THE VOLUME OF AN ISOLATED SINGULARITY

SEBASTIEN BOUCKSOM, TOMMASO DE FERNEX, AND CHARLES FAVRE

Abstract. We introduce a notion of volume of a normal isolated singularity that generalizes Wahl's characteristic number of surface singularities to arbitrary dimensions. We prove a basic monotonicity property of this volume under finite morphisms. We draw several consequences regarding the existence of non-invertible finite endomorphisms fixing an isolated singularity. Using a cone construction, we deduce that the anticanonical divisor of any smooth projective variety carrying a non-invertible polarized endomorphism is pseudoeffective.

Our techniques build on Shokurov's b-divisors. We define the notion of nef Weil b-divisors, and of nef envelopes of b-divisors. We relate the latter notion to the pull-back of Weil divisors introduced by de Fernex and Hacon. Using the subadditivity theorem for multiplier ideals with respect to pairs recently obtained by Takagi, we carry over to the isolated singularity case the intersection theory of nef Weil b-divisors formerly developed by Boucksom, Favre and Jonsson in the smooth case.

Contents

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Introduction

Wahl's *characteristic number* [Wah90] is a topological invariant of a normal surface singularity. Its simple behavior under finite morphisms enables one to characterize surface singularities that carry finite non-invertible endomorphisms. Our main goal is to generalize Wahl's invariant to higher dimensional isolated normal singularities, and to present a few applications to the description of singularities admitting non-trivial finite endomorphisms. Our main result can be stated as follows.

Theorem A. *To any normal isolated singularity $(X, 0)$ is associated a non-negative real number* $\mathrm{Vol}(X, 0)$ *that we call its* volume*, satisfying the following properties:*

(i) *For every finite morphism $\phi : (X, 0) \to (Y, 0)$ of degree $e(\phi)$ we have*

$$\mathrm{Vol}(X, 0) \geq e(\phi)\,\mathrm{Vol}(Y, 0),$$

and equality holds when ϕ is étale in codimension one.

(ii) *If* $\dim X = 2$ *then* $\mathrm{Vol}(X, 0)$ *coincides with Wahl's characteristic number.*

(iii) *If X is $\mathbb{Q}$-Gorenstein then* $\mathrm{Vol}(X, 0) = 0$ *if and only if X has log-canonical (=lc) singularities.*

Our result generalizes in particular the well-known fact that $\mathbb{Q}$-Gorenstein lc singularities are preserved under finite morphism (see for instance [Kol97, Proposition 3.16]).

Just as in dimension 2 one infers restrictions on isolated singularities admitting finite endomorphisms.

Theorem B. *Suppose $\phi : (X, 0) \to (X, 0)$ is a finite non-invertible endomorphism of an isolated singularity. Then* $\mathrm{Vol}(X, 0) = 0$.

If X is $\mathbb{Q}$-Gorenstein then X has lc singularities, and it furthermore has klt singularities if ϕ is not étale in codimension one.

To obtain a more precise classification of singularities carrying finite endomorphisms one would need to get deeper into the structure of singularities with $\mathrm{Vol}(X, 0) = 0$. This can be done in dimension 2, see [Wah90, Fav10], but unfortunately, this task seems very difficult at the moment in arbitrary dimension. To illustrate the previous result, we construct however several classes of (non-necessarily $\mathbb{Q}$-Gorenstein) isolated normal singularities carrying finite endomorphisms, see §6.2-6.3 below. Our examples include quotient singularities, Tsuchihashi's cusp singularities [Oda, Tsu83], toric singularities, and certain simple singularities obtained from cone or deformation constructions.

In dimension 2, the conclusion of Theorem B plays a key role in the classification of projective surfaces admitting non-invertible endomorphisms, which is by now essentially complete, see [FN05, Naka08]. In higher dimensions, classifying projective varieties carrying a non-invertible endomorphism has recently attracted quite a lot of attention, see [dqZ06] and the references therein, but the general problem remains largely open.

The assumption on the singularity being isolated in Theorem B is too strong to be directly useful in this perspective. Nevertheless we observe that Theorem B has some consequences in the more rigid case of so-called *polarized endomorphisms*. Recall that an endomorphism $\phi : V \to V$ of a projective variety is said to be polarized if there exists an ample line bundle L on V such that $\phi^* L = dL$ in $\mathrm{Pic}(V)$ for some $d \geq 1$ (cf. [swZ06] for a nice survey). By looking at the affine cone over X induced by L, we obtain:

Theorem C. *If V is a smooth projective variety carrying a non-invertible polarized endomorphism ϕ then $-K_V$ is pseudoeffective.*

Observe that the ramification formula implies $K_V \cdot L^{n-1} \leq 0$. If K_V is pseudoeffective then $K_V \equiv 0$ and (V, ϕ) is then an endomorphism of an abelian variety up to finite étale cover (see [Fakh03, Theorem 4.2]). If K_V is not pseudoeffective then V is uniruled by [BDPP04], and our result then puts further constraints on the geometry of V.

Throughout the paper, we insist on working with arbitrary non $\mathbb{Q}$-Gorenstein singularities. This degree of generality is crucial to obtain Theorem C since the cone over V is $\mathbb{Q}$-Gorenstein iff $\pm K_V$ is either $\mathbb{Q}$-linearly trivial or ample, see Example 2.28 below.

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In order to understand our construction, and the difficulties that one has to overcome to define the volume above, let us recall briefly Wahl's definition for a normal surface singularity $(X, 0)$.

Pick any *log-resolution* $\pi : Y \to X$ of $(X, 0)$, i.e. a birational morphism which is an isomorphism above $X \setminus \{0\}$, and such that Y is smooth and the scheme-theoretic inverse image $\pi^{-1}(0)$ is a divisor with simple normal crossing support E. Let K_X be a canonical divisor on X and let K_Y be the induced canonical divisor on Y. Denote by $\pi^* K_X$ Mumford's *numerical pull-back* of K_X to Y, which is uniquely determined as a $\mathbb{Q}$-divisor by the conditions $\pi_*(\pi^* K_X) = K_X$ and $\pi^* K_X \cdot C = 0$ for any π-exceptional curve C. The *log-discrepancy* divisor is then defined by the relation $A_{Y/X} := K_Y + E - \pi^* K_X$. Recall that X is (numerically) lc iff $A_{Y/X} \geq 0$ while X is (numerically) klt iff $A_{Y/X} > 0$ on the whole of E.

Wahl's invariant measures the degree of positivity of the log-discrepancy divisor. The positivity is here relative to the contraction morphism $Y \to X$, and it is thus natural to consider the *relative Zariski decomposition* $A_{Y/X} = P + N$ in the sense of [Sak77], where N is the smallest effective π-exceptional $\mathbb{Q}$-divisor such that $P = A_{Y/X} - N$ is π-nef. Finally one sets:

$$\mathrm{Vol}(X, 0) := -P^2 \in \mathbb{Q}_{\geq 0} \ . \tag{1}$$

Two (related) difficulties arise in generalizing Wahl's construction to higher dimensions: first, one needs to introduce a notion of pull-back for Weil divisors; and second one needs to find a replacement for the relative Zariski decomposition. These problems have already been addressed in [dFH09], and in [BFJ08, KuMa08] respectively. Building on these works our first objective is to explain how these difficulties can be conveniently addressed using Shokurov's language of *b-divisors*. In §§1-3, we define and study the notion of nef Weil b-divisor in the general setting of a normal variety X. This leads to the notion of nef envelope and relative Zariski decomposition as follows.

Let us recall some terminology. A *Weil b-divisor* W over X is the data of Weil divisors W_π on all birational models $\pi : X_\pi \to X$ of X that are compatible under push-forward. A *Cartier b-divisor* C is a Weil b-divisor determined by some π in the sense that its incarnations $C_{\pi'}$ on all higher models $\pi' \geq \pi$ are obtained by pulling-back C_π (all the divisors we consider for the time being have $\mathbb{R}$-coefficients).

A Cartier b-divisor C is said to be *nef* (relatively to a given projective morphism $X \to S$) if C_π is nef for one (hence any) determination π of C. Generalizing [BFJ08, KuMa08] we say that a Weil b-divisor W is nef iff its numerical class is a limit of classes of nef Cartier b-divisors, or equivalently iff W_π lies in the closed movable cone $\overline{\mathrm{Mov}}(X_\pi/S)$ for all smooth models X_π (cf. Lemma 2.9 below).

In §2, we prove that the following definitions make sense (under suitable conditions), and introduce the following two notions of *nef envelopes*.

- The nef envelope $\mathrm{Env}_X(D)$ of a Weil divisor D on X is the largest nef Weil b-divisor Z that is both relatively nef over X and satisfies $Z_X \leq D$.
- The nef envelope $\mathrm{Env}_{\mathfrak{X}}(W)$ of a Weil b-divisor W is the largest nef Weil b-divisor Z that is both relatively nef over X and satisfies $Z \leq W$.

In dimension two, nef envelopes recover the notions of numerical pull-back and relative Zariski decomposition. Specifically, if D is a divisor on a normal surface X then the incarnation $\mathrm{Env}_X(D)_\pi$ on a given model X_π coincides with the numerical pull-back of D by π, while if D is a divisor on a smooth model X_π over X, then the nef part of D in its relative Zariski decomposition is given by $\mathrm{Env}_{\mathfrak{X}}(\overline{D})_\pi$ where $\overline{D}$ is the Cartier b-divisor induced by D.

In higher dimensions $D \mapsto \mathrm{Env}_X(D)$ is non-linear in general, and $\mathrm{Env}_X(D)_\pi$ coincides up to sign with the pull-back π^*D defined in [dFH09]. It is however this approach via b-divisors and nef envelopes that brings to light the crucial positivity properties of the pull-back of Weil divisors.

We are now in a position to generalize the log-discrepancy divisor and its relative Zariski decomposition. The choice of a canonical divisor K_X on X induces a canonical divisor K_Y for each model $Y \to X$, hence a *canonical b-divisor* $K_{\mathfrak{X}}$ over X. The *log-discrepancy b-divisor* is then defined as

$$A_{\mathfrak{X}/X} := K_{\mathfrak{X}} + 1_{\mathfrak{X}/X} + \mathrm{Env}_X(-K_X)\ ,$$

where the incarnation of $1_{\mathfrak{X}/X}$ in any model is equal to the reduced exceptional divisor over X. The log-discrepancy b-divisor is exceptional over X and does not depend on the choice of K_X. Its coefficients are given by the (usual) log-discrepancies of X when the latter is $\mathbb{Q}$-Gorenstein. The role of the nef part of $A_{\mathfrak{X}/X}$ in its relative Zariski decomposition is in turn played by the nef envelope

$$P := \mathrm{Env}_{\mathfrak{X}}(A_{\mathfrak{X}/X})\ .$$

To generalize (1), we now face the problem of defining the intersection product of nef b-divisors. This step is non-trivial. The intersection of Cartier b-divisors is defined as their intersection in a common determination. However it cannot be extended to a multilinear intersection product on the space of *Weil* b-divisors having reasonable continuity properties. As it turns out, it is nevertheless possible to extend it to a multilinear intersection pairing on *nef* Weil b-divisors lying over a point $0 \in X$. This is done following the approach of [BFJ08], in which multiplier ideals appear as a prominent tool.

Assume from now on that $(X, 0)$ is an n-dimensional *isolated* normal singularity. For all (relatively) nef b-divisors $W_1, ..., W_n$ above 0, we set:

$$W_1 \cdot ... \cdot W_n := \inf\{C_1 \cdot ... \cdot C_n \mid C_j \text{ nef Cartier, } C_j \geq W_j\} \in [-\infty, 0] \ .$$

To develop a reasonable calculus of these intersection numbers, *additivity* in each variable is a desirable property. We obtain this result as a consequence of the fact that any nef envelope of a Cartier b-divisor is the *decreasing* limit of a sequence of nef Cartier b-divisors C_k.

Let us explain how to get this crucial approximation property. The first observation is that the nef envelope of a Cartier b-divisor C is a limit of the graded sequence of ideals $\mathfrak{a}_m := \mathcal{O}_X(mC)$, $m \geq 0$ (see §2.1). For any fixed $c > 0$, we use the general notion of (asymptotic) multiplier ideal $\mathcal{J}(X; \mathfrak{a}_\bullet^c)$ introduced in [dFH09] for any ambient variety X with normal singularities. As was shown in [dFH09] this multiplier ideal can also be computed using *compatible boundaries*: namely, there exist effective $\mathbb{Q}$-boundaries Δ such that $\mathcal{J}(X; \mathfrak{a}_\bullet^c)$ coincides with the standard (asymptotic) multiplier ideal $\mathcal{J}((X, \Delta); \mathfrak{a}_\bullet^c)$ with respect to the pair (X, Δ).

This connection enables us to make use of a recent result of Takagi [Tak10], which extends the usual subadditivity property of multiplier ideals [DEL00] to multiplier ideals with respect to a pair (X, Δ), up to an (inevitable) error term involving Δ and the Jacobian ideal of X. The approximation we are looking for then follows by taking the nef Cartier b-divisor C_k associated to $\mathcal{J}(X; \mathfrak{a}_\bullet^k)$.

Now that we have defined the intersection product of nef Weil b-divisors, we can come back to the definition of the volume. We set

$$\mathrm{Vol}(X, 0) := -\operatorname{Env}_{\mathfrak{X}}(A_{\mathfrak{X}/X})^n,$$

which is shown to be finite (and non-negative). Once the volume is defined, the properties stated in Theorem A follow smoothly from transformation laws of envelopes under finite morphisms, see Proposition 2.17.

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The volume relates to other kind of invariants that were previously defined and are connected to growth rate of pluricanonical forms.

In the 2-dimensional case, we first note that the definition (1) admits an equivalent formulation in terms of the growth rate of a certain quotient of sections. It was indeed shown in [Wah90] that if X is a surface then

$$\dim \left(H^0(X \setminus \{0\}, mK_X)/H^0(Y, m(K_Y + E)) \right) = \frac{m^2}{2} \mathrm{Vol}(X, 0) + o(m^2) \ ,$$

where the left-hand side is independent of the choice of Y and is equal by definition to the m-th *log-plurigenus* $\lambda_m(X,0)$ in the sense of Morales [Mora87], a notion which makes sense in all dimensions.

In line with this point of view M. Fulger [Fulg] has recently considered the following invariant of an isolated singularity $(X,0)$:

$$\mathrm{Vol}_F(X,0) := \limsup_m \frac{n!}{m^n} \dim \left(H^0(X \setminus \{0\}, mK_X)/H^0(Y, m(K_Y+E))\right).$$

It measures by definition the growth rate of $\lambda_m(X,0)$, or equivalently that of Watanabe's L^2*-plurigenera* $\delta_m(X,0)$ [Wat80, Wat87], and yields a finite number since

$$\delta_m(X,0) = \lambda_m(X,0) + O(m^{n-1}) = O(m^n)$$

(see [Ish90], which contains a thorough introduction to these notions, and §5.2 below).

The notion of volume considered by Fulger also behaves well under finite morphisms, and the analog of Theorem A holds true. Moreover, in contrast to our volume, $\mathrm{Vol}_F(X,0)$ is more accessible to explicit computations. On the other hand, our volume $\mathrm{Vol}(X,0)$ relates more closely to lc singularities (see question (c) below).

Fulger explores in [Fulg] how the two approaches compare to one another, proving that $\mathrm{Vol}(X,0) \geq \mathrm{Vol}_F(X,0)$ for any isolated normal singularity $(X,0)$. Equality holds when X is $\mathbb{Q}$-Gorenstein, but can fail otherwise (cf. Example 5.4).

* * *

This paper leaves several questions open.

(a) Is $\mathrm{Vol}(X,0)$ a topological invariant of the link of the singularity?
(b) Does there exists a *positive* lower bound, only depending on the dimension, for the volume of isolated Gorenstein singularities with positive volume?
(c) Is it true that $\mathrm{Vol}(X,0) = 0$ implies the existence of an effective $\mathbb{Q}$-boundary Δ such that the pair (X,Δ) is log-canonical? (the converse being easily shown).

As explained in [Wah90], question (a) has a positive answer in dimension two. And question (b) was solved by Ganter [Gan] for surfaces. In case of a positive answer to (a) in general, it would follow (as in the two-dimensional case) that the volume $\mathrm{Vol}(X,0)$ is a characteristic number of the link of the singularity. It is to be noted that (c) fails with $\mathrm{Vol}_F(X,0)$ in place of $\mathrm{Vol}(X,0)$, again by Example 5.4.

It seems likely that there should be examples where the volume $\mathrm{Vol}(X,0)$ is irrational. In [Urb10] Urbinati constructs examples where the log-discrepancy takes irrational values, and in [Fulg] Fulger shows that similar examples have irrational volume $\mathrm{Vol}_F(X,0)$ in his sense.

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The plan of our paper is the following. In the first four sections, we work over a normal algebraic variety. Section 1 contains basics on b-divisors. The notion of envelope is analyzed in detail in §2. In this section we also formalize a measure of the failure of a Weil divisor to be Cartier in terms of certain *defect ideals*, which are related to the notion of compatible boundary. In §3 we turn to the definition of the log-discrepancy b-divisor and of multiplier ideals. The key result of this section is the subadditivity theorem (Theorem 3.13) that we deduce from Takagi's work.

The rest of the paper deals with normal isolated singularities. We define the volume of such a singularity and prove Theorem A (i) and (iii) in §4. In §5 we complete the proof of Theorem A, and compare our notion with the approaches via plurigenera and Fulger's work. Finally §6 focuses on endomorphisms, and contains a proof of Theorem B and C.

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1. Shokurov's b-divisors

In this section X denotes a normal variety defined over an algebraically closed field of characteristic 0 and we set $n := \dim X$. The goal of this section is to gather general properties of Shokurov's b-divisors over X, for which [Isk03] and [Cor] constitute general references. Proposition 1.11 seems to be new.

1.1. **The Riemann-Zariski space.** The set of all proper birational morphisms $\pi : X_\pi \to X$ modulo isomorphism is (partially) ordered by $\pi' \geq \pi$ iff π' factors through π, and the order is inductive (i.e. any two proper birational morphisms to X can be dominated by a third one). The *Riemann-Zariski space* of X is defined as the projective limit

$$\mathfrak{X} = \varprojlim_\pi X_\pi,$$

taken in the category of locally ringed topological spaces, each X_π being viewed as a scheme with its Zariski topology (note that $\mathfrak{X}$ itself is not a scheme anymore).

