A posteriori error estimations for mixed finite-element approximations to the Navier-Stokes equations

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November 15, 2010

Abstract

A posteriori estimates for mixed finite element discretizations of the Navier-Stokes equations are derived. We show that the task of estimating the error in the evolutionary Navier-Stokes equations can be reduced to the estimation of the error in a steady Stokes problem. As a consequence, any available procedure to estimate the error in a Stokes problem can be used to estimate the error in the nonlinear evolutionary problem. A practical procedure to estimate the error based on the so-called postprocessed approximation is also considered. Both the semidiscrete (in space) and the fully discrete cases are analyzed. Some numerical experiments are provided.

1 Introduction

We consider the incompressible Navier–Stokes equations

$$\begin{aligned} u_t - \Delta u + (u\cdot\nabla)u + \nabla p &= f, \\ \mathrm{div}(u) &= 0, \end{aligned} \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary subject to homogeneous Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. In (1), u is the velocity field, p the pressure, and f a given force field. For simplicity in the exposition we assume, as in [8], [27], [28], [29], [33], that the fluid density and

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viscosity have been normalized by an adequate change of scale in space and time.

Let u_h and p_h be the semi-discrete (in space) mixed finite element (MFE) approximations to the velocity u and pressure p, respectively, solution of (1) corresponding to a given initial condition

$$u(\cdot,0)=u_0. \tag{2}$$

We study the a posteriori error estimation of these approximations in the L^2 and H^1 norm for the velocity and in the $L^2/\mathbb{R}$ norm for the pressure. To do this for a given time $t^*>0$, we consider the solution $(\tilde{u},\,\tilde{p})$ of the Stokes problem

$$\left.\begin{aligned} -\Delta\tilde{u}+\nabla\tilde{p}=f-\tfrac{d}{dt}u_h(t^*)-(u_h(t^*)\cdot\nabla)u_h(t^*)\\ \operatorname{div}(\tilde{u})=0\end{aligned}\right\}\quad \text{in } \Omega, \qquad \tilde{u}=0, \quad \text{on } \partial\Omega. \tag{3}$$

We prove that $\tilde{u}$ and $\tilde{p}$ are approximations to u and p whose errors decay by a factor of $h\,|\log(h)|$ faster than those of u_h and p_h (h being the mesh size). As a consequence, the quantities $\tilde{u}-u_h$ and $\tilde{p}-p_h$, are asymptotically exact indicators of the errors $u-u_h$ and $p-p_h$ in the Navier-Stokes problem (1)–(2).

Furthermore, the key observation in the present paper is that (u_h,p_h) is also the MFE approximation to the solution $(\tilde{u},\tilde{p})$ of the Stokes problem (3). Consequently, any available procedure to a posteriori estimate the errors in a Stokes problem can be used to estimate the errors $\tilde{u}-u_h$ and $\tilde{p}-p_h$ which, as mentioned above, coincide asymptotically with the errors $u-u_h$ and $p-p_h$ in the evolutionary NS equations. Many references address the question of estimating the error in a Stokes problem, see for example [2], [6], [7], [32], [35], [39], [40] and the references therein. In this paper we prove that any efficient or asymptotically exact estimator of the error in the MFE approximation (u_h,p_h) to the solution of the *steady* Stokes problem (3) is also an efficient or asymptotically exact estimator, respectively, of the error in the MFE approximation (u_h,p_h) to the solution of the *evolutionary* Navier-Stokes equations (1)–(2).

For the analysis in the present paper we do not assume to have more than second-order spatial derivatives bounded in $L^2(\Omega)^d$ up to initial time $t=0$, since demanding further regularity requires the data to satisfy nonlocal compatibility conditions unlikely to be fulfilled in practical situations [27], [28]. The analysis of the errors $u-\tilde{u}$ and $p-\tilde{p}$ follows closely [16] where MFE approximations to the Stokes problem (3) (the so-called postprocessed approximations) are considered with the aim of getting improved approximations to the solution of (1)–(2) at any fixed time $t^*>0$. In this paper we will also refer to $(\tilde{u},\,\tilde{p})$ as postprocessed approximations although they are of course not computable in practice and they are only considered for the analysis of a posteriori error estimators. The postprocessed approximations to the Navier-Stokes equations were first developed for spectral methods in [23], [24], [18], [36] and also developed for MFE methods for the Navier-Stokes equations in [4], [5], [16].

For the sake of completeness, in the present paper we also analyze the use of the computable postprocessed approximations of [16] for a posteriori error estimation. The use of this kind of postprocessing technique to get a posteriori error estimations has been studied in [19], [20] and [15] for nonlinear parabolic equations excluding the Navier-Stokes equations. We refer also to [33] where the so-called Stokes reconstruction is used to a posteriori estimate the errors of the semi-discrete in space approximations to a linear time-dependent Stokes problem. We remark that the Stokes reconstruction of [33] is exactly the post-processing approximation $(\tilde u, \tilde p)$ in the particular case of a linear model.

In the second part of the paper we consider a posteriori error estimations for the fully discrete MFE approximations $U_h^n \approx u_h(t_n)$ and $P_h^n \approx p_h(t_n)$, $(t_n = t_{n-1}\Delta t_{n-1}$ for $n = 1, 2, \ldots, N)$ obtained by integrating in time with either the backward Euler method or the two-step backward differentiation formula (BDF). For this purpose, we define a Stokes problem similar to (3) but with the right-hand-side depending now on the fully discrete MFE approximation U_h^n (problem (70)–(71) in Section 4 below). We will call time-discrete postprocessed approximation to the solution $(\widetilde U^n, \widetilde P^n)$ of this new Stokes problem. As before, $(\widetilde U^n, \widetilde P^n)$ is not computable in practice and it is only considered for the analysis of a posteriori error estimation.

Observe that in the fully discrete case (which is the case in actual computations) the task of estimating the the error $u(t_n) - U_h^n$ of the MFE approximation becomes more difficult due to the presence of time discretization errors $e_h^n = u_h(t_n) - U_h^n$, which are added to the spatial discretization errors $u(t_n) - u_h(t_n)$. However we show in Section 4 that if temporal and spatial errors are not very different in size, the quantity $\widetilde U^n - U_h^n$ correctly esimates the spatial error because the leading terms of the temporal errors in $\widetilde U^n$ and U_h^n get canceled out when subtracting $\widetilde U^n - U_h^n$, leaving only the spatial component of the error. This is a very convenient property that allows to use independent procedures for the tasks of estimating the errors of the spatial and temporal discretizations. We remark that the temporal error can be routinely controlled by resorting to well-known ordinary differential equations techniques. Analogous results were obtained in [15] for fully discrete finite element approximations to evolutionary convection-reaction-diffusion equations using the backward Euler method.

