, \tag{63}$$

where the normalization constant provides the equality $\int_0^\pi d\phi\,\mathcal{P}(\phi) = 1$. Thus, the distribution of phases of eigenfunctions in a long weakly disordered chain at the center of band ($k = \pi/2$) is not isotropic but has maxima at $\phi = \pm\frac{\pi}{4}$. According to the interpretation Eq.(40) of the amplitude-phase variables, this implies the tendency towards smaller difference between $|\psi_j|^2$ and $|\psi_{j+1}|^2$, i.e. larger localization length. This phenomenon has been coined as *"the center-of-band-anomaly"*.

Now we proceed with our much more difficult task: studying not an ordinary but the partial differential equation (59) for the generating function $\Phi(u, \phi)$ of two variables.

5.1. Hidden symmetry and separation of variables

The integrability of the stationary equation (61) is shown in three steps. The step one is to pass from (u, ϕ) to a new set of variables (u, v) with $v = u\cos(2\phi)$, and to introduce a new function

$$\tilde{\Phi}(u, v) = \frac{1}{u}\Phi(u, \phi)|_{\cos(2\phi)=v/u} \,. \tag{64}$$

In these variables the zero-mode equation Eq.(59) takes a very symmetric form:

$$\sqrt{u^2 - v^2}\,\left\{\partial_u\,\sqrt{u^2 - v^2}\,\partial_u + \partial_v\,\sqrt{u^2 - v^2}\,\partial_v\right\}\,\tilde{\Phi} = \frac{u}{2}\,\tilde{\Phi}\,. \tag{65}$$

It is remarkable that the L.H.S. of this equation can be represented as $[D_1^2 + D_3^2]\,\tilde{\Phi}$ where the operators D_1 and D_3 belong to the family of three operators from the representation of the sl_2 algebra:

$$D_1 = \sqrt{u^2 - v^2}\,\partial_u \;\;;\;\; D_2 = u\,\partial_v + v\,\partial_u \;\;;\;\; D_3 = -\sqrt{u^2 - v^2}\,\partial_v \tag{66}$$

with the commutation relations:

$$[D_1, D_2] = -D_3,\; [D_3, D_1] = D_2,\; [D_2, D_3] = D_1. \tag{67}$$

Now it is clear that there is a hidden order in a set of coefficients in Eq.(59) resulting from the sl_2 symmetry. The latter frequently manifests in various scattering problems. However, Eq.(59) is connected with even higher symmetry. Introducing a set of three additional (mutually commuting) operators

$$B_1 = v,\; B_2 = \sqrt{u^2 - v^2},\; B_3 = u\,, \tag{68}$$

we can represent Eq.(59) in the form $[D_1^2 + D_3^2 - B_3/2]\,\tilde{\Phi} = 0$. The operators D_i and B_i constitute an algebra $D \bigoplus B$ with the commutative subalgebra B and commutation relations: $[D, D] = D$ (see Eq.(67)), $[B, B] = 0$, and $[D, B] = B$; in more detail, the latter relation looks like: $[D_i, B_i] = 0$ and

$$\begin{aligned}
&[D_1, B_2] = B_3 \;\;;\;\; [D_2, B_1] = B_3 \;\;;\;\; [D_3, B_1] = -B_2 \;\;; \\
&[D_1, B_3] = B_2 \;\;;\;\; [D_2, B_3] = B_1 \;\;;\;\; [D_3, B_2] = B_1\,.
\end{aligned} \tag{69}$$

It is tempting to interpret $D \bigoplus B$ as the algebra of generators of rotations (D) and translations (B) of the 3d pseudo-euclidian space $R^{1,2}$. However, the question of a constructive application of this symmetry to the considered problem remains open. Below we follow a more prosaic way.

The next step is to introduce a function

$$\Psi(u, v) = (u^2 - v^2)^{\frac{1}{4}}\,\tilde{\Phi}(u, v) \tag{70}$$

to transform Eq.(65) to the Schrödinger-like equation for the function $\Psi(u,v)$:

$$H\Psi \equiv -(\partial_u^2 + \partial_v^2)\,\Psi + U(u,v)\,\Psi = 0, \tag{71}$$

$$U(u,v) = -\frac{3}{4}\,\frac{u^2+v^2}{(u^2-v^2)^2} + \frac{1}{2}\,\frac{u}{u^2-v^2}\,. \tag{72}$$

Finally we introduce the variables

$$\xi = \frac{u+v}{2} = u\,\cos^2\phi\,, \qquad \eta = \frac{u-v}{2} = u\,\sin^2\phi\,. \tag{73}$$

It is easy to see that in these variables the operator in Eq.(71) splits into two *identical one-dimensional* Hamiltonians

$$[\hat{H}_\xi + \hat{H}_\eta]\Psi(\xi,\eta) = 0\,, \tag{74}$$

where $\hat{H}_\xi$ is given by:

$$\hat{H}_\xi = -\partial_\xi^2 - \frac{3}{16}\,\frac{1}{\xi^2} + \frac{1}{4\xi}. \tag{75}$$

Thus, in new variables Eq.(73) the partial differential equation (61) for the generating function at $k = \pi/2$ is separable and can be reduced to the two ordinary differential equations of the Schrödinger type

$$\hat{H}_\xi\psi_\Lambda(\xi) = \Lambda\psi_\Lambda(\xi)\ \ ;\ \ \hat{H}_\eta\psi_{-\Lambda}(\eta) = -\Lambda\psi_{-\Lambda}(\eta)\,, \tag{76}$$

where the opposite sign of the two eigenvalues guarantees the zero-energy solution to Eq.(74).