As a topological space $\mathfrak{X}$ may alternatively be viewed as the set of all valuation subrings $V \subset k(X)$ with non-empty center on X, endowed with the Krull-Zariski topology. Indeed given a Krull valuation V the center $c_\pi(V)$ of V on X_π is non-empty for each π by the valuative criterion for properness, and the collection of all scheme-theoretic points $c_\pi(V)$ defines a point in $c(V)$ in $\mathfrak{X}$. By [ZS, p.122 Theorem 41] the mapping $V \mapsto c(V)$ so defined is a homeomorphism.

1.2. **Divisors on the Riemann-Zariski space.** Following Shokurov we define the group of *Weil b-divisors* over X (where b stands for birational) as

$$\operatorname{Div}(\mathfrak{X}) := \varprojlim_\pi \operatorname{Div}(X_\pi)$$

where $\operatorname{Div}(X_\pi)$ denotes the group of Weil divisors of X_π. It can alternatively be thought of as the group of Weil divisors on the Riemann-Zariski space $\mathfrak{X}$ (hence the notation).

The group of *Cartier b-divisors* over X is in turn defined as

$$\operatorname{CDiv}(\mathfrak{X}) := \varinjlim_\pi \operatorname{CDiv}(X_\pi)$$

with $\operatorname{CDiv}(X_\pi)$ denoting the group of Cartier divisors of X_π. One can easily check that

$$\operatorname{CDiv}(\mathfrak{X}) = H^0(\mathfrak{X}, \mathcal{M}^*_\mathfrak{X}/\mathcal{O}^*_\mathfrak{X})$$

is indeed the group of Cartier divisors of the locally ringed space $\mathfrak{X}$.

There is an injection $\mathrm{CDiv}(\mathfrak{X}) \hookrightarrow \mathrm{Div}(\mathfrak{X})$ determined by the cycle maps on birational models X_π.

An element of $\mathrm{Div}_{\mathbb{R}}(\mathfrak{X}) := \mathrm{Div}(\mathfrak{X}) \otimes \mathbb{R}$ (resp. $\mathrm{CDiv}_{\mathbb{R}}(\mathfrak{X}) := \mathrm{CDiv}(\mathfrak{X}) \otimes \mathbb{R}$) will be called an $\mathbb{R}$-Weil b-divisor (resp. $\mathbb{R}$-Cartier b-divisor), and similarly with $\mathbb{Q}$ in place of $\mathbb{R}$.

Let us now interpret these definitions in more concrete terms. A Weil divisor W on $\mathfrak{X}$ consists of a family of Weil divisors $W_\pi \in \mathrm{Div}(X_\pi)$ that are compatible under push-forward, i.e. such that $W_\pi = \mu_* W_{\pi'}$ whenever π' factors through a morphism $\mu\colon X_{\pi'} \to X_\pi$. We say that W_π (also denoted by W_{X_π}) is the *incarnation* of W on the model X_π. By contrast, a Cartier divisor C on $\mathfrak{X}$ is determined by its incarnation on a high enough model, i.e. there exists π such that $C_{\pi'} = \mu^* C_\pi$ for every $\pi' \geq \pi$, where $\mu\colon X_{\pi'} \to X_\pi$ is the induced morphism. We shall say that C is *determined on* X_π (or *by* π).

Weil b-divisors can also be interpreted as certain functions on the set of *divisorial valuations* of X. Recall first that a divisorial valuation of X is a rank 1 valuation of transcendence degree $\dim X - 1$ of the function field $k(X)$, whose center on X is non-empty. By a classical result of Zariski (see e.g. [KoMo, Lemma 2.45]) the divisorial valuations on X are exactly those of the form $\nu = t\,\mathrm{ord}_E$ where $t \in \mathbb{R}_+^*$ and E is a prime divisor on some birational model X_π over X.

Given an $\mathbb{R}$-Weil b-divisor W over X we can then define $(t\,\mathrm{ord}_E)(W)$ as t times the coefficient of E in W_π. Setting $g_W(\nu) := \nu(W)$ yields an identification $W \mapsto g_W$ between $\mathrm{Div}_{\mathbb{R}}(\mathfrak{X})$ and the space of all real-valued 1-homogeneous functions g on the set of divisorial valuations of X satisfying the following finiteness property: the set of prime divisors $E \subset X$ (or equivalently on X_π for any given π) such that $g(\mathrm{ord}_E) \neq 0$ is finite.

The topology of pointwise convergence therefore induces a *topology of coefficient-wise* convergence on $\mathrm{Div}_{\mathbb{R}}(\mathfrak{X})$, for which $\lim_j W_j = W$ iff $\lim_j \mathrm{ord}_E(W_j) = \mathrm{ord}_E(W)$ for each prime divisor E over X.

1.3. **Examples of b-divisors.** We introduce the main types of b-divisors we shall consider.

Example 1.1. The choice of a non-zero rational form ω of top degree on X induces a *canonical b-divisor* $K_{\mathfrak{X}}$ whose incarnation on X_π is equal to the canonical divisor determined by ω on X_π.

Example 1.2. A Cartier divisor D on a given model X_π induces a Cartier b-divisor $\overline{D}$, its *pull-back* to $\mathfrak{X}$. It is simply defined by pulling-back D to all models dominating X_π. By definition all Cartier b-divisors are actually obtained this way.

Example 1.3. Given a coherent fractional ideal sheaf $\mathfrak{a}$ on X we denote by $Z(\mathfrak{a})$ the Cartier b-divisor determined on the normalized blow-up X_π of X along $\mathfrak{a}$ by

$$\mathfrak{a} \cdot \mathcal{O}_{X_\pi} = \mathcal{O}_{X_\pi}(Z(\mathfrak{a})_\pi).$$

In particular we have $Z(f)_\pi = -\pi^* \operatorname{div}(f)$ when f is a rational function on X. Note that with this convention $Z(\mathfrak{a})$ is anti-effective when $\mathfrak{a}$ is an actual ideal sheaf.

We record the following easy properties.

Lemma 1.4. *Let $\mathfrak{a}, \mathfrak{b}$ be two coherent fractional ideal sheaves on X.*

- $Z(\mathfrak{a}) \leq Z(\mathfrak{b})$ *whenever $\mathfrak{a} \subset \mathfrak{b}$.*

- $Z(\mathfrak{a}\cdot\mathfrak{b}) = Z(\mathfrak{a}) + Z(\mathfrak{b})$.
- $Z(\mathfrak{a}+\mathfrak{b}) = \max\{Z(\mathfrak{a}), Z(\mathfrak{b})\}$, *where the maximum is defined coefficient-wise.*
- $Z(\mathfrak{a}) = Z(\mathfrak{b})$ *iff the integral closures of* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are equal.*

Remark 1.5. Given an ideal sheaf $\mathfrak{a}$ and a positive number $s > 0$ we set $Z(\mathfrak{a}^s) := sZ(\mathfrak{a})$. Then (basically by definition) we have $Z(\mathfrak{a}^s) = Z(\mathfrak{b}^t)$ iff the 'ℝ-ideals' $\mathfrak{a}^s$ and $\mathfrak{b}^t$ are equivalent in the sense of Teissier and Kawakita.

Definition 1.6. *Let W be an ℝ-Weil b-divisor over X. We denote by $\mathcal{O}_X(W)$ the fractional ideal sheaf of X whose sections on an open set $U \subset X$ are the rational functions f such that $Z(f) \le W$ over U.*

We emphasize that the sheaf of $\mathcal{O}_X$-modules $\mathcal{O}_X(W)$ is *not* coherent in general, since we are imposing infinitely many (even uncountably many) conditions on f (compare [Isk03]). Note that $\pi_*\mathcal{O}_{X_\pi}(W_\pi) \subset \tau_*\mathcal{O}_{X_\tau}(W_\tau)$ whenever $\pi \ge \tau$ and

$$\mathcal{O}_X(W) = \bigcap_\pi \pi_*\mathcal{O}_{X_\pi}(W_\pi).$$

However if C is an ℝ-Cartier b-divisor then we have $\mathcal{O}_X(C) = \pi_*\mathcal{O}_{X_\pi}(C_\pi)$ for each determination π of C, and $\mathcal{O}_X(C)$ is in particular coherent in that case.

Cartier b-divisors associated with coherent fractional ideal sheaves can be characterized as follows:

Lemma 1.7. *A Cartier b-divisor $C \in \mathrm{CDiv}(\mathfrak{X})$ is of the form $Z(\mathfrak{a})$ for some coherent fractional ideal sheaf $\mathfrak{a}$ on X iff C is relatively globally generated over X.*

In particular the Cartier divisors $Z(\mathfrak{a})$ with $\mathfrak{a}$ ranging over all coherent (fractional) ideal sheaves of X generate $\mathrm{CDiv}(\mathfrak{X})$ *as a group.*

Here we say that C is relatively globally generated over X iff so is C_π for one (hence any) determination π of C.

Proof. Let C be a Cartier b-divisor determined by π. To say that C is relatively globally generated over X means by definition that the evaluation map

$$\pi^*\pi_*\mathcal{O}_{X_\pi}(C_\pi) \to \mathcal{O}_{X_\pi}(C_\pi)$$

is surjective. If this is the case we thus see that $C = Z(\mathfrak{a})$ with $\mathfrak{a} := \pi_*\mathcal{O}_{X_\pi}(C_\pi) = \mathcal{O}_X(C)$, while the converse direction is equally clear. The second assertion now follows from the fact that any Cartier divisor on a given model X_π can be written as a difference of two π-very ample (hence π-globally generated) Cartier divisors. □

1.4. **Numerical classes of b-divisors.** Let $X \to S$ be a projective morphism. We define the space of 1-*codimensional numerical classes* by

$$N^1(\mathfrak{X}/S) := \varinjlim_\pi N^1(X_\pi/S)$$

where the maps are given by pulling-back. We define in turn the space of $(n-1)$-*dimensional numerical classes* by

$$N_{n-1}(\mathfrak{X}/S) := \varprojlim_\pi N^1(X_\pi/S)$$

where the maps are given by pushing-forward and π now runs over all *smooth* (or at least ℚ-factorial) birational models of X – so that the push-forward map $N^1(X_{\pi'}/S) \to N^1(X_\pi/S)$ is well-defined for $\pi' \ge \pi$.

Each $N^1(X_\pi/S)$ is a finite dimensional $\mathbb{R}$-vector space and we endow $N^1(\mathfrak{X}/S)$ and $N_{n-1}(\mathfrak{X}/S)$ with their natural inductive and projective limit topologies respectively. The cycle maps induce a natural continuous injection with dense image $N^1(\mathfrak{X}/S) \to N_{n-1}(\mathfrak{X}/S)$. There are also natural surjections $\mathrm{CDiv}_\mathbb{R}(\mathfrak{X}) \to N^1(\mathfrak{X}/S)$ and $\mathrm{Div}_\mathbb{R}(\mathfrak{X}) \to N_{n-1}(\mathfrak{X}/S)$, but one should be careful that the latter map is *not* continuous with respect to coefficient-wise convergence in general.

Example 1.8. Consider an infinite sequence C_j of (-1)-curves on $X = \mathbb{P}^2$ blown-up at 9 points. We then have $C_j \to 0$ coefficient-wise but the numerical classes $[C_j] \in N^1(X)$ do not tend to zero since $C_j^2 = -1$ for each j.

Lemma 1.9. *Let $\pi : X_\pi \to X$ be a birational model of X and let $\alpha \in N^1(X_\pi/X)$. Then there exists at most one π-exceptional $\mathbb{R}$-Cartier divisor D on X_π whose numerical class is equal to α.*

Proof. Let D be a π-exceptional and π-numerically trivial $\mathbb{R}$-Cartier divisor. We are to show that $D = 0$. Upon pulling-back D to a higher birational model, we may assume that π is the blow-up of X along a subscheme of codimension at least two. If we denote by E_j the π-exceptional divisors we then have on the one hand $D = \sum_j d_j E_j$ and on the other hand there exists positive integers a_j such that $F := \sum_j a_j E_j$ is π-antiample. Now set $t := \max_j d_j/a_j$. If we assume by contradiction that $D \neq 0$ then upon replacing D by $-D$ we may assume that $t > 0$. Now $D - tF$ is effective and there exists j such that E_j is not contained in its support. If $C \subset E_j$ is a general curve in a fiber of π we then have $(D - tF) \cdot C \geq 0$ since C is not contained in the support of the effective divisor $D - tF$, which contradicts the fact that $D - tF$ is π-ample. $\square$

Even assuming that X_π is smooth, it is not true in general that any class $\alpha \in N^1(X_\pi/X)$ can be represented by a π-exceptional $\mathbb{R}$-divisor (since π might for instance be small, i.e. without any π-exceptional divisor). It is however true when X is $\mathbb{Q}$-factorial, and for any normal X when $\dim X = 2$ thanks to Mumford's numerical pull-back.

Using these remarks we may now prove the following simple lemma which enables to circumvent the discontinuity of the quotient map $\mathrm{Div}_\mathbb{R}(\mathfrak{X}) \to N_{n-1}(\mathfrak{X}/S)$.

Lemma 1.10.

(a) *Let W_j be a sequence (or net) of $\mathbb{R}$-Weil b-divisors which converges to an $\mathbb{R}$-Weil b-divisor W coefficient-wise. If there exists a fixed finite dimensional vector space V of $\mathbb{R}$-Weil divisors on X such that $W_{j,X} \in V$ for all j then $[W_j] \to [W]$ in $N_{n-1}(\mathfrak{X}/S)$.*

(b) *Let conversely $\alpha_j \to \alpha$ be a convergent sequence (or net) in $N_{n-1}(\mathfrak{X}/S)$. Then there exist representatives $W_j, W \in \mathrm{Div}_\mathbb{R}(\mathfrak{X})$ of α_j and α respectively and a finite dimensional vector space V of $\mathbb{R}$-Weil divisors on X such that*
   - *$W_j \to W$ coefficient-wise.*
   - *$W_{j,X} \in V$ for all j.*

   If $\alpha_j \in N^1(\mathfrak{X}/S)$ then W_j can be chosen to be $\mathbb{R}$-Cartier.

Proof. For each smooth model π the existence of V yields a finite dimensional space V_π of $\mathbb{R}$-divisors on X_π such that $W_{j,\pi} \in V_\pi$ for all j. The natural linear map $V_\pi \to N^1(X_\pi/S)$ is of course continuous since both spaces are finite dimensional, and it follows that $[W_{j,\pi}] \to$

$[W_\pi]$ in $N^1(X_\pi/S)$ for each smooth model. Since smooth models are cofinal in the family of all models we conclude as desired that $[W_j] \to [W]$ in $N_{n-1}(\mathfrak{X}/S)$.

We now consider the converse. Let X_π be a fixed smooth model of X. For each j $\alpha_j - \overline{\alpha}_{j,\pi}$ (resp. $\alpha - \overline{\alpha}_\pi$) is exceptional over X_π. By the above remarks it is thus *uniquely* represented by an $\mathbb{R}$-Weil b-divisor Z_j (resp. Z) that is exceptional over X_π. Since $(\alpha_j - \overline{\alpha}_{j,\pi})_{\pi'}$ converges to $(\alpha - \overline{\alpha}_\pi)_{\pi'}$ in $N^1(X_{\pi'}/X_\pi)$ for each $\pi' \geq \pi$ it follows by uniqueness of Z_j that $Z_j \to Z$ coefficient-wise.

On the other hand since $N^1(X_\pi/S)$ is finite dimensional there exists a finite dimensional $\mathbb{R}$-vector space V of $\mathbb{R}$-divisors on X_π such that $V \to N^1(X_\pi/X)$ is surjective. This map is therefore open and we may thus find representatives $C_j \in V$ of $\alpha_{j,\pi}$ converging to a representative $C \in V$ of α_π. Setting $W_j := Z_j + \overline{C}_j$ concludes the proof. □

1.5. **Functoriality.** Given a morphism $\phi : X \to Y$ between normal varieties it is immediate to see that pushing forward and pulling back respectively induce homomorphisms $\phi_* : \mathrm{Div}(\mathfrak{X}) \to \mathrm{Div}(\mathfrak{Y})$ and $\phi^* : \mathrm{CDiv}(\mathfrak{Y}) \to \mathrm{CDiv}(\mathfrak{X})$ in a functorial way.

In what follows we furthermore assume that $\phi : X \to Y$ is dominant and generically finite.

Proposition 1.11. *Let $\phi : X \to Y$ be a dominant generically finite morphism. Then $\phi_* \mathrm{CDiv}(\mathfrak{X}) \subset \mathrm{CDiv}(\mathfrak{Y})$.*

Proof. The assertion is obvious when ϕ is birational because we are just shifting models in that case. Using the Stein factorization of ϕ we may thus assume that ϕ is finite (and still dominant). By Lemma 1.7 it is then enough to show that for every coherent fractional ideal sheaf $\mathfrak{a}$ on X there exists a coherent fractional ideal sheaf $\mathfrak{b}$ on Y such that $\phi_* Z(\mathfrak{a}) = Z(\mathfrak{b})$. In fact we claim that

$$\phi_* Z(\mathfrak{a}) = Z(N_{X/Y}(\mathfrak{a})) \tag{2}$$

where $N_{X/Y}(\mathfrak{a})$ denotes the image of $\mathfrak{a}$ under the *norm homomorphism* (compare [EGA4, Définition 21.5.5]).

More precisely pick an affine chart $U \subset Y$. Since the restriction $\phi^{-1}(U) \to U$ is finite, $\phi^{-1}(U)$ is affine and $\mathfrak{a}$ is thus generated by its global sections g on $\phi^{-1}(U)$. For each such g its norm is defined by setting

$$N_{X/Y}(g)(x) = \prod_{\phi(y)=x} g(y)$$

for every smooth point $x \in U$ over which ϕ is étale and by extending it to a regular function on U by normality. We then define $N_{X/Y}(\mathfrak{a})(U)$ as the $\mathcal{O}_U$-module generated by all $N_{X/Y}(g)$ with g as above.

