Valuation Extensions of Algebras Defined by Monic Gröbner Bases *

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Abstract. Let K be a field, $\mathcal{O}_v$ a valuation ring of K associated to a valuation v: $K \rightarrow \Gamma \cup \{\infty\}$, and $\mathbf{m}_v$ the unique maximal ideal of $\mathcal{O}_v$. Consider an ideal $\mathcal{I}$ of the free K-algebra $K\langle X\rangle = K\langle X_1, ..., X_n\rangle$ on $X_1, ..., X_n$. If $\mathcal{I}$ is generated by a subset $\mathcal{G} \subset \mathcal{O}_v\langle X\rangle$ which is a monic Gröbner basis of $\mathcal{I}$ in $K\langle X\rangle$, where $\mathcal{O}_v\langle X\rangle = \mathcal{O}_v\langle X_1, ..., X_n\rangle$ is the free $\mathcal{O}_v$-algebra on $X_1, ..., X_n$, then the valuation v induces naturally an exhaustive and separated Γ-filtration F^vA for the K-algebra $A = K\langle X\rangle/\mathcal{I}$, and moreover $\mathcal{I} \cap \mathcal{O}_v\langle X\rangle = \langle \mathcal{G}\rangle$ holds in $\mathcal{O}_v\langle X\rangle$; it follows that, if furthermore $\mathcal{G} \not\subset \mathbf{m}_v O_v\langle X\rangle$ and $k\langle X\rangle/\langle \overline{\mathcal{G}}\rangle$ is a domain, where $k = \mathcal{O}_v/\mathbf{m}_v$ is the residue field of $\mathcal{O}_v$, $k\langle X\rangle = k\langle X_1, ..., X_n\rangle$ is the free k-algebra on $X_1, ..., X_n$, and $\overline{\mathcal{G}}$ is the image of $\mathcal{G}$ under the canonical epimorphism $\mathcal{O}_v\langle X\rangle \rightarrow k\langle X\rangle$, then F^vA determines a valuation function $A \rightarrow \Gamma \cup \{\infty\}$, and thereby v extends naturally to a valuation function on the (skew-)field Δ of fractions of A provided Δ exists.

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1. Introduction

In the so-called noncommutative algebraic geometry, the class of schematic algebras in the sense of ([VOW1, [VOW3]) has provided an ample stage to play on. Among others of the topics concerning noncommutative geometric objects associated to schematic algebras, noncommutative valuations are applied to obtain tools for an equivalent of divisor theory in noncommutative geometry (the reader is referred to [VO] for details on this aspect). In the study of extending commutative valuations to noncommutative valuations, filtered-graded structural methods have been used successfully to obtain sufficient conditions assuring the existence of an extension ([LVO2], [MVO], [Li1], [VOW2], [VO], [BVO]). More precisely, let K be a field and $\mathcal{O}_v$ a valuation ring of K associated to a valuation v: $K \rightarrow \Gamma \cup \{\infty\}$, where Γ is a totally ordered abelian additive group. Then v determines an exhaustive and separated Γ-filtration $F^vK = \{F^v_\gamma K\}_{\gamma\in\Gamma}$ for K, where $F^v_\gamma K = \{\lambda \mid \lambda \in K,\ v(\lambda) \geq -\gamma)\}$, such that $F^v_{\gamma_1}K \cdot F^v_{\gamma_2}K = F^v_{\gamma_1+\gamma_2}K$ for all $\gamma_1, \gamma_2 \in \Gamma$, i.e., F^vK is a strong Γ-filtration in the sense of [LVO1]. Consider an affine K-algebra $A = K[a_1, ..., a_n]$ with the (finite or infinite) set of defining relations $\mathcal{G}$, that is, $A \cong K\langle X\rangle/\mathcal{I}$ with $\mathcal{I} = \langle\mathcal{G}\rangle$, where $K\langle X\rangle = K\langle X_1, ..., X_n\rangle$ is the free K-algebra on $X_1, ..., X_n$. From loc. cit. we have learnt that the key points of naturally extending the given valuation v of K to A and further to the (skew-)field Δ of fractions of A (provided Δ exists) are to assure that

(1) the valuation Γ-filtration F^vK of K induces an exhaustive Γ-filtration $F^vA = \{F^v_\gamma A\}_{\gamma\in\Gamma}$ for A in a natural way, i.e., $F^v_\gamma K = K \cap F^v_\gamma A$ for every $\gamma \in \Gamma$, such that $F^v_0 A = \mathcal{O}_v\langle X\rangle + \mathcal{I}/\mathcal{I}$ and $F^v_{<0}A = \mathbf{m}_v F^v_0 A$, where $\mathcal{O}_v\langle X\rangle = \mathcal{O}_v\langle X_1, ..., X_n\rangle$ is the free $\mathcal{O}_v$-algebra on $X_1, ..., X_n$, and $\mathbf{m}_v$ is the unique maximal ideal of $\mathcal{O}_v$;

(2) the Γ-filtration F^vA obtained in (1) above is separated, i.e., $0 \neq a \in A$ implies that there is some $\Gamma \in \Gamma$ such that $a \in F^v_\gamma A - F^v_{<\gamma}A$, in particular $1 \in F^v_0 A - F^v_{<0}A$, where $F^v_{<\gamma}A = \cup_{\gamma'<\gamma} F^v_{\gamma'}A$; and

(3) if $\mathcal{G} \subset \mathcal{O}_v\langle X\rangle$ then $\mathcal{O}_v\langle X\rangle \cap \mathcal{I} = \langle\mathcal{G}\rangle$ holds in $\mathcal{O}_v\langle X\rangle$. In loc. cit. this property is referred to as saying that the $\mathcal{O}_v$-algebra $\mathcal{O}_v\langle X\rangle + \mathcal{I}/\mathcal{I}$ defines a good reduction for the K-algebra A.

For a connected positively $\mathbb{N}$-graded K-algebra A, it was shown in ([VO], Theorem 4.3.7; [BVO], Theorem 2.2) that the Γ-filtration F^vA constructed in loc. cit. may have the properties (1) – (2) provided A has a PBW K-bsis in the classical sense; while the property (3) may be derived under the so-called v-comaximal condition assumed on the ideal $I = \langle\mathcal{G}\rangle$ of $\mathcal{O}_v\langle X\rangle$, i.e., $I \cap (F^v_\gamma K)\langle X\rangle = (F^v_\gamma K)I$ for every $\gamma \in \Gamma$ ([MVO], Lemma 2.1; [VO], Lemmaa 4.3.2).

From ([Li2], CH.III Theorem 1.5; [Li3], Theorem 3.1) we know that, for algebras of the type $A = K\langle X\rangle/\mathcal{I}$ as considered above, the property that A has a classical PBW

K-basis may be equivalent to the property that $\mathcal{I}$ is generated by a (finite or infinite) Gröbner basis of special type. For instance, all the concrete algebras quoted in ([MVO], [Li1], [VO], [BVO]) are indeed defined by Gröbner bases that give rise to PBW K-bases (cf. [Li2], [Li3]). Inspired by such a fact, we aim to demonstrate the following main result in this paper:

- If $\mathcal{G} \subset \mathcal{O}_v\langle X\rangle$ forms a monic Gröbner basis for the ideal $\mathcal{I}$ in $K\langle X\rangle$, where "monic" means that the leading coefficient of every element in $\mathcal{G}$ is 1 (see Section 2 for details), then A has the three properties (1) – (3) described above. It follows that, if furthermore $\mathcal{G} \not\subset \mathbf{m}_v\mathcal{O}_v\langle X\rangle$ and the k-algebra $k\langle X\rangle/\langle\overline{\mathcal{G}}\rangle$ is a domain, where $k = \mathcal{O}_v/\mathbf{m}_v$ is the residue field of $\mathcal{O}_v$, $k\langle X\rangle = k\langle X_1, ..., X_n\rangle$ is the free k-algebra on $X_1, ..., X_n$, and $\overline{\mathcal{G}}$ is the canonical image of $\mathcal{G}$ in $\mathcal{O}_v\langle X\rangle/\mathbf{m}_v\mathcal{O}_v\langle X\rangle$, then F^vA determines a valuation function $A \to \Gamma \cup \{\infty\}$, and thereby v extends naturally to a valuation function on the (skew-)field Δ of fractions of A provided Δ exists.