As in the semidiscrete case, a key point in our results is again the fact that the fully discrete MFE approximation (U_h^n, P_h^n) to the Navier-Stokes problem (1)–(2) is also the MFE approximation to the solution $(\widetilde U^n, \widetilde P^n)$ of the Stokes problem (70)–(71). As a consequence, we can use again any available error estimator for the Stokes problem to estimate the spatial error of the fully discrete MFE approximations (U_h^n, P_h^n) to the Navier-Stokes problem (1)–(2).

Computable mixed finite element approximations to $(\widetilde U^n, \widetilde P^n)$, the so-called fully discrete postprocessed approximations, were studied and analyzed in [17] where we proved that the fully discrete postprocessed approximations maintain the increased spatial accuracy of the semi-discrete approximations. The analysis in the second part of the present paper borrows in part from [17]. Also, we propose a computable error estimator based on the fully discrete postprocessed approximation of [17] and show that it also has the excellent property of separating spatial and temporal errors.

The rest of the paper is as follows. In Section 2 we introduce some preliminaries and notation. In Section 3 we study the a posteriori error estimation of semi-discrete in space MFE approximations. In Section 4 we study a posteriori error estimates for fully discrete approximations. Finally, some numerical experiments are shown in Section 5.

2 Preliminaries and notations

We will assume that Ω is a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, of class $\mathcal{C}^m$, for $m \geq 2$. When dealing with linear elements ($r = 2$ below) Ω may also be a convex polygonal or polyhedral domain. We consider the Hilbert spaces

$$H = \left\{ u \in L^2(\emptyset)^d \mid \operatorname{div}(u) = 0,\, u \cdot n_{|\partial\Omega} = 0 \right\},$$
$$V = \left\{ u \in H_0^1(\emptyset)^d \mid \operatorname{div}(u) = 0 \right\},$$

endowed with the inner product of $L^2(\emptyset)^d$ and $H_0^1(\emptyset)^d$, respectively. For $l \geq 0$ integer and $1 \leq q \leq \infty$, we consider the standard spaces, $W^{l,q}(\Omega)^d$, of functions with derivatives up to order l in $L^q(\Omega)$, and $H^l(\Omega)^d = W^{l,2}(\Omega)^d$. We will denote by $\|\cdot\|_l$ the norm in $H^l(\Omega)^d$, and $\|\cdot\|_{-l}$ will represent the norm of its dual space. We consider also the quotient spaces $H^l(\Omega)/\mathbb{R}$ with norm $\|p\|_{H^l/\mathbb{R}} = \inf\{\|p + c\|_l \mid c \in \mathbb{R}\}$.

We recall the following Sobolev's imbeddings [1]: For $q \in [1, \infty)$, there exists a constant $C = C(\Omega, q)$ such that

$$\|v\|_{L^{q'}} \leq C\|v\|_{W^{s,q}}, \qquad \frac{1}{q'} \geq \frac{1}{q} - \frac{s}{d} > 0, \quad q < \infty, \quad v \in W^{s,q}(\Omega)^d. \tag{4}$$

For $q' = \infty$, (4) holds with $\frac{1}{q} < \frac{s}{d}$.

The following inf-sup condition is satisfied (see [25]), there exists a constant $\beta > 0$ such that

$$\inf_{q \in L^2(\Omega)/\mathbb{R}} \sup_{v \in H_0^1(\Omega)^d} \frac{(q, \nabla \cdot v)}{\|v\|_1 \|q\|_{L^2/\mathbb{R}}} \geq \beta, \tag{5}$$

where, here and in the sequel, $(\cdot, \cdot)$ denotes the standard inner product in $L^2(\Omega)$ or in $L^2(\Omega)^d$.

Let $\Pi : L^2(\emptyset)^d \longrightarrow H$ be the $L^2(\emptyset)^d$ projector onto H. We denote by A the Stokes operator on $\emptyset$:

$$A : \mathcal{D}(A) \subset H \longrightarrow H, \quad A = -\Pi\Delta, \quad \mathcal{D}(A) = H^2(\emptyset)^d \cap V.$$

Applying Leray's projector Π to (1), the equations can be written in the form

$$u_t + Au + B(u, u) = \Pi f \quad \text{ in } \emptyset,$$

where $B(u, v) = \Pi(u \cdot \nabla)v$ for u, v in $H_0^1(\Omega)^d$.

We shall use the trilinear form $b(\cdot,\cdot,\cdot)$ defined by

$$b(u,v,w) = (F(u,v),w) \quad \forall u,v,w \in H_0^1(\Omega)^d,$$

where

$$F(u,v) = (u\cdot\nabla)v + \frac{1}{2}(\nabla\cdot u)v \quad \forall u,v \in H_0^1(\Omega)^d.$$

It is straightforward to verify that b enjoys skew-symmetry:

$$b(u,v,w) = -b(u,w,v) \quad \forall u,v,w \in H_0^1(\Omega)^d. \tag{6}$$

Let us observe that $B(u,v) = \Pi F(u,v)$ for $u \in V$, $v \in H_0^1(\Omega)^d$.

Let us consider for $\alpha \in \mathbb{R}$ and $t > 0$ the operators A^α and e^{-tA}, which are defined by means of the spectral properties of A (see, e.g., [13, p. 33], [21]). Notice that A is a positive self-adjoint operator with compact resolvent in H. An easy calculation shows that

$$\|A^\alpha e^{-tA}\|_0 \le (\alpha e^{-1})^\alpha t^{-\alpha}, \qquad \alpha \ge 0,\ t > 0, \tag{7}$$

where, here and in what follows, $\|\cdot\|_0$ when applied to an operator denotes the associated operator norm.