Let us now prove (2). Pick a prime divisor E on a birational model $Y' \to Y$ and choose a birational model $X' \to X$ such that ϕ lifts to a morphism $\phi' \colon X' \to Y'$. Note that ϕ' is proper and generically finite. Let $(E_i)_i$ be the (finitely many) prime divisors of X' dominating E, so that $\phi_* E_i = c_i E$ for some positive integer c_i. Then we have

$$\mathrm{ord}_E(\phi_* Z(\mathfrak{a})) = \sum_i c_i \,\mathrm{ord}_{E_i} Z(\mathfrak{a}) = -\sum_i c_i \,\mathrm{ord}_{E_i}(\mathfrak{a})$$

by definition of ϕ_*. On the other hand, let $V \subset Y'$ be an affine chart containing a point of E. The ideal sheaf $N_{X/Y}(\mathfrak{a}) \cdot \mathcal{O}_{Y'}$ is generated, over V, by the functions $N_{X'/Y'}(g)$ where

g ranges over all global sections of $\mathfrak{a} \cdot \mathcal{O}_{X'}$ on $(\phi')^{-1}(V)$. We have

$$\operatorname{ord}_E N_{X'/Y'}(g) = \sum c_i \operatorname{ord}_{E_i}(g)$$

hence

$$\begin{aligned}\operatorname{ord}_E N_{X/Y}(\mathfrak{a}) &= \min\{\operatorname{ord}_E N_{X'/Y'}(g),\, g \in H^0((\phi')^{-1}(V), \mathfrak{a} \cdot \mathcal{O}_{X'})\} \\ &= \min\{\sum c_i \operatorname{ord}_{E_i}(f),\, f \in \mathfrak{a}\}\end{aligned}$$

which proves the claim since we have $\operatorname{ord}_{E_i}(f) = \operatorname{ord}_{E_i}(\mathfrak{a})$ for each i if $f \in \mathfrak{a}$ is a general element. □

Let us now consider the pull-back operator $\phi^* : \operatorname{CDiv}(\mathfrak{Y}) \to \operatorname{CDiv}(\mathfrak{X})$. Still assuming that ϕ is generically finite and dominant we are going to show that ϕ^* extends in a natural way to $\operatorname{Div}(\mathfrak{Y}) \to \operatorname{Div}(\mathfrak{X})$. Indeed given a divisorial valuation ν on X it is well-known that the valuation $\phi_*\nu$ defined by

$$(\phi_*\nu)(f) := \nu(f \circ \phi)$$

is a divisorial valuation on Y (since the restriction of the valuation ring of ν to $\mathbb{C}(Y)$ has transcendence degree $\dim Y - 1$ by [ZS, VI.6, Corollary 1]).

Given a prime divisor E over X we thus have $\phi_* \operatorname{ord}_E = b \operatorname{ord}_F$ for some positive number b and some prime divisor F over Y. The coefficient b is actually an integer and can be described as follows: there exist birational models $X' \to X$ and $Y' \to Y$ such that E (resp. F) is a prime divisor on X' (resp. Y') and such that the rational lift $\phi' : X' \dashrightarrow Y'$ of ϕ sends the generic point of E to that of F. We then have $b = \operatorname{ord}_E((\phi')^*F)$ (the pull-back is well-defined since it is only considered at the generic point of F and Y' is regular in codimension 1).

Definition 1.12. *Let $\phi : X \to Y$ be a generically finite dominant morphism. If W is a Weil b-divisor over Y we define its pull-back ϕ^*W to be the b-divisor over X characterized by $\nu(\phi^*W) = (\phi_*\nu)(W)$ for every divisorial valuation ν.*

This is indeed a Weil b-divisor since each prime divisor E on X such that $(\phi_* \operatorname{ord}_E)(W) \neq 0$ is either mapped to a prime divisor F on Y such that $\operatorname{ord}_F(W) \neq 0$ or is contracted by ϕ, so that the set of all such prime divisors E is finite.

Proposition 1.13. *Suppose $\phi : X \to Y$ is a generically finite dominant morphism of normal varieties, and let $e(\phi) \in \mathbb{N}^*$ be its degree. Then we have*

$$\phi_*\phi^*W = e(\phi)\, W$$

for every $W \in \operatorname{Div}(\mathfrak{Y})$.

The proof is left to the reader.

2. Nef envelopes

In this section X still denotes an arbitrary normal variety (over an algebraically closed field of characteristic zero). We reinterpret the pull-back construction of [dFH09] as a *nef envelope*, which shows in particular that it coincides with Mumford's numerical pull-back on surfaces. Section 2.5 introduces the *defect ideal* of a Weil divisor, measuring its failure to be Cartier, and a precise description of the defect ideal is obtained.

2.1. **Graded sequences and nef envelopes.** Recall that $\mathfrak{a}_\bullet = (\mathfrak{a}_m)_{m\geq 0}$ is a *graded sequence of fractional ideal sheaves* if $\mathfrak{a}_0 = \mathcal{O}_X$, each $\mathfrak{a}_m$ is a coherent fractional ideal sheaf of X and $\mathfrak{a}_k \cdot \mathfrak{a}_m \subset \mathfrak{a}_{k+m}$ for every k, m. We shall say that $\mathfrak{a}_\bullet$ has *linearly bounded denominators* if there exists a (fixed) Weil divisor D on X such that $\mathcal{O}_X(mD) \cdot \mathfrak{a}_m \subset \mathcal{O}_X$ for all m.

Let us first attach an $\mathbb{R}$-Weil b-divisor to any graded sequence of ideal sheaves with linearly bounded denominators:

Proposition 2.1. *Suppose that $\mathfrak{a}_\bullet = (\mathfrak{a}_m)_{m\geq 0}$ is a graded sequence of fractional ideals sheaves $\mathfrak{a}_m$ with linearly bounded denominators. Then we have*

$$\tfrac{1}{l} Z(\mathfrak{a}_l) \leq \tfrac{1}{m} Z(\mathfrak{a}_m)$$

for every m divisible by l and the sequence $\frac{1}{m} Z(\mathfrak{a}_m)$ converges coefficient-wise to an $\mathbb{R}$-Weil b-divisor $Z(\mathfrak{a}_\bullet)$.

Proof. All this follows from the super-additivity property

$$Z(\mathfrak{a}_m) + Z(\mathfrak{a}_n) \leq Z(\mathfrak{a}_{m+n})$$

since the condition that $\mathfrak{a}_\bullet$ has linearly bounded denominators guarantees that the sequence $\frac{1}{m} \operatorname{ord}_E Z(\mathfrak{a}_m)$ is bounded below for each prime divisor E over X and even identically zero for all but finitely many prime divisors E on X. □

Lemma 2.2. *Let $\mathfrak{a}_\bullet$ be a graded sequence of fractional ideal sheaves on X with linearly bounded denominators. Then we have $Z(\mathfrak{a}_\bullet) = \frac{1}{m_0} Z(\mathfrak{a}_{m_0})$ for some m_0 iff the graded $\mathcal{O}_X$-algebra $\bigoplus_{m\geq 0} \overline{\mathfrak{a}_m}$ of integral closures is finitely generated.*

Proof. Assume that $Z(\mathfrak{a}_\bullet) = \frac{1}{m_0} Z(\mathfrak{a}_{m_0})$ for a given m_0, so that $Z(\mathfrak{a}_{km_0}) = kZ(\mathfrak{a}_{m_0})$ for all k. Let π be the normalized blow-up of X along $\mathfrak{a}_{m_0}$. One then easily checks that

$$\overline{\mathfrak{a}_{km_0}} = \pi_* \mathcal{O}_{X_\pi} \left(kZ\left(\mathfrak{a}_{m_0}\right)\right)$$

for all k. Since $Z(\mathfrak{a}_{m_0})_\pi$ is π-globally generated this implies that the $\mathcal{O}_X$-algebra $\bigoplus_k \overline{\mathfrak{a}_{km_0}}$ is finitely generated, hence so is its finite integral extension $\bigoplus_m \overline{\mathfrak{a}_m}$. The converse implication is left to the reader. □

Definition 2.3. *Let D be an $\mathbb{R}$-Weil divisor on X_π for a given π. The* nef envelope $\operatorname{Env}_\pi(D)$ *of D is defined as the $\mathbb{R}$-Weil b-divisor associated with the graded sequence $\pi_* \mathcal{O}_{X_\pi}(mD)$, $m \geq 0$.*

When π is the identity we write Env_X for Env_π.

Remark 2.4. If D is an $\mathbb{R}$-Weil divisor on X then $-\operatorname{Env}_X(-D)_\pi$ coincides by definition with $\pi^* D$ in the sense of [dFH09, Definition 2.9].

Proposition 2.5. *Let D, D' be two $\mathbb{R}$-Weil divisors on a model X_π. Then we have:*

- $\operatorname{Env}_\pi(D + D') \geq \operatorname{Env}_\pi(D) + \operatorname{Env}_\pi(D')$*.*
- $\operatorname{Env}_\pi(tD) = t \operatorname{Env}_\pi(D)$ *for each* $t \in \mathbb{R}_+$

Proof. For each $m \geq 0$ we have

$$(\pi_* \mathcal{O}_{X_\pi}(mD)) \cdot (\pi_* \mathcal{O}_{X_\pi}(mD')) \subset \pi_* \mathcal{O}_{X_\pi}(m(D + D'))$$

whence the first point.

In order to prove the second point we may assume that D is effective (since we may add to D the pull-back of an appropriate Cartier divisor of X to make it effective). Now observe that $\mathrm{Env}_\pi(mD) = m\,\mathrm{Env}_\pi(D)$ for each positive integer m since $\mathrm{Env}_\pi(D) = \lim \frac{1}{m} Z(\pi_* \mathcal{O}_{X_\pi}(mD))$, hence $\mathrm{Env}_\pi(tD) = t\,\mathrm{Env}_\pi(D)$ for each $t \in \mathbb{Q}_+^*$. On the other hand $D \mapsto \mathrm{Env}_\pi(D)$ is obviously non-decreasing, so if we pick $t \in \mathbb{R}_+^*$ and approximate it from below and from above by rational numbers s_j, t_j we get

$$s_j\,\mathrm{Env}_\pi(D) = \mathrm{Env}_\pi(s_j D) \le \mathrm{Env}_\pi(tD) \le \mathrm{Env}_\pi(t_j D) = t_j\,\mathrm{Env}_\pi(D)$$

hence the result. □

Linearity of nef envelopes fails in general. The obstruction to linearity will be studied in greater detail in Section 2.5 (see also Example 2.21 and [dFH09]).

Corollary 2.6. *For every finite dimensional vector space V of $\mathbb{R}$-Weil divisors on X_π and every divisorial valuation ν the map $D \mapsto \nu(\mathrm{Env}_\pi(D))$ is continuous on V.*

Proof. Proposition 2.5 implies that $D \mapsto \nu(\mathrm{Env}_\pi(D))$ is a concave function on V and the result follows. □

Proposition 2.7. *For every $\mathbb{R}$-Weil divisor D on X the incarnation $(\mathrm{Env}_X(D))_X$ of $\mathrm{Env}_X(D)$ on X coincides with D.*

Proof. If D is a Weil divisor on X then we have $Z(\mathcal{O}_X(D))_X = D$. Indeed this means that $\mathrm{ord}_E\,\mathcal{O}_X(D) = -\,\mathrm{ord}_E\,D$ for each prime divisor E of X, which holds true since X, being normal, is regular at the generic point of E.

As a consequence we get $D = (\mathrm{Env}_X(D))_X$ when D is a $\mathbb{Q}$-Weil divisor on X, and the general case follows by density, using Corollary 2.6. □

2.2. **Variational characterization of nef envelopes.** Let $X \to S$ be a projective morphism. In the usual theory of b-divisors one says that an $\mathbb{R}$-Cartier b-divisor C is relatively nef over S (or S-nef for short) if C_π is S-nef for one (hence any) determination π of C. Following [BFJ08, KuMa08] we extend this definition to arbitrary $\mathbb{R}$-Weil b-divisors:

Definition 2.8. *Let $X \to S$ be a projective morphism. We define $\mathrm{Nef}(\mathfrak{X}/S) \subset N_{n-1}(\mathfrak{X}/S)$ as the closed convex cone generated by all S-nef classes $\beta \in N^1(\mathfrak{X}/S)$, i.e. all classes of S-nef $\mathbb{R}$-Cartier b-divisors.*

Since the usual notion of nefness is preserved by pull-back, it is immediate to check that S-nef classes in the sense of the above definition are also preserved by pull-back. On the other hand nefness is in general not preserved under push-forward when $\dim X > 2$, and the incarnations W_π of an S-nef $\mathbb{R}$-Weil b-divisor are therefore *not* S-nef in general.

Given a projective morphism $Y \to S$ recall that the *S-movable cone* $\overline{\mathrm{Mov}}(Y/S) \subset N^1(Y/S)$ is the closed convex cone $\overline{\mathrm{Mov}}(Y/S)$ generated by the numerical classes of all Cartier divisors D on Y whose S-base locus has codimension at least two.

We now have the following alternative description of nef b-divisors:

Lemma 2.9. *Let $X \to S$ be a projective morphism. Then we have*

$$\mathrm{Nef}(\mathfrak{X}/S) = \operatorname{proj}\lim_{\pi} \overline{\mathrm{Mov}}(X_\pi/S)$$

where the limit is taken over all smooth (or $\mathbb{Q}$-facforial) models X_π. In other words an $\mathbb{R}$-Weil b-divisor W is S-nef iff W_π is S-movable on each smooth (or $\mathbb{Q}$-factorial) model

X_π. In particular the restriction of (the class of) W_π to any prime divisor of X_π is S-pseudoeffective.

Proof. Let $\alpha \in N_{n-1}(\mathfrak{X}/S)$. Since the latter is endowed with the product topology the sets

$$V_{\pi,U} := \{\beta \in N_{n-1}(\mathfrak{X}/S),\ \beta_\pi \in U\}$$

where π ranges over all smooth models of X and $U \subset N^1(X_\pi/S)$ ranges over all conical open neighborhoods of α_π form a neighborhood basis of α.

We infer by definition that α is S-nef iff for every π and U there exists an S-nef class $\beta \in N^1(\mathfrak{X}/S)$ such that $\beta_\pi \in U$. On the other hand since U is conical it is immediate to see that β may be assumed to be the class of an S-globally generated Cartier b-divisor, and the result follows. □

The next result is a limiting case of Lemma 1.7.

Lemma 2.10. *Let $\mathfrak{a}_\bullet$ be a graded linearly bounded denominators. Then the $\mathbb{R}$-Weil b-divisor $Z(\mathfrak{a}_\bullet)$ is X-nef.*

Proof. Since $\mathfrak{a}_\bullet$ has linearly bounded denominators it is in particular clear that there exists a finite dimensional vector space V of $\mathbb{R}$-Weil divisors on X such that $Z(\mathfrak{a}_m) \in V$ for all m. By Lemma 1.10 it thus follows that $[\frac{1}{m}Z(\mathfrak{a}_m)]$ converges to $[Z(\mathfrak{a}_\bullet)]$ in $N_{n-1}(\mathfrak{X}/X)$. But each $Z(\mathfrak{a}_m)$ is X-globally generated by Lemma 1.7, and we thus conclude that $Z(\mathfrak{a}_\bullet)$ is X-nef □

Proposition 2.11 (Negativity Lemma)**.** *Let W be an X-nef $\mathbb{R}$-Weil b-divisor over X. Then for each π we have $W \le \mathrm{Env}_\pi(W_\pi)$.*

The following argument provides in particular an alternative proof of the well-known *negativity lemma* [KoMo, Lemma 3.39].

Proof. Let X_π be a fixed model of X.

Step 1. Let C be an X-globally generated Cartier b-divisor, determined on some model X_τ that may be assumed to dominate X_π. As in the proof of Lemma 1.7 we have $C = Z(\mathcal{O}_X(C))$ since C is X-globally generated, and we infer that $C \le \mathrm{Env}_\pi(C_\pi)$ since $\tau \ge \pi$ implies

$$\mathcal{O}_X(C) = \tau_*\mathcal{O}_{X_\tau}(C_\tau) \subset \pi_*\mathcal{O}_{X_\pi}(C_\pi).$$

Step 2. Let C be an X-nef $\mathbb{R}$-Cartier b-divisor, determined on a model X_τ that may again be assumed to dominate X_π. We may then find a sequence of X-very ample Cartier divisors A_j on X_τ and a sequence $t_j \in \mathbb{R}_+^*$ such that $t_jA_j \to C_\tau$ coefficient-wise while staying in a fixed finite dimensional vector space of $\mathbb{R}$-divisors on X_τ. By Step 1 and Proposition 2.5 we have $t_j\overline{A_j} \le \mathrm{Env}_\pi(t_j(\overline{A_j})_\pi)$ for each j, hence $C \le \mathrm{Env}_\pi(C_\pi)$ by Corollary 2.6. This step recovers in particular the usual statement of the negativity lemma.

Step 3. Let W be an arbitrary X-nef $\mathbb{R}$-Weil b-divisor. By Lemma 1.10 there exists a net W_j of X-nef $\mathbb{R}$-Cartier divisors such that $W_j \to W$ coefficient-wise and $W_{j,X}$ stays in a fixed finite dimensional space of $\mathbb{R}$-Weil divisors on X. The result now follows by another application of Corollary 2.6. □

As a consequence we get the following variational characterization of nef envelopes.

Corollary 2.12. *If D is an $\mathbb{R}$-Weil divisor on X_π then $\operatorname{Env}_\pi(D)$ is the largest X-nef $\mathbb{R}$-Weil b-divisor W such that $W_\pi \le D$. In particular we have:*

- $\operatorname{Env}_\pi(D) = \overline{D}$ *if D is $\mathbb{R}$-Cartier and X-nef.*
- *The b-divisor* $\operatorname{Env}_\pi(D)$ *is $\mathbb{R}$-Cartier, determined by a given $\tau \ge \pi$, iff the incarnation of* $\operatorname{Env}_\pi(D)$ *on X_τ is $\mathbb{R}$-Cartier and X-nef.*

Proof. The $\mathbb{R}$-Weil b-divisor $\operatorname{Env}_\pi(D)$ is X-nef by Lemma 2.10. We also clearly have $\frac{1}{m}Z(\pi_*\mathcal{O}_{X_\pi}(mD))_\pi \le D$, hence $\operatorname{Env}(D)_\pi \le D$ in the limit. Conversely if Z is an X-nef $\mathbb{R}$-Weil b-divisor such that $Z_\pi \le D$ then $Z \le \operatorname{Env}_\pi(Z_\pi) \le \operatorname{Env}_\pi(D)$ by the negativity lemma. □

As an illustration we now prove:

Proposition 2.13. *Assume that X has klt singularities in the sense of* [dFH09]*, i.e. there exists an effective $\mathbb{Q}$-Weil divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and (X, Δ) is klt. Then* $\operatorname{Env}_X(D)$ *is an $\mathbb{R}$-Cartier b-divisor for every $\mathbb{R}$-Weil divisor D on X. When D has $\mathbb{Q}$-coefficients we even have* $\operatorname{Env}_X(D) = \frac{1}{m}Z(\mathcal{O}_X(mD))$ *for some m.*

The result easily follows from [Kol08, Exercise 109], but we provide some details for the convenience of the reader.