The result mentioned above will be reached by deriving several results for R-algebras over an arbitrary commutative ring R, where the filtration considered will be Γ-filtration with Γ a totally ordered (commutative or noncommutative) monoid. That is, the results obtained in Sections 3 – 5 may be of independent interest, for instance, they may be used to study valuation extensions of commutative algebras defined by monic Gröbner bases (see the remark given at the end of this paper), and they may also be used to study more general reductions of algebras over a field K as specified in [LVO3].

By the algorithmic Gröbner basis theory for free K-algebras over a field K ([Mor], [Gr]), in principle every finitely presented algebra $A = K\langle X\rangle/\mathcal{I}$ has the defining ideal $\mathcal{I}$ generated by a (finite or infinite) Gröbner basis $\mathcal{G}$ which can always be assumed to be monic. Furthermore, by [Li3] (or see Proposition 2.7 in Section 2 below), if D is a subring of K with the same multiplicative identity 1, then $\mathcal{G} \subset D\langle X\rangle = D\langle X_1, ..., X_n\rangle$ is a monic Gröbner basis for the ideal $I = \langle\mathcal{G}\rangle$ in $D\langle X\rangle$ if and only if $\mathcal{G}$ is a monic Gröbner basis for the ideal $\mathcal{I} = \langle\mathcal{G}\rangle$ in $K\langle X\rangle$ with respect to the same monomial ordering on both $K\langle X\rangle$ and $D\langle X\rangle$. In this sense, the work of this paper may be viewed as a computational approach to solving the valuation extension problem. So, the contents of this paper are organized as follows.

1. Introduction
2. Monic Gröbner bases over rings
3. Extending FR naturally to FA by Gröbner bases over F_0R
4. Realizing the separability of FA by Gröbner bases over F_0R
5. Realizing good reductions for A by Gröbner bases over $D \subset R$
6. Realizing valuation extensions of A by Gröbner bases over $\mathcal{O}_v$

Unless otherwise stated, rings considered in this paper are associative rings with multiplicative identity 1, ideals are meant two-sided ideals, and modules are unitary left modules. For a subset U of a ring S, we write $\langle U\rangle$ for the ideal generated by U. Moreover, we use $\mathbb{N}$, respectively $\mathbb{Z}$, to denote the set of nonnegative integers, respectively the set of integers. Moreover, valuations of a (skew-)field Δ are in the sense of O. Schilling [Sc].

2. Monic Gröbner Bases over Rings

For the reader's convenience, in this section we briefly recall from [Li3] some basics on monic Gröbner bases in free algebras over rings. Classical Gröbner basis theory for free algebras over a field K is referred to [Mor] and [Gr].

Let R be an arbitrary commutative ring, $R\langle X\rangle = R\langle X_1, ..., X_n\rangle$ the free R-algebra of n generators, and $\mathcal{B}_R$ the standard R-basis of $R\langle X\rangle$ consisting of monomials (words in alphabet $X = \{X_1, ..., X_n\}$, including empty word which is identified with the multiplicative identity element 1 of $R\langle X\rangle$). Unless otherwise stated, monomials in $\mathcal{B}_R$ are denoted by lower case letters $u, v, w, s, t, \cdots$. By a *monomial ordering* on $\mathcal{B}_R$ (or on $R\langle X\rangle$) we mean a well-ordering $\prec$ on $\mathcal{B}_R$ which satisfies:

(M1) For $w, u, v, s \in \mathcal{B}_R$, $u \prec v$ implies $wus \prec wvs$;

(M2) For $w, u, v \in \mathcal{B}_R$, $w = uv$ implies $u \preceq w$ and $v \preceq w$.

In particular, by an $\mathbb{N}$-*graded monomial ordering* on $\mathcal{B}_R$, denoted $\prec_{gr}$, we mean a monomial ordering on $\mathcal{B}_R$ which is defined subject to a well-ordering $\prec$ on $\mathcal{B}_R$, that is, for $u, v \in \mathcal{B}_R$, $u \prec_{gr} v$ if either $\deg(u) < \deg(v)$ or $\deg(u) = \deg(v)$ but $u \prec v$, where deg( ) denotes the degree function on elements of $R\langle X\rangle$ with respect to a fixed *weight* $\mathbb{N}$-*gradation* of $R\langle X\rangle$ (i.e. each X_i is assigned a positive degree n_i, $1 \le i \le n$). For instance, the usual $\mathbb{N}$-graded (reverse) lexicographic ordering is a popularly used $\mathbb{N}$-graded monomial ordering.

If $\prec$ is a monomial ordering on $\mathcal{B}_R$ and $f = \sum_{i=1}^{s} \lambda_i w_i \in R\langle X\rangle$, where $\lambda_i \in R - \{0\}$ and $w_i \in \mathcal{B}_R$, such that $w_1 \prec w_2 \prec \cdots \prec w_s$, then the *leading monomial* of f is defined as $\mathbf{LM}(f) = w_s$ and the *leading coefficient* of f is defined as $\mathbf{LC}(f) = \lambda_s$. For a subset $H \subset R\langle X\rangle$, we write $\mathbf{LM}(H) = \{\mathbf{LM}(f) \mid f \in H\}$ for the set of leading monomials of S. We say that a subset $G \subset R\langle X\rangle$ is *monic* if $\mathbf{LC}(g) = 1$ for every $g \in G$. Moreover, for $u, v \in \mathcal{B}_R$, we say that v divides u, denoted $v|u$, if $u = wvs$ for some $w,\ s \in \mathcal{B}_R$.

With notation and all definitions as above, it is easy to see that a division algorithm by a monic subset G is valid in $R\langle X\rangle$ with respect to any fixed monomial ordering $\prec$ on $\mathcal{B}_R$. More precisely, let $f \in R\langle X\rangle$. Noticing $\mathbf{LC}(g) = 1$ for every $g \in G$, if $\mathbf{LM}(g)|\mathbf{LM}(f)$ for some $g \in G$, then f can be written as $f = \mathbf{LC}(f)ugv + f_1$ with $u,\ v \in \mathcal{B}_R,\ f_1 \in R\langle X\rangle$

satisfying $\mathbf{LM}(f_1) \prec \mathbf{LM}(f)$; if $\mathbf{LM}(g) \nmid \mathbf{LM}(f)$ for all $g \in G$, then $f = f_1 + \mathbf{LC}(f)\mathbf{LM}(f)$ with $f_1 = f - \mathbf{LC}(f)\mathbf{LM}(f)$ satisfying $\mathbf{LM}(f_1) \prec \mathbf{LM}(f)$. Next, consider the divisibility of $\mathbf{LM}(f_1)$ by $\mathbf{LM}(g)$ with $g \in G$, and so forth. Since $\prec$ is a well-ordering, after a finite number of successive division by elements in G in this way, we see that f can be written as

$$\begin{aligned} f \;\; = \;\; & \textstyle\sum_{i,j} \lambda_{ij} u_{ij} g_j v_{ij} + r_f, \text{ where } \lambda_{ij} \in R,\ u_{ij}, v_{ij} \in \mathcal{B}_R,\ g_j \in G, \\ & \text{and } r_f = \textstyle\sum_p \lambda_p w_p \text{ with } \lambda_p \in R,\ w_p \in \mathcal{B}_R, \\ & \text{satisfying } \mathbf{LM}(u_{ij} g_j v_{ij}) \preceq \mathbf{LM}(f) \text{ whenever } \lambda_{ij} \neq 0, \\ & \mathbf{LM}(r_f) \preceq \mathbf{LM}(f) \text{ and } \mathbf{LM}(g) \nmid w_p \text{ for every } g \in G \text{ whenever } \lambda_p \neq 0. \end{aligned}$$

If, $r_f = 0$ in the representation of f obtained above, then we say that f *is reduced to* 0 *by division by* G, and we write $\overline{f}^G = 0$ for this property. The validity of such a division algorithm by G leads to the following definition.