5.2. Problem of "degeneracy" of zero-mode solutions

Although Eqs.(76) look like the ordinary Schrödinger equations in a potential well, the problem we are solving is very different from quantum mechanics. The point is that the basic property of a Hamiltonian is its Hermiticity which insures that eigenvalues are real. For singular Hamiltonians like the one in Eq.(76) this property is not given for granted. It requires (i) that the boundary term

$$\psi_1(\xi)\partial_\xi\psi_2(\xi) - \psi_2(\xi)\partial_\xi\psi_1(\xi)|_{\xi=0}^{\infty} = 0. \tag{77}$$

is zero for any two functions $\psi_1(\xi)$ and $\psi_2(\xi)$ from the Hilbert space, and (ii) the diagonal matrix element

$$\int d\xi\,\psi^*(\xi)\,\hat{H}\,\psi(\xi) \tag{78}$$

is finite. For functions $\psi(\xi) \sim \xi^\mu$ which behave as power law at $\xi \to 0$, these two conditions imply $\mu > \frac{1}{2}$.

There is no such a restriction in our problem where the eigenvalue Λ plays an auxiliary role (the zero mode arises as a result of cancelation $\Lambda + (-\Lambda) = 0$) and does not have a meaning of an observable. That is why Λ is allowed to be complex.

Solutions $\psi_\Lambda(\xi)$ can be expressed via the Whittaker function $W_{-\lambda,\,\mu}(\xi)$ which obeys the Weber's differential equation (see, e.g. Ref. [21], 9.220):

$$\frac{d^2}{dx^2}\,W_{-\lambda,\,\mu}(x) + \left(-\frac{1}{4} - \frac{\lambda}{x} + \frac{\frac{1}{4}-\mu^2}{x^2}\right)W_{-\lambda,\,\mu}(x) = 0 \tag{79}$$

and decays at $x \to \infty$:

$$W_{-\lambda,\,\mu}(x) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu+\lambda)} M_{-\lambda,\,\mu}(x) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu+\lambda)} M_{-\lambda,\,-\mu}(x)\,; \tag{80}$$

$$M_{-\lambda,\,\mu}(x) = x^{\frac{1}{2}+\mu} \mathrm{e}^{-x/2} {}_1F_1(\frac{1}{2}+\mu+\lambda, 2\mu+1; x)\,, \tag{81}$$

$$M_{-\lambda,\,-\mu}(x) = x^{\frac{1}{2}-\mu} \mathrm{e}^{-x/2} {}_1F_1(\frac{1}{2}-\mu+\lambda, -2\mu+1; x)\,, \tag{82}$$

where ${}_1F_1$ is the confluent hypergeometric function, and $\Re\,\lambda \geq 0$. For $\Lambda < 0$, the equation (75) is mapped on Eq.(79) by the following identification:

$$\xi = \frac{x}{2\sqrt{-\Lambda}} \qquad \lambda = \frac{1}{8\sqrt{-\Lambda}}\ \ ;\ \ \mu = \frac{1}{4}\,. \tag{83}$$

Note that the solution $\psi_\Lambda(\xi)$ to Eqs.(76),(79) at $\Lambda < 0$ which decays at $\xi \to \infty$, contains both a part $\sim \xi^{\frac{1}{4}}$ and a part $\sim \xi^{\frac{3}{4}}$ at $\xi \to 0$. It clearly violates the condition of Hermiticity $\mu > \frac{1}{2}$ and should be ruled out in quantum mechanics. That is why the singular Hamiltonian Eq.(75) does not have bound states. In contrast to that, our problem has a *continuous spectrum*. Moreover, the spectrum is complex, as Eq.(83) can be easily extended to complex Λ with the convention that $\sqrt{z} > 0$ at $z > 0$ and has a cut along the semi-axis $z < 0$. As a result, the general solution to Eq.(71) can be represented as a superposition

$$\Psi(\xi,\eta) = \int_{\Re\,\lambda\geq 0} d^2\lambda\ c(\lambda,\bar{\lambda})\ W_{-\lambda,\,\frac{1}{4}}\left(\frac{\xi}{4\lambda}\right)\ W_{i\lambda,\,\frac{1}{4}}\left(\frac{i\eta}{4\lambda}\right)\,. \tag{84}$$

We mention for completeness, that the Whittaker functions with the second index $\mu = 1/4$ which emerges in our problem, constitute a special class. They can be expressed in terms of the *parabolic cylinder functions* $D_\kappa(x)$: $W_{\lambda,\,\frac{1}{4}}(x) = 2^{-\lambda}(2x)^{1/4} D_{2\lambda-\frac{1}{2}}(\sqrt{2x})$, which reflects the possibility of mapping the problem Eq.(76) on the problem of two "harmonic oscillators" but with an "upside down" potential for one of them. We will not explore this correspondence in the present paper.

There is also a special eigenfunction $\psi_0(\xi)$ of the operator Eq.(75)

$$\psi_0(\xi) = \xi^{1/4} \exp\left[-\sqrt{\xi}\right], \tag{85}$$

which corresponds to the eigenvalue $\Lambda = 0$, when the mapping Eq.(83) becomes singular. However, as will be shown in the next section, the corresponding solution of Eq.(76)

$$\Psi_0(\xi,\eta) = (\xi\eta)^{1/4} \exp\left[-(\sqrt{\xi}+\sqrt{\eta})\right] \tag{86}$$

does not meet physical requirements of smoothness of $\Phi(u,\phi)$ as a function of ϕ and thus it must be ignored.

Note that Eq.(84) possesses a huge degeneracy, due to an arbitrary choice of the function $c(\lambda,\bar{\lambda})$. This is in contradiction with an intuitive expectation that the statistics of wave functions in an infinite disordered chain should be unique and independent of the boundary conditions. Below we show that the natural physical requirement of smoothness of $\Phi(u,\phi)$ as a function of ϕ helps to determine the solution for the generating function up to a constant pre-factor which can be further fixed using the relation Eq.(45) and the normalization condition $\int_0^\pi \mathcal{P}^{(an)}(\phi)d\phi = 1$ for the anomalous phase distribution function $\mathcal{P}^{(an)}(\phi)$.