Note that the analogous result for $\operatorname{Env}_\pi(D)$, D being a Weil divisor on a higher model X_π, fails even when X is smooth (cf. [Cut00, Kür03] for an explicit example).

Proof. Since (X, Δ) is klt it follows from [BCHM10] that there exists a $\mathbb{Q}$-*factorialization* $\pi : X_\pi \to X$, i.e. a small birational morphism π such that X_π is $\mathbb{Q}$-factorial. Denote by $\hat{\Delta}_\pi$ and $\hat{D}_\pi$ the strict transforms on X_π of Δ and D respectively. Since π is small we have $\pi^*(K_X + \Delta) = K_{X_\pi} + \hat{\Delta}_\pi$, which shows that $(X_\pi, \hat{\Delta}_\pi)$ is klt, hence so is $(X_\pi, \hat{\Delta}_\pi + \varepsilon\hat{D}_\pi)$ for $0 < \varepsilon \ll 1$. By applying [BCHM10] to $\varepsilon\hat{D}_\pi$, which is π-numerically equivalent to $K_{X_\pi} + \hat{\Delta} + \varepsilon\hat{D}$ as well as π-big (since π is birational) we infer the existence of a new $\mathbb{Q}$-factorialization $\tau : X_\tau \to X$ such that the strict transform $\hat{D}_\tau$ of D on X_τ is furthermore X-nef. Since τ is small it is easily seen that $\tau_*\mathcal{O}_{X_\tau}(m\hat{D}_\tau) = \mathcal{O}_X(mD)$ for all m, hence $\operatorname{Env}_\tau(\hat{D}_\tau) = \operatorname{Env}_X(D)$, and it follows by Corollary 2.12 that $\operatorname{Env}_X(D)$ is the $\mathbb{R}$-Cartier b-divisor determined by $\hat{D}_\tau$.

When D has rational coefficients the base-point free theorem shows that $\hat{D}_\tau$ is X-globally generated, so that

$$\bigoplus_{m\ge 0} \mathcal{O}_X(mD) = \bigoplus_{m\ge 0} \tau_*\mathcal{O}_{X_\tau}(m\hat{D}_\tau)$$

is finitely generated over $\mathcal{O}_X$. We thus have $\operatorname{Env}_X(D) = \frac{1}{m}Z(\mathcal{O}_X(mD))$ for some m. □

2.3. **Nef envelopes of Weil b-divisors.** The next result is a variant in the relative case of [BFJ08, Proposition 2.13] and [KuMa08, Theorem D]:

Proposition 2.14. *Let W be an $\mathbb{R}$-Weil b-divisor. If the set of X-nef $\mathbb{R}$-Weil b-divisors Z such that $Z \le W$ is non-empty then it admits a largest element.*

We shall say that *the nef envelope of W is well-defined* if the assumption of the lemma holds. We then denote the largest element in question by $\operatorname{Env}_{\mathfrak{X}}(W)$ and call it the *nef envelope* of W.

Proof. Every Z as in the lemma satisfies $Z \leq \mathrm{Env}_\pi(W_\pi)$ for all π by Corollary 2.12, which also implies that $\pi \mapsto \mathrm{Env}_\pi(W_\pi)$ is non-increasing, i.e.

$$\mathrm{Env}_{\pi'}(W_{\pi'}) \leq \mathrm{Env}_\pi(W_\pi)$$

whenever $\pi' \geq \pi$. If there exists at least one Z as above then it follows that $\mathrm{Env}_{\mathfrak{X}}(W) := \lim_\pi \mathrm{Env}_\pi(W_\pi)$ is well-defined as a b-divisor and satisfies $\mathrm{Env}_{\mathfrak{X}}(W) \geq Z$ for every such Z. There remains to show that $\mathrm{Env}_{\mathfrak{X}}(W)$ is X-nef and satisfies $\mathrm{Env}_{\mathfrak{X}}(W) \leq W$. But the existence of Z guarantees the existence a finite dimensional vector space V of $\mathbb{R}$-Weil divisors on X such that $\mathrm{Env}_\pi(W_\pi)_X \in V$ for all π. Since $\mathrm{Env}_\pi(W_\pi)$ converges to $\mathrm{Env}_{\mathfrak{X}}(W)$ coefficient-wise, we conclude as before by Lemma 1.10 that $\mathrm{Env}_{\mathfrak{X}}(W)$ is X-nef, whereas $\mathrm{Env}_{\mathfrak{X}}(W) \leq W$ follows from $\mathrm{Env}_\pi(W_\pi)_\tau \leq W_\tau$ for $\tau \leq \pi$ by letting $\pi \to \infty$. □

Remark 2.15. Note that the proof gives:

$$\mathrm{Env}_{\mathfrak{X}}(W) = \inf_\pi \mathrm{Env}_\pi(W_\pi)\ .$$

If W is an $\mathbb{R}$-Cartier b-divisor then we have

$$\mathrm{Env}_{\mathfrak{X}}(W) = \mathrm{Env}_\pi(W_\pi)$$

for each determination π.

Proposition 2.16. *Let $(W_i)_{i\in I}$ be a net of b-divisors decreasing to W such that $\mathrm{Env}_{\mathfrak{X}}(W)$ is well-defined. Then $\mathrm{Env}_{\mathfrak{X}}(W_i)$ decreases to $\mathrm{Env}_{\mathfrak{X}}(W)$.*

Proof. Since $W_i \geq W$ for all i, the net $\mathrm{Env}_{\mathfrak{X}}(W_i)$ decreases to a b-divisor $Z \geq \mathrm{Env}_{\mathfrak{X}}(W)$. Pick any π. Since $W_{i,\pi} \to W_\pi$, we have $\mathrm{Env}_{\mathfrak{X}}(W_i) \leq \mathrm{Env}_\pi(W_{i,\pi}) \to \mathrm{Env}_\pi(W_\pi)$. Letting $i \to \infty$, we get $Z \leq \mathrm{Env}_\pi(W_\pi)$. We conclude using the preceding remark. □

Proposition 2.17. *Suppose $\phi\colon X \to Y$ is a finite dominant morphism of normal varieties. Let W be any $\mathbb{R}$-Weil b-divisor over Y whose nef envelope* $\mathrm{Env}_{\mathfrak{Y}}(W)$ *is well-defined. Then* $\mathrm{Env}_{\mathfrak{X}}(\phi^* W)$ *is also well-defined and we have*

$$\mathrm{Env}_{\mathfrak{X}}(\phi^* W) = \phi^* \mathrm{Env}_{\mathfrak{Y}}(W).$$

We similarly have

$$\mathrm{Env}_X(\phi^* D) = \phi^* \mathrm{Env}_Y(D)$$

for every $\mathbb{R}$-Weil divisor D on Y.

Proof. Since $\mathrm{Env}_{\mathfrak{Y}}(W)$ is Y-nef, its pull-back $\phi^* \mathrm{Env}_{\mathfrak{Y}}(W)$ is Y-nef as well, hence also X-nef. Since we have $\phi^* \mathrm{Env}_{\mathfrak{Y}}(W) \leq \phi^* W$ this shows that $\mathrm{Env}_{\mathfrak{X}}(\phi^* W)$ is well-defined and satisfies $\phi^* \mathrm{Env}_{\mathfrak{Y}}(W) \leq \mathrm{Env}_{\mathfrak{X}}(\phi^* W)$ by Proposition 2.14.

Conversely, Lemma 2.18 below shows that $\phi_* \mathrm{Env}_{\mathfrak{X}}(\phi^* W)$ is Y-nef. Since $\phi_* \mathrm{Env}_{\mathfrak{X}}(\phi^* W) \leq \phi_* \phi^* W = e(\phi) W$ by Proposition 1.13 it follows that

$$\phi_* \mathrm{Env}_{\mathfrak{X}}(\phi^* W) \leq e(\phi) \mathrm{Env}_{\mathfrak{Y}}(W) = \phi_* \phi^* \mathrm{Env}_{\mathfrak{Y}}(W)$$

by Proposition 1.13 again, and we conclude by applying Lemma 2.19 below to $Z := \mathrm{Env}_{\mathfrak{X}}(\phi^* W) - \phi^* \mathrm{Env}_{\mathfrak{Y}}(W)$. □

Lemma 2.18. *Let $\phi : X \to Y$ be a* finite *dominant morphism between normal varieties and let W be an X-nef $\mathbb{R}$-Weil b-divisor over X. Then $\phi_* W$ is Y-nef.*

Proof. By Lemma 1.10 there exists a net W_j of X-nef $\mathbb{R}$-Cartier b-divisors such that $W_j \to W$ coefficient-wise and $W_{j,X}$ stays in a fixed finite dimensional vector of $\mathbb{R}$-Weil divisors on X. It follows that the divisors $(\phi_* W_j)_Y$ also stay in a fixed finite dimensional vector space of $\mathbb{R}$-Weil divisors on Y, and it is immediate to check from the definition that $\phi_* W_j \to \phi_* W$ coefficient-wise. It thus follows that $[\phi_* W_j] \to [\phi_* W]$ in $N_{n-1}(\mathfrak{Y}/Y)$ and we are thus reduced to the case where W is $\mathbb{R}$-Cartier.

Now let π be a determination of W. By Corollary 2.12 we have in particular $W = \operatorname{Env}_\pi(W_\pi)$, so that the fractional ideals $\mathfrak{a}_m := \pi_* \mathcal{O}(mW_\pi)$ satisfy $W = \lim \frac{1}{m} Z(\mathfrak{a}_m)$ coefficient-wise, and it is clear that the $Z(\mathfrak{a}_m)_X$ stay in a fixed finite dimensional vector space by monotonicity. We are now reduced to the case where $W = Z(\mathfrak{a})$ for some fractional ideal, in which case we have $\phi_* Z(\mathfrak{a}) = Z(N_{X/Y}(\mathfrak{a}))$ by (the proof of) Proposition 1.11. We conclude that $\phi_* Z(\mathfrak{a})$ is Y-globally generated, hence in particular Y-nef, by Lemma 1.7. □

Lemma 2.19. *Let $\phi\colon X \to Y$ be a generically finite dominant morphism. Suppose $Z \geq 0$ is an $\mathbb{R}$-Weil b-divisor over X. Then $\phi_* Z = 0$ only if $Z = 0$.*

Proof. Suppose that there is a prime divisor E lying in some model X' over X such that $\operatorname{ord}_E Z > 0$. Since ϕ is generically finite, we can choose a model Y' over Y such that $E' = \phi'(E)$ is a prime divisor in Y' (where ϕ' is the rational lift of ϕ). Then $\operatorname{ord}_{E'}(\phi_* Z) \geq \operatorname{ord}_E Z > 0$, hence $\phi_* Z$ cannot be zero. □

2.4. **The case of surfaces and toric varieties.**

Theorem 2.20. *Let X be a normal surface and let $\pi : X_\pi \to X$ be a smooth (or at least $\mathbb{Q}$-factorial) model.*

(i) *If D is an $\mathbb{R}$-divisor on X_π then the b-divisor $\operatorname{Env}_\pi(D)$ is $\mathbb{R}$-Cartier, determined on X_π, and*
$$D = \operatorname{Env}_\pi(D)_\pi + (D - \operatorname{Env}_\pi(D)_\pi)$$
coincides with the relative Zariski decomposition of D.

(ii) *If D is an $\mathbb{R}$-Weil divisor on X then $\operatorname{Env}_X(D) = \overline{\pi^* D}$ where $\pi^* D$ is the numerical pull-back of D in the sense of Mumford.*

Recall that the *numerical pull-back* of D is defined as the orthogonal projection of the strict transform of D parallel to the space of π-exceptional divisors, i.e. the unique $\mathbb{R}$-divisor D' on X_π such that $\pi_* D' = D$ and $D' \cdot E = 0$ for all π-exceptional divisors E.

Proof. Let us prove (i). The first assertion follows from Corollary 2.12, since each movable class is nef when $\dim X = 2$.

The divisor $P := \operatorname{Env}_\pi(D)_\pi$ is an X-nef $\mathbb{R}$-divisor on X_π such that $D \geq P$ and $P \geq Q$ for every X-nef divisor Q on X_π such that $D \geq Q$, by Corollary 2.12 again. This is one of the characterizations of the (relative) Zariski decomposition, which concludes the proof of (i).

Let us now prove (ii). Let $\pi^* D$ be the numerical pull-back of D to X_π. Since $\pi^* D$ is π-nef it follows that $C := \overline{\pi^* D}$ is X-nef and satisfies $C_X = D$, hence $C \leq \operatorname{Env}_X(D)$ by Corollary 2.12. Conversely set $D' := \operatorname{Env}_X(D)_\pi$. We claim that $D' = \pi^* D$. Taking this for granted for the moment we then get $\operatorname{Env}_X(D) \leq C$ by the negativity lemma and the result follows.

Since we have $\pi_* D' = D$ by Proposition 2.7, the claim will follow if we show that $D' \cdot E = 0$ for each π-exceptional prime divisor E on X_π. This is a consequence of the variational characterization of $\mathrm{Env}_X(D)$. Indeed note that $D' \cdot E \geq 0$ since D' is π-nef by Lemma 2.9. If we assume by contradiction that $D' \cdot E > 0$ then $D' + \varepsilon E$ is still π-nef for $0 < \varepsilon \ll 1$ and $C := \overline{D' + \varepsilon E}$ is then an X-nef b-divisor with $C_X = D$. It follows that $C \leq \mathrm{Env}_X(D)$ by Corollary 2.12, hence $D' + \varepsilon E \leq D'$, a contradiction. □

Let us now decribe the case of toric varieties. We refer to [Fult, Oda] for basics on toric varieties. Let N be a free abelian group of rank n, and suppose we are given two rational polyhedral fans Δ, Δ' in N such that $\Delta \subset \Delta'$. For the sake of simplicity we assume Δ and Δ' have the same support S. Denote by $X(\Delta)$ and $X(\Delta')$ the corresponding toric varieties. Since Δ is a subset of Δ', we have an induced birational map $\pi : X(\Delta') \to X(\Delta)$.

Let D be an $\mathbb{R}$-Weil toric divisor on $X(\Delta)$. It is given by a real valued function h_D on the set of primitive vectors $\Delta(1)$ generating the 1-dimensional faces of Δ, and D is $\mathbb{R}$-Cartier iff h_D extends to a continuous function on S that is linear on each face. In that case D is π-nef iff h_D is convex on the union S_0 of all faces of Δ' that contain a ray in $\Delta'(1) \setminus \Delta(1)$. By Corollary 2.12 it follows that the function attached to $\mathrm{Env}_\pi(D)_\pi$ is the supremum of all 1-homogeneous functions on the convex set S such that $g \leq h_D$ on $\Delta(1)$ and g is convex on the subset S_0.

Example 2.21. Take Δ in $\mathbb{R}^3$ the fan having a single 3-dimensional cone generated by the four rays $(1,0,0), (0,1,0), (0,0,1), (1,1,-1)$. Then $X(\Delta)$ is an affine variety having an isolated singularity at the origin and is locally isomorphic to a quadratic cone there.

Let Δ' be the regular fan having $(1,0,0), (0,1,0), (0,0,1), (1,1,-1), (1,1,0)$ as vertices. The natural map $X(\Delta') \to X(\Delta)$ is a proper birational map which gives a (non-minimal) desingularization of $X(\Delta)$. Denote by E_v the divisor associated to the corresponding ray $v \in \mathbb{R}^3$ either in $X(\Delta)$ or $X(\Delta')$.

Now take $D_1 = E_{100} + E_{010} + E_{001}$, and $D_2 = E_{100} + E_{001} + E_{11-1}$. Then $D_1 + D_2$ is a Cartier divisor on $X(\Delta)$ whose support function is given by $2x_1 + x_2 + 2x_3$ in the standard coordinates $(x_1, x_2, x_3) \in \mathbb{R}^3$. Hence $\mathrm{ord}_{E_{110}} \mathrm{Env}_X(D_1 + D_2) = 3$. On the other hand, for any convex function g having value 1 at $(0,0,1)$ and 0 at $(1,1,-1)$, we have $g(1,1,0) \leq 1$, hence $\mathrm{ord}_{E_{110}} \mathrm{Env}_X(D_1) \leq 1$. The same argument shows that $\mathrm{ord}_{E_{110}} \mathrm{Env}_X(D_2) \leq 1$, hence $\mathrm{ord}_{E_{110}} \mathrm{Env}_X(D_1) + \mathrm{ord}_{E_{110}} \mathrm{Env}_X(D_2) < \mathrm{ord}_{E_{110}}(\mathrm{Env}_X(D_1 + D_2))$.

2.5. **Defect ideals.**

Definition 2.22. *The* defect ideal *of an $\mathbb{R}$-Weil divisor D on X is defined as*

$$\mathfrak{d}(D) := \mathcal{O}_X(D) \cdot \mathcal{O}_X(-D).$$

Note that $\mathfrak{d}(D) \subset \mathcal{O}_X(D - D) = \mathcal{O}_X$ is an ideal sheaf. The following proposition summarizes immediate properties of defect ideals.

Proposition 2.23. *Let D, D' be $\mathbb{R}$-Weil divisors on X. Then we have:*

(i) $\mathfrak{d}(D + C) = \mathfrak{d}(D)$ *for every Cartier divisor C.*

(ii)
$$\mathfrak{d}(D) \cdot \mathcal{O}_X(D + D') \subset \mathcal{O}_X(D) \cdot \mathcal{O}_X(D') \subset \mathcal{O}_X(D + D').$$

(iii)
$$\phi^{-1}\mathfrak{d}_X(D) \cdot \mathcal{O}_Y(\phi^* D) \subset \phi^{-1}\mathcal{O}_X(D) \cdot \mathcal{O}_Y \subset \mathcal{O}_Y(\phi^* D)$$

for every finite dominant morphism $\phi : Y \to X$.

It follows in particular that $\mathfrak{d}_\bullet(D) = (\mathfrak{d}(mD))_{m\geq 0}$ is a graded sequence of ideals. By definition we get
$$Z(\mathfrak{d}_\bullet(D)) = \operatorname{Env}_X(D) + \operatorname{Env}_X(-D).$$

Definition 2.24. *We shall say that an $\mathbb{R}$-Weil divisor D on X is* numerically Cartier *if* $\operatorname{Env}_X(-D) = -\operatorname{Env}_X(D)$*. In the special case where $D = K_X$ we shall say that X is* numerically Gorenstein *if K_X is numerically Cartier.*

By Proposition 2.5 it is straightforward to see that numerically Cartier divisors form an $\mathbb{R}$-vector space. We also have:

Lemma 2.25. *Let D be an $\mathbb{R}$-Weil divisor on X. Then D is numerically Cartier iff*
$$\operatorname{Env}_X(D + D') = \operatorname{Env}_X(D) + \operatorname{Env}_X(D')$$
for every $\mathbb{R}$-Weil divisor D' on X.