2.1. Definition Let $\prec$ be a fixed monomial ordering on $\mathcal{B}_R$, and I an ideal of $R\langle X\rangle$. A *monic Gröbner basis* of I is a subset $\mathcal{G} \subset I$ satisfying:
(1) $\mathcal{G}$ is monic; and
(2) $f \in I$ and $f \neq 0$ implies $\mathbf{LM}(g)|\mathbf{LM}(f)$ for some $g \in \mathcal{G}$.

By the division algorithm presented above, it is clear that a monic Gröbner basis of I is first of all a generating set of the ideal I, i.e., $I = \langle \mathcal{G}\rangle$, and moreover, a monic Gröbner basis of I can be characterized as follows.

2.2. Proposition Let $\prec$ be a fixed monomial ordering on $\mathcal{B}_R$, and I an ideal of $R\langle X\rangle$. For a monic subset $\mathcal{G} \subset I$, the following statements are equivalent:
(i) $\mathcal{G}$ is a monic Gröbner basis of I;
(ii) Each nonzero $f \in I$ has a Gröbner representation:

$$\begin{aligned} f \;\; = \;\; & \textstyle\sum_{i,j} \lambda_{ij} u_{ij} g_j v_{ij}, \text{ where } \lambda_{ij} \in R,\ u_{ij}, v_{ij} \in \mathcal{B}_R,\ g_j \in G, \\ & \text{satisfying } \mathbf{LM}(u_{ij} g_j v_{ij}) \preceq \mathbf{LM}(f) \text{ whenever } \lambda_{ij} \neq 0, \end{aligned}$$

or equivalently, $\overline{f}^{\mathcal{G}} = 0$;
(iii) $\langle \mathbf{LM}(\mathcal{G})\rangle = \langle \mathbf{LM}(I)\rangle$. □

Let $\prec$ be a monomial ordering on the standard R-basis $\mathcal{B}_R$ of $R\langle X\rangle$, and let G be a monic subset of $R\langle X\rangle$. We call an element $f \in R\langle X\rangle$ a *normal element* (mod G) if $f = \sum_j \mu_j v_j$ with $\mu_j \in R$, $v_j \in \mathcal{B}_R$, and f has the property that $\mathbf{LM}(g) \nmid v_j$ for every $g \in G$ and every $\mu_j \neq 0$. The set of normal monomials in $\mathcal{B}_R$ (mod G) is denoted by

$N(G)$, i.e.,

$$N(G)=\{u\in\mathcal{B}_R\mid \mathbf{LM}(g)\nmid u,\ g\in G\}.$$

Thus, an element $f\in R\langle X\rangle$ is normal (mod G) if and only if $f\in\sum_{u\in N(G)}Ru$.

2.3. Proposition Let $\mathcal{G}$ be a monic Gröbner basis of the ideal $I=\langle\mathcal{G}\rangle$ in $R\langle X\rangle$ with respect to some monomial ordering $\prec$ on $\mathcal{B}_R$. Then each nonzero $f\in R\langle X\rangle$ has a finite presentation

$$f=\sum_{i,j}\lambda_{ij}s_{ij}g_iw_{ij}+r_f,\quad \lambda_{ij}\in R,\ s_{ij},w_{ij}\in\mathcal{B}_R,\ g_i\in\mathcal{G},$$

where $\mathbf{LM}(s_{ij}g_iw_{ij})\preceq\mathbf{LM}(f)$ whenever $\lambda_{ij}\neq 0$, and either $r_f=0$ or r_f is a unique normal element (mod $\mathcal{G}$). Hence, $f\in I$ if and only if $r_f=0$, solving the "membership problem" for I.

□

The foregoing results enable us to obtain further characterization of a monic Gröbner basis $\mathcal{G}$, which, in turn, gives rise to the fundamental decomposition theorem of the R-module $R\langle X\rangle$ by the ideal $I=\langle\mathcal{G}\rangle$, and thereby yields a free R-basis for the R-algebra $R\langle X\rangle/I$.

2.4. Theorem Let $I=\langle\mathcal{G}\rangle$ be an ideal of $R\langle X\rangle$ generated by a monic subset $\mathcal{G}$. With notation as above, the following statements are equivalent.
(i) $\mathcal{G}$ is a monic Gröbner basis of I.
(ii) The R-module $R\langle X\rangle$ has the decomposition

$$R\langle X\rangle=I\oplus\sum_{u\in N(\mathcal{G})}Ru=\langle\mathbf{LM}(I)\rangle\oplus\sum_{u\in N(\mathcal{G})}Ru.$$

(iii) The canonical image $\overline{N(\mathcal{G})}$ of $N(\mathcal{G})$ in $R\langle X\rangle/\langle\mathbf{LM}(I)\rangle$ and $R\langle X\rangle/I$ forms a free R-basis for $R\langle X\rangle/\langle\mathbf{LM}(I)\rangle$ and $R\langle X\rangle/I$ respectively.

□

Before mentioning a version of the termination theorem in the sense of ([Mor], [Gr]) for verifying an LM-reduced monic Gröbner basis in $R\langle X\rangle$ (see the definition below), we need a little more preparation.

Given a monomial ordering $\prec$ on $\mathcal{B}_R$, we say that a subset $G\subset R\langle X\rangle$ is *LM-reduced* if

$$\mathbf{LM}(g_i)\nmid\mathbf{LM}(g_j)\ \text{for all}\ g_i,g_j\in G\ \text{with}\ g_i\neq g_j.$$

If a subset $G \subset R\langle X\rangle$ is both LM-reduced and monic, then we call G an *LM-reduced monic subset*. Thus we have the notion of an *LM-reduced monic Gröbner basis.*

Let I be an ideal of $R\langle X\rangle$. If $\mathcal{G}$ is a monic Gröbner basis of I and $g_1, g_2 \in \mathcal{G}$ such that $g_1 \neq g_2$ but $\mathbf{LM}(g_1)|\mathbf{LM}(g_2)$, then clearly g_2 can be removed from $\mathcal{G}$ and the remained subset $\mathcal{G} - \{g_2\}$ is again a monic Gröbner basis for I. Hence, in order to have a better criterion for monic Gröbner basis we need only to consider the subset which is both LM-reduced and monic.

Let $\prec$ be a monomial ordering on $\mathcal{B}_R$. For two monic elements $f,\ g \in R\langle X\rangle - \{0\}$, including $f = g$, if there are monomials $u, v \in \mathcal{B}_R$ such that

(1) $\mathbf{LM}(f)u = v\mathbf{LM}(g)$, and

(2) $\mathbf{LM}(f) \not| v$ and $\mathbf{LM}(g) \not| u$,

then the element

$$o(f, u;\ v, g) = f \cdot u - v \cdot g$$

is called an *overlap element* of f and g. From the definition it is clear that

$$\mathbf{LM}((o(f, u;\ v, g)) \prec \mathbf{LM}(fu) = \mathbf{LM}(vg),$$

and moreover, there are only finitely many overlap elements for each pair (f, g) of monic elements in $R\langle X\rangle$. So, for a finite subset of monic elements $\mathcal{G} \subset R\langle X\rangle$, actually as in the classical case ([Mor], [Gr]), the termination theorem below enables us to check effectively whether $\mathcal{G}$ is a Gröbner basis of I or not.