6. Solution for the generating function at $k = \pi/2$

The stationary generating function

$$\Phi(u,\phi) \equiv \{\Phi(\xi,\eta)\}_{\xi=u\cos^2\phi\,,\eta=u\sin^2\phi} = \left\{\frac{\xi+\eta}{(\xi\eta)^{1/4}}\Psi(\xi,\eta)\right\}_{\xi=u\cos^2\phi\ ,\ \eta=u\sin^2\phi}, \quad (87)$$

which determines the moments Eq.(38), should obey the following requirements:

1. It should vanish at $u \to \infty$.

2. It should be periodic in ϕ (with the period π). Moreover, for the considered case ($k = \pi/2$) the coefficients of Eq.(59) are periodic functions with the period $\pi/2$ and we impose the requirement $\Phi(u, \phi + \pi/2) = \Phi(u, \phi)$ on the stationary solution, too.

3. $\Phi(u, \phi)$ should be a smooth function of ϕ together with all the derivatives with respect to ϕ. It should have no jumps, cusps, etc.

The first requirement restricts the integration domain in Eq.(84) to the first quadrant ($\Re\lambda \geq 0$, $\Im\lambda \geq 0$) of the complex plane λ. The second requirement is equivalent to $\Psi(\xi,\eta) = \Psi(\eta,\xi)$ and can be achieved by the symmetrization of the integrand in Eq.(84) with respect to the replacement $\xi \leftrightarrow \eta$. Accounting for the third, extremely important requirement is not as simple. It will be postponed till subsection 6.2.

6.1. From plane to contour integral

Similarly to the usual coherent states, the set of partial solutions ψ_λ is overcomplete, and actually the 2d integration domain in Eq.(84) (i.e., the first quadrant of the complex plane λ) can be reduced without losses to a 1d integration contour. First of all, note that $F(\lambda;\xi,\eta) \equiv W_{-\lambda,\frac{1}{4}}\left(\frac{\xi}{4\lambda}\right)\ W_{i\lambda,\frac{1}{4}}\left(\frac{i\eta}{4\lambda}\right)$ is a *holomorphic function* of λ in the first quadrant, i.e., $F(\lambda;\xi,\eta)$ depends only on $\lambda = \rho e^{i\sigma}$ but not on $\bar{\lambda} = \rho e^{-i\sigma}$. In polar coordinates (ρ,σ) we have:

$$\Psi(\xi,\eta) = \int_0^{\pi/2} d\sigma \int_{C_0} \rho\, d\rho\, c(\rho,\sigma)\, F(\rho e^{i\sigma};\xi,\eta) = \int_0^{\pi/2} d\sigma \int_{C_\sigma} \rho\, d\rho\, c(\rho,\sigma)\, F(\rho e^{i\sigma};\xi,\eta) \quad (88)$$

where the integration contour C_0 coincides with the semi-axis $\lambda \geq 0$, and the second equality is the realization of the possibility to rotate the contour C_0 providing only that the argument ρ of F remains in the first quadrant. By our choice, the contour C_σ corresponds to the rotation of C_0 by the angle $-\sigma$, so that the variable ρ on C_σ is represented as $|\rho|\mathrm{e}^{-i\sigma}$. Changing, at a fixed σ, the variable ρ in the internal integral: $\rho = t\mathrm{e}^{-i\sigma}$, where real t runs from 0 to $+\infty$, we arrive at:

$$\Psi(\xi,\eta) = \int_0^{\pi/2} d\sigma \int_0^{+\infty} dt\, \mathrm{e}^{-2i\sigma}\, t\, c(t\mathrm{e}^{-i\sigma},\sigma)\, F(t;\xi,\eta) = \int_0^{+\infty} d\lambda\, C(\lambda)\, F(\lambda;\xi,\eta)\,, \quad (89)$$

where we have changed the order of integrations, introduced a new weight function

$$C(t) \equiv t\int_0^{\pi/2} d\sigma\, \mathrm{e}^{-2i\sigma}\, c(t\mathrm{e}^{-i\sigma},\sigma) \quad (90)$$

and changed back the notation $t \to \lambda$. Thus without loss of generality we have expressed a general solution to the zero-mode equation (61) in terms of a *contour* integral.

$$\Psi(\xi,\eta) = \int_{C_0} d\lambda\, C(\lambda)\, F(\lambda)\,. \tag{91}$$

Note again, that the key condition for this transformation is the fact that the solution to Eq.(79) is a holomorphic function of λ.

Another choice of the integration contour can make the expression more symmetric. Namely, rotating the contour C_0 by the angle $\pi/4$, so that $\lambda \to |\lambda| \mathrm{e}^{i\pi/4}$, and introducing a new real variable λ' by $\lambda = \lambda' \mathrm{e}^{i\pi/4}$, we obtain (omitting the prime and re-defining the arbitrary function $C(\lambda)$) for the generating function $\Phi(u,\phi)$ Eq.(87):

$$\Phi(\xi,\eta) = \frac{\xi+\eta}{(\xi\eta)^{1/4}} \int_0^\infty d\lambda\, C(\lambda) \left[W_{-\lambda\epsilon,\frac{1}{4}}\left(\frac{\bar{\epsilon}\xi}{4\lambda}\right) W_{-\lambda\bar{\epsilon},\frac{1}{4}}\left(\frac{\epsilon\eta}{4\lambda}\right) + c.c. \right]. \tag{92}$$

Here $\epsilon = e^{i\pi/4}$, $\bar{\epsilon} = e^{-i\pi/4}$, and the complex conjugate counterpart is added to symmetrize the integrand with respect to the permutation $\xi \leftrightarrow \eta$ in order to fulfill the formulated above requirement 2; the function $C(\lambda)$ is a real (without loss of generality) function yet to be determined. Up to now we have used only the following loose assumptions on its properties:

1^o. $C(\lambda)$ has no singularities in the first quadrant of the complex λ plane;

2^o. at $|\lambda| \to \infty$, the integrand in Eq.(92) decays faster than $1/\lambda$; this justifies rotations of the contour neglecting contributions of distant arcs.