Proof. Assume that D is numerically Cartier, so that $\operatorname{Env}_X(-D) = -\operatorname{Env}_X(D)$. Then we have on the one hand $\operatorname{Env}_X(D + D') \geq \operatorname{Env}_X(D) + \operatorname{Env}_X(D')$ and on the other hand $\operatorname{Env}_X(-D) + \operatorname{Env}_X(D+D') \leq \operatorname{Env}_X(D')$, and additivity follows. The converse is equally easy and left to the reader. □

Example 2.26 (Surfaces)*.* Since Mumford's pull-back of Weil divisors on surfaces is linear, it follows from Theorem 2.20 that all $\mathbb{R}$-Weil divisors on a normal surface X are numerically Cartier.

Example 2.27 (Toric varieties)*.* If D is a toric $\mathbb{R}$-Weil divisor on a toric variety X then it follows from the discussion from the last section that D is numerically Cartier iff D is already $\mathbb{R}$-Cartier.

Example 2.28 (Cone singularities)*.* Let (V, L) be a smooth projective variety endowed with an ample line bundle L. Recall that the affine cone over (V, L) is the algebraic variety defined by
$$X = C(V, L) := \operatorname{Spec}\left(\bigoplus_{m\geq 0} H^0(V, mL)\right).$$
It has an isolated normal singularity at its vertex $0 \in X$, and is obtained by blowing-down the zero section $E \simeq V$ in the total space Y of the dual bundle L^*. We denote by $\pi : Y \to X$ the contraction map, which is isomorphic to the blow-up of X at 0. Every divisor D on V induces a Weil divisor $C(D)$ on X, and the map $D \mapsto C(D)$ induces an isomorphism $\operatorname{Pic}(V)/\mathbb{Z}\,L \simeq \operatorname{Cl}(X)$ onto the divisor class group of X.

Lemma 2.29*. Let (V, L) be a smooth polarized variety and let D be an $\mathbb{R}$-Weil divisor on V.*

1. *$C(D)$ is $\mathbb{R}$-Cartier iff D and L are $\mathbb{R}$-linearly proportional in* $\operatorname{Pic}(X)\otimes\mathbb{R}$*.*
2. *$C(D)$ is numerically Cartier iff D and L are numerically proportional in $N^1(V)$.*

Proof. (1) follows from the description of the divisor class group of $X = C(V)$ recalled above. Let us prove (2). Let $\pi : Y \to X$ be the blow-up of X at its vertex 0. The restriction to $E \simeq V$ of the strict transform $C(D)'$ is linearly equivalent to D. If D is numerically Cartier then the restriction to E of $\operatorname{Env}_X(-C(D))_Y = -\operatorname{Env}_X(C(D))_Y$ is both pseudoeffective and anti-pseudoeffective by Lemma 2.9, so $\operatorname{Env}_X(C(D))_Y$ is numerically equivalent to 0 in $N^1(Y/X)$. But $\mathrm{Env}_X(C(D))_Y - C(D)'$ is π-exceptional, hence proportional to E, and we conclude as desired that $D \equiv C(D)'|_E$ is proportional to $L \equiv -E|E$ in $N^1(V)$.

Conversely assume that $D \equiv aL$ are proportional in $N^1(V)$. Then $C(D)'$ and E are proportional in $N^1(Y/X)$, hence there exists $t \in \mathbb{R}$ such that $\mathrm{Env}_X(C(D))_Y \equiv -tE$ in $N^1(Y/X)$. Since $-E$ is X-ample and the numerical class of $\mathrm{Env}_X(C(D))_Y$ is in the X-movable cone it follows that $t \geq 0$, which implies that $\mathrm{Env}_X(C(D))_Y$ is X-nef. This in turn shows as in the proof of Theorem 2.20 that the b-divisor $\mathrm{Env}_X(C(D))$ is $\mathbb{R}$-Cartier, determined on Y by $C(D)' - aE$. If we replace D by $-D$ then we get that $\mathrm{Env}_X(C(D))$ is determined on Y by $C(-D)' + aE = -(C(D)' - aE)$, i.e. $\mathrm{Env}_X(-C(D)) = -\mathrm{Env}_X(C(D))$ holds as desired. □

We now give a more precise description of defect ideals, which is basically an elaboration of [dFH09, Theorem 5.4]. As a matter of terminology we introduce:

Definition 2.30. *We say that a determination π of an $\mathbb{R}$-Cartier b-divisor C is a* log-resolution *of C if X_π is smooth, the exceptional locus* $\mathrm{Exc}(\pi)$ *has codimension one and* $\mathrm{Exc}(\pi) + C_\pi$ *has SNC support.*

Another $\mathbb{R}$-Cartier b-divisor C' is then said to be transverse *to π and C if π is also a log-resolution of $C + C'$ and C'_π has no common component with* $\mathrm{Exc}(\pi) + C_\pi$.

Every $\mathbb{R}$-Cartier b-divisor admits a log-resolution by Hironaka's theorem.

Proposition 2.31. *Let D be a Weil divisor on X and assume that X is quasi-projective. Then we have*

$$\mathfrak{d}(D) = \sum_E \mathcal{O}_X(-E)$$

where the sum is taken over the set of all prime divisors E of X such that $D - E$ is Cartier (and this set is in particular non-empty).

Given a Cartier b-divisor C and a joint log-resolution π of C and $\mathcal{O}_X(D)$ the sum can be further restricted to those E such that $Z(\mathcal{O}_X(E))$ is transverse to π and C.

Proof. Observe first that

$$\mathcal{O}_X(-E) \subset \mathcal{O}_X(-E) \cdot \mathcal{O}_X(E) = \mathfrak{d}(E) = \mathfrak{d}(D)$$

for all effective Weil divisors E such that $D - E$ is Cartier.

Since X is quasi-projective there exists a line bundle L on X such that $L \otimes \mathcal{O}_X(D)$ is generated by a finite dimensional vector space of global sections V, which we view as rational sections of L. For each $s \in V$ set $E_s := D + \mathrm{div}(s)$, which is an effective Weil divisor congruent to D modulo Cartier divisors.

We claim that there exists a (non-empty) Zariski open subset U of V such that

$$\mathfrak{d}(D) = \sum_{s \in U} \mathcal{O}_X(-E_s) \tag{3}$$

and

- E_s is a prime divisor on X,
- $Z(\mathcal{O}_X(E_s))$ is transverse to π and C,

for each $s \in U$, which will conclude the proof of Proposition 2.31.

Since π dominates the blow-up of $\mathcal{O}_X(D)$ it is easily seen that the effective divisors

$$M_s := Z(\mathcal{O}_X(E_s))_\pi = Z(\mathcal{O}_X(D))_\pi + \pi^* \operatorname{div}(s)$$

move in a base-point free linear system on X_π as s moves in V. We may thus find a non-empty Zariski open subset U of V such that for each $s \in U$ we have

- M_s has no common component with $\operatorname{Exc}(\pi) + C_\pi$,
- M_s is smooth and irreducible,
- $M_s + \operatorname{Exc}(\pi) + C_\pi$ has SNC support,

where the last two points follow from Bertini's theorem. Since $\pi_* M_s = Z(\mathcal{O}_X(D))_X + \operatorname{div}(s) = E_s$ by Proposition 2.7, we see in particular that E_s is a prime divisor for each $s \in U$ and $Z(\mathcal{O}_X(E_s))$ is transverse to π and C. There remains to show (3). Observe that

$$s \cdot \mathcal{O}_X(-D) \subset L \otimes \mathcal{O}_X(-\operatorname{div}(s)) \cdot \mathcal{O}_X(-D) = L \otimes \mathcal{O}_X(-E_s)$$

for each $s \in V$. Since V generates $L \otimes \mathcal{O}_X(D)$ and U is open in V we obtain

$$L \otimes \mathfrak{d}(D) = L \otimes \mathcal{O}_X(D) \cdot \mathcal{O}_X(-D)$$
$$= \sum_{s \in U} s \cdot \mathcal{O}_X(-D) \subset L \otimes \sum_{s \in U} \mathcal{O}_X(-E_s)$$

and the result follows since L is invertible. □

3. Multiplier ideals and approximation

In this section X still denotes a normal variety. Our main goal here is to show how to obtain from Takagi's subadditivity theorem for multiplier ideals of pairs a similar statement for the general multiplier ideals defined in [dFH09]. This result will in turn enable us to approximate nef envelopes of Cartier divisors from above by nef Cartier divisors, in the spirit of [BFJ08].

3.1. **Log-discrepancies.** We shall say that an $\mathbb{R}$-Weil divisor Δ on X is an $\mathbb{R}$-*boundary* (resp. a $\mathbb{Q}$-boundary, resp. an m-boundary) if $K_X + \Delta$ is $\mathbb{R}$-Cartier (resp. $K_X + \Delta$ is $\mathbb{Q}$-Cartier, resp. $m(K_X + \Delta)$ is Cartier).

Let ω be a rational top-degree form on X and consider the associated canonical b-divisor $K_{\mathfrak{X}}$. Given an $\mathbb{R}$-boundary Δ on X we define the *relative canonical b-divisor* of (X, Δ) by

$$K_{\mathfrak{X}/(X,\Delta)} = K_{\mathfrak{X}} - \overline{K_X + \Delta},$$

which is independent of the choice of ω. If E is a prime divisor above X then $\operatorname{ord}_E K_{\mathfrak{X}/(X,\Delta)}$ is nothing but the *discrepancy* of the pair (X, Δ) along E. Following [dFH09] we introduce on the other hand:

Definition 3.1. *The* m-limiting relative canonical b-divisor *is defined by*

$$K_{m,\mathfrak{X}/X} := K_{\mathfrak{X}} + \tfrac{1}{m} Z(\mathcal{O}_X(-mK_X))$$

and the relative canonical b-divisor *is*

$$K_{\mathfrak{X}/X} = K_{\mathfrak{X}} + \operatorname{Env}_X(-K_X).$$

They are both independent of the choice of ω and are exceptional over X by Proposition 2.7. Note that $K_{m,\mathfrak{X}/X} \to K_{\mathfrak{X}/X}$ coefficient-wise as $m \to \infty$.

Recall that the *log-discrepancy* of a pair (X, Δ) along a prime divisor E above X is defined by adding 1 to the discrepancy. Let us reformulate this by introducing the 'pseudo b-divisor' $1_{\mathfrak{X}}$, i.e. the homogeneous function on the set of divisorial valuations of X such that

$$(t \operatorname{ord}_E)(1_{\mathfrak{X}}) = t$$

for each divisorial valuation $t \operatorname{ord}_E$, so that $\operatorname{ord}_E(K_{\mathfrak{X}/(X,\Delta)} + 1_{\mathfrak{X}})$ is now equal to the log-discrepancy of (X, Δ) along E. We also consider the reduced exceptional b-divisor $1_{\mathfrak{X}/X}$, which takes value 1 on the prime divisors that are exceptional over X, and value zero on the prime divisors contained in X.

The following well-known properties show that $K_{\mathfrak{X}} + 1_{\mathfrak{X}}$ is better behaved than $K_{\mathfrak{X}}$.

Lemma 3.2. *Assume that X is smooth and let E be a reduced SNC divisor on X. Then we have $K_{\mathfrak{X}} + 1_{\mathfrak{X}} \geq \overline{K_X + E}$.*

This result is [Kol97, Lemma 3.11], whose proof we reproduce for the convenience of the reader.

Proof. Let F be a smooth irreducible divisor in some model $\pi : X_\pi \to X$. We may choose local coordinates $(x_1, ..., x_n)$ near the generic point of $\pi(F)$ such that the local equation of E writes $x_1 \dots x_p = 0$ for some $p = 0, ..., n$, and we let z be a local equation of F at its generic point. We then have $\pi^* x_i = z^{b_i} u_i$ where u_i is a unit at the generic point of F and $b_i \in \mathbb{N}$ vanishes for $i > p$. It follows that $\pi^* dx_i = b_i z^{b_i - 1} u_i dz + z^{b_i} du_i$, hence

$$\operatorname{ord}_F(K_{\mathfrak{X}} - \pi^* K_X) = \operatorname{ord}_F(K_{X_\pi/X})$$

$$= \operatorname{ord}_F(\pi^*(dx_1 \wedge ... \wedge dx_n)) \geq -1 + \sum_i b_i = -1 + \operatorname{ord}_F \overline{E}.$$

□

Lemma 3.3. *Let $\phi : X \to Y$ be a generically finite dominant morphism between normal varieties. Let ω_Y be a rational top-degree form on Y, ω_X be its pull-back to X and $K_{\mathfrak{Y}}$, $K_{\mathfrak{X}}$ be the associated canonical b-divisors. Then we have*

$$K_{\mathfrak{X}} + 1_{\mathfrak{X}} = \phi^*(K_{\mathfrak{Y}} + 1_{\mathfrak{Y}}).$$

Proof. Let F be a prime divisor on a smooth birational model $Y' \to Y$ and pick a smooth birational model $X' \to X$ such that ϕ lifts to a morphism $\phi' : X' \to Y'$ and such that there exists a prime divisor E on X' with $\phi'(E) = F$. We then have $\phi_* \operatorname{ord}_E = b \operatorname{ord}_F$ with $b := \operatorname{ord}_E(\phi'^* F)$. The same computation as above shows that the ramification order of ϕ' at the generic point of E is equal to $b - 1$, so that we have

$$\operatorname{ord}_E(K_{X'} - (\phi')^* K_{Y'}) = b - 1.$$

It follows that

$$\operatorname{ord}_E(K_{X'}) = b \operatorname{ord}_F(K_{Y'}) + b - 1,$$

i.e.

$$\operatorname{ord}_E(K_{\mathfrak{X}} + 1_{\mathfrak{X}}) = (b \operatorname{ord}_F)(K_{\mathfrak{Y}} + 1_{\mathfrak{Y}})$$

as was to be shown. □

Definition 3.4. *The* m-limiting log-discrepancy b-divisor $A_{m,\mathfrak{X}/X}$ *and the* log-discrepancy b-divisor $A_{\mathfrak{X}/X}$ *are the Weil b-divisors defined by*

$$A_{m,\mathfrak{X}/X} := K_{m,\mathfrak{X}/X} + 1_{\mathfrak{X}/X}$$

and

$$A_{\mathfrak{X}/X} := K_{\mathfrak{X}/X} + 1_{\mathfrak{X}/X}.$$

Note that $\lim_{m\to\infty} A_{m,\mathfrak{X}/X} = A_{\mathfrak{X}/X}$ coefficient-wise.

If $\phi : X \to Y$ is a finite dominant morphism recall that the *ramification divisor* R_ϕ is the effective Weil divisor on X such that

$$K_X = \phi^* K_Y + R_\phi,$$

where K_Y and K_X are defined by ω_Y and $\phi^*\omega_Y$ respectively, the divisor R_ϕ being again independent of the choice of ω_Y.

Corollary 3.5. *Let* $\phi : X \to Y$ *be a finite dominant morphism between normal varieties. Then we have*

$$0 \le \mathrm{Env}_X(R_\phi) \le \phi^* A_{\mathfrak{Y}/Y} - A_{\mathfrak{X}/X} \le - \mathrm{Env}_X(-R_\phi)$$

and the second (resp. third) inequality is an equality when X *(resp.* Y*) is numerically Gorenstein.*

Proof. Since ϕ is finite, we have

$$\phi^* A_{\mathfrak{Y}/Y} - A_{\mathfrak{X}/X} = \phi^*(K_{\mathfrak{Y}/Y} + 1_{\mathfrak{Y}}) - (K_{\mathfrak{X}/X} + 1_{\mathfrak{X}})$$
$$= \phi^* \mathrm{Env}_Y(-K_Y) - \mathrm{Env}_X(-K_X) = \mathrm{Env}_X(-\phi^* K_Y) - \mathrm{Env}_X(-K_X)$$

by Lemma 3.3 and Proposition 2.17. Now we have on the one hand

$$\mathrm{Env}_X(-\phi^* K_Y) = \mathrm{Env}_X(-K_X + R_\phi) \ge \mathrm{Env}_X(-K_X) + \mathrm{Env}_X(R_\phi)$$

and this is an equality when X is numerically Gorenstein by Lemma 2.25. On the other hand

$$\mathrm{Env}_X(-K_X) = \mathrm{Env}_X(-\phi^* K_Y - R_\phi) \ge \mathrm{Env}_X(-\phi^* K_Y) + \mathrm{Env}_X(-R_\phi)$$

which is an equality if Y is numerically Gorenstein by Proposition 2.17 and Lemma 2.25. The result follows, noting that $\mathrm{Env}(R_\phi) \ge 0$ since $R_\phi \ge 0$. ☐

3.2. **Multiplier ideals.** The following definition is a straightforward extension of the usual notion of multiplier ideal with respect to a pair.

Definition 3.6. *Let* Δ *be an effective* $\mathbb{R}$*-boundary on* X *and let* C *be an* $\mathbb{R}$*-Cartier b-divisor. We define the multiplier ideal sheaf of* C *with respect to* (X, Δ) *as the fractional ideal sheaf*

$$\mathcal{J}((X, \Delta); C) := \mathcal{O}_X\left(\lceil K_{\mathfrak{X}/(X,\Delta)} + C \rceil\right).$$

We have in particular

$$\mathcal{J}((X, \Delta); C) \subset \mathcal{O}_X(\lceil C_X - \Delta_X \rceil),$$

which shows that the (fractional) multiplier ideal is an actual ideal as soon as $C_X \le 0$. By Lemma 3.2 we have

$$\mathcal{J}((X, \Delta); C) = \pi_* \mathcal{O}_{X_\pi}\left(\lceil K_{X_\pi} - \pi^*(K_X + \Delta) + C_\pi \rceil\right)$$

for each joint log-resolution π of (X,Δ) and C. This shows in particular that $\mathcal{J}((X,\Delta);C)$ is coherent, and in case $C = Z(\mathfrak{a}^c)$ for a coherent ideal sheaf $\mathfrak{a}$ and $c>0$ we recover

$$\mathcal{J}((X,\Delta);Z(\mathfrak{a}^c)) = \mathcal{J}((X,\Delta);\mathfrak{a}^c)$$

where the right-hand side is defined in [Laz, Definition 9.3.56]. We similarly introduce the following straightforward generalization of the notion of multiplier ideal defined in [dFH09]:

Definition 3.7. *Let C be an $\mathbb{R}$-Cartier b-divisor over X.*

- *For each positive integer m the m-*limiting multiplier ideal sheaf *of C is the fractional ideal sheaf*
$$\mathcal{J}_m(C) := \mathcal{O}_X\left(\lceil K_{m,\mathfrak{X}/X} + C\rceil\right).$$
- *The* multiplier ideal sheaf *$\mathcal{J}(C)$ is the unique maximal element in the family of fractional ideal sheaves $\mathcal{J}_m(C)$, $m \geq 1$.*

Here again Lemma 3.2 implies that

$$\mathcal{J}_m(C) = \pi_*\mathcal{O}_{X_\pi}\left(\lceil K_{X_\pi} + \tfrac{1}{m}Z(\mathcal{O}_X(-mK_X))_\pi + C_\pi\rceil\right)$$

for each joint log-resolution π of $\mathcal{O}_X(-mK_X)$ and C, which shows in particular that $\mathcal{J}_m(C)$ is coherent. We also have

$$\mathcal{J}_m(C) \subset \mathcal{O}_X(\lceil C_X\rceil),$$

which implies the existence of a unique maximal element in the set of fractional ideals $\{\mathcal{J}_m(C),\, m \geq 1\}$, by using as usual

$$\frac{1}{lm}Z(\mathcal{O}_X(-lmK_X)) \geq \max\left(\frac{1}{m}Z(\mathcal{O}_X(-mK_X)), \frac{1}{l}Z(\mathcal{O}_X(-lK_X))\right).$$

As in [dFH09] we now relate the above two notions of multiplier ideals, obtaining in particular a more precise version of [dFH09, Theorem 5.4].