2.5. Theorem (Termination theorem) Let $\prec$ be a fixed monomial ordering on $\mathcal{B}_R$. If $\mathcal{G}$ is an LM-reduced monic subset of $R\langle X\rangle$, then $\mathcal{G}$ is an LM-reduced monic Gröbner basis for the ideal $I = \langle \mathcal{G}\rangle$ if and only if for each pair $g_i, g_j \in \mathcal{G}$, including $g_i = g_j$, every overlap element $o(g_i, u;\ v, g_j)$ of g_i, g_j has the property $\overline{o(g_i, u;\ v, g_j)}^{\mathcal{G}} = 0$, that is, by division by $\mathcal{G}$, every $o(g_i, u;\ v, g_j)$ is reduced to zero.

□

Remark (i) Obviously, if $\mathcal{G} \subset R\langle X\rangle$ is an LM-reduced subset with the property that each $g \in \mathcal{G}$ has the leading coefficient $\mathbf{LC}(g)$ which is invertible in R, then Theorem 2.5 is also valid for $\mathcal{G}$.

(ii) It is obvious as well that Theorem 2.5 does not necessarily induce an analogue of the Buchberger algorithm as in the classical case.

(iii) It is not difficult to see that all results we presented so far are valid for getting monic Gröbner bases in a commutative polynomial ring $R[x_1, ..., x_n]$ over an arbitrary commutative ring R where overlap elements are replaced by S-polynomials.

By virtue of Theorem 2.5 (or more precisely, its proof given in [Li3]), the following two propositions are obtained.

2.6. Proposition Let $K\langle X\rangle = K\langle X_1, ..., X_n\rangle$ be the free algebra of n generators over a field K, and let $R\langle X\rangle = R\langle X_1, ..., X_n\rangle$ be the free algebra of n generators over an arbitrary commutative ring R. With notation as before, fixing the same monomial ordering $\prec$ on both $K\langle X\rangle$ and $R\langle X\rangle$, the following statements hold.
(i) If a monic subset $\mathcal{G} \subset K\langle X\rangle$ is a Gröbner basis for the ideal $\langle \mathcal{G}\rangle$ in $K\langle X\rangle$, then, taking a counterpart of $\mathcal{G}$ in $R\langle X\rangle$ (if it exists), again denoted by $\mathcal{G}$, $\mathcal{G}$ is a monic Gröbner basis for the ideal $\langle \mathcal{G}\rangle$ in $R\langle X\rangle$.
(ii) If a monic subset $\mathcal{G} \subset R\langle X\rangle$ is a Gröbner basis for the ideal $\langle \mathcal{G}\rangle$ in $R\langle X\rangle$, then, taking a counterpart of $\mathcal{G}$ in $K\langle X\rangle$ (if it exists), again denoted by $\mathcal{G}$, $\mathcal{G}$ is a Gröbner basis for the ideal $\langle \mathcal{G}\rangle$ in $K\langle X\rangle$.

□

2.7. Proposition Let R be a commutative ring and R' a subring of R with the same identity element 1. If we consider the free R-algebra $R\langle X\rangle = R\langle X_1, ..., X_n\rangle$ and the free R'-algebra $R'\langle X\rangle = R'\langle X_1, ..., X_n\rangle$, then the following two statements are equivalent for a subset $\mathcal{G} \subset R'\langle X\rangle$:
(i) $\mathcal{G}$ is an LM-reduced monic Gröbner basis for the ideal $I = \langle \mathcal{G}\rangle$ in $R'\langle X\rangle$ with respect to some monomial ordering $\prec$ on the standard R'-basis $\mathcal{B}_{R'}$ of $R'\langle X\rangle$;
(ii) $\mathcal{G}$ is an LM-reduced monic Gröbner basis for the ideal $J = \langle \mathcal{G}\rangle$ in $R\langle X\rangle$ with respect to the monomial ordering $\prec$ on the standard R-basis $\mathcal{B}_R$ of $R\langle X\rangle$, where $\prec$ is the same monomial ordering used in (i).

□

Let K be a field. From the literature we know that numerous well-known K-algebras, such as Weyl algebras over K, enveloping algebras of K-Lie algebras, exterior K-algebras, Clifford K-algebras, down-up K-algebras, quantum binomial K-algebras, most popularly studied quantum groups over K, etc., all have defining relations that form an LM-reduced monic Gröbner basis in free K-algebras (cf. [Li2], [Laf], [G-I]). Hence, by Proposition 2.6, if the field K is replaced by a commutative ring R, then all of these R-algebras (if they exist) have defining relations that form an LM-reduced monic Gröbner basis in a free R-algebra. The reader is referred to [Li3] for more details on this topic and for more concrete examples.

3. Extending FR Naturally to FA by Gröbner Bases over F_0R

Let R be an arbitrary *commutative ring*, and Γ a totally ordered (commutative or non-commutative) *monoid* with the total ordering $<$. To make the notation uniform in our context, we first fix the convention:

- From now on in this paper we use $+$ to denote the binary operation of Γ, and we use 0 to denote the neutral element of Γ (though Γ is not necessarily commutative).
- The definition of an exhaustive Γ-filtration defined for R below applies to every R-algebra (ring) considered in this paper.

We say that R is equipped with an exhaustive Γ-filtration $FR = \{F_\gamma R\}_{\gamma\in\Gamma}$, if each $F_\gamma R$ is an additive subgroup of R and FR satisfies

(F1) $R = \cup_{\gamma\in\Gamma} F_\gamma R$;

(F2) $F_{\gamma_1}R \subseteq F_{\gamma_2}R$ whenever $\gamma_1 < \gamma_2$;

(F3) $F_\gamma R \cdot F_\tau R \subseteq F_{\gamma+\tau}R$ for all $\gamma, \tau \in \Gamma$;

(F4) $1 \in F_0R$.

Note that F_0R is a subring of R with the same identity element 1. To simplify notation, we write R_0 for F_0R. Let $R\langle X\rangle = R\langle X_1, ..., X_n\rangle$ be the free R-algebra on $X_1, ..., X_n$, and $\mathcal{I}$ an ideal of $R\langle X\rangle$. Considering the R-algebra $A = R\langle X\rangle/\mathcal{I}$, if $\mathcal{G}$ is a monic Gröbner basis of $\mathcal{I}$ in $R\langle X\rangle$ with respect to a monomial ordering on the standard R-basis B_R of $R\langle X\rangle$, and if $N(\mathcal{G})$ denotes the set of normal monomials in $\mathcal{B}_R$ (mod $\mathcal{G}$) (see Section 2), then, by Theorem 2.4, the canonical image $\overline{N(\mathcal{G})}$ of $N(\mathcal{G})$ in $A = R\langle X\rangle/\langle\mathcal{I}\rangle = R\langle X\rangle/\langle\mathcal{G}\rangle$ forms a free R-basis for A. Bearing this preliminary in mind, we are able to establish the following result.

3.1. Theorem Let the commutative ring R be equipped with an exhaustive Γ-filtration $FR = \{F_\gamma R\}_{\gamma\in\Gamma}$. With notation as fixed above, suppose that the ideal $\mathcal{I}$ is generated by a subset $\mathcal{G} \subset R_0\langle X\rangle$ which is a monic Gröbner basis of $\mathcal{I}$ in $R\langle X\rangle$ with respect to a monomial ordering $\prec$ on the standard R-basis $\mathcal{B}_R$ of $R\langle X\rangle$, where $R_0\langle X\rangle = R_0\langle X_1, ..., X_n\rangle$ is the free R_0-algebra on $X_1, ..., X_n$, and without loss of generality we assume that $\mathbf{LM}(g) \neq 1$ for every $g \in \mathcal{G}$. Then the R-algebra $A = R\langle X\rangle/\mathcal{I} = R\langle X\rangle/\langle\mathcal{G}\rangle$ can be endowed with the exhaustive Γ-filtration $FA = \{F_\gamma A\}_{\gamma\in\Gamma}$ by putting

$$F_\gamma A = \left\{ a = \sum_i \lambda_i \bar{w}_i \;\middle|\; \lambda_i \in F_\gamma R,\ \bar{w}_i \in \overline{N(\mathcal{G})} \right\},\ \gamma \in \Gamma,$$

such that $F_\gamma R = R \cap F_\gamma A,\ \gamma \in \Gamma$, that is, FR extends naturally to FA.