6.2. Equation for $C(\lambda)$

We begin by determining the behavior of the function $C(\lambda)$ at $\lambda \to 0$. To this end we note that according to the relations Eqs.(45) and (63), the generating function $\Phi(\xi,\eta)$ Eq.(92) must tend to a finite limit as $\xi \to 0$ and $\eta \to 0$.

Re-scaling in Eq.(92) the integration variable $\lambda \to u\lambda$, we find at $u \to 0$

$$\Phi(u\to 0,\phi) \sim \frac{u^{3/2}}{|\cos\phi\sin\phi|^{1/2}} \int_0^\infty d\lambda\, C(u\lambda) \left[W_{0,\frac{1}{4}}\left(\frac{\bar{\epsilon}\cos^2\phi}{4\lambda}\right) W_{0,\frac{1}{4}}\left(\frac{\epsilon\sin^2\phi}{4\lambda}\right) + c.c. \right] \tag{93}$$

To provide a finite value of the expression (93) in the limit of vanishing u, we should require that (see Appendix A):

$$C(\lambda) = \frac{\tilde{C}(\lambda)}{\lambda^{\frac{3}{2}}}\,, \quad \tilde{C}(0) = \mathrm{const}\,. \tag{94}$$

A crucial role in further restricting the possible choice of the function $\tilde{C}(\lambda)$ is played by the requirement of smoothness of $\Phi(u,\phi)$ as a function of ϕ (requirement 3 of the previous subsection). The generating function Eq.(92) is periodic in ϕ with the period $\frac{\pi}{2}$ and it is continuous at the end points $\phi = 0$ (i.e. $\eta = 0$), and $\phi = \pi/2$ (i.e. $\xi = 0$) of the interval of the periodicity $(0, \pi/2)$. This is guaranteed by the $c.c.$ term in Eq.(92), equivalent to the permutation $\xi \leftrightarrow \eta$. What is not automatically guaranteed is that $\Phi(\xi,\eta)$ is *smooth* as a function of ϕ at $\phi = 0$ and $\phi = \pi/2$; the smoothness implies the continuity of all the derivatives. Amazingly, the requirement of *smoothness* is sufficient to determine the function $\tilde{C}(\lambda)$ up to a constant pre-factor. As we will see this happens because of the special property of the solution Eq.(92) encoded in the certain identity for the confluent hypergeometric functions in Eqs.(80)-(82).

Consider, for instance, the behavior of $\Phi(u,\phi)$ at $\phi \to 0$, (i.e. $\eta \to 0$, while $\xi \to u$). A discontinuity of derivatives at $\phi = 0$ may arise from the branching of the expression in Eq.(92) at small η. Indeed, according to the representation of the Whittaker function Eq.(80) in terms of M-functions Eqs.(81) and (82), we see that

$$W_{-\lambda\bar{\epsilon},\frac{1}{4}}\left(\frac{\epsilon\eta}{4\lambda}\right) = \left(\frac{\epsilon\eta}{4\lambda}\right)^{\frac{1}{4}} \left[f_1(\lambda,\eta) + \sqrt{\eta}\, f_2(\lambda,\eta)\right] , \tag{95}$$

where f_1 and f_2 are analytic functions of η in the vicinity of $\eta = 0$. The common factor $\eta^{1/4}$ is canceled by the pre-factor in front of the integral in Eq.(92). The first term in square brackets of Eq.(95) is regular in the vicinity of $\eta = 0$, while the second one $\sim \sqrt{\eta} \sim |\phi|$, is not analytical at $\eta = 0$. As such a non-analytical behavior is in conflict with the requirement 3 of smoothness, the corresponding part of the solution must identically vanish. Extracting this singular ($\propto \sqrt{\eta}$ in the domain $\eta < \xi$) part $\Phi_{sing}(\xi,\eta)$ of the general solution Eq.(92), we obtain:

$$\Phi_{sing}(\xi,\eta) \sim \int_0^{+\infty} d\lambda\, \frac{\tilde{C}(\lambda)}{\lambda^{3/2}} \left[W_{-\lambda\epsilon,\,\frac{1}{4}}\left(\frac{\bar{\epsilon}\xi}{4\lambda}\right) M_{-\lambda\bar{\epsilon},\frac{1}{4}}\left(\frac{\epsilon\eta}{4\lambda}\right) \frac{1}{\Gamma(\frac{1}{4}+\bar{\epsilon}\lambda)} \right.$$
$$\left. + M_{-\lambda\epsilon,\,\frac{1}{4}}\left(\frac{\bar{\epsilon}\eta}{4\lambda}\right) W_{-\lambda\bar{\epsilon},\,\frac{1}{4}}\left(\frac{\epsilon\xi}{4\lambda}\right) \frac{1}{\Gamma(\frac{1}{4}+\epsilon\lambda)} \right] = 0 \,, \tag{96}$$

which should be fulfilled for any $\eta < \xi$. Vanishing of $\Phi_{sing}(\xi,\eta)$ is equivalent to the homogeneous integral equation for the real weight function $\tilde{C}(\lambda)$ with the boundary condition $\tilde{C}(\lambda \to 0) \to \text{const}$.

Similarly, the presence of non-analytical terms $\sqrt{\eta}$ and $\sqrt{\xi}$ in the special solution Eq.(86) is the reason why this solution should be ignored.