Theorem 3.8. *Assume that X is quasi-projective, let C be an $\mathbb{R}$-Cartier b-divisor and let $m \geq 2$. Then we have*

$$\mathfrak{d}(mK_X) = \sum_\Delta \mathcal{O}_X(-m\Delta)$$

where Δ ranges over the set of all effective m-boundaries such that

$$\mathcal{J}_m(C) = \mathcal{J}((X,\Delta);C),$$

(so that this set is in particular non-empty).

Proof. Let π be a joint log-resolution of $\mathfrak{a}$ and $\mathcal{O}_X(-mK_X)$. By Proposition 2.31 applied to $-mK_X$ we have

$$\mathfrak{d}(mK_X) = \sum_E \mathcal{O}_X(-E)$$

where E ranges over all prime divisors such that $mK_X + E$ is Cartier and $Z(\mathcal{O}_X(E))$ is transverse to π and C. There remains to set $\Delta := \frac{1}{m}E$ and to observe that $\lfloor\Delta\rfloor = 0$, so that $\mathcal{J}_m(C) = \mathcal{J}((X,\Delta);C)$ by Lemma 3.9 below. □

Lemma 3.9. *Let C be an $\mathbb{R}$-Cartier b-divisor, let π be a joint log-resolution of C and $\mathcal{O}_X(-mK_X)$ and let Δ be an effective m-boundary.*

- *We have*
$$\mathcal{J}((X,\Delta);C) \subset \mathcal{J}_m(C).$$
- *If* $\lfloor \Delta \rfloor = 0$ *and* $Z(\mathcal{O}_X(m\Delta))$ *is transverse to* π *and* C *then*
$$\mathcal{J}((X,\Delta);C) = \mathcal{J}_m(C).$$

Proof. Since $m(K_X+\Delta)$ is Cartier we have
$$\mathcal{O}_X(-mK_X)) = \mathcal{O}_X(m\Delta) \cdot \mathcal{O}_X(-m(K_X+\Delta))$$
hence
$$\frac{1}{m} Z(\mathcal{O}_X(-mK_X)) = \frac{1}{m} Z(\mathcal{O}_X(m\Delta)) - \overline{K_X+\Delta} \tag{4}$$
and the first point follows because $Z(\mathcal{O}_X(m\Delta)) \geq 0$.

Assume now that $\lfloor \Delta \rfloor = 0$ and that $Z(\mathcal{O}_X(m\Delta))$ is transverse to π and C. By (4) we have
$$\lceil K_{X_\pi} - \pi^*(K_X+\Delta) + C_\pi \rceil = \lceil K_{X_\pi} + \tfrac{1}{m} Z(\mathcal{O}_X(-mK_X))_\pi + C_\pi \rceil - \lfloor \tfrac{1}{m} Z(\mathcal{O}_X(m\Delta))_\pi \rfloor.$$
Indeed, by the transversality assumption $\frac{1}{m} Z(\mathcal{O}_X(m\Delta)_\pi$ has no common component with C_π and no common component with $K_{X_\pi} + \frac{1}{m} Z(\mathcal{O}_X(-mK_X))_\pi$, the latter being π-exceptional by Proposition 2.7. But by transversality we also have $\frac{1}{m} Z(\mathcal{O}_X(m\Delta))_\pi = \widehat{\Delta}_\pi$, the strict transform of Δ on X_π, and the result follows since $\lfloor \widehat{\Delta}_\pi \rfloor = 0$. □

As a consequence we get the following extension of [dFH09, Corollary 5.5] to b-divisors.

Corollary 3.10. *Let* X *be a normal quasi-projective variety and let* C *be an* $\mathbb{R}$*-Cartier* b*-divisor.*

- *The* m*-limiting multiplier ideal* $\mathcal{J}_m(C)$ *is the largest element of the set of multiplier ideals* $\mathcal{J}((X,\Delta);C)$ *where* Δ *ranges over all effective* m*-boundaries on* X*.*
- *The multiplier ideal* $\mathcal{J}(C)$ *is the largest element of the set of multiplier ideals* $\mathcal{J}((X,\Delta);C)$ *where* Δ *ranges over all effective* $\mathbb{Q}$*-boundaries on* X*.*

3.3. **Subadditivity and approximation.** Recall that the Jacobian ideal sheaf $\mathrm{Jac}_X \subset \mathcal{O}_X$ of X is defined as the n-th Fitting ideal $\mathrm{Fitt}^n(\Omega^1_X)$ with $n = \dim X$. Locally, if $X \subset \mathbb{C}^N$ is defined by equations $h_1 = \cdots = h_m = 0$, where $h_i \in \mathbb{C}[x_1,\ldots,x_N]$, then Jac_X is generated by the $(r \times r)$-minors of the matrix $(\partial h_i / \partial x_j)$, where $r = N - n$.

Takagi obtained in [Tak10] the following general subadditivity result for multiplier ideals with respect to a pair:

Theorem 3.11. [Tak10] *Let* X *be a normal variety and let* Δ *be an effective* $\mathbb{Q}$*-Weil divisor such that* $m(K_X+\Delta)$ *is Cartier. If* $\mathfrak{a}, \mathfrak{b}$ *are two coherent ideal sheaves on* X *and* $c, d > 0$ *then we have*
$$\tfrac{1}{m} Z(\mathcal{O}_X(-m\Delta)) + Z(\mathrm{Jac}_X) + Z(\mathcal{J}((X,\Delta);\mathfrak{a}^c \cdot \mathfrak{b}^d)) \leq Z(\mathcal{J}((X,\Delta);\mathfrak{a}^c)) + Z(\mathcal{J}((X,\Delta);\mathfrak{b}^d)).$$

When X is smooth and $\Delta = 0$ the 'error term' $\frac{1}{m} Z(\mathcal{O}_X(-m\Delta)) + Z(\mathrm{Jac}_X)$ vanishes and the statement reduces (up to integral closure) to the original subadditivity theorem of [DEL00]. We now show how to deduce from Takagi's result a subadditivity theorem for multiplier ideals in the sense of [dFH09].

Theorem 3.12 (Subadditivity). *Let X be a normal variety. If $\mathfrak{a}, \mathfrak{b}$ are two coherent ideal sheaves on X and $c, d > 0$ then we have*

$$Z(\mathfrak{d}_\bullet(K_X)) + Z(\mathrm{Jac}_X) + Z(\mathcal{J}(\mathfrak{a}^c\mathfrak{b}^d)) \leq Z(\mathcal{J}(\mathfrak{a}^c)) + Z(\mathcal{J}(\mathfrak{b}^d)).$$

The results in [Tak06, Sch09], combined, suggest the possibility that the term $Z(\mathfrak{d}_\bullet(K_X))$ might be superfluous.

Proof. The result is local so we may assume that X is affine (and in particular quasi-projective). We have by definition

$$\mathcal{J}(\mathfrak{a}^c \cdot \mathfrak{b}^d) = \mathcal{J}_m(\mathfrak{a}^c \cdot \mathfrak{b}^d)$$

for all sufficiently large and divisible m. Since $Z(\mathfrak{a}_\bullet(K_X))$ is the limit of $\frac{1}{m}Z(\mathfrak{d}(mK_X))$ it is thus enough to show that

$$\tfrac{1}{m}Z(\mathfrak{d}(mK_X)) + Z(\mathrm{Jac}_X) + Z(\mathcal{J}_m(\mathfrak{a}^c \cdot \mathfrak{b}^d)) \leq Z(\mathcal{J}_m(\mathfrak{a}^c)) + Z(\mathcal{J}_m(\mathfrak{b}^d)).$$

Now Theorem 3.8 yields $\mathfrak{d}(mK_X) = \sum_\Delta \mathcal{O}_X(-m\Delta)$, with Δ ranging over all m-boundaries such that

$$\mathcal{J}((X,\Delta); \mathfrak{a}^c \cdot \mathfrak{b}^d) = \mathcal{J}_m(\mathfrak{a}^c \cdot \mathfrak{b}^d)$$

Since $\mathcal{J}((X,\Delta); \mathfrak{a}^c) \subset \mathcal{J}_m(\mathfrak{a}^c)$ and the similar statement for $\mathfrak{b}^d$ hold at any rate by Lemma 3.9, the desired result follows from Takagi's theorem. □

Given a graded sequence of ideal sheaves $\mathfrak{a}_\bullet = (\mathfrak{a}_m)_{m\geq 0}$ we define the *asymptotic multiplier ideal* $\mathcal{J}(\mathfrak{a}_\bullet^c)$ as the unique maximal element of $\{\mathcal{J}(\mathfrak{a}_m^{c/m}),\ m \geq 1\}$ – which exists by noetherianity since $\frac{c}{lm}Z(\mathfrak{a}_{lm}) \geq \max(\frac{c}{l}Z(\mathfrak{a}_l), \frac{c}{m}Z(\mathfrak{a}_m))$.

Theorem 3.13. *Let X be a normal variety and let $\mathfrak{a}_\bullet$ be a graded sequence of ideal sheaves on X. Then we have*

$$Z(\mathfrak{d}_\bullet(K_X)) + Z(\mathrm{Jac}_X) \leq Z(\mathcal{J}(\mathfrak{a}_\bullet)) - Z(\mathfrak{a}_\bullet) \leq A_{\mathfrak{X}/X}.$$

In particular $\frac{1}{k}Z(\mathcal{J}(\mathfrak{a}_\bullet^k)) \to Z(\mathfrak{a}_\bullet)$ coefficient-wise as $k \to \infty$, uniformly with respect to $\mathfrak{a}_\bullet$.

This result is an extension to the singular case of [BFJ08, Proposition 3.18], which was in turn a direct elaboration of the main result of [ELS03].

Proof. For each $k \geq 1$ we have

$$Z(\mathcal{J}(\mathfrak{a}_k^{1/k})) \leq \tfrac{1}{k}Z(\mathfrak{a}_k) + A_{\mathfrak{X}/X}$$

by definition of multiplier ideals and the right-hand inequality follows.

On the other hand set $W := Z(\mathfrak{d}_\bullet(K_X)) + Z(\mathrm{Jac}_X)$. Theorem 3.12 yields

$$(k-1)W + kZ(\mathcal{J}(\mathfrak{a}_k)) \leq kZ(\mathcal{J}(\mathfrak{a}_k)^{1/k}),$$

and the left-hand inequality follows by using the trivial inequality

$$Z(\mathfrak{a}_k) + Z(\mathcal{J}(\mathcal{O}_X)) \leq Z(\mathcal{J}(\mathfrak{a}_k)),$$

dividing by k and letting $k \to \infty$. □

4. Normal isolated singularities

From now on X has an *isolated* normal singularity at a given point $0 \in X$, and $\mathfrak{m} \subset \mathcal{O}_X$ denotes the maximal ideal of 0. We first show how to extend to this setting the intersection theory of nef b-divisors introduced in the smooth case in [BFJ08]. The main ingredient to do so is the approximation theorem from the previous section. We next define the *volume* of $(X, 0)$ as the self-intersection of the nef envelope of the log-canonical b-divisor.

4.1. **b-divisors over 0.** Observe that every Weil b-divisor W over X decomposes in a unique way as a sum

$$W = W^0 + W^{X \setminus 0},$$

where all irreducible components of W^0 have center 0, and none of $W^{X\setminus 0}$ have center 0. If $W = W^0$, then we say that W *lies over* 0 and we denote by

$$\mathrm{Div}(\mathfrak{X}, 0) \subset \mathrm{Div}(\mathfrak{X})$$

the subspace of all Weil b-divisors over $0 \in X$. An element of $\mathrm{Div}_{\mathbb{R}}(\mathfrak{X}, 0)$ is the same thing as a real-valued homogeneous function on the set of divisorial valuations on X centered at 0.

Example 4.1. For every coherent ideal sheaf $\mathfrak{a}$ on X we have

$$Z(\mathfrak{a})^0 = \lim_{k\to\infty} Z(\mathfrak{a} + \mathfrak{m}^k).$$

On the other hand we say that a Cartier b-divisor $C \in \mathrm{CDiv}(\mathfrak{X})$ is *determined* over 0 if it admits a determination π which is an *isomorphism away from* 0, and we say that C is a *Cartier b-divisor over* 0 if C furthermore lies over 0. We denote by $\mathrm{CDiv}(\mathfrak{X}, 0)$ the space of Cartier b-divisors over 0. There is an inclusion

$$\mathrm{CDiv}(\mathfrak{X}, 0) \subset \mathrm{CDiv}(\mathfrak{X}) \cap \mathrm{Div}(\mathfrak{X}, 0)$$

but this is in general not an equality. The following example was kindly suggested to us by Fulger.

Example 4.2. Consider $(X, 0) = (\mathbb{C}^n, 0)$. It is a toric variety defined by the regular fan Δ_0 in $\mathbb{R}^n$ having the canonical basis as vertices. Any proper birational toric modification $\pi : X(\Delta) \to \mathbb{C}^n$ is determined by a refinement Δ of Δ_0. We assume $X(\Delta)$ to be smooth. Denote by $V(\sigma)$ the torus invariant subvariety of $X(\Delta)$ associated to a face σ of Δ.

For any vertex v of Δ, let $D(v)$ be the Cartier b-divisor determined in $X(\Delta)$ by the divisor $V(\mathbb{R}_+ v)$. Observe that for any face σ of Δ, we have $\pi(V(\sigma)) = 0$ iff σ is included in the open cone $(\mathbb{R}^*)^n_+$. Whence $D(v)$ lies over 0 iff $v \in (\mathbb{R}^*)^n_+$. And $D(v)$ is determined over 0 iff any face of Δ containing v is included in $(\mathbb{R}^*)^n_+$.

Example 4.3. Let $\mathfrak{a} \subset \mathcal{O}_X$ be an ideal. Then $Z(\mathfrak{a})$ is determined over 0 as soon as $\mathfrak{a}$ is locally principal outside 0 since the normalized blow-up of X along $\mathfrak{a}$ is then an isomorphism away from 0. If $\mathfrak{a}$ is furthermore $\mathfrak{m}$-primary then $Z(\mathfrak{a})$ is a Cartier b-divisor over 0.

Definition 4.4. *We shall say than an $\mathbb{R}$-Weil b-divisor W over* 0 *is* bounded below *if there exists $c > 0$ such that $W \geq cZ(\mathfrak{m})$.*

Recall that $Z(\mathfrak{m}) \leq 0$, so that the condition means that the function $\nu \mapsto \nu(W)/\nu(\mathfrak{m})$ is bounded below on the set of divisorial valuations centered at 0.

Proposition 4.5. *$(A_{\mathfrak{X}/X})^0$ is bounded below.*

Proof. First note that $A_{\mathfrak{X}/X} \geq A_{1,\mathfrak{X}/X}$ by general properties of nef envelopes, and hence it suffices to check that $(A_{1,\mathfrak{X}/X})^0$ is bounded below. Let π be a resolution of the singularity of X, chosen to be an isomorphism away from 0. For each divisorial valuation ν centered at 0 we have

$$\nu(A_{1,\mathfrak{X}/X}) = \nu\left((K_{\mathfrak{X}} + 1_{\mathfrak{X}}) - \overline{K_{X_\pi}}\right) + \nu\left(\overline{K_{X_\pi}} + Z(\mathcal{O}_X(-K_X))\right).$$

The first term in the right-hand side is non-negative since it is equal to the log-discrepancy of the smooth variety X_π along ν. On the other hand the Cartier b-divisor $\left(\overline{K_{X_\pi}} + Z(\mathcal{O}_X(-K_X))\right)$ is determined over 0 since $\mathcal{O}_X(-K_X)$ is locally principal outside 0 by assumption (cf. Example 4.3) and it also lies over 0 by Proposition 2.7. We thus see that

$$\left(\overline{K_{X_\pi}} + Z(\mathcal{O}_X(-K_X))\right) \in \mathrm{CDiv}(\mathfrak{X}, 0)$$

and we conclude by Lemma 4.6 below. □

Lemma 4.6. *Every $C \in \mathrm{CDiv}(\mathfrak{X}, 0)$ is bounded below.*

Proof. Let π be a determination of C which is an isomorphism away from 0. The result follows directly from the fact that $Z(\mathfrak{m})_\pi$ contains every π-exceptional prime divisor E in its support (since ord_E is centered at 0). □

4.2. **Nef b-divisors over 0.** We shall that an $\mathbb{R}$-Weil b-divisor over 0 is *nef* if its class in $N^1(\mathfrak{X}/X)$ is X-nef. If W is an $\mathbb{R}$-Weil b-divisor over 0 that is bounded below then $\mathrm{Env}_{\mathfrak{X}}(W)$ is well-defined, nef, and it lies over 0.

By a result of Izumi [Izu81] for every two divisorial valuations ν, ν' on X centered at 0 there is a constant $c = c(\nu, \nu') > 0$ such that

$$c^{-1}\nu(f) \leq \nu'(f) \leq c\nu(f)$$

for every $f \in \mathcal{O}_X$. This result extends to nef b-divisors by approximation:

Theorem 4.7. *Given two divisorial valuations ν, ν' centered at 0 there exists $c > 0$ such that*

$$c\nu(W) \leq \nu'(W) \leq c^{-1}\nu(W)$$

for every X-nef $\mathbb{R}$-Weil b-divisor W such that $W \leq 0$ (which amounts to $W_X \leq 0$ by the negativity lemma).