We shall assume that the solution u of (1)-(2) satisfies

$$\|u(t)\|_1 \le M_1, \quad \|u(t)\|_2 \le M_2, \quad 0 \le t \le T, \tag{8}$$

for some constants M_1 and M_2. We shall also assume that there exists a constant $\tilde{M}_2$ such that

$$\|f\|_1 + \|f_t\|_1 + \|f_{tt}\|_1 \le \tilde{M}_2, \quad 0 \le t \le T. \tag{9}$$

Finally, we shall assume that for some $k \ge 2$

$$\sup_{0\le t\le T} \big\|\partial_t^{\lfloor k/2\rfloor} f\big\|_{k-1-2\lfloor k/2\rfloor} + \sum_{j=0}^{\lfloor (k-2)/2\rfloor} \sup_{0\le t\le T} \big\|\partial_t^j f\big\|_{k-2j-2} < +\infty,$$

so that, according to Theorems 2.4 and 2.5 in [27], there exist positive constants M_k and K_k such that the following bounds hold:

$$\|u(t)\|_k + \|u_t(t)\|_{k-2} + \|p(t)\|_{H^{k-1}/\mathbb{R}} \le M_k \tau(t)^{1-k/2}, \tag{10}$$

$$\int_0^t \sigma_{k-3}(s)\big(\|u(s)\|_k^2 + \|u_s(s)\|_{k-2}^2 + \|p(s)\|_{H^{k-1}/\mathbb{R}}^2 + \|p_s(s)\|_{H^{k-3}/\mathbb{R}}^2\big)\,\mathrm{d}s \le K_k^2, \tag{11}$$

where $\tau(t) = \min(t,1)$ and $\sigma_n = e^{-\alpha(t-s)}\tau^n(s)$ for some $\alpha > 0$. Observe that for $t \le T < \infty$, we can take $\tau(t) = t$ and $\sigma_n(s) = s^n$. For simplicity, we will take these values of τ and σ_n.

Let $\mathcal{T}_h = (\tau_i^h, \phi_i^h)_{i\in I_h}$, $h > 0$ be a family of partitions of suitable domains Ω_h, where h is the maximum diameter of the elements $\tau_i^h \in \mathcal{T}_h$, and ϕ_i^h are the mappings of the reference simplex τ_0 onto τ_i^h.

Let $r \geq 2$, we consider the finite-element spaces

$$S_{h,r} = \left\{ \chi_h \in \mathcal{C}\left(\overline{\emptyset}_h\right) \mid \chi_h|_{\tau_i^h} \circ \phi_i^h \in P^{r-1}(\tau_0) \right\} \subset H^1(\emptyset_h),\ S_{h,r}^0 = S_{h,r} \cap H_0^1(\emptyset_h),$$

where $P^{r-1}(\tau_0)$ denotes the space of polynomials of degree at most $r-1$ on τ_0. As it is customary in the analysis of finite-element methods for the Navier-Stokes equations (see e. g., [8], [27], [28], [29], [30]) we restrict ourselves to quasiuniform and regular meshes $\mathcal{T}_h$, so that as a consequence of [12, Theorem 3.2.6], the following inverse inequality holds for each $v_h \in (S_{h,r}^0)^d$

$$\|v_h\|_{W^{m,q}(\Omega_h)^d} \leq Ch^{l-m-d\left(\frac{1}{q'}-\frac{1}{q}\right)} \|v_h\|_{W^{l,q'}(\Omega_h)^d}, \tag{12}$$

where $0 \leq l \leq m \leq 1$, $1 \leq q' \leq q \leq \infty$.

We shall denote by $(X_{h,r}, Q_{h,r-1})$ the so-called Hood–Taylor element [9, 31], when $r \geq 3$, where

$$X_{h,r} = \left(S_{h,r}^0\right)^d, \quad Q_{h,r-1} = S_{h,r-1} \cap L^2(\emptyset_h)/\mathbb{R}, \quad r \geq 3,$$

and the so-called mini-element [10] when $r = 2$, where $Q_{h,1} = S_{h,2} \cap L^2(\emptyset_h)/\mathbb{R}$, and $X_{h,2} = (S_{h,2}^0)^d \oplus \mathbb{B}_h$. Here, $\mathbb{B}_h$ is spanned by the bubble functions b_τ, $\tau \in \mathcal{T}_h$, defined by $b_\tau(x) = (d+1)^{d+1}\lambda_1(x) \cdots \lambda_{d+1}(x)$, if $x \in \tau$ and 0 elsewhere, where $\lambda_1(x), \ldots, \lambda_{d+1}(x)$ denote the barycentric coordinates of x. For these elements a uniform inf-sup condition is satisfied (see [9]), that is, there exists a constant $\beta > 0$ independent of the mesh grid size h such that

$$\inf_{q_h \in Q_{h,r-1}} \sup_{v_h \in X_{h,r}} \frac{(q_h, \nabla \cdot v_h)}{\|v_h\|_1 \|q_h\|_{L^2/\mathbb{R}}} \geq \beta. \tag{13}$$

We remark that our analysis can also be applied to other pairs of LBB-stable mixed finite elements (see [16, Remark 2.1]).

The approximate velocity belongs to the discrete divergence-free space

$$V_{h,r} = X_{h,r} \cap \left\{ \chi_h \in H_0^1(\emptyset_h)^d \mid (q_h, \nabla \cdot \chi_h) = 0 \quad \forall q_h \in Q_{h,r-1} \right\},$$

which is not a subspace of V. We shall frequently write V_h instead of $V_{h,r}$ whenever the value of r plays no particular role.

Let $\Pi_h : L^2(\emptyset)^d \longrightarrow V_{h,r}$ be the discrete Leray's projection defined by

$$(\Pi_h u, \chi_h) = (u, \chi_h) \quad \forall \chi_h \in V_{h,r}.$$

We will use the following well-known bounds

$$\|(I - \Pi_h)u\|_j \leq Ch^{l-j}\|u\|_l, \quad 1 \leq l \leq 2, \quad j = 0, 1. \tag{14}$$

We will denote by $A_h : V_h \to V_h$ the discrete Stokes operator defined by

$$(\nabla v_h, \nabla \phi_h) = (A_h v_h, \phi_h) = \left(A_h^{1/2} v_h, A_h^{1/2} \phi_h\right) \quad \forall v_h, \phi_h \in V_h.$$

Let $(u, p) \in (H^2(\Omega)^d \cap V) \times (H^1(\Omega)/\mathbb{R})$ be the solution of a Stokes problem with right-hand side g, we will denote by $s_h = S_h(u) \in V_h$ the so-called Stokes projection (see [28]) defined as the velocity component of solution of the following Stokes problem: find $(s_h, q_h) \in (X_{h,r}, Q_{h,r-1})$ such that

$$(\nabla s_h, \nabla \phi_h) + (\nabla q_h, \phi_h) = (g, \phi_h) \qquad \forall \phi_h \in X_{h,r}, \tag{15}$$

$$(\nabla \cdot s_h, \psi_h) = 0 \qquad \forall \psi_h \in Q_{h,r-1}. \tag{16}$$

The following bound holds for $2 \le l \le r$:

$$\|u - s_h\|_0 + h\|u - s_h\|_1 \le C h^l \big(\|u\|_l + \|p\|_{H^{l-1}/\mathbb{R}}\big). \tag{17}$$