The integral equation Eq.(96) imposes severe constraints on the function $\tilde{C}(\lambda)$, because Eq.(96) must be satisfied *for arbitrary* η and ξ (at $\eta < \xi$). Thus the requirement of smoothness lifts a huge degeneracy and arbitrariness in the possible choice of $\tilde{C}(\lambda)$ reducing the situation to the point where the existence of even a single (non-zero) solution for $\tilde{C}(\lambda)$ is not evident. We will show below that the solution to Eq.(96) does exist and is unique up to the constant pre-factor.

6.3. Solution for $\tilde{C}(\lambda)$

Rotating the integration contours independently for each of the two terms in the integrand of Eq.(96) and changing $\lambda \to t\bar{\epsilon}$ and $\lambda \to t\epsilon$, respectively, one can make the Whittaker function real and take it out of the square brackets. Thus, the integral equation Eq.(96) takes the form:

$$\int_0^{+\infty} \frac{dt}{t^{9/4}} W_{-\lambda,\frac{1}{4}}\left(\frac{\xi}{4t}\right) \left[\frac{\tilde{C}(\bar{\epsilon}t)\exp\left(-i\frac{\eta}{8t}\right)}{\Gamma(\frac{1}{4}-it)} {}_1F_1\left(\frac{3}{4}-it,\frac{3}{2};\frac{i\eta}{4t}\right) - \frac{\tilde{C}(\epsilon t)\exp\left(i\frac{\eta}{8t}\right)}{\Gamma(\frac{1}{4}+it)} {}_1F_1\left(\frac{3}{4}+it,\frac{3}{2};\frac{-i\eta}{4t}\right)\right] = 0\,, \quad (97)$$

where the dependence of the integrand on ξ and η is factorized. The only possibility to satisfy this equation for arbitrary ξ and η is to require the square bracket to vanish identically.

The crucial observation for the possibility to fulfil this condition is an identity for the confluent hypergeometric functions [21]:

$$e^{-z/2}\,{}_1F_1\left(\frac{3}{4}-it,\frac{3}{2},z\right) = e^{z/2}\,{}_1F_1\left(\frac{3}{4}+it,\frac{3}{2},-z\right). \quad (98)$$

With the help of Eq.(98), we find that the square bracket in Eq.(97) vanishes identically *for all* η if and only if the function $\tilde{C}(t)$ obeys (for positive t) the condition

$$\frac{\tilde{C}(\bar{\epsilon}t)}{\Gamma(\frac{1}{4}-it)} = \frac{\tilde{C}(\epsilon t)}{\Gamma(\frac{1}{4}+it)}. \quad (99)$$

Now one can immediately guess *a solution* for $\tilde{C}(\lambda)$:

$$\tilde{C}_0(\lambda) = \Gamma\left(\frac{1}{4}+\epsilon\lambda\right)\Gamma\left(\frac{1}{4}+\bar{\epsilon}\lambda\right). \quad (100)$$

It is easily seen that the solution Eq.(100) obeys the both conditions formulated at the end of the subsection 6.1. The function $C_0(\lambda) \equiv \tilde{C}_0(\lambda)/\lambda^{3/2}$ (see definition (94)) is an analytical function in the domain of our interest ($\Re\,\lambda > 0$). Though $C_0(\lambda)$ grows at $|\lambda| \to \infty$, one can check that the integrand in Eq.(92) decays as $1/|\lambda|^3$ for $|\lambda| \to \infty$. This provides the convergence of the integral Eq.(92) at large λ and justifies rotations of integration contours neglecting contributions of infinitely remote arcs. Note also that $\tilde{C}_0(\lambda)$ is real at the semi-axis $\lambda > 0$.

Now, looking for *a general* solution to Eq.(99) in the form

$$\tilde{C}(\lambda) = C_0(\lambda)S(\lambda)\,, \quad (101)$$

we obtain the following functional equation for $S(\lambda)$ at $\lambda > 0$:

$$S(\epsilon\lambda) = S(\bar{\epsilon}\lambda)\,, \quad (102)$$

which is also supposed to be an analytic function at $\Re\lambda \geq 0$. This equation requires $S(\lambda)$ to be an analytical function of $z = \lambda^4$:

$$S(\lambda) = \sum_{n=0}^{\infty} s_n \lambda^{4n}\,. \quad (103)$$

Therefore, being regular in the domain $\Re\,\lambda > 0$ (it is even sufficient to require analyticity within a sector $|\arg(\lambda)| \leq \pi/4$), the function $S(\lambda)$ must be regular in all the complex plane λ, i.e. it should be an *entire function.* Now we apply the condition of convergence

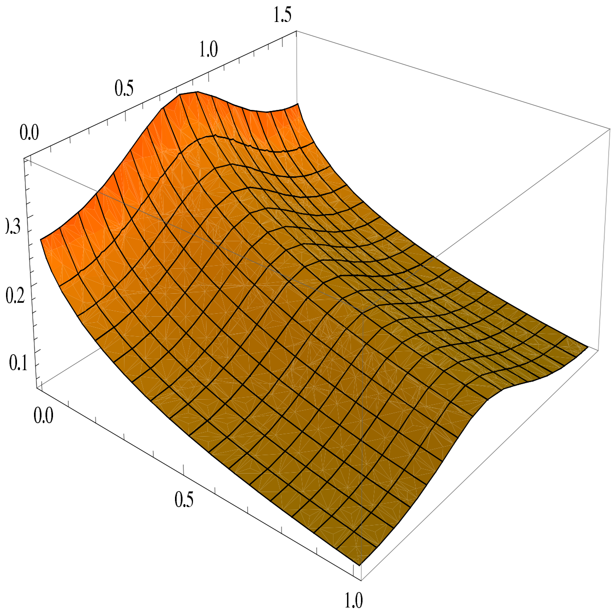

Figure 1. (color online) The function $\Phi^{(an)}(u,\phi)$, in the range $u \in [0,1]$, $\phi \in [0,\pi/2]$.