Proof. Since $\mathrm{Env}_\pi(W_\pi)$ decreases coefficient-wise to W as $\pi \to \infty$ by Proposition 2.14, it is enough to treat the case where $W = \mathrm{Env}_{\mathfrak{X}}(C)$ for some $\mathbb{R}$-Cartier b-divisor $C \leq 0$. But we then have

$$W = \lim_{m\to\infty} \tfrac{1}{m} Z(\mathcal{O}_X(mC))$$

with $\mathcal{O}_X(mC) \subset \mathcal{O}_X$ so we are reduced to the case of an ideal, for which the result directly follows from Izumi's theorem. □

Corollary 4.8. *For each X-nef $\mathbb{R}$-Weil b-divisor W such that $W \leq 0$ and $W^0 \neq 0$ there exists $\varepsilon > 0$ such that*

$$W \leq \varepsilon Z(\mathfrak{m}).$$

Proof. Since $W^0 \neq 0$ there exists a divisorial valuation ν_0 centered at 0 such that $\nu_0(W) < 0$, and it follows that $\nu(W) < 0$ for *all* divisorial valuations centered at 0 by Theorem 4.7.

Now let π be the normalized blow-up of $\mathfrak{m}$. Since W_π contains each π-exceptional prime in its support there exists $\varepsilon > 0$ such that $W_\pi \leq \varepsilon Z(\mathfrak{m})_\pi$ and the result follows by the negativity lemma. □

For nef envelopes of Weil divisors with integer coefficients this result can be made uniform as follows:

Theorem 4.9. *There exists $\varepsilon > 0$ only depending on X such that*

$$\mathrm{Env}_X(-D) \leq \varepsilon Z(\mathfrak{m})$$

for all effective Weil divisors (with integer coefficients) D on X containing 0.

Proof. By Hironaka's resolution of singularities we may choose a smooth birational model X_π which dominates the blow-up of $\mathfrak{m}$ and is isomorphic to X away from 0, and such that there exists a π-ample and π-exceptional Cartier divisor A on X_π. If we denote by $E_1, ..., E_r$ the π-exceptional prime divisors then $A = -\sum_j a_j E_j$ with $a_j \geq 1$ by the negativity lemma.

By the negativity lemma the desired result means that there exists $\varepsilon > 0$ such that for each effective Weil divisor D through 0 on X we have

$$\mathrm{Env}_X(-D)_\pi \leq \varepsilon Z(\mathfrak{m})_\pi.$$

If we set $c_j(D) := -\operatorname{ord}_{E_j} \mathrm{Env}_X(-D)$ then in view of Theorem 4.7 this amounts to proving the existence of $\varepsilon > 0$ such that

$$\max_{1 \leq j \leq r} c_j(D) \geq \varepsilon$$

for each D. Note that

$$\sum_j c_j(D) E_j = -\mathrm{Env}_X(-D)_\pi - \widehat{D}_\pi \tag{5}$$

by Proposition 2.7. Now we have on the one hand

$$\begin{aligned} -A^{n-1} \cdot \mathrm{Env}_X(-D)_\pi &= \sum a_j E_j \cdot A^{n-2} \cdot \mathrm{Env}_X(-D)_\pi \\ &= \sum_j a_j (A|_{E_j})^{n-2} \cdot (\mathrm{Env}_X(-D)_\pi|_{E_j}) \geq 0 \end{aligned}$$

since $A|_{E_j}$ is ample and $\mathrm{Env}_X(-D)_\pi|_{E_j}$ is pseudo-effective by Lemma 2.9. On the other hand

$$-A^{n-1} \cdot \widehat{D}_\pi = \sum_j a_j (A|_{E_j})^{n-2} \cdot (\widehat{D}_\pi|_{E_j}) \geq 1$$

since $\widehat{D}_\pi|_{E_j}$ is an effective Cartier divisor on E_j, and is non-zero for at least one j. We thus get $\sum_j c_j(D)(E_j \cdot A^{n-1}) \geq 1$ from (5) and we infer that

$$\max_j c_j(D) \geq \varepsilon := 1/\max_j (E_j \cdot A^{n-1}).$$

□

We conclude this section by the following crucial consequence of Theorem 3.13.

Theorem 4.10. *Let $C \in \mathrm{CDiv}(\mathfrak{X}, 0)$ and set $W := \mathrm{Env}_{\mathfrak{X}}(C)$. Then there exists a sequence of $\mathfrak{m}$-primary ideals $\mathfrak{b}_k$ and a sequence of positive rational numbers $c_k \to 0$ such that:*

- *$c_k Z(\mathfrak{b}_k) \geq W$ for all k.*
- *$\lim_{k\to\infty} c_k Z(\mathfrak{b}_k) = W$ coefficient-wise.*

Proof. Consider the graded sequence of $\mathfrak{m}$-primary ideals $\mathfrak{a}_m := \mathcal{O}_X(mW) = \mathcal{O}_X(mC)$ and set $\mathfrak{b}_k := \mathcal{J}(\mathfrak{a}_\bullet^k)$. By Theorem 3.13 we have in particular

$$Z(\mathfrak{b}_k) \geq kW + Z(\mathfrak{d}(K_X)) + Z(\mathrm{Jac}_X)$$

and $\frac{1}{k} Z(\mathfrak{b}_k) \to W$ coefficient-wise. Since $0 \in X$ is an isolated singularity we see that both $\mathfrak{d}(K_X)$ and Jac_X are $\mathfrak{m}$-primary ideals and Lemma 4.6 yields $c > 0$ such that

$$Z(\mathfrak{d}(K_X)) + Z(\mathrm{Jac}_X) \geq cZ(\mathfrak{m}).$$

On the other hand there exists $\varepsilon > 0$ such that $W \leq \varepsilon Z(\mathfrak{m})$ by Corollary 4.8 and we conclude that there exists $c > 0$ such that

$$Z(\mathfrak{b}_k) \geq kW + cW$$

for all k. There remains to set $c_k := 1/(k + c)$. □

4.3. **Intersection numbers of nef b-divisors.** We indicate in this section how to extend to the singular case the local intersection theory of nef b-divisors introduced in [BFJ08, §4] in the smooth case. The main point is to replace the approximation result [BFJ08, Proposition 3.13] by Theorem 4.10.

Let $C_1, ..., C_n$ be $\mathbb{R}$-Cartier b-divisors over 0. Pick a common determination π which is an isomorphism away from 0 and set

$$C_1 \cdot ... \cdot C_n := C_{1,\pi} \cdot ... \cdot C_{n,\pi}.$$

The right-hand side is well-defined since $C_{1,\pi}$ has compact support and it does not depend on the choice of π.

Proposition 4.11. *Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \subset \mathcal{O}_X$ be $\mathfrak{m}$-primary ideals. Then*

$$-Z(\mathfrak{a}_1) \cdot ... \cdot Z(\mathfrak{a}_n) = e(\mathfrak{a}_1, ..., \mathfrak{a}_n)$$

where $e(\mathfrak{a}_1, ..., \mathfrak{a}_n)$ denotes the mixed multiplicity (see e.g. [Laz, p.91] *for a definition).*

This result if for instance a direct consequence of [Ram73].

The intersection numbers of *nef* $\mathbb{R}$-Cartier b-divisors $C_1, ..., C_n$, $C'_1, ..., C'_n$ over 0 satisfy the monotonicity property:

$$C_1 \cdot ... \cdot C_n \leq C'_1 \cdot ... \cdot C'_n$$

if $C_i \leq C'_i$ for each i.

Definition 4.12. *If $W_1, ..., W_n$ are arbitrary nef $\mathbb{R}$-Weil b-divisors over 0 we set*

$$W_1 \cdot ... \cdot W_n := \inf_{C_i \geq W_i} (C_1 \cdot ... \cdot C_n) \in [-\infty, +\infty[$$

where the infimum is taken over all nef $\mathbb{R}$-Cartier b-divisors C_i over 0 such that $C_i \geq W_i$ for each i.

Note that $(W_1 \cdot ... \cdot W_n)$ is finite when all W_i are bounded below. This is for instance the case if each W_i is the nef envelope of a Cartier b-divisor by Lemma 4.6.

The next theorem summarizes the main properties of the intersection product. The non-trivial part of the assertion is additivity, which requires the approximation theorem.

Theorem 4.13. *The intersection product $(W_1, \dots, W_n) \mapsto W_1 \cdot ... \cdot W_n$ of nef $\mathbb{R}$-Weil b-divisors over 0 is symmetric, upper semi-continuous, and continuous along monotonic families (for the topology of coefficient-wise convergence).*

It is also homogeneous, additive, and non-decreasing in each variable. Furthermore, $W_1 \cdot ... \cdot W_n < 0$ if $W_i \neq 0$ for each i.

Proof. We follow the same lines as [BFJ08, Proposition 4.4]. Symmetry, homogeneity and monotonicity are clear. If $W_i \neq 0$ for all i then there exists $\varepsilon > 0$ such that $W_i \leq \varepsilon Z(\mathfrak{m})$ for all i by Corollary 4.8, hence

$$W_1 \cdot ... \cdot W_n \leq \varepsilon^n Z(\mathfrak{m})^n = -\varepsilon^n e(\mathfrak{m}) < 0$$

where $e(\mathfrak{m})$ is the Samuel multiplicity of $\mathfrak{m}$.

Let us prove the semi-continuity. Suppose that $W_i \neq 0$ for all i, and pick $t \in \mathbb{R}$ such that $W_1 \cdot ... \cdot W_n < t$. By definition there exist nef $\mathbb{R}$-Cartier b-divisors C_i over 0 such that $W_i \leq C_i$ and $C_1 \cdot ... \cdot C_n < t$. Replacing each C_i by $(1-\varepsilon)C_i$ we may assume $C_i \neq W_i$ while still preserving the previous conditions. Now consider the set U_i of all nef b-divisors W_i' such that $W_i' \leq C_i$. This is a neighborhood of W_i in the topology of coefficient-wise convergence and $(W_1' \cdot ... \cdot W_n') < t$ for all $W_i' \in U_i$. This proves the upper semi-continuity.

As a consequence we get the following continuity property: for all families $W_{j,k}$ such that

- $W_{j,k} \geq W_j$ for all j, k and
- $\lim_k W_{j,k} = W_j$ for all j

we have $\lim_k W_{1,k} \cdot ... \cdot W_{n,k} = W_1 \cdot ... \cdot W_n$. Indeed $W_{1,k} \cdot ... \cdot W_{n,k} \geq W_1 \cdot ... \cdot W_n$ holds by monotonicity and the claim follows by upper semi-continuity.

We now turn to additivity. Assume first that W', $W_1, W_2, ..., W_n$ are nef envelopes of Cartier b-divisors over 0. By Theorem 4.10 there exist two sequences C_k' and $C_{j,k}$ of nef Cartier divisors above 0 such that $C_{j,k} \geq W_j$ and $C_{j,k} \to W_j$ as $k \to \infty$, and similarly for C_k' and W'. Since $C_{1,k} + C_k' \geq W_1 + W'$ also converges to $W_1 + W'$ the above remark yields

$$(C_{1,k} + C_k') \cdot C_{2,k} \cdot ... \cdot C_{n,k} \to (W_1 + W') \cdot W_2 \cdot ... \cdot W_n$$

On the other hand we have

$$(C_{1,k} + C_k') \cdot C_{2,k} \cdot ... \cdot C_{n,k} = (C_{1,k} \cdot C_{2,k} \cdot ... \cdot C_{n,k}) + (C_k' \cdot C_{2,k} \cdot ... \cdot C_{n,k})$$

where

$$(C_{1,k} \cdot C_{2,k} \cdot ... \cdot C_{n,k}) \to (W_1 \cdot W_2 \cdot ... \cdot W_n) \text{ and } (C_{1,k} \cdot C_{2,k} \cdot ... \cdot C_{n,k}) \to (W' \cdot W_2 \cdot ... \cdot W_n)$$

so we get additivity for nef envelopes.

In the general case let $W', W_1, W_2, ..., W_n$ be arbitrary nef b-divisors over 0. We then have $\mathrm{Env}_\pi(W_{j,\pi}) \geq W_j$ and $\mathrm{Env}_\pi(W_{j,\pi}) \to W_j$ as $\pi \to \infty$ by Proposition 2.14 so we may argue exactly as above to get the result.

Finally, the continuity along non-decreasing sequences is a direct adaptation of the corresponding result in [BFJ11]. □

The expected local Khovanskii-Teissier inequality holds:

Theorem 4.14. *For all nef* $\mathbb{R}$*-Weil b-divisors* $W_1, ..., W_n$ *above* 0 *we have*

$$|W_1 \cdot ... \cdot W_n| \le |W_1^n|^{1/n} \dots |W_n^n|^{1/n}. \tag{6}$$

In particular we have

$$|(W_1 + W_2)^n|^{1/n} \le |W_1^n|^{1/n} + |W_2^n|^{1/n} \ .$$

Proof. Arguing as in the proof of Theorem 4.13 we may use Theorem 4.10 to reduce to the case where $W_i = Z(\mathfrak{a}_i)$ for some $\mathfrak{m}$-primary ideals $\mathfrak{a}_i$. In that case the result follows from Proposition 4.11 and the local Khovanskii-Teissier inequality (cf. [Laz, Theorem 1.6.7 (iii)]). $\square$

Proposition 4.15. *Suppose* $\phi\colon (X, 0) \to (Y, 0)$ *is a finite map of degree* $e(\phi)$*. Then for all nef* $\mathbb{R}$*-Weil b-divisors* $W_1, \dots, W_n$ *over* $0 \in Y$ *we have:*

$$(\phi^* W_1) \cdot ... \cdot (\phi^* W_n) = e(\phi)\, W_1 \cdot \cdots \cdot W_n. \tag{7}$$

Proof. Arguing as in the proof of Theorem 4.13 by successive approximation relying on Theorem 4.10, we reduce to the case where each W_j is $\mathbb{R}$-Cartier over 0. Let $\pi : Y' \to Y$ be a common determination of the W_j which is an isomorphism away from 0. Since $\phi^{-1}(0) = 0$ there exists a birational morphism $\mu : X' \to X$ which is an isomorphism away from 0 such that ϕ lifts as a morphism $\phi' : X' \to Y'$, whose degree is still equal to $e(\phi)$ and the result follows. $\square$

Remark 4.16. For every graded sequence $\mathfrak{a}_\bullet$ of $\mathfrak{m}$-primary ideals we have

$$-Z(\mathfrak{a}_\bullet)^n = \lim_{k \to \infty} \frac{\dim_{\mathbb{C}}(\mathcal{O}_X / \mathfrak{a}_k)}{k^n / n!}.$$

Indeed it was shown by Lazarsfeld and Mustaţă [LM09, Theorem 3.8] that the right-hand side limit exists and coincides with $\lim_{k \to \infty} e(\mathfrak{a}_k)/k^n$ (which corresponds to a local version of the Fujita approximation theorem). On the other hand $Z(\mathfrak{a}_\bullet)$ is the non-decreasing limit of $\frac{1}{k!} Z(\mathfrak{a}_{k!})$ hence $Z(\mathfrak{a}_\bullet)^n = \lim_{k \to \infty} Z(\mathfrak{a}_k)^n / k^n$ by using the continuity of intersection numbers along non-decreasing sequence and the claim follows in view of Proposition 4.11.

4.4. **The volume of an isolated singularity.** By Proposition 4.5 the log-discrepancy divisor $A_{\mathfrak{X}/X}$ is always bounded below. Its nef envelope $\mathrm{Env}_{\mathfrak{X}}(A_{\mathfrak{X}/X})$ is therefore well-defined and bounded below as well, and we may introduce:

Definition 4.17. *The volume of a normal isolated singularity* $(X, 0)$ *is defined as*

$$\mathrm{Vol}(X, 0) := -\,\mathrm{Env}_{\mathfrak{X}} \left(A_{\mathfrak{X}/X} \right)^n \ .$$

We have the following characterization of singularities with zero volume:

Proposition 4.18. $\mathrm{Vol}(X, 0) = 0$ *iff* $A_{\mathfrak{X}/X} \ge 0$*. When* X *is* $\mathbb{Q}$*-Gorenstein,* $\mathrm{Vol}(X, 0) = 0$ *iff it has log-canonical singularities.*

Proof. By Theorem 4.13 we have $\mathrm{Vol}(X, 0) = 0$ iff $\mathrm{Env}_{\mathfrak{X}}(A_{\mathfrak{X}/X}) = 0$, which is equivalent to $A_{\mathfrak{X}/X} \ge 0$ since every X-nef b-divisor over 0 is antieffective by the negativity lemma.

When X is $\mathbb{Q}$-Gorenstein, then $A_{\mathfrak{X}/X} = A_{m,\mathfrak{X}/X}$ for any integer m such that mK_X is Cartier. We conclude recalling that X is log-canonical if the incarnation of the log-discrepancy divisor $A_{m,\mathfrak{X}/X}$ in one (or equivalently any) log-resolution of X is effective. $\square$

Example 4.19. Let $0 \in X$ be the affine cone over a polarized variety (V, L) as in Example 2.28, and denote by $\pi : X_\pi \to X$ the blow-up at 0, with exceptional divisor $E \simeq V$. If $\mathrm{Vol}(X, 0) = 0$ then we claim that $-K_V$ is pseudoeffective. Indeed we then have $a = \mathrm{ord}_E(A_{\mathfrak{X}/X}) \geq 0$ by Proposition 4.18 and

$$K_{X_\pi} + E + \mathrm{Env}_X(-K_X)_\pi = aE$$

since E is the only π-exceptional divisor. Now $\mathrm{Env}_X(-K_X)_\pi$ restricts to a pseudoeffective class in $N^1(E)$ by Lemma 2.9. The pseudoeffectivity of $-K_E$ follows by adjunction, and we also see that $-K_E$ is big if the 'generalized log-discrepancy' a is positive.

The volume satisfies the following basic monotonicity property:

Theorem 4.20. *Let $\phi : (X, 0) \to (Y, 0)$ be a finite morphism between normal isolated singularities. Then we have*

$$\mathrm{Vol}(X, 0) \geq e(\phi) \, \mathrm{Vol}(Y, 0),$$

with equality if ϕ is étale in codimension 1.

