A DIFFEOMORPHISM WITH GLOBAL DOMINATED SPLITTING CAN NOT BE MINIMAL

PENGFEI ZHANG

Abstract. Let M be a closed manifold and f be a diffeomorphism on M. We show that if f has a nontrivial dominated splitting $TM = E \oplus F$, then f can not be minimal. The proof mainly use Mañé's argument and Liao's selecting lemma.

1. Introduction

In [4] Herman constructed a (family of) C^∞ diffeomorphism on a compact manifold that is minimal and has positive topological entropy simultaneously. So positive entropy is insufficient to guarantee the nonminimality. This draws forth the problem to find some nature structure of the system that incompatible with the minimality. In [7] Mañé gave an argument to locate some nonrecurrent point if the map admits some invariant expanding foliation (also see [1]). In particular this argument shows that a partially hyperbolic diffeomorphism always has some nonrecurrent point and hence can not be minimal. In this note we show that a global dominated splitting is sufficient to exclude the minimality of the system.

Let M be a closed Riemannian manifold and $f : M \to M$ be a C^1 diffeomorphism on M. The map f is said to have a global dominated splitting on M if there exist an invariant splitting $TM = E \oplus F$, two numbers $\lambda \in (0,1)$ and $C \geq 1$ such that

$$\|Df^n|_{E(x)}\| \cdot \|Df^{-n}|_{F(f^n x)}\| < C\lambda^n \text{ for all } n \geq 1, x \in M. \tag{1.1}$$

A Riemannian metric on M is said to be *adapted* to the dominated splitting if we can take $C = 1$ in (1.1) with respect to this metric. Adapted metric always exists, see [3] for details.

Although restriction of the dominated splitting is much weaker than (partially) hyperbolic splitting, we show the restriction is strong enough to exclude the possibility of minimality. Recall that the map f is said to be minimal if for each $x \in M$, the orbit $\mathcal{O}_f(x) = \{f^n x : n \in \mathbb{Z}\}$ is a dense subset in M. The following is our main result.

Main Theorem. *Let M be a closed Riemannian manifold and $f : M \to M$ be a diffeomorphism on M. If f has a global dominated splitting, then f can not be minimal.*

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There are vast results in the case that the dimension of M is 2. Pujals and Sambarino gave several good characteristics in [8, 9] for the topology of invariant set having some dominated splitting for C^2 diffeomorphisms on surfaces. On the other hand, Xia proved in [11] that for all compact surface M with nonzero Euler characteristic, any homeomorphism on M admits some periodic points. In [12] X. Zhang considered a diffeomorphism f on a closed surface and an f-invariant set Λ with dominated splitting. He used Liao's selecting lemma and Crovisier's central models to find a periodic orbit near Λ. Our result follows from an observation from [12] combining with Mañé's argument and Liao's sifting lemma (or use Liao's selecting lemma).

2. Partial hyperbolicity and quasi-hyperbolic strings

In this section let's review several useful results for later discussions. Let M be a compact manifold and assume $f : M \to M$ has a dominated splitting $TM = E \oplus F$. We always assume the Riemannian metric on M is chosen to be *adapted*. That is, there exists $\lambda \in (0, 1)$ with

$$\|Df|_{E(x)}\| \cdot \|Df^{-1}|_{F(fx)}\| < \lambda, \text{ for all } x \in M.$$

In this case we say $TM = E \oplus F$ is a λ-dominated splitting. Generally we have $\|Df^n|_{E(x)}\| \leq \prod_{k=0}^{n-1} \|Df|_{E(f^k x)}\|$ for all $n \geq 1$. We have similar observations for the subbundle F.

The first result we recall is the argument of Mañé which can locate a *nonrecurrent* point (see [7, Lemma 5.2]. Also see [1, Corollary 1]).

Proposition 2.1. *Let f be a diffeomorphism on M and $\mathcal{W}$ be an f-invariant foliation tangent to a distribution $E \subset TM$ such that Df is uniformly expanding (or uniformly contracting) on E. Then there exists a nonrecurrent point of f. Moreover the set $\{z \in M : z \notin \omega(z)\}$ of points that are nonrecurrent in the future is dense in every leaf of $\mathcal{W}$.*