Proof We show that FA satisfies the conditions (F1) – (F4) required by an exhaustive Γ-filtration. Obviously each $F_\gamma A$ is an additive subgroup of A. Also it is clear that $F_{\gamma_1}A \subseteq F_{\gamma_2}A$ whenever $\gamma_1 < \gamma_2$. If $0 \neq a \in A$, then a can be written uniquely as $a = \sum_{j=1}^m \mu_j \bar{w}_j$ with $\mu_j \in R$, $\bar{w}_j \in \overline{N(\mathcal{G})}$. Since $R = \cup_{\gamma\in\Gamma} F_\gamma R$, we may assume that $\mu_j \in F_{\gamma_j}R$, $1 \leq j \leq m$, and that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_m$ in the totally ordered monoid Γ. It follows that $\mu_j \in F_{\gamma_1}R$, $1 \leq j \leq m$, and thereby $a \in F_{\gamma_1}A$. This shows that $A = \cup_{\gamma\in\Gamma} F_\gamma^v A$. If $a = \sum_i \lambda_i \bar{w}_i \in F_\gamma A$, $b = \sum_j \mu_j \bar{w}_j \in F_\rho A$, then $ab = \sum_{i,j} \lambda_i\mu_j \bar{w}_i\bar{w}_j$ with $\lambda_i\mu_j \in F_{\gamma+\rho}^v R$. Since the monic Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ is contained in $R_0\langle X\rangle = (F_0R)\langle X\rangle$, if we run the the division algorithm of each w_iw_j by $\mathcal{G}$ in $R\langle X\rangle$, it is indeed implemented in $R_0\langle X\rangle$. It turns out that

$$w_iw_j = \sum_{p,q} \xi_{pq}u_{pq}g_qv_{pq} + \sum_m \eta_m s_m, \text{ where } \xi_{pq}, \eta_m \in R_0 = F_0R,\ u_{pq}, v_{pq} \in \mathcal{B}_R,\ s_m \in N(\mathcal{G}).$$

Considering the residue classes in $R\langle X\rangle/\mathcal{I}$, we have $\bar{w}_i\bar{w}_j = \sum_m \eta_m \bar{s}_m$ with $\eta_m \in F_0R$ and $\bar{s}_m \in \overline{N(\mathcal{G})}$, which implies $ab \in F_{\gamma+\rho}^v A$. Thereby $F_\gamma A \cdot F_\rho A \subseteq F_{\gamma+\rho}A$ for all $\gamma, \rho \in \Gamma$. Moreover, since $1 \in N(\mathcal{G})$ by our assumption on $\mathcal{G}$, it is easy to see that $1 \in F_0A$. This shows that FA defines an exhaustive Γ-filtration for A.

Finally, noticing the fact that $1 \in N(\mathcal{G})$, it is straightforward that $F_\gamma R \subseteq R \cap F_\gamma A \subseteq F_\gamma R$, i.e., $F_\gamma R = R \cap F_\gamma A$ for every $\gamma \in \Gamma$, as desired. □

4. Realizing the Separability of FA by Gröbner Bases over F_0R

Let R be an arbitrary commutative ring, and Γ a totally ordered (commutative or noncommutative) monoid with the total ordering $<$. Suppose that R is equipped with an exhaustive Γ-filtration $FR = \{F_\gamma R\}_{\gamma\in\Gamma}$ in the sense of Section 3. We say that the Γ-filtration FR is *separated*, if $0 \neq \lambda \in R$ implies that there is a $\gamma \in \Gamma$ such that

$$\lambda \in F_\gamma R - F_{<\gamma}R, \text{ where } F_{<\gamma}R = \cup_{\gamma'<\gamma} F_{\gamma'}R.$$

Let $R\langle X\rangle = R\langle X_1, ..., X_n\rangle$ be the free R-algebra on $X_1, ..., X_n$, $\mathcal{I}$ an ideal of $R\langle X\rangle$, and $A = R\langle X\rangle/\mathcal{I}$. With every definition and all notations as in Section 3, especially with $R_0 = F_0R$, in this section we aim to show the next Theorem.

4.1. Theorem Suppose that the ideal $\mathcal{I}$ is generated by a subset $\mathcal{G} \subset R_0\langle X\rangle$ which is a monic Gröbner basis of $\mathcal{I}$ in $R\langle X\rangle$ with respect to a monomial ordering $\prec$ on the standard R-basis $\mathcal{B}_R$ of $R\langle X\rangle$, where $R_0\langle X\rangle = R_0\langle X_1, ..., X_n\rangle$ is the free R_0-algebra on $X_1, ..., X_n$, and without loss of generality we assume that $\mathbf{LM}(g) \neq 1$ for every $g \in \mathcal{G}$. Then the R-algebra $A = R\langle X\rangle/\mathcal{I} = R\langle X\rangle/\langle\mathcal{G}\rangle$ can be endowed with the Γ-filtration FA as constructed in Theorem 3.1, and if the Γ-filtration FR of R is separated, then FA is separated, i.e., if $0 \neq a \in A$, then there is a $\gamma \in \Gamma$ such that $a \in F_\gamma A - F_{<\gamma}A$, where $F_{<\gamma}A = \cup_{\gamma'<\gamma}F_{\gamma'}A$. In particular, if $1 \in F_0R - F_{<0}R$ then $1 \in F_0A - F_{<0}A$.

Proof By the assumption on $\mathcal{I}$ and $\mathcal{G}$, Theorem 3.1 assures the existence of the Γ-filtration FA of A. Let $N(\mathcal{G})$ be the set of normal monomials in $\mathcal{B}_R$ (mod $\mathcal{G}$). Then, by Theorem 2.4, the canonical image $\overline{N(\mathcal{G})}$ of $N(\mathcal{G})$ in $A = R\langle X\rangle/\mathcal{I} = R\langle X\rangle/\langle\mathcal{G}\rangle$ forms a free R-basis for A, and moreover, $1 \in \overline{N(\mathcal{G})}$. If $0 \neq a \in A$, then a can be written uniquely as $a = \sum_{j=1}^m \mu_j\bar{w}_j$ with $\mu_j \in R$, $\bar{w}_j \in \overline{N(\mathcal{G})}$. Since FR is separated by the assumption, there are $\gamma_1, \gamma_2, ..., \gamma_m \in \Gamma$ such that $\mu_j \in F_{\gamma_j}R - F_{<\gamma_j}R$, $1 \leq j \leq m$. Assuming $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_m$ in the totally ordered monoid Γ, we have $\mu_j \in F_{\gamma_1}R$, $1 \leq j \leq m$, and hence $a \in F_{\gamma_1}A$. If there were some $\tau \in \Gamma$ with $\tau < \gamma_1$, such that $a \in F_\tau A$, then $a = \sum_{i=1}^n \lambda_i\bar{w}_i$ with $\lambda_i \in F_\tau R$ and $\bar{w}_i \in \overline{N(\mathcal{G})}$. Comparing the coefficients on both sides of the equality

$$\sum_{j=1}^m \mu_j\bar{w}_j = a = \sum_{i=1}^n \lambda_i\bar{w}_i,$$

we would get $\mu_j \in F_\tau R$, $1 \leq j \leq m$, in particular $\mu_1 \in F_\tau R$ with $\tau < \gamma_1$, which is a contradiction. Therefore $a \in F_{\gamma_1}A - F_{<\gamma_1}A$. This shows that FA is separated.