Proof. We have $A_{\mathfrak{X}/X} \leq \phi^* A_{\mathfrak{Y}/Y}$ by Corollary 3.5, and equality holds if and only if $R_\phi = 0$, i.e. iff ϕ is étale in codimension 1. The result follows immediately using Theorem 2.17 and Proposition 4.15. □

5. Comparison with other invariants of isolated singularities

5.1. **Wahl's characteristic number.** As recalled in the introduction, Wahl defined in [Wah90] the *characteristic number* of a normal surface singularity $(X, 0)$ as $-P^2$ of the nef part P in the Zariski decomposition of $K_{X_\pi} + E$, where $\pi : X_\pi \to X$ is any log-resolution of $(X, 0)$ and E is the reduced exceptional divisor of π. The following result proves that the volume defined above extends Wahl's invariant to all isolated normal singularities.

Proposition 5.1. *If $(X, 0)$ is a normal surface singularity then* $\mathrm{Vol}(X, 0)$ *coincides with Wahl's characteristic number.*

Proof. Let $\pi : X_\pi \to X$ be log-resolution of $(X, 0)$ and let E be its reduced exceptional divisor. By Theorem 2.20 we see that $\mathrm{Env}_\pi(A_{X_\pi/X})$ coincides with the nef part of $K_{X_\pi} + E - \pi^* K_X$. Since the latter is π-numerically equivalent to $K_{X_\pi} + E$ it follows that $\mathrm{Env}_\pi(A_{X_\pi/X})$ is π-numerically equivalent to the nef part P of $K_{X_\pi} + E$, so that

$$-P^2 = -\mathrm{Env}_\pi(A_{X_\pi/X})^2.$$

On the other hand we claim that $\mathrm{Env}_\pi(A_{X_\pi/X}) = \mathrm{Env}_{\mathfrak{X}}(A_{\mathfrak{X}/X})$, which will conclude the proof. Indeed on the one hand we have

$$\mathrm{Env}_{\mathfrak{X}}(A_{\mathfrak{X}/X}) \leq \mathrm{Env}_\pi(A_{X_\pi/X})$$

as for any Weil b-divisor. On the other hand Lemma 3.2 implies that

$$K_{\mathfrak{X}} + 1_{\mathfrak{X}} \geq \overline{K_{X_\pi} + E}$$

over 0, hence $A_{\mathfrak{X}/X} \geq \overline{A_{X_\pi/X}}$, and we infer $\mathrm{Env}_{\mathfrak{X}}(A_{\mathfrak{X}/X}) \geq \mathrm{Env}_\pi(A_{X_\pi/X})$ as desired. □

Proof of Theorem A. The definition of the volume is given in §4.4. Theorem A (i) is precisely Theorem 4.20. Statement (ii) is Proposition 5.1. Statement (iii) is Proposition 4.18. □

5.2. **Plurigenera and Fulger's volume.** Let $0 \in X$ be (a germ of) an isolated singularity and let $\pi : X_\pi \to X$ be a log-resolution with reduced exceptional SNC divisor E. One may then consider the following plurigenera (see [Ish90] for a review).

- Knöller's plurigenera [Knö73], defined by
$$\gamma_m(X,0) := \dim H^0(X_\pi \backslash E, mK_{X_\pi})/H^0(X_\pi, mK_{X_\pi}).$$
- Watanabe's L^2-plurigenera [Wat80], defined by
$$\delta_m(X,0) := \dim H^0(X_\pi \backslash E, mK_{X_\pi})/H^0(X_\pi, mK_{X_\pi} + (m-1)E).$$
- Morales' log-plurigenera [Mora87, Definition 0.5.4], defined by
$$\lambda_m(X,0) := \dim H^0(X_\pi \backslash E, mK_{X_\pi})/H^0(X_\pi, m(K_{X_\pi} + E)).$$

These numbers do not depend on the choice of log-resolution. They satisfy
$$\lambda_m(X,0) \leq \delta_m(X,0) \leq \gamma_m(X,0) = O(m^n),$$
and one may use them to define various notions of Kodaira dimension of an isolated singularity.

In a recent work, Fulger [Fulg] has explored in more detail the growth of these numbers. His framework is the following. Given a Cartier divisor D on X_π, consider the local dimension
$$h^0_{\mathrm{loc}}(D) := \dim H^0(X_\pi \backslash E, D)/H^0(X_\pi, D) = \dim \mathcal{O}_X(\pi_* D)/\mathcal{O}_X(D).$$
Observe that $\gamma_m(X,0) = h^0_{\mathrm{loc}}(mK_{X_\pi})$ and $\lambda_m(X,0) = h^0_{\mathrm{loc}}(m(K_{X_\pi} + E))$. Fulger proves that $h^0_{\mathrm{loc}}(mD) = O(m^n)$ and defines the *local volume* of D by setting
$$\mathrm{vol}_{\mathrm{loc}}(D) := \limsup_{m\to\infty} \frac{n!}{m^n} h^0_{\mathrm{loc}}(mD) \ .$$
When the Cartier divisor D lies over 0 one has:

Proposition 5.2. *Suppose D is a Cartier divisor in X_π lying over* 0*. Then*
$$\mathrm{vol}_{\mathrm{loc}}(D) = -\operatorname{Env}_{\mathfrak{X}}(\overline{D})^n \ .$$

Proof. We may assume $D \leq 0$. The envelope of D is the b-divisor associated to the graded sequence of $\mathfrak{m}$-primary ideals $\mathcal{O}_X(-mD)$. The result follows from Remark 4.16. □

Fulger [Fulg] then introduces an alternative notion of volume of an isolated singularity by setting:
$$\mathrm{Vol}_F(X,0) := \mathrm{vol}_{\mathrm{loc}}(K_{X_\pi} + E).$$

Proposition 5.3. $\mathrm{Vol}(X,0) = \mathrm{Vol}_F(X,0)$ *if X is $\mathbb{Q}$-Gorenstein.*

Proof. For any integer m such that mK_X is Cartier, one has $A_{\mathfrak{X}/X} = A_{m,\mathfrak{X}/X}$. Pick any log-resolution $\pi : X_\pi \to X$. Then Lemma 3.2 applied to X_π shows that $\overline{A_{X_\pi/X}} \leq A_{\mathfrak{X}/X}$. In particular, these b-divisors share the same envelope. We conclude by Proposition 5.2 above. □

In general, Fulger proves that there is always an inequality
$$\mathrm{Vol}(X,0) \geq \mathrm{Vol}_F(X,0).$$
We know by [Wah90] that in dimension two these volumes always coincide. In higher dimension these two invariants may however differ, as shown by the following example.

Example 5.4. Let (V, L) be a smooth polarized variety, let $0 \in X$ be the affine cone over it and let $\pi : X_\pi \to X$ be the blow-up of 0, with exceptional divisor E.

If V is *uniruled* (i.e. if K_V is not pseudoeffective) then we have $\delta_m(X, 0) = 0$ for all m since $H^0(X_\pi, mK_{X_\pi} + pE)/H^0(Y, mK_{X_\pi} + (p-1)E)$ embeds in $H^0(E, mK_E + (p - m)E|E) \simeq H^0(V, mK_V - (p-m)L)$, which vanishes if $p \geq m$ since L is ample and K_V is not pseudoeffective. If we choose V uniruled such that such that $-K_V$ is however not pseudoeffective (for example $V = C \times \mathbb{P}^1$ where C is a curve of genus at least 2) then Example 4.19 shows on the other hand that $\mathrm{Vol}(X, 0) > 0$. We thus get an example where $\delta_m(X, 0) = 0$ for all m (hence $\mathrm{Vol}_F(X, 0) = 0$) but $\mathrm{Vol}(X, 0) > 0$.

6. Endomorphisms

We apply the previous analysis to the study of normal isolated singularities admitting endomorphisms.

6.1. **Proofs of Theorems B and C.** We start by proving the following result.

Theorem 6.1. *Assume that X is numerically Gorenstein and let $\phi : (X, 0) \to (X, 0)$ is a finite endomorphism of degree $e(\phi) \geq 2$ such that $R_\phi \neq 0$. Then there exists $\varepsilon > 0$ such that $A_{\mathfrak{X}/X} \geq -\varepsilon Z(\mathfrak{m})$.*

Remark 6.2. When X is $\mathbb{Q}$-Gorenstein or $\dim X = 2$, the condition $A_{\mathfrak{X}/X} \geq -\varepsilon Z(\mathfrak{m})$ for some $\varepsilon > 0$ is equivalent to $A_{m,\mathfrak{X}/X} > 0$ for some m. By Corollary 3.10 the latter condition means in turn that X has klt singularities in the sense of [dFH09], i.e. there exists a $\mathbb{Q}$-boundary Δ such that (X, Δ) is klt. In a forthcoming work [BdFF] we shall prove this result unconditionnally.

Remark 6.3. Tsuchihashi's cusp singularities (see below) show that the assumption $R_\phi \neq 0$ is essential even when K_X is Cartier.

Proof. Since X is numerically Gorenstein $R_{\phi^k} = K_X - (\phi^k)^* K_X$ is numerically Cartier for each k and Corollary 3.5 yields

$$(\phi^k)^* A_{\mathfrak{X}/X} = A_{\mathfrak{X}/X} + \mathrm{Env}_X(R_{\phi^k}).$$

On the other hand observe that $R_{\phi^k} = \sum_{j=0}^{k-1} (\phi^j)^* R_\phi$ by the chain-rule. Each $(\phi^j)^* R_\phi$ is numerically Cartier as well, so that

$$\mathrm{Env}_X(R_{\phi^k}) = \sum_{j=0}^{k-1} (\phi^j)^* \,\mathrm{Env}_X(R_\phi)$$

by Lemma 2.25 and Proposition 2.17. Using Proposition 4.5 and Theorem 4.9 we thus obtain $c_1, c_2 > 0$ such that

$$(\phi^k)^*(A_{\mathfrak{X}/X}) \geq c_1 Z(\mathfrak{m}) - c_2 \sum_{j=0}^{k-1} (\phi^j)^* Z(\mathfrak{m})$$

for all divisorial valuations ν centered at 0 and all k. Since we have $(\phi^j)^*\mathfrak{m} \subset \mathfrak{m}$ it follows that

$$(\phi^k)^* A_{\mathfrak{X}/X} \geq -Z(\mathfrak{m})(kc_2 - c_1).$$

But the action of ϕ^k on divisorial valuations centered at 0 is surjective by [ZS]. We furthermore have $\nu\left((\phi^k)^* A_{\mathfrak{X}/X}\right) = \nu\left((\phi^k)^*\mathfrak{m}\right)\nu\left(A_{\mathfrak{X}/X}\right)$ for each divisorial valuation ν centered at 0 and there exists $c_k > 0$ such that $\nu((\phi^k)^*\mathfrak{m}) \le c_k \nu(\mathfrak{m})$ for all ν by Lemma 4.6. We thus get $A_{\mathfrak{X}/X} \ge -\varepsilon_k Z(\mathfrak{m})$ with

$$\varepsilon_k := \frac{kc_2 - c_1}{c_k} > 0$$

as soon as $k > c_1/c_2$. □

Proof of Theorem B. If $\phi : X \to X$ is a finite endomorphism with $e(\phi) \ge 2$, then Theorem A implies $\mathrm{Vol}(X,0) \ge 2\,\mathrm{Vol}(X,0)$ hence $\mathrm{Vol}(X,0) = 0$. When X is $\mathbb{Q}$-Gorenstein and ϕ is not étale in codimension 1, then X is klt by the previous theorem and Remark 6.2. □

Proof of Theorem C. By assumption, there exists an endomorphism $F : V \to V$ and an ample line bundle L such that $F^*L \simeq dL$ for some $d \ge 2$. The composite map

$$H^0(V, mL) \overset{F^*}{\to} H^0(V, mF^*L) \simeq H^0(V, dmL)$$

induces an endomorphism of the finitely generated algebra $\bigoplus_{m\ge 0} H^0(V, mL)$ (which does not preserve the grading). Since the spectrum of this algebra is equal to $X = C(V)$, we get an induced endomorphism $C(F)$ on $C(V)$. It is clear that $C(F)$ is finite, fixes the vertex $0 \in X$, and is not an automorphism. We conclude that $\mathrm{Vol}(X,0) = 0$, which implies that $-K_V$ is pseudoeffective by Example 4.19. □

6.2. **Simple examples of endomorphisms.** Any quotient singularity admits finite endomorphisms of degree ≥ 2, and any toric singularity as well. We saw above examples of endomorphisms on cone singularities. One can modify this construction to get examples on other kind of simple singularities.

Consider a smooth projective morphism $f : Z \to C$ to a smooth pointed curve $0 \in C$ and suppose given a non-invertible endomorphism ϕ such that $f \circ \phi = f$ that. Note that ϕ is automatically finite since the injective endomorphism ϕ^* of $N^1(Z/C)$ has to be bijective.

Assume that $D \subset Z_0$ is a smooth irreducible ample divisor of the fiber Z_0 over 0 that does not intersect the ramification locus of ϕ and such that $\phi(D) \subset D$. Denote by $Y \to Z$ be the blow-up of Z along D. Then ϕ lifts to a rational self-map of Y over C, and the fact that ϕ is étale around D implies that the indeterminacy locus of this rational lift is contained in $\mu^{-1}(\phi^{-1}(D) \setminus D)$ hence in the strict transform E of Z_0 on Y.

Since the conormal bundle of E in Y is ample, E contracts to a simple singularity $0 \in X$ by [Gra62] (we are therefore dealing with an analytic germ $0 \in X$ in that case). The above discussion shows that ϕ induces a finite endomorphism of $(X,0)$, which is furthermore not invertible since ϕ was assumed not to be an automorphism.

Basic examples of this construction include deformations of abelian varieties having a section, with ϕ the multiplication by a positive integer.

6.3. **Endomorphisms of cusp singularities.** Our basic references are [Oda, Tsu83]. Let $C \subset \mathbb{R}^n$ be an open convex cone in which is strict (i.e. its closure contains no line) and let $\Gamma \subset \mathrm{SL}(n,\mathbb{Z})$ be a subgroup leaving C invariant, whose action on $C/\mathbb{R}_+^*$ is properly discontinuous without fixed point, and has compact quotient. Denote by

$$M := \Gamma\backslash C/\mathbb{R}_+^*$$

the corresponding $(n-1)$-dimensional orientable manifold.

Consider the convex envelope Θ of $C \cap \mathbb{Z}^n$. It is proved in [Tsu83] that the faces of $\overline{\Theta}$ are convex polytopes contained in C and with integral vertices. Since Θ is Γ-invariant the cones over the faces of Θ therefore give rise to a Γ-invariant rational fan Σ of $\mathbb{R}^n$ with $|\Sigma| = C \cup \{0\}$. This fan is infinite but is finite modulo Γ since M is compact.

The (infinite type) toric variety $X(\Sigma)$ comes with a Γ-action which preserves the toric divisor $D := X(\Sigma) \setminus (\mathbb{C}^*)^n$ as well the inverse image of C by the map $\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$ defined by

$$\mathrm{Log}(z_1, ..., z_n) = (\log |z_1|, ..., \log |z_n|).$$

The Γ-invariant set $U := \mathrm{Log}^{-1}(C) \cup D$ is open in $X(\Sigma)$ and the action of Γ is properly discontinuous and without fixed point on U. One then shows that the divisor $E := D/\Gamma \subset U/\Gamma =: Y$, which is compact since Σ is a finite fan modulo Γ, admits a strictly pseudoconvex neighbourhood in Y, so that it can be contracted to a normal singularity $0 \in X$, which is furthermore isolated since $Y - E$ is smooth. Note that Y, though possibly not smooth along E, has at most rational singularities since U does, being an open subset of a toric variety. The isolated normal singularity $(X, 0)$ is called the *cusp singularity* attached to (C, Γ). It is shown in [Tsu83] that (C, Γ) is determined up to conjugation in $\mathrm{GL}(n, \mathbb{Z})$ by the (analytic) isomorphism type of the germ $(X, 0)$.

Lemma 6.4. *The canonical divisor K_X is Cartier, X is lc but not klt.*

Remark 6.5. Cusp singularities are however not Cohen-Macaulay in general, hence not Gorenstein.

Proof. The n-form $\Omega = \frac{dz_1}{z_1} \wedge ... \wedge \frac{dz_n}{z_n}$ on the torus $(\mathbb{C}^*)^n$ extends to $X(\Sigma)$ with poles of order one along D. It is Γ-invariant since Γ is a subgroup of $\mathrm{SL}(n, \mathbb{Z})$ thus it descends to a meromorphic form on U/Γ with order one poles along D/Γ. We conclude K_X is zero and that X is lc but not klt since $\pi : (Y, E) \to X$ is crepant and $(X(\Sigma), D)$ is lc but not klt as for any toric variety. □

Now let $A \in \mathrm{GL}(n, \mathbb{R})$ with integer coefficient which preserves C and commutes with Γ (e.g. a homothety). Then Z induces a regular map on U that descends to the quotient Y and preserves the divisors E and we get a finite endomorphism $\phi : (X, 0) \to (X, 0)$ whose topological degree is equal to $|\det A|$.

Example 6.6 (Hilbert modular cusp singularities). Let K be a totally real number field of degree n over $\mathbb{Q}$ and let N be a free $\mathbb{Z}$-submodule of K of rank n (for instance $N = \mathcal{O}_K$). Using the n distinct embeddings of K into $\mathbb{R}$ we get a canonical identification $K \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^n$ and we may view N as a lattice in $\mathbb{R}^n$. Now set $C := (\mathbb{R}_+^*)^n \subset N_{\mathbb{R}}$ and consider the group Γ_N^+ of totally positive units of $u \in \mathcal{O}_K^*$ such that $uN = N$, where u is said to be totally positive if its image under any embedding of K in $\mathbb{R}$ is positive. By Dirichlet's unit theorem, Γ_N^+ is isomorphic to $\mathbb{Z}^{n-1}$, and there is a canonical injective homomorphism $\Gamma_N^+ \hookrightarrow \mathrm{SL}(N)$. For any subgroup $\Gamma \subset \Gamma_N^+$ of finite index, the triple (N, C, Γ) then satisfies the requirements of the definition of a cusp singularities. The singularities obtained by this construction are called Hilbert modular cusp singularities.

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CNRS - Institut de Mathématiques de Jussieu, 4 place Jussieu, 75252 Paris Cedex, France
E-mail address: `boucksom@math.jussieu.fr`

Department of Mathematics, University of Utah, 155 South 1400 East, Salt Lake City, UT 48112-0090, USA
E-mail address: `defernex@math.utah.edu`

CNRS - Centre de Mathématiques Laurent Schwartz, École Polytechnique, 91128 Palaiseau Cedex, France
E-mail address: `favre@math.polytechnique.fr`