Let's sketch the proof here. First we show that each leaf $W(x)$ contains at most one periodic point. To this end, we assume E is ν-expanding for some $\nu > 1$. Suppose that there are two periodic points $p, q \in W(x)$ for some $x \in M$. Pick some $n \geq 1$ with $f^n p = p$ with $f^n q = q$. Let γ be a smooth curve in $W(x)$ connecting p and q with length $|\gamma| \leq \nu^{1/2} \cdot d_W(p, q)$. Then $f^{-n}\gamma$ is also a path connecting p and q with length $d_W(p, q) \leq |f^{-n}\gamma| \leq \nu^{-n} \cdot |\gamma| \leq \nu^{-1/2} \cdot d_W(p, q)$. This is impossible unless $p = q$. This shows that each leaf $W(x)$ contains at most one periodic point. Let $N \geq 1$ such that $\lambda^N \geq 5$. Then for a nonperiodic point $y \in W(x)$ we pick $\epsilon > 0$ small enough such that $f^k B(y, \epsilon)$, $0 \leq k \leq N$ are pairwisely disjoint. Then choose $\delta > 0$ to be much smaller than ϵ. We define inductively a sequence of closed disks $D_n \subset f^n W(y, \delta)$ for $n \geq 1$ with $D_{n+1} \subset f D_n$ and $D_n \cap B(y, 2\delta) = \emptyset$. Then each point $z \in \bigcap_{n \geq 1} f^{-n} D_n$ satisfies $f^n z \notin B(y, 2\delta)$ for all $n \geq 1$. Such z is nonrecurrent and we finishes the proof. See [1, Lemma 5.2] for the construction of D_n and more details.

The second one is Liao's *Sifting Lemma* which helps us to locate some *periodic* point. See [5, 6] for example.

Proposition 2.2. *Let $1 \le I \le d-1$ and Λ be a compact invariant set of f with a λ-dominated splitting $T_\Lambda M = E \oplus F$ of index I. Assume*

(1) *There is a point $b \in \Lambda$ satisfying $\prod_{k=0}^{n-1} \|Df|_{E(f^k b)}\| \ge 1$ for all $n \ge 1$.*
(2) *(The tilde condition.) There are λ_1 and λ_2 with $\lambda < \lambda_1 < \lambda_2 < 1$ such that if a point $x \in \Lambda$ satisfies $\prod_{k=0}^{n-1} \|Df|_{E(f^k x)}\| \ge \lambda_2^n$ for all $n \ge 1$, then the omega-set $\omega(x)$ contains a point c satisfying $\prod_{k=0}^{n-1} \|Df|_{E(f^k c)}\| \le \lambda_1^n$ for all $n \ge 1$.*

Then for each $\lambda_3 \in (\lambda_2, 1)$ and each $l \in \mathbb{N}$, there are l positive integers $n_1 < n_2 < \cdots < n_l$ with the following property: for every $j = 1, \cdots, l-1$ and every $k = n_j + 1, \cdots, n_{j+1}$,

$$\prod_{i=n_j}^{k-1} \|Df|_{E(f^i b)}\| \le \lambda_3^{k-n_j} \text{ and } \prod_{i=k-1}^{n_{j+1}-1} \|Df|_{E(f^i b)}\| \ge \lambda_2^{n_{j+1}-k+1}. \tag{2.1}$$

See [10, Lemma 2.2] for a proof. In the following we sketch how to find a hyperbolic periodic point near Λ. Proposition 2.2 shows that there are many 'double' uniform stings when $f^n b$ approaches some 'good' point $c \in \omega(b)$: for each $\lambda_3 \in (\lambda_2, 1)$ and each $l \in \mathbb{N}$, there are l positive integers $n_1 < n_2 < \cdots < n_l$ with the following property: for every $j = 1, \cdots, l-1$ and every $k = n_j + 1, \cdots, n_{j+1}$,

$$\prod_{i=n_j}^{k-1} \|Df|_{E(f^i b)}\| \le \lambda_3^{k-n_j} \text{ and } \prod_{i=k-1}^{n_{j+1}-1} \|Df|_{E(f^i b)}\| \ge \lambda_2^{n_{j+1}-k+1}.$$

Then by λ-domination assumption we have that

$$\prod_{i=k-1}^{n_{j+1}-1} \|Df^{-1}|_{F(f^{i+1} b)}\| \le \prod_{i=k-1}^{n_{j+1}-1} \frac{\lambda}{\|Df|_{E(f^i b)}\|} \le \left(\frac{\lambda}{\lambda_2}\right)^{n_{j+1}-k+1}$$

for every $j = 1, \cdots, l-1$ and every $k = n_j + 1, \cdots, n_{j+1}$. Let $\tilde{\lambda} = \max\{\sqrt{\lambda}, \lambda_3, \lambda/\lambda_2\}$. Then $\tilde{\lambda} < 1$ and $(f^{n_j} b, f^{n_{j+1}} b)$ forms a '$\tilde{\lambda}$-quasi-hyperbolic string' for each $j = 1, \cdots, l-1$.