The proof of (17) for $\Omega = \Omega_h$ can be found in [28]. For the general case, Ω_h must be such that the value of $\delta(h) = \max_{x\in\partial\Omega_h} \operatorname{dist}(x, \partial\Omega)$ satisfies $\delta(h) = O(h^{2(r-1)})$. This can be achieved if, for example, $\partial\Omega$ is piecewise of class $\mathcal{C}^{2(r-1)}$, and superparametric approximation at the boundary is used [3]. Under the same conditions, the bound for the pressure is [25]

$$\|p - q_h\|_{L^2/\mathbb{R}} \le C_\beta h^{l-1} \big(\|u\|_l + \|p\|_{H^{l-1}/\mathbb{R}}\big), \tag{18}$$

where the constant C_β depends on the constant β in the inf-sup condition (13). We will assume that the domain Ω is of class $\mathcal{C}^m$, with $m \ge r$ so that standard bounds for the Stokes problem [3], [22] imply that

$$\left\|A^{-1}\Pi g\right\|_{2+j} \le \|g\|_j, \qquad -1 \le j \le m-2. \tag{19}$$

For a domain Ω of class $\mathcal{C}^2$ we also have the bound (see [11])

$$\|p\|_{H^1/\mathbb{R}} \le c\|g\|_0. \tag{20}$$

In what follows we will apply the above estimates to the particular case in which (u, p) is the solution of the Navier–Stokes problem (1)–(2). In that case $s_h = S_h(u)$ is the discrete velocity in problem (15)–(16) with $g = f - u_t - (u\cdot\nabla u)$. Note that the temporal variable t appears here merely as a parameter, and then, taking the time derivative, the error bound (17) can also be applied to the time derivative of s_h changing u, p by u_t, p_t.

Since we are assuming that Ω is of class $\mathcal{C}^m$ and $m \ge 2$, from (17) and standard bounds for the Stokes problem [3, 22], we deduce that

$$\left\|\left(A^{-1}\Pi - A_h^{-1}\Pi_h\right) f\right\|_j \le C h^{2-j}\|f\|_0 \quad \forall f \in L^2(\Omega)^d, \quad j = 0, 1. \tag{21}$$

We consider the semi-discrete finite-element approximation (u_h, p_h) to (u, p), solution of (1)–(2). That is, given $u_h(0) = \Pi_h u_0$, we compute $u_h(t) \in X_{h,r}$ and $p_h(t) \in Q_{h,r-1}$, $t \in (0, T]$, satisfying

$$(\dot u_h, \phi_h) + (\nabla u_h, \nabla \phi_h) + b(u_h, u_h, \phi_h) + (\nabla p_h, \phi_h) = (f, \phi_h) \quad \forall\, \phi_h \in X_{h,r}, \tag{22}$$

$$(\nabla \cdot u_h, \psi_h) = 0 \qquad \forall\, \psi_h \in Q_{h,r-1}. \tag{23}$$

For $2 \le r \le 5$, provided that (17)–(18) hold for $l \le r$, and (10)–(11) hold for $k = r$, then we have

$$\|u(t) - u_h(t)\|_0 + h\|u(t) - u_h(t)\|_1 \le C\frac{h^r}{t^{(r-2)/2}}, \quad 0 \le t \le T, \tag{24}$$

(see, e.g., [16, 27, 28]), and also,

$$\|p(t) - p_h(t)\|_{L^2/\mathbb{R}} \le C\frac{h^{r-1}}{t^{(r'-2)/2}}, \quad 0 \le t \le T, \tag{25}$$

where $r' = r$ if $r \le 4$ and $r' = r + 1$ if $r = 5$.

see [29, Proposition 3.2].

3 A posteriori error estimations. Semidiscrete case

Let us consider the MFE approximation (u_h, p_h) at any time $t^* \in (0, T]$ to $(u(t^*), p(t^*))$ obtained by solving (22)–(23). We consider the postprocessed approximation $(\tilde u(t^*), \tilde p(t^*))$ in $(V, L^2(\Omega)/\mathbb{R})$ which is the solution of the following Stokes problem written in weak form

$$\begin{aligned} \big(\nabla \tilde u(t^*), \nabla\phi\big) + \big(\nabla \tilde p(t^*), \phi\big) &= (f, \phi) - b(u_h(t^*), u_h(t^*), \phi) - (\dot u_h(t^*), \phi), & (26)\\ \big(\nabla\cdot \tilde u(t^*), \psi\big) &= 0, & (27)\end{aligned}$$

for all $\phi \in H_0^1(\Omega)^d$ and $\psi \in L^2(\Omega)/\mathbb{R}$. We remark that the MFE approximation $(u_h(t^*), p_h(t^*))$ to $(u(t^*), p(t^*))$ is also the MFE approximation to the solution $(\tilde u(t^*), \tilde p(t^*))$ of the Stokes problem (26)–(27). In Theorems 1 and 2 below we prove that the postprocessed approximation $(\tilde u(t^*), \tilde p(t^*))$ is an improved approximation to the solution (u, p) of the evolutionary Navier-Stokes equations (1)–(2) at time t^*. Although, as it is obvious, $(\tilde u(t^*), \tilde p(t^*))$ is not computable in practice, it is however a useful tool to provide a posteriori error estimates for the MFE approximation (u_h, p_h) at any desired time $t^* > 0$. In Theorem 1 we obtain the error bounds for the velocity and in Theorem 2 the bounds for the pressure. The improvement is achieved in the $H^1(\Omega)^d$ norm when using the mini-element ($r = 2$) and in both the $L^2(\Omega)^d$ and $H^1(\Omega)^d$ norms in the cases $r = 3, 4$.

In the sequel we will use that for a forcing term satisfying (9) there exists a constant $\tilde M_3 > 0$, depending only on $\tilde M_2$, $\|A_h u_h(0)\|_0$ and $\sup_{0\le t\le T}\|u_h(t)\|_1$, such that the following bound hold for $0 \le t \le T$:

$$\|A_h u_h(t)\|_0^2 \le \tilde M_3^2, \tag{28}$$

The following inequalities hold for all $v_h, w_h \in V_h$ and $\phi \in H_0^1(\Omega)^d$, see [29, (3.7)]:

$$\begin{aligned} |b(v_h, v_h, \phi)| &\le c\|v_h\|_1^{3/2}\|A_h v_h\|_0^{1/2}\|\phi\|_0, & (29)\\ |b(v_h, w_h, \phi)| + |b(w_h, v_h, \phi)| &\le c\|v_h\|_1\|A_h w_h\|_0\|\phi\|_0. & (30)\end{aligned}$$

The proof of Theorem 1 requires some previous results which we now state and prove.