of the integral over λ in Eq.(92) at large λ to find the allowed asymptotic behavior of $S(\lambda)$ at $|\lambda| \to \infty$. Substituting Eq.(101) into Eq.(92) and using the asymptotics of the Whittaker and Γ-functions we find that the integrand behaves as $\lambda^{-3}S(\lambda)$ at $\lambda \to \infty$. This means that $|S(\lambda)|$ should increase not faster than λ^2. There is only one such entire function with the structure of Eq.(103): this is a constant $S(\lambda) = s_0 = \text{const}$. Thus we have proven the uniqueness of the solution Eq.(100) up to a constant factor. This factor has to be determined from the relation Eq.(45) and the normalization condition for the phase distribution function $\mathcal{P}(\phi)$. Now we may write down the solution for the *anomalous* (at the center of the band) generating function $\Phi^{(an)}(u,\phi)$ in the final form :

$$\Phi^{(an)}(u,\phi) = \frac{u^{1/2}}{2\Gamma^4\left(\frac{1}{4}\right)|\cos\phi\,\sin\phi|^{1/2}} \int_0^\infty d\lambda\, \frac{\Gamma\left(\frac{1}{4}+\epsilon\lambda\right)\Gamma\left(\frac{1}{4}+\bar{\epsilon}\lambda\right)}{\lambda^{3/2}}$$
$$\left[W_{-\lambda\epsilon,\frac{1}{4}}\left(\frac{\bar{\epsilon}\xi}{4\lambda}\right) W_{-\lambda\bar{\epsilon},\frac{1}{4}}\left(\frac{\epsilon\eta}{4\lambda}\right) + c.c.\right], \quad (104)$$

where $\xi = u\cos^2\phi$, $\eta = u\sin^2\phi$; $\epsilon = \mathrm{e}^{i\pi/4}$, $\bar{\epsilon} = \mathrm{e}^{-i\pi/4}$. In the appendix we demonstrate explicitly that the obtained solution Eq.(104) does obey the relation $\Phi^{(an)}(u=0,\phi) = \mathcal{P}^{(an)}(\phi)$, where $\mathcal{P}^{(an)}(\phi)$ is given by Eq.(63). The 3D plot of the function $\Phi(u,\phi)$ is given in Fig.1. In the next section we apply the solution Eq.(104) for studying moments of the wave function distribution.

7. Physical applications. Moments of the wave function distribution

The exact expression Eq.(104) for the anomalous (at the center of the band) generation function is our main analytic result. It determines statistics of wave function distribution at the center-of-band anomaly. Although extensive physical applications go beyond the framework of the present work, here we briefly discuss the applicability of the one-parameter scaling description for the anomalous statistics.

As has been mentioned in the introduction, the Lyapunov exponent $\gamma(E)$, Eq.(6), sharply decreases in a narrow vicinity of the band center $E = 0$ [11, 12]. Being dependent only on the phase distribution function, the Lyapunov exponent can be easily calculated using Eq.(63) at the center-of-band anomaly $E = 0$ and the trivial homogeneous phase distribution $\mathcal{P}^{(norm)}(\phi) = 1/\pi$ close to but outside the anomalous region. The ratio of the two corresponding Lyapunov exponents $\gamma_{an}(E = 0)$ and $\gamma_{norm}(E \approx 0)$ is given by: [11, 12]:

$$\frac{\gamma_{an}(E=0)}{\gamma_{norm}(E\approx 0)} = \int_0^{\pi} [1+\cos(4\phi)]\, \mathcal{P}^{(an)}(\phi) = \frac{8\,\Gamma^2\left(\frac{3}{4}\right)}{\Gamma^2\left(\frac{1}{4}\right)} \approx 0.9139. \qquad (105)$$

According to Eq.(6), this can be interpreted as an increasing localization length at the anomaly: $\ell_0 \to \ell_{an} = 1.094\,\ell_0$. However, the variation in the localization length is not the only manifestation of the statistical anomaly. We will show below that, unlike the moments I_m^{norm} away from the band center which require only one parameter ℓ_0 for their full description (see Eq.(57)), at the anomaly more than one parameter is needed.

To this end, we analyze the behavior of moments I_m $(m > 2)$ of the anomalous (at the center of band, $E = 0$, $k = \pi/2$) wave function distribution. With the definition Eq.(46), the expression Eq.(38) in the limit of a long chain takes the form:

$$I_m^{(an)}(E=0) = \frac{4\pi}{(m-2)![\ell_0]^{m-1}} \int_0^{\pi/2} d\phi\, \cos^{2m}(\phi) \int_0^{\infty} du\, u^{m-2} [\Phi^{(an)}(u,\phi)]^2 \,. \qquad (106)$$

One might expect that the behavior of anomalous moments is similar to Eq.(57) but with the localization length ℓ_0 replaced by some other length scale ℓ_{sc} (in the simplest case $\ell_{sc} = \ell_{an}$). This would be the scenario of *one-parameter scaling* which appears to fail at the band center.

A convenient way to present the results is to plot the *reduced moments* $R_m \equiv I_m^{(an)}(E=0)/I_m^{(norm)}(E \approx 0)$,

$$R_m = \frac{2^{2m}\,\pi}{(m-1)!(m-2)!} \int_0^{\infty} du \int_0^{\pi/2} d\phi\, \cos^{2m}(\phi)\, u^{m-2}\, [\Phi^{(an)}(u,\phi)]^2. \qquad (107)$$

Plugging the solution Eq.(104) into Eq.(106) we arrive at the exact expression for anomalous moments written in quadratures. Unfortunately, we were not able to do the corresponding integrations analytically. Instead, we evaluated the reduced moments R_m numerically up to $m = 10$. The results are given in Fig.2.