Let $L \ge 1$ and d_0 given by [2, Theorem 1.1] with respect to $\tilde{\lambda}$. For $\epsilon \in (0, d_0]$ let's pick an integer $l = l(\epsilon) \ge 1$ large enough such that given arbitrary l points $x_1, \cdots, x_l$ in M, there exists $1 \le i < j \le l$ such that $d(x_i, x_j) < \epsilon$. For this l we let $n_1 < \cdots < n_l$ be given by Proposition 2.2 such that (2.1) holds. Then $d(f^{n_i} b, f^{n_j} b) < \epsilon$ for some $1 \le i < j \le l$. Finally we apply Liao–Gan's shadowing lemma (see [2, Theorem 1.1]) to find a hyperbolic periodic point $L\epsilon$-shadowing the periodic pseudo-orbit $\{(f^{n_i} b, f^{n_{i+1}} b), (f^{n_{i+1}} b, f^{n_{i+2}} b), \cdots, (f^{n_{j-1}} b, f^{n_j} b)\}$. This finishes the proof. For more details see [2, 10]. Also see Liao's Selecting Lemma [10, Lemma 2.3] for more information.

3. DOMINATED SPLITTING AND MINIMALITY

With the preparations in previous section let's prove the main theorem that if the map f has a global dominated splitting, then it can not be minimal.

Proof of Main Theorem. Let $TM = E \oplus F$ be a λ-dominated splitting for some $\lambda \in (0,1)$ with respect some adapted Riemannian metric. If F is uniformly expanding, then F is uniquely integrable and tangent to the strongly unstable foliation $\mathcal{W}^{su}$. By Proposition 2.1 there exists some nonrecurrent point of f and hence f can not be minimal on M. Similarly if E is uniformly contracting, then f can not be minimal either. Then we are left with the case that neither E is uniformly contracting, nor F is uniformly expanding. In this case we have that:

(1) since E is not uniformly contracting, there exists $p \in M$ such that $\|Df^n|_{E(p)}\| \geq 1$ for all $n \geq 1$,
(2) since F is not uniformly expanding, there exists $q \in M$ such that $\|Df^{-n}|_{F(f^n q)}\| \geq 1$ for all $n \geq 1$.

We first observe that, by the λ-domination assumption, for all $n \geq 1$,

$$\prod_{k=0}^{n-1} \|Df|_{E(f^k q)}\| \leq \prod_{k=0}^{n-1} \frac{\lambda}{\|Df^{-1}|_{F(f^{k+1}q)}\|} \leq \frac{\lambda^n}{\|Df^{-n}|_{F(f^n q)}\|} \leq \lambda^n. \tag{3.1}$$

Also note that the first condition in Proposition 2.2 is already satisfied if we take $\Lambda = M$ and $b = p$ since $\prod_{k=0}^{n-1} \|Df|_{E((f^k p)}\| \geq \|Df^n|_{E(p)}\| \geq 1$ for each $n \geq 1$. Then we divide the discussion into two subcases:

Subcase 1. The *tilde condition* in Proposition 2.2 holds on M for some λ_1, λ_2 with $\lambda < \lambda_1 < \lambda_2 < 1$. Then by Proposition 2.2 and Liao–Gan's shadowing lemma (Theorem 1.1 in [2]), there does exist a hyperbolic periodic point of f: the map f can not be minimal.

Subcase 2. The *tilde condition* fails. So for each pair λ_1, λ_2 with $\lambda < \lambda_1 < \lambda_2 < 1$, there exists some point $\tilde{x} \in M$ such that

- $\prod_{k=0}^{n-1} \|Df|_{E(f^k \tilde{x})}\| \geq \lambda_2^n$ for all $n \geq 1$.
- for each $y \in \omega(\tilde{x})$, there exists some $n(y) \geq 1$ with $\prod_{k=0}^{n(y)-1} \|Df|_{E(f^k y)}\| \geq \lambda_1^{n(y)}$.

According to (3.1), we see that $q \notin \omega(\tilde{x})$ since $\lambda < \lambda_1$. So $\omega(\tilde{x}) \subsetneq M$ for some point $\tilde{x} \in M$ and the map f is not minimal either.

This finishes the verification for both subcases and ends the proof of theorem.

Remark 1. The result is not true if we consider invariant subsets instead of the whole manifold, since there are various kinds of minimal subsets on which the map f is dominated. For example let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ be the induced diffeomorphism. Let $R : \mathbb{T} \to \mathbb{T}$ be an irrational rotation. Then f_A is Anosov with a fixed point $o \in \mathbb{T}^2$ and R is minimal. Moreover the product system $(R, f_A) : \mathbb{T} \times \mathbb{T}^2 \to \mathbb{T} \times \mathbb{T}^2$ is partially hyperbolic with an invariant minimal subset $\Lambda = \mathbb{T} \times \{o\}$.

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Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, P. R. China,
and CEMA, Central University of Finance and Economics, Beijing 100081, China

E-mail address: pfzh311@gmail.com