We will use the fact that $\|A_h^{1/2}w_h\|_0 = \|\nabla w_h\|_0$ for $w_h \in V_h$, from where it follows that

$$C^{-1}\|A_h^{-1/2}w_h\|_0 \le \|w_h\|_{-1} \le C\|A_h^{-1/2}w_h\|_0 \quad \forall w_h \in V_h, \tag{31}$$

where the constant C is independent of h.

Lemma 1 *Let (u,p) be the solution of (1)–(2) and fix $\alpha > 0$. Then there exists a positive constant $C = C(M_2, \alpha)$ such that for $w_h^1, w_h^2 \in V_h$ satisfying the threshold condition*

$$\|w_h^l - u\|_j \le \alpha h^{3/2-j}, \quad j = 0, 1, \quad l = 1, 2, \tag{32}$$

the following inequalities hold for $j = 0, 1$:

$$\|A_h^{-j/2}\Pi_h(F(w_h^1, w_h^1) - F(w_h^2, w_h^2))\|_0 \le C\|A_h^{(1-j)/2}(w_h^1 - w_h^2)\|_0, \tag{33}$$
$$\|A_h^{-j/2}\Pi_h(F(w_h^1, w_h^1) - F(u, u))\|_0 \le C\|w_h^1 - u\|_{1-j}. \tag{34}$$

Proof Due to the equivalence (31) and and since $\|\Pi_h f\|_0 \le \|f\|_0$ for $f \in L^2(\Omega)^d$ it is sufficient to prove

$$\|F(w_h^1, w_h^1) - F(w, w)\|_{-j} \le C\|w_h^1 - w\|_{1-j}, \quad j = 0, 1, \tag{35}$$

for $w = w_h^2$ or $w = u$. We follow the proof [5, Lemma 3.1] where a different threshold assumption is assumed. We do this for $w = w_h^2$, since the case $w = u$ is similar but yet simpler. We write

$$F(w_h^1, w_h^1) - F(w_h^2, w_h^2) = F(w_h^1, e_h) + F(e_h, w_h^2), \tag{36}$$

where $e_h = w_h^1 - w_h^2$. We first observe that

$$\begin{aligned} \|F(e_h, w_h^2)\|_0 &= \sup_{\|\phi\|_0=1} \left|(e_h \cdot \nabla w_h^2), \phi) + \frac{1}{2}((\nabla \cdot e_h)w_h^2), \phi)\right| \\ &\le C\|e_h\|_{L^{2d}}\|\nabla w_h^2\|_{L^{2d/(d-1)}} + C\|e_h\|_1\|w_h^2\|_{L^\infty} \\ &\le C\big(\|\nabla w_h^2\|_{L^{2d/(d-1)}} + \|w_h^2\|_{L^\infty}\big)\|e_h\|_1, \end{aligned}$$

where, in the last inequality, we have used that thanks to Sobolev's inequality (4) we have $\|e_h\|_{L^{2d}} \le C\|e_h\|_1$. Similarly,

$$\begin{aligned} \|F(w_h^1, e_h)\|_0 &\le C\|w_h^1\|_{L^\infty}\|e_h\|_1 + C\|\nabla w_h^1\|_{L^{2d/(d-1)}}\|e_h\|_{L^{2d}} \\ &\le C\big(\|w_h^1\|_{L^\infty} + \|\nabla w_h^1\|_{L^{2d/(d-1)}}\big)\|e_h\|_1. \end{aligned}$$

The proof of the case $j = 0$ in (35) is finished if we show that for $l = 1, 2$, both $\|w_h^l\|_{L^\infty}$ and $\|\nabla w_h^l\|_{L^{2d/(d-1)}}$ are bounded in terms of M_2 and the value α in the threshold assumption (32). To do this, we will use the inverse inequality (12) and the fact that the Stokes projection $s_h = S_h(u)$ satisfies that

$$\|s_h\|_{L^\infty} \le C_s, \qquad \|\nabla s_h\|_{L^{2d}} \le C_s$$

for some constant $C_s = C_s(M_2)$ (see for example the proof of Lemma 3.1 in [5]). We have

$$\|w_h^l\|_{L^\infty} \le \|w_h^l - s_h\|_{L^\infty} + \|s_h\|_{L^\infty} \le Ch^{-d/2}\|w_h^l - s_h\|_0 + \|s_h\|_{L^\infty},$$

where in the last inequality we have applied (12), and, similarly,

$$\begin{aligned}\|\nabla w_h^l\|_{L^{2d/(d-1)}} &\le \|\nabla(w_h^l - s_h)\|_{L^{2d/(d-1)}} + \|\nabla s_h\|_{L^{2d/(d-1)}}\\ &\le Ch^{-1/2}\|\nabla(w_h^l - s_h)\|_0 + \|\nabla s_h\|_{L^{2d}},\end{aligned}$$

where we also have used that $\|\cdot\|_{L^p} \le \|\cdot\|_{L^{p'}}$ for $p < p'$. Now the threshold assumption (32) and (17) show the boundedness of $\|w_h^l\|_{L^\infty}$ and $\|\nabla w_h^l\|_{L^{2d/(d-1)}}$.

Finally, the proof of the case $j = 1$ in (35) is, with obvious changes, that of the equivalent result in [5, Lemma 3.1]. □

In the sequel we consider the auxiliary function $v_h : [0, T] \to V_h$ solution of

$$\dot v_h + A_h v_h + \Pi_h F(u, u) = \Pi_h f, \qquad v_h(0) = \Pi_h u_0. \tag{37}$$

According to [16, Remark 4.2] we have

$$\max_{0\le t\le T} \|v_h(t) - \Pi_h u(t)\|_0 \le C|\log(h)|h^2, \tag{38}$$

for some constant $C = C(M_2)$. The following lemma provides a superconvergence result.