One can see that the behavior of reduced moments R_m with relatively small $m < 6$, $R_m \approx (\ell_0/\ell_{sc})^{m-1}$, is, indeed, compatible with a simple renormalization of the localization length. The best exponential fit gives $\ell_0 \to \ell_{sc} \approx 1.252\,\ell_0$ which reflects the same tendency of increasing the localization length as Eq.(105). Note that $\ell_{sc} > \ell_{an}$. This is already an indication that one-parameter scaling may fail to describe eigenfunction statistics at the anomaly. However even more interesting phenomenon takes place for large moments. At $m > 6$ one can see a significant enhancement of

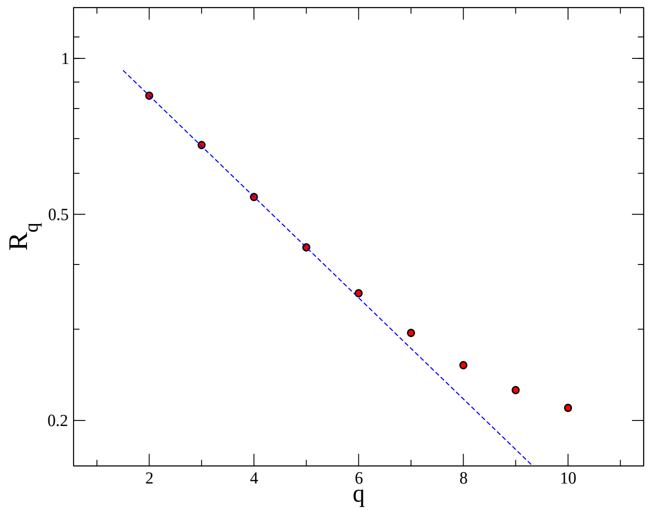

Figure 2. (color line)Reduced moments R_m (red points) in the log-linear scale. The dashed line is the exponential fit $R_m = (\ell_0/\ell_{sc})^{m-1}$ with $l_{sc}/\ell_0 = 1.252$.

the moments compared to their value extrapolated from the exponential dependence of R_m at small m. In order to describe the crossover to this new regime one needs more parameters. A possible physical meaning of these new parameters are discussed in a short publication [25]

8. Conclusions and open problems

Eq.(104) is the main result of the paper. It gives an exact and unique solution (in quadratures) to the stationary *partial* differential equation Eq.(59) for the generating function $\Phi(u, \phi)$ at the center of the energy band ($k = \pi/2$, $E(k) = 0$) of a weakly disordered chain. The variables u and ϕ are associated with slowly varying (squared) amplitude and phase of localized eigenfunctions. The generating function we obtained can be used to compute all local statistics of normalized wave functions in the one-dimensional Anderson model in the bulk of a long ($L \gg \ell$) chain. The solution of this problem goes beyond the known problem of the Lyapunov exponent and related quantities (e.g. conductance) [11, 12, 14, 15], which are completely determined by the distribution function of phase $\mathcal{P}(\phi)$. As we have shown, $\mathcal{P}(\phi)$ is a descender of the generating function $\Phi(u, \phi)$ and is related with it in a simple way: $\mathcal{P}(\phi) = \Phi(u = 0, \phi)$ (see Eq.(45)).

The integrability of the partial differential equation Eq.(59) for the generating function $\Phi(u, \phi)$ which we discovered is a remarkable evidence of a hidden symmetry of the problem at $k = \pi/2$. Although in the course of derivation we mentioned about the sl_2 algebra of operators Eq.(66), which are the building blocks for Eq.(65), and about even more extended algebra Eq.(69), we did not exploit this algebraic content explicitly.

What would be highly useful is to find the symmetry transformation which enables the separation of variables. Moreover, the hidden symmetry which survives violation of the chiral (or sublattice) symmetry [24, 2] by on-site disorder, could be important for other systems (like edge states in Quantum Hall effect or in topological insulators) where the chiral symmetry is broken by disorder. Speculating on its nature we may surmise that this symmetry might be more naturally formulated in the three dimensional space rather than in the two-dimensional space (ξ, η) and that it may have something to do with the symmetry of the 3d harmonic oscillator. This conjecture is fed by an analogy between our main result Eq.(92) and the expression for the Green's function of the 3d harmonic oscillator problem [22]. The analogy concerns the parameter (λ in our problem and k in Ref.[22]) entering both in the argument and in the first index of the Whittaker functions in a mutually reciprocal way, as well as the second index of the Whittaker functions being $\frac{1}{4}$ in both cases. Establishing this symmetry would also be useful for studying higher order anomalies at $k = \pi m/n$ with $n > 2$ (see in Ref.[23] the preliminary analysis of the "devil's staircase" of these higher-order anomalies).

We would like to note that the studied anomalies are inherent not only to the problem of a disordered chain but might occur in other physical situations where there is a periodic perturbation of random amplitude. We mention an analogy between the 1d localization and the classical system of kicked oscillator studied recently in Ref.[13]. According to this analogy the energy-dependent de-Broglie wavelength λ_E of a particle on a chain is encoded in the frequency of the oscillator while the lattice constant a determines the period of the δ-function pulses of the external force ("kicks"), their amplitude being proportional to disorder. From this point of view, statistical anomalies arise due to sequences of several kicks with correlated amplitudes. Correlated amplitudes of kicks correspond to exclusive configurations of the local disorder, hence the anomalies are weak (for weak disorder) and exhibit only in narrow windows around selected energies. Remarkably, the variables ξ and η, which allowed us to factorize the equation for the generating function of the Anderson model, play a role of the co-ordinate and the momentum of the kicked oscillator.