Finally, noticing $1 \in \overline{N(\mathcal{G})}$, if $1 \in F_0R - F_{<0}R$, then by the construction of F_0A and a similar argument as above we get $1 \in F_0A - F_{<0}A$. □

5. Realizing Good Reductions for A by Gröbner Bases over $D \subset R$

Let R be an arbitrary commutative ring, $R\langle X\rangle = R\langle X_1, ..., X_n\rangle$ the free R-algebra on $X_1, ..., X_n$, $\mathcal{I}$ an ideal of $R\langle X\rangle$, and $A = R\langle X\rangle/\mathcal{I}$. In this section we generalize the notion of a good reduction (in the sense of [MVO]) to the R-algebra A, and we realize this property by using monic Gröbner bases.

Let D be *any subring* of R which has the same identity element 1 as that of R, and let $D\langle X\rangle = D\langle X_1, ..., X_n\rangle$ be the free D-algebra on $X_1, ..., X_n$. In what follows we use B_R, respectively B_D, to denote the standard R-basis of $R\langle X\rangle$, respectively the standard

D-basis of $D\langle X\rangle$. Considering the D-subalgebra

$$\Lambda = D\langle X\rangle + \mathcal{I}/\mathcal{I}$$

of A, we have $R\Lambda = A$ and $\Lambda \cong D\langle X\rangle/\mathcal{I}\cap D\langle X\rangle$. Observe that the exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow R\langle X\rangle \xrightarrow{\pi} A \longrightarrow 0$$

and the canonical D-algebra epimorphism π_D: $D\langle X\rangle \rightarrow \Lambda$ give rise to the exact sequence

$$0 \longrightarrow \mathcal{I}\cap D\langle X\rangle \longrightarrow D\langle X\rangle \xrightarrow{\pi_D} \Lambda \longrightarrow 0$$

Let ω be *any proper ideal* of D and $k = D/\omega$. Then the k-algebra epimorphism π_ω: $D\langle X\rangle/\omega D\langle X\rangle \rightarrow \Lambda/\omega\Lambda$ induced by π_D yields the exact sequence

$$0 \longrightarrow \frac{\mathcal{I}\cap D\langle X\rangle + \omega D\langle X\rangle}{\omega D\langle X\rangle} \longrightarrow D\langle X\rangle/\omega D\langle X\rangle \xrightarrow{\pi_\omega} \Lambda/\omega\Lambda \longrightarrow 0$$

It is clear that if the ideal $\mathcal{I}$ of $R\langle X\rangle$ is generated by a subset $\mathcal{G}\subset D\langle X\rangle$ but $\mathcal{G}\not\subset \omega D\langle X\rangle$, such that $\mathcal{I}\cap D\langle X\rangle = \langle\mathcal{G}\rangle$ as an ideal of $D\langle X\rangle$, then $\Lambda/\omega\Lambda \cong k\langle X\rangle/\langle\overline{\mathcal{G}}\rangle$ as k-algebras, where $\overline{\mathcal{G}}$ is the canonical image of $\mathcal{G}$ in $D\langle X\rangle/\omega D\langle X\rangle$.

5.1. Definition (Compare with the definition given in [MVO] Section 2) Let $\mathcal{I} = \langle\mathcal{G}\rangle$ be the ideal of $R\langle X\rangle$ generated by a subset $\mathcal{G}\subset D\langle X\rangle$. If, as an ideal of $D\langle X\rangle$, $\mathcal{I}\cap D\langle X\rangle = \langle\mathcal{G}\rangle$, then we say that the D-algebra $\Lambda = D\langle X\rangle + \mathcal{I}/\mathcal{I}$ defines a *good reduction* for the R-algebra $A = R\langle X\rangle/\mathcal{I}$.

5.2. Theorem With the notation as before, if the ideal $\mathcal{I}$ is generated by a subset $\mathcal{G}\subset D\langle X\rangle$ which is a monic Gröbner basis of $\mathcal{I}$ in $R\langle X\rangle$ with respect to a monomial ordering $\prec$ on the standard R-basis $\mathcal{B}_R$ of $R\langle X\rangle$, then the following statements hold.
(i) $\mathcal{G}$ is a Gröbner basis for the ideal $\mathcal{I}\cap D\langle X\rangle$ in $D\langle X\rangle$ with respect to the same monomial ordering $\prec$ on the standard D-basis $\mathcal{B}_D$ of $D\langle X\rangle$. Hence the ideal $\mathcal{I}\cap D\langle X\rangle$ of $D\langle X\rangle$ is generated by $\mathcal{G}$, i.e., $D\langle X\rangle\cap\mathcal{I} = \langle\mathcal{G}\rangle$ holds in $D\langle X\rangle$. Moreover, the set of normal monomials in $\mathcal{B}_D$ (mod $\mathcal{G}$) is the same as the set of normal monomials in $\mathcal{B}_R$ (mod $\mathcal{G}$), denoted $N(\mathcal{G})$.
(ii) The D-algebra $\Lambda = D\langle X\rangle + \mathcal{I}/\mathcal{I}$ defines a good reduction for the R-algebra $A = R\langle X\rangle/\mathcal{I}$.
(iii) For any ideal ω of D such that $\mathcal{G}\not\subset\omega D\langle X\rangle$, we have $\Lambda/\omega\Lambda \cong k\langle X\rangle/\langle\overline{\mathcal{G}}\rangle$ as k-algebras.

Proof Although (i) is a consequence of Proposition 2.7, we prefer giving a direct proof here. First note that $\mathcal{B}_R = \mathcal{B}_D$. If $f\in\mathcal{I}\cap D\langle X\rangle$, say $f = \sum_i d_iu_i$ with $d_i\in D$ and

$u_i \in \mathcal{B}_D$, then since $\mathcal{G} \subset D\langle X\rangle$ is a monic Gröbner basis for the ideal $\mathcal{I}$ in $K\langle X\rangle$ with respect to the monomial ordering $\prec$ on $\mathcal{B}_R$, we have $\overline{f}^{\mathcal{G}} = 0$ in $D\langle X\rangle$ with respect to the same monomial ordering $\prec$ on $\mathcal{B}_D$, that is, the division of f by $\mathcal{G}$ produces a Gröbner representation $f = \sum_{i,j} d_{ij}w_{ij}g_jv_{ij}$ with $d_{ij} \in D$, $w_{ij}, v_{ij} \in \mathcal{B}_D = \mathcal{B}_R$, and $g_j \in \mathcal{G}$. Hence $\mathcal{G}$ is a Gröbner basis for the ideal $\mathcal{I} \cap D\langle X\rangle$ in $D\langle X\rangle$ with respect to the same monomial ordering $\prec$ on $\mathcal{B}_D$. It turns out that $\mathcal{I} \cap D\langle X\rangle = \langle \mathcal{G}\rangle$ in $D\langle X\rangle$, and that the set of normal monomials in $\mathcal{B}_D$ (mod $\mathcal{G}$) is the same as the set of normal monomials in $\mathcal{B}_R$ (mod $\mathcal{G}$) .

(ii) and (iii) are clear enough by (i) and the discussion made before Definition 5.1. $\square$

6. Realizing Valuation Extensions of A by Gröbner Bases over $\mathcal{O}_v$

In this section, we apply the results of previous sections to proving the main result of this paper.