Lemma 2 *Let (u, p) be the solution of (1)–(2). Then, there exists a positive constant C such that the solution v_h of (37) and the Galerkin approximation u_h satisfy the following bound,*

$$\|v_h(t) - u_h(t)\|_1 \le C|\log(h)|^2h^2, \quad t \in (0, T]. \tag{39}$$

Proof Since for $y_h = A_h^{1/2}(v_h - u_h)$ we have

$$\dot y_h + A_h y_h + A_h^{1/2}\Pi_h(F(v_h, v_h) - F(u_h, u_h)) = A_h^{1/2}\rho_h,$$

where $\rho_h = \Pi_h(F(v_h, v_h) - F(u, u))$, it follows that

$$\begin{aligned}\|y_h(t)\|_0 \le &\int_0^t \left\|A_h^{1/2}e^{-(t-s)A_h}\right\|_0 \|\Pi_h(F(v_h, v_h) - F(u_h, u_h))\|_0\\ &+ \int_0^t \left\|A_h e^{-(t-s)A_h}(A_h^{-1/2}\rho_h(s))\right\|_0 ds.\end{aligned}$$

Applying (33) we have $\|\Pi_h(F(v_h,v_h)-F(u_h,u_h))\|_0 \le C\,\|y_h\|_0$, so that taking into account that

$$\left\|A_h^{1/2}e^{-(t-s)A_h}\right\|_0 \le (2e(t-s))^{-1/2}, \tag{40}$$

it follows that

$$\|y_h(t)\|_0 \le \frac{1}{\sqrt{2e}}\int_0^t \frac{\|y_h(s)\|_0}{\sqrt{t-s}} + \int_0^t \left\|A_h e^{-(t-s)A_h}(A_h^{-1/2}\rho_h(s))\right\|_0 ds.$$

Since applying [16, Lemma 4.2] we obtain

$$\int_0^t \left\|A_h e^{-(t-s)A_h}(A_h^{-1/2}\rho_h(s))\right\|_0 ds \le C|\log(h)| \max_{0\le s\le t} \|\rho_h(s)\|_0,$$

a generalized Gronwall lemma [26, pp. 188-189], together with (33) allow us to conclude

$$\|v_h-u_h\|_1 \le C|\log(h)|\|v_h-u\|_0.$$

Then by writing $\|v_h-u\|_0 \le \|v_h-\Pi_h u\|_0 + \|\Pi_h u-u\|_0$ and applying (14) and (38), the proof is finished if we check that the threshold condition (32) holds for $w_h^1 = u_h$ and $w_h^2 = v_h$. In view of (38), (14) and the inverse inequality (12) we have indeed that $\|v_h-u\|_j = o(h^{3/2-j})$, for $j=0,1$. In the case of u_h the threshold condition holds due to (24). □

Lemma 3 *Let (u,p) be the solution of* (1)–(2). *Then there exists a positive constant C such that*

$$\|\dot v_h(t)-\dot u_h(t)\|_{-1} \le C|\log(h)|^2 h^2, \quad t\in(0,T], \tag{41}$$

where v_h and u_h are defined by (37) *and* (22)-(23) *respectively.*

Proof The difference v_h-u_h satisfies that $\dot v_h-\dot u_h = A_h(v_h-u_h)+\Pi_h(B(u,u)-B(u_h,u_h)$, so that multiplying by $A_h^{-1/2}$ and taking norms, thanks to (34), we have

$$\left\|A_h^{-1/2}(\dot v_h-\dot u_h)\right\|_0 \le \left\|A_h^{1/2}(v_h-u_h)\right\|_0 + C\,\|u-u_h\|_0\,.$$

Now we write

$$\|u-u_h\|_0 \le \|u-\Pi_h u\|_0 + \|\Pi_h u-v_h\|_0 + \|v_h-u_h\|_0\,,$$

so that (14), (38) and (39), allow us to write,

$$\left\|A_h^{-1/2}(\dot v_h-\dot u_h)\right\|_0 \le C\,|\log(h)|^2\,h^2.$$

Then, applying (31) the proof is finished. □

Lemma 4 *Let (u,p) be the solution of* (1)–(2). *Then there exists a positive constant C such that*

$$\|u_t-\dot u_h(t)\|_{-1} \le \frac{C}{t^{(r-1)/2}}h^r\,|\log(h)|^{r'}, \quad t\in(0,T], \quad r=2,3,4, \tag{42}$$

where $r'=2$ when $r=2$ and $r'=1$ otherwise.

Proof The case $r = 3, 4$ is proved in [16, Lemma 5.1]. For the case $r = 2$ we write

$$u_t - \dot{u}_h = (u_t - \Pi_h u_t) + (\Pi_h u_t - \dot{v}_h) + (\dot{v}_h - \dot{u}_h). \tag{43}$$

A simple duality argument and the fact that $\|u_t - \Pi_h u_t\|_0 \le Ch\|u_t\|_1$, easily show that

$$\|(I - \Pi_h)u_t\|_{-1} \le Ch^2\|u_t\|_1 \le C\frac{M_3}{t^{1/2}}h^2.$$

The bound of the third term on the right-hand side of (43) is given in Lemma 3, so that, thanks to the equivalence (31) we are left with estimating

$$y_h = t^{1/2}A_h^{-1/2}(\Pi_h u_t - \dot{v}_h).$$

We notice that

$$\dot{y}_h + A_h y_h = t^{1/2}A_h^{1/2}\dot{\theta}_h + \frac{1}{2}t^{-1/2}A_h^{-1/2}(\Pi_h u_t - \dot{v}_h),$$

where $\theta_h = (\Pi_h - S_h)u$. Thus,

$$\begin{aligned} y_h(t) &= \int_0^t s^{-1/2}A_h^{1/2}e^{-(t-s)A_h}\left(s\dot{\theta}_h\right) ds \\ &\quad + \frac{1}{2}\int_0^t s^{-1/2}A_h^{1/2}e^{-(t-s)A_h}A_h^{-1}(\Pi_h u_s - \dot{v}_h)\, ds. \end{aligned}$$

Recalling (40) by means of the change of variables $\tau = s/t$ it is easy to show that

$$\int_0^t s^{-1/2}\left\|A_h^{1/2}e^{-(t-s)A_h}\right\|_0 ds \le \frac{1}{\sqrt{2e}}B(\frac{1}{2}, \frac{1}{2}), \tag{44}$$

where B is the Beta function (see e. g., [14]). Thus, we have

$$\|y_h\|_0 \le CB(\frac{1}{2}, \frac{1}{2}) \max_{0\le s\le t}\left(s\|\dot{\theta}_h\|_0 + \|A_h^{-1}(\Pi_h u_s - \dot{v}_h)\|_0\right).$$

The first term on the right-hand side above is bounded by CM_4h^2. For the second one we notice hat

$$A_h^{-1}(\Pi_h u_t - \dot{v}_h) = \theta_h - (\Pi_h u - v_h)$$

so that using (14), (17) and (38) it is bounded by $M_2h^2\,|\log(h)|$. □

Theorem 1 *Let (u, p) be the solution of (1)-(2). Then, there exists a positive constant C such that the postprocessed velocity $\tilde{u}$, defined in* (26)-(27)*, satisfies the following bounds:*