Finally, we applied the exact solution for the generating function to analyze the behavior of moments of the wave function distribution at the center-of-band anomaly. We have found that relatively small moments behave similar to those outside the anomalous region but with a re-scaled localization length $\ell_0 \to \ell_{sc}$, while the larger moments deviate significantly from this dependence. This fact together with the appreciable enhancement of the "participation" length $\ell_{sc} \approx 1.252\,\ell_0$ with respect to the "Lyapunov" length (inverse Lyapunov exponent) $\ell_{an} \approx 1.094\,\ell_0$, implies a significant change of the form of the "average" eigenfunction at the center-of-band anomaly and simultaneously a failure of one-parameter description of eigenfunction statistics.

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Appendix A. Normalization of $\Phi^{(an)}(u=0,\phi)$: explicit check of the relation $\Phi^{(an)}(u=0,\phi)=\mathcal{P}^{(an)}(\phi)$ for Eqs. (104) and (63).

In order to take the limit $u\to 0$ in Eq.(104), we re-scale the integration variable $\lambda\to u\lambda$ and use the identity $W_{0,\nu}(z)=\sqrt{\frac{z}{\pi}}K_\nu(\frac{z}{2})$. Introducing a new integration variable $x=1/(8\lambda)$, we arrive at the following expression for $\Phi^{(an)}(u=0,\,\phi)$:

$$\Phi^{(an)}(u=0,\,\phi)=\frac{4}{\pi\Gamma^2\left(\frac{1}{4}\right)}\sqrt{|\sin{(2\phi)}|}\,\Re I\,, \tag{A.1}$$

where (see [21])

$$I\equiv\int_0^\infty\sqrt{x}dxK_{\frac{1}{4}}(\bar{\epsilon}x\cos^2\phi)K_{\frac{1}{4}}(\epsilon x\sin^2\phi) \tag{A.2}$$

$$=\frac{\sqrt{2}\pi^2 i}{\Gamma^2\left(\frac{1}{4}\right)}\frac{\sin^{\frac{1}{2}}\phi}{|\cos\phi|^{\frac{7}{2}}}F(1,\,\frac{3}{4},\,\frac{3}{2};\,1+\tan^4\phi)\,. \tag{A.3}$$

The latter expression is rather complicated. To find its real part one has to use nontrivial identities for the hypergeometric function. Obviously this brute force approach does not exploit efficiently the symmetry of the problem.

It is more advantageous and instructive to exploit the symmetry and perform the integration in Eq.(A.2) in several elementary steps. Using the representation ([21])

$$K_{\frac{1}{4}}(z)=\left(\frac{z}{2}\right)^{1/4}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}\int_0^\infty dt\ \mathrm{e}^{-z\cosh t}\sqrt{\sinh t}\,, \tag{A.4}$$

one can perform an elementary integration over x in Eq.(A.2) arriving at

$$\Re I=\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\sqrt{2\pi}}|\sin\phi\cos\phi|^{1/2}\int_0^\infty dt_1\,dt_2\sqrt{\sinh t_1\sinh t_2}\;\Re\frac{1}{[\bar{\epsilon}\cos^2\phi\,\cosh t_1+\epsilon\sin^2\phi\,\cosh t_2]^2}\,. \tag{A.5}$$

Using the identity

$$\Re\frac{1}{[\bar{\epsilon}x+\epsilon y]^2}=\frac{2xy}{[x^2+y^2]^2} \tag{A.6}$$

(for real x and y) and introducing new variables:

$$y_1=\cos^2\phi\,\sinh t_1\ \ ;\ \ y_2=\sin^2\phi\,\sinh t_2\,, \tag{A.7}$$

we obtain:

$$\Re I = \frac{\Gamma^2\left(\frac{1}{4}\right)}{\pi\sqrt{|\sin(2\phi)|}}\int_0^\infty \frac{\sqrt{y_1 y_2}\, dy_1\, dy_2}{[\cos^4\phi\ + \sin^4\phi\ + y_1^2 + y_2^2]^2}\,. \tag{A.8}$$

Re-scaling the variables $y_{1(2)} = y'_{1(2)}\sqrt{\cos^4\phi + \sin^4\phi}$ and introducing the polar coordinates ρ and $\alpha \in (0, \pi/2)$ as $y'_1 = \rho\cos\alpha$ and $y'_2 = \rho\sin\alpha$, we get

$$\Re I = \frac{\Gamma^2\left(\frac{1}{4}\right)}{\pi\sqrt{|\sin(2\phi)|}}\frac{I_\alpha I_\rho}{\sqrt{\cos^4\phi\ + \sin^4\phi}}\,, \tag{A.9}$$

where

$$I_\alpha = \int_0^{\pi/2}\sqrt{\sin\alpha\,\cos\alpha} = \frac{1}{2}B(\frac{3}{4},\frac{3}{4}) = \frac{2\pi^{3/2}}{\Gamma^2\left(\frac{1}{4}\right)}\,; \tag{A.10}$$

$$I_\rho = \int_0^\infty \frac{\rho^2\, d\rho}{[1+\rho^2]^2} = \frac{\pi}{4}\,. \tag{A.11}$$

Collecting things together we arrive at the following final expression for $\Phi^{(an)}(u = 0, \phi)$, Eq.(A.1),:

$$\Phi^{(an)}(u = 0, \phi) = \frac{2\sqrt{\pi}}{\Gamma^2\left(\frac{1}{4}\right)}\frac{1}{\sqrt{\cos^4\phi\ + \sin^4\phi}} = \frac{4\sqrt{\pi}}{\Gamma^2\left(\frac{1}{4}\right)}\frac{1}{\sqrt{3+\cos(4\phi)}}\,. \tag{A.12}$$

This expression coincides with the anomalous probability distribution of phase $\mathcal{P}^{(an)}(\phi)$ Eq.(63) and thus proves the correct choice of the numerical pre-factor in Eq.(104).

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