We first recall some basics on valuations, especially some fundamental results concerning valuation extensions via filtered-graded structures (cf. [Sc], [Coh], [LVO2], [MVO], [VO]). Let $(\Gamma, +;\ <)$ be a totally ordered *abelian additive group* with the neutral element 0 and the total ordering $<$, and let A be an arbitrary (commutative or noncommutative) ring with 1. A valuation v of A is a surjective function $A \to \Gamma \cup \{\infty\}$, where the symbol ∞ plays conventionally the role such that $\gamma < \infty$, $\infty + \gamma = \gamma + \infty = \infty$ for every $\gamma \in \Gamma$, $\infty + \infty = \infty$, and $\infty - \infty = 0$, such that for $a, b \in A$, the following conditions are satisfied:

(V1) $v(a) = \infty$ if and only if $a = 0$;

(V2) $v(ab) = v(a) + v(b)$;

(V3) $v(a + b) \geq \min(v(a), v(b))$.

If A is an Ore domain with a valuation function v as above, then v can be extended in a unique way to the (skew-)field Δ of fractions of A subject to the rule:

$$v(ab^{-1}) = v(a) - v(b) \text{ for } a, b \in A \text{ with } b \neq 0.$$

Valuation theory is closely related to filtered-graded structures. If a ring A has a valuation v: $A \to \Gamma \cup \{\infty\}$, then v determines an exhaustive Γ-filtration $F^vA = \{F^v_\gamma A\}_{\gamma\in\Gamma}$ for A by putting

$$F^v_\gamma A = \{a \in A \mid v(a) \geq -\gamma\},\ \gamma \in \Gamma,$$

i.e., F^vA satisfies (F1) – (F4) as described in the beginning of Section 3. For the convenience of later use we also note three more properties of F^vA as follows.

6.1. Proposition The exhaustive Γ-filtration F^vA of A defined above has the following properties:
(i) For $0 \neq a \in A$, $v(a) = -\gamma$ if and only if $a \in F^v_\gamma A - F^v_{<\gamma}A$, where $F^v_{<\gamma}A = \cup_{\gamma'<\gamma}F^v_{\gamma'}A$. Hence F^vA is separated. In particular, $1 \in F_0A - F_{<0}A$.
(ii) If $a \in F^v_\gamma A - F^v_{<\gamma}A$ and a is invertible, then $a^{-1} \in F^v_{-\gamma}A - F_{<-\gamma}A$.
(iii) Let $G(A) = \oplus_{\gamma\in\Gamma}G(A)_\gamma$ be the associated Γ-graded ring determined by FA, where $G(A)_\gamma = F_\gamma A/F_{<\gamma}A$ for every $\gamma \in \Gamma$. Then $G(A)$ is a domain and thereby A is a domain. □

Conversely, let A be a Γ-filtered ring with an exhaustive Γ-filtration $FA = \{F_\gamma A\}_{\gamma\in\Gamma}$ in the sense of Section 3. If FA is separated, i.e., $0 \neq a \in A$ implies that there is a $\gamma \in \Gamma$ such that $a \in F_\gamma A - F_{<\gamma}A$, where $F_{<\gamma}A = \cup_{\gamma'<\gamma}F_{\gamma'}A$, in particular, we insist that $1 \in F_0A - F_{<0}A$. then the *degree function* on A can be defined by setting

$$\begin{array}{rccl} d: & A & \longrightarrow & \Gamma \cup \{\infty\} \\ & a & \mapsto & \begin{cases} \gamma, & \text{if } a \in F_\gamma A - F_{<\gamma}A, \\ -\infty, & \text{if } a = 0 \end{cases} \end{array}$$

Furthermore, consider the associated Γ-graded ring $G(A) = \oplus_{\gamma\in\Gamma}G(A)_\gamma$ of A determined by FA, where $G(A)_\gamma = F_\gamma A/F_{<\gamma}A$ for every $\gamma \in \Gamma$. If $G(A)$ is a domain (hence A is a domain), then the function defined by setting

$$\begin{array}{rccl} v: & A & \longrightarrow & \Gamma \cup \{\infty\} \\ & a & \mapsto & -d(a) \end{array}$$

is a valuation function on A.

In conclusion, the next theorem summarizes the principle of valuation extensions via filtered-graded structures.

6.2. Theorem Let A be a Γ-filtered ring with an exhaustive and separated filtration $FA = \{F_\gamma A\}_{\gamma\in\Gamma}$ such that $1 \in F_0A - F_{<0}A$. Then the following statements hold.
(i) The degree function $d(x)$ on A defines a valuation function $v(x) = -d(x)$ on A if and only if $G(A)$ is a domain.
(ii) Suppose that $G(A)$ is a domain (hence A is a domain) and the (skew-)field Δ of fractions of A exists, then v can be uniquely extended to a valuation function on Δ, or equivalently, the Γ-filtration FA can be extended to an exhaustive and separated Γ-filtration $F\Delta$ such that $F_\gamma A = A \cap F_\gamma\Delta$ for every $\gamma \in \Gamma$, and moreover, $G(\Delta)$ is a Γ-graded (skew-)field in the sense that every nonzero homogenous element of $G(\Delta)$ is invertible. □

Now, let K be a field v: $K \to \Gamma \cup \{\infty\}$ be a valuation of K. Then, by the foregoing discussion, v determines an exhaustive and separated Γ-filtration $F^vK = \{F^v_\gamma K\}_{\gamma\in\Gamma}$ with $F^v_\gamma K = \{\lambda \in K \mid v(\lambda) \geq -\gamma\}$ for K, which has all properties as described in Proposition 6.1. Moreover, since K is a field, it follows from Proposition 6.1 that F^vK is a strong Γ-filtration in the sense of [LVO1], i.e., $F^v_{\gamma_1}K \cdot F^v_{\gamma_2}K = F^v_{\gamma_1+\gamma_2}K$, in particular, $F^v_\gamma K \cdot F^v_{-\gamma}K = F^v_0 K$, for all $\gamma_1, \gamma_2, \gamma \in \Gamma$, and this leads to the fact that the associated graded ring $G(K)$ of K is a strongly Γ-graded ring in the sense of [NVO], i.e., $G(K)_{\gamma_1} \cdot G(K)_{\gamma_2} = G(K)_{\gamma_1+\gamma_2}$, in particular, $G(K)_\gamma \cdot G(K)_{-\gamma} = G(K)_0$, for all $\gamma_1, \gamma_2, \gamma \in \Gamma$. Noting that the valuation ring of K associated to v is local ring $\mathcal{O}_v = \{\lambda \in K \mid v(\lambda) \geq 0\}$ with the unique maximal ideal $\mathbf{m}_v = \{\lambda \in K \mid v(\lambda) > 0\}$, by the definition of F^vK we have $F^v_0K = \mathcal{O}_v$ and $F^v_{<0}K = \mathbf{m}_v$. Thus, $G(K)_0 = k = \mathcal{O}_v/\mathbf{m}_v$ is the residue field of $\mathcal{O}_v$ and $G(K)$ is indeed a commutative Γ-graded field in the sense that every nonzero homogeneous element of $G(K)$ is invertible.

Next, consider the free K-algebra $K\langle X\rangle = K\langle X_1, ..., X_n\rangle$ on $X_1, ..., X_n$. Let $\mathcal{I}$ be an ideal of $K\langle X\rangle$, $A = K\langle X\rangle/\mathcal{I}$, and $\Lambda = \mathcal{O}_v\langle X\rangle + \mathcal{I}/\mathcal{I}$, where $\mathcal{O}_v\langle X\rangle = \mathcal{O}_v\langle X_1, ..., X_n\rangle$ is the free $\mathcal{O}_v$-algebra on $X_1, ..., X_n$. If $\mathcal{G}$ is a Gröbner basis for $\mathcal{I}$ in $K\langle X\rangle$ with respect to a monomial ordering $\prec$ on the standard K-basis $\mathcal{B}_K$ of $K\langle X\rangle$, as before we write $N(\mathcal{G})$ for the set of normal monomials in $\mathcal{B}_K$ (mod $\mathcal{G}$), and we write $\overline{N(\mathcal{G})}$ for the canonical image of $N(\mathcal{G})$ in A, which is known a K-basis for A (Theorem 2.4).