(i) If $r = 2$ then

$$\|u(t^*) - \tilde{u}(t^*)\|_1 \le \frac{C}{t^{*(1/2)}}h^2|\log(h)|^2. \tag{45}$$

(ii) If $r = 3, 4$ then

$$\|u(t^*) - \tilde{u}(t^*)\|_j \le \frac{C}{t^{*(r-1)/2}}h^{r+1-j}|\log(h)|, \quad j = 0, 1. \tag{46}$$

Proof The proof follows the same steps as [16, Theorem 5.2]. Subtracting (26) from (1), standard duality arguments show that

$$\|\tilde{u}(t^*)-u(t^*)\|_1 \le C\big(\|F(u(t^*),u(t^*))-F(u_h(t^*),u_h(t^*))\|_{-1}+\|u_t(t^*)-\dot{u}_h(t^*)\|_{-1}\big).$$

To bound the second term on the right-hand side above we apply Lemma 4, whereas for the second we apply (35) to get

$$\|F(u(t^*),u(t^*)) - F(u_h(t^*),u_h(t^*))\|_{-1} \le C\|u(t^*) - u_h(t^*)\|_0. \tag{47}$$

so that applying (24) the proof of (45) and the case $j = 1$ of (46) are finished.

We now get the error bounds in the L^2 norm. It is easy to see that

$$A(\tilde{u}(t^*) - u(t^*)) = \Pi(F(u(t^*),u(t^*)) - F(u_h(t^*),u_h(t^*))) + \Pi(u_t(t^*) - \dot{u}_h(t^*)).$$

Then, by applying A^{-1} to both sides of the above equations, we obtain

$$\begin{aligned}\|\tilde{u}(t^*) - u(t^*)\|_0 \le &\|A^{-1}\Pi(F(u(t^*),u(t^*)) - F(u_h(t^*),u_h(t^*)))\|_0\\ &+ \|A^{-1}\Pi(u_t(t^*) - \dot{u}_h(t^*))\|_0.\end{aligned}$$

As regards the nonlinear term, applying [16, Lemma 4.1] we obtain

$$\begin{aligned}&\|A^{-1}\Pi(F(u(t^*),u(t^*)) - F(u_h(t^*),u_h(t^*)))\|_0 \le\\ &\qquad C(\|u(t^*) - u_h(t^*)\|_{-1} + \|u(t^*) - u_h(t^*)\|_1\|u(t^*) - u_h(t^*)\|_0).\end{aligned}$$

To bound the second term on the right-hand side above we apply (24), whereas the first one is bounded in the proof of [16, Theorem 5.2] by

$$\|u(t^*) - u_h(t^*)\|_{-1} \le \frac{C}{t^{*(r-2)/2}} h^{r+1}|\log(h)|.$$

Finally, to bound $\|A^{-1}\Pi(u_t(t^*) - \dot{u}_h(t^*))\|_0$ we apply [16, Lemma 5.1] to obtain

$$\|A^{-1}\Pi(u_t(t^*) - \dot{u}_h(t^*))\|_0 \le \frac{C}{t^{*(r-1)/2}} h^{r+1}|\log(h)|,$$

which concludes the proof. □

In the following theorem we obtain the error bounds for the pressure $\tilde{p}$.

Theorem 2 *Let (u,p) be the solution of (1)-(2). Then, there exists a positive constant C such that the postprocessed pressure, $\tilde{p}$, satisfies the following bounds:*

$$\|p(t^*) - \tilde{p}(t^*)\|_{L^2/\mathbb{R}} \le \frac{C}{t^{*(r-1)/2}} h^r|\log(h)|^{r'}, \tag{48}$$

where $r' = 2$ if $r = 2$ and $r' = 1$ if $r = 3, 4$.

Proof The proof follows the same steps as [16, Theorem 5.3]. Applying the inf-sup condition (5) it is easy to see that

$$\begin{aligned}\beta\|p(t^*)-\tilde p(t^*)\|_{L^2/\mathbb{R}} &\le \|\tilde u(t^*)-u(t^*)\|_1+\|u_t(t^*)-\dot u_h(t^*)\|_{-1}\\ &\quad +\|F(u_h(t^*),u_h(t^*))-F(u(t^*),u(t^*))\|_{-1}.\end{aligned}$$

Applying now (45) and (46) to bound the first term and reasoning as in the proof of Theorem 1 to bound the other two terms we conclude (48). □

Remark 1 As a consequence of Theorems 1 and 2 we obtain in the proof of Theorem 3 that $(\tilde u - u_h)$ is an asymptotically exact estimator of the error $(u-u_h)$ while $(\tilde p-p_h)$ is an asymptotically exact estimator of the error $(p-p_h)$. However, as we have already observed $\tilde u$ and $\tilde p$ are not computable in practice. In Theorems 3, 4 and 6 we present different procedures to get computable error estimators.

As we pointed out before the MFE approximations (u_h, p_h) to the velocity and the pressure of the solution (u, p) of the evolutionary Navier-Stokes equations (1)-(2) at any fixed time t^* are also the approximations to the velocity and pressure of the steady Stokes problem (26)-(27). In Theorem 3 we show that any a posteriori error estimator of the error in the steady Stokes problem (26)-(27) gives us an a posteriori indicator of the error in the approximations to the evolutionary Navier-Stokes equations.

Using the notation of [33] we will denote in the sequel by $\xi_{\rm vel}((u_h,p_h),f,H^j)$, $j=0,1$, any a posteriori error estimator of the error $u_h-\tilde u$ in the norm of $H^j(\Omega)^d$ in the approximation to the velocity in the steady Stokes problem (26)-(27). We will denote by $\xi_{\rm pres}((u_h,p_h),f,L^2/\mathbb{R})$ any error estimator of the quantity $\|p_h-\tilde p\|_{L^2/\mathbb{R}}$.

Theorem 3 *Let (u, p) be the solution of* (1)-(2) *and fix any positive time $t^*>0$. Assume that the Galerkin approximation (u_h, p_h) satisfies, for h small enough and $r = 2, 3, 4$,*

$$\|u(t^*)-u_h(t^*)\|_j \ge C_r h^{r-j}, \quad j=0,1. \tag{49}$$

for some positive constant $C_r = C_r(t^*)$.