6.3. Theorem With all notations as we fixed so far, suppose that the ideal $\mathcal{I}$ is generated in $K\langle X\rangle$ by a monic Gröbner basis $\mathcal{G} \subset \mathcal{O}_v\langle X\rangle$ with respect to a monomial ordering $\prec$ on $\mathcal{B}_K$, and without loss of generality we assume that $\mathbf{LM}(g) \neq 1$ for every $g \in \mathcal{G}$. Then the following statements hold.
(i) The K-algebra $A = K\langle X\rangle/\mathcal{I} = K\langle X\rangle/\langle\mathcal{G}\rangle$ can be endowed with the exhaustive Γ-filtration $F^vA = \{F^v_\gamma A\}_{\gamma\in\Gamma}$ by putting

$$F^v_\gamma A = \left\{ a = \sum_i \lambda_i \bar{w}_i \;\middle|\; \lambda_i \in F^v_\gamma K,\ \bar{w}_i \in \overline{N(\mathcal{G})} \right\},\ \gamma \in \Gamma,$$

such that $F_\gamma K = K \cap F_\gamma A$, $\gamma \in \Gamma$, that is, F^vK extends naturally to F^vA.
(ii) The Γ-filtration F^vA obtained in (i) is separated, i.e., if $0 \neq a \in A$, then there is a $\gamma \in \Gamma$ such that $a \in F^v_\gamma A - F^v_{<\gamma}A$, in particular $1 \in F^v_0A - F^v_{<0}A$, where $F^v_{<\gamma}A = \cup_{\gamma'<\gamma}F^v_{\gamma'}A$.
(iii) The Γ-filtration F^vA obtained in (i) has

$$F^v_0A = \mathcal{O}_v\langle X\rangle/\langle\mathcal{G}\rangle \cong \mathcal{O}_v\langle X\rangle + \mathcal{I}/\mathcal{I} = \Lambda,$$
$$F^v_{<0}A = \mathbf{m}_vF^v_0A = \mathbf{m}_v\Lambda.$$

Hence the associated Γ-graded K-algebra $G(A) = \oplus_{\gamma\in\Gamma}G(A)_\gamma$ of A determined by F^vA has $G(A)_0 = F^v_0A/F^v_{<0}A = \Lambda/\mathbf{m}_v\Lambda$

(iv) The Γ-filtration F^vA obtained in (i) is a strong filtration, i.e., $F^v_{\gamma_1}A\cdot F^v_{\gamma_2}A=F^v_{\gamma_1+\gamma_2}A$ for all $\gamma_1,\gamma_2\in\Gamma$, and the associated Γ-graded K-algebra $G(A)=\oplus_{\gamma\in\Gamma}G(A)_\gamma$ with $G(A)_\gamma=F^v_\gamma A/F^v_{<\gamma}A$ is strongly Γ-graded, i.e., $G(A)_{\gamma_1}\cdot G(A)_{\gamma_2}=G(A)_{\gamma_1+\gamma_2}$ for all $\gamma_1,\gamma_2\in\Gamma$.
(v) $\mathcal{G}$ is a Gröbner basis for the ideal $\mathcal{I}\cap D\langle X\rangle$ of $D\langle X\rangle$ with respect to the same monomial ordering $\prec$ on the standard $\mathcal{O}_v$-basis $\mathcal{B}_{\mathcal{O}_v}$ of $\mathcal{O}_v\langle X\rangle$. It follows that $D\langle X\rangle\cap\mathcal{I}=\langle\mathcal{G}\rangle$ holds in $D\langle X\rangle$, and thereby the $\mathcal{O}_v$-algebra $\Lambda=\mathcal{O}_v\langle X\rangle+\mathcal{I}/\mathcal{I}$ defines a good reduction for the K-algebra $A=K\langle X\rangle/\mathcal{I}$ in the sense of Definition 5.1.
(vi) If $\mathcal{G}\not\subset\mathbf{m}_v\mathcal{O}_v\langle X\rangle$, then $\Lambda/\mathbf{m}_v\Lambda\cong k\langle X\rangle/\langle\overline{\mathcal{G}}\rangle$ as k-algebras, where $k=\mathcal{O}_v/\mathbf{m}_v$ is the residue field of $\mathcal{O}_v$, $k\langle X\rangle=k\langle X_1,...,X_n\rangle$ is the free k-algebra on $X_1,...,X_n$, and $\overline{\mathcal{G}}$ is the canonical image of $\mathcal{G}$ in $\mathcal{O}_v\langle X\rangle/\mathbf{m}_v\mathcal{O}_v\langle X\rangle$
(vii) If $\mathcal{G}\not\subset\mathbf{m}_v\mathcal{O}_v\langle X\rangle$ and $k\langle X\rangle/\langle\overline{\mathcal{G}}\rangle$ is a domain, then $G(A)$ is a domain and thereby A is a domain. It follows that F^vA determines a valuation function $A\rightarrow\Gamma\cup\{\infty\}$, and thereby v extends naturally to a valuation function on the (skew-)field Δ of fractions of A provided Δ exists.

Proof Note that the Γ-filtration F^vK of K determined by the valuation v: $K\rightarrow\Gamma\cup\{\infty\}$ is exhaustive and separated. Moreover, $F^v_0K=\mathcal{O}_v$ and $F^v_{<0}K=\mathbf{m}_v$.

(i) and (ii) follow from Theorem 3.1, Theorem 4.1, and Proposition 6.1.

(iv) follows from (ii), (iii), and Proposition 6.1.

(iii), (v), and (vi) follow from Theorem 5.2.

(vii) By the foregoing (iii), (vi) and (iv), $G(A)$ is now a strongly Γ-graded algebra with $G(A)_0=\Lambda/\mathbf{m}_v\Lambda\cong k\langle X\rangle/\langle\overline{\mathcal{G}}\rangle$. If $k\langle X\rangle/\langle\overline{\mathcal{G}}\rangle$ is a domain, then $G(A)$ is a domain and thereby A is a domain. It follows from Theorem 6.2. that the last assertion holds. □

Let $K[x_1,...,x_n]$ be the commutative polynomial K-algebra in n variables over a field K. Noticing $K[x_1,...,x_n]\cong K\langle X_1,...,X_n\rangle/\langle\mathcal{G}\rangle$ with $\mathcal{G}=\{X_jX_i-X_iX_j\mid 1\le i<j\le n\}$ a Gröbner basis for the ideal $\langle\mathcal{G}\rangle$, Theorem 6.3 has an immediate application to $K[x_1,...,x_n]$.

6.4. Corollary Let K be a field. Then every valuation v on K extends naturally to a valuation function on $K[x_1,...,x_n]$ and further to a valuation function on the field of rational functions $K(x_1,...,x_n)$.

□

More generally, as it was pointed out in ([Li3], Section 1, Remark(iv)), Proposition 2.7 of previous Section 2 is valid for getting monic Grobner bases in a commutative polynomial ring $R[x_1,...,x_n]$ over an arbitrary commutative ring R when overlap elements are replaced by S-polynomials. It follows that the results of Sections 3 – 5 and Theorem 6.3 are also valid for commutative algebras over a field K after replacing $K\langle X\rangle$ by $K[x_1,...,x_n]$. For instance, let $\mathcal{O}_v$ be a valuation ring of K associated to a valuation v of K, and let $A = K[x_1,...,x_n]/I$ be the coordinate ring of an affine variety $V(I) \subset K^n$. If the ideal I is generated by a subset $\mathcal{G} \subset \mathcal{O}_v[x_1,...,x_n]$ which is a Gröbner basis of I in $K[x_1,...,x_n]$ with respect to a monomial ordering on $K[x_1,...,x_n]$, then Theorem 6.3 holds for A after replacing $K\langle X\rangle$ by $K[x_1,...,x_n]$.

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