(i) If $\xi_{\rm vel}((u_h(t^*),p_h(t^*)),f,H^j)$, $j=0,1$, is an efficient error indicator of the error in the MFE approximation to the steady Stokes problem (26)-(27). *That is, if there exist positive constants, C_1 and C_2, that are independent of the mesh size h, such that the following bound holds*

$$C_1 \le \frac{\xi_{\rm vel}((u_h(t^*),p_h(t^*)),f,H^j)}{\|\tilde u(t^*)-u_h(t^*)\|_j} \le C_2, \quad j=0,1, \tag{50}$$

then $\xi_{\rm vel}((u_h(t^*),p_h(t^*)),f,H^j)$, $j=0,1$, it is also an efficient error indicator of the error in the MFE approximation to the evolutionary Navier-Stokes equations, i.e. there exist positive constants C_3 and C_4 that are independent of the mesh size h such that the following bound holds for h small enough

$$C_3 \le \frac{\xi_{\rm vel}((u_h(t^*),p_h(t^*)),f,H^j)}{\| \ u(t^*)-u_h(t^*)\|_j} \le C_4, \quad j=0,1. \tag{51}$$

(ii) If $\xi_{\rm vel}((u_h(t^*),p_h(t^*)),f,H^j)$, $j=0,1$ is an asymptotically exact error estimator of the error in the steady Stokes problem then it is also an asymptotically exact error estimator of the error in the evolutionary Navier-Stokes equations.

(iii) Analogous results are obtained in the approximations to the pressure. In the case $r=2$, the results are valid only in the H^1 norm.

Proof For simplicity in the exposition we will concentrate on the cases $r=3,4$ in the approximations to the velocity, the proof for the approximations to the pressure and for the case $r=2$ being the same except for obvious changes.

Let us first observe that

$$\|u_h(t^*)-u(t^*)\|_j \le \|u_h(t^*)-\tilde u(t^*)\|_j + \|\tilde u(t^*)-u(t^*)\|_j, \quad j=0,1.$$

Dividing by $\|u_h(t^*)-u(t^*)\|_j$, using (49) and applying Theorem 1 we obtain

$$1 \le \frac{\|u_h(t^*)-\tilde u(t^*)\|_j}{\|u_h(t^*)-u(t^*)\|_j} + \frac{C{t^*}^{-((r-1)/2)}}{C_r} h|\log(h)|.$$

Now, using (50) we get

$$\begin{aligned}\frac{\|u_h(t^*)-\tilde u(t^*)\|_j}{\|u_h(t^*)-u(t^*)\|_j} &= \frac{\|u_h(t^*)-\tilde u(t^*)\|_j}{\|u_h(t^*)-u(t^*)\|_j}\frac{\xi_{\rm vel}((u_h(t^*),p_h(t^*)),f,H^j)}{\xi_{\rm vel}((u_h(t^*),p_h(t^*)),f,H^j)}\\ &\le \frac{1}{C_1}\frac{\xi_{\rm vel}((u_h(t^*),p_h(t^*)),f,H^j)}{\|u_h(t^*)-u(t^*)\|_j}.\end{aligned}$$

Taking h small enough so that $\frac{C{t^*}^{-((r-1)/2)}}{C_r}h|\log(h)| \le 1/2$, we get

$$\frac{C_1}{2} \le \frac{\xi_{\rm vel}((u_h(t^*),p_h(t^*)),f,H^j)}{\|u_h(t^*)-u(t^*)\|_j}. \tag{52}$$

Now, we use the decomposition

$$\|u_h(t^*)-\tilde u(t^*)\|_j \le \|u_h(t^*)-u(t^*)\|_j + \|u(t^*)-\tilde u(t^*)\|_j, \quad j=0,1. \tag{53}$$

Reasoning as before we get

$$\frac{\|u_h(t^*)-\tilde u(t^*)\|_j}{\|u_h(t^*)-u(t^*)\|_j} \le 1 + \frac{C{t^*}^{-((r-1)/2)}}{C_r} h|\log(h)|.$$

Since

$$\frac{\|u_h(t^*)-\tilde u(t^*)\|_j}{\|u_h(t^*)-u(t^*)\|_j} \ge \frac{1}{C_2}\frac{\xi_{\rm vel}((u_h(t^*),p_h(t^*)),f,H^j)}{\|u_h(t^*)-u(t^*)\|_j},$$

we finally reach

$$\frac{\xi_{\rm vel}((u_h(t^*),p_h(t^*)),f,H^j)}{\|u_h(t^*)-u(t^*)\|_j} \le \frac{3C_2}{2}. \tag{54}$$

From (52) and (54) we conclude (51) with $C_3=C_1/2$ and $C_4=3C_2/2$.

Let us now assume that $\xi_{\rm vel}((u_h(t^*),p_h(t^*)),f,H^j)$ is an asymptotically exact error estimator. Using again the decomposition (53) we have

$$\lim_{h\to 0}\frac{\|u_h(t^*)-\tilde u(t^*)\|_j}{\|u_h(t^*)-u(t^*)\|_j}=1+\lim_{h\to 0}\frac{\|u(t^*)-\tilde u(t^*)\|_j}{\|u_h(t^*)-u(t^*)\|_j}=1,$$

the last equality being a consequence of Theorem (1) and the saturation hypothesis (49). As we pointed out before, this limit implies that $(\tilde u-u_h)$ is an asymptotically exact estimator of the error $(u-u_h)$. Then

$$\lim_{h\to 0}\frac{\xi_{\rm vel}((u_h(t^*),p_h(t^*)),f,H^j)}{\|u_h(t^*)-u(t^*)\|_j}$$
$$=\lim_{h\to 0}\frac{\xi_{\rm vel}((u_h(t^*),p_h(t^*)),f,H^j)}{\|u_h(t^*)-\tilde u(t^*)\|_j}\frac{\|u_h(t^*)-\tilde u(t^*)\|_j}{\|u_h(t^*)-u(t^*)\|_j}=1,$$

and $\xi_{\rm vel}((u_h(t^*),p_h(t^*)),f,H^j)$ is also an asymptotically exact estimator of the error in the approximation to the velocity of the evolutionary Navier-Stokes equations. □

Remark 2 We remark that with hypothesis (49) we are merely assuming that the term of order h^{r-j} is really present in the asymptotic expansion of the Galerkin error. Let us also notice that the constant C_r in (49) is, in general $O(t^{*-(r-2)/2})$, so that the ratio $t^{*-((r-1)/2)}/C_r$ in the proof of Theorem 3 is, in general, $O(t^{*(-1/2)})$.

The key point in Theorem 3 comes from the observation that if we decompose

$$u-u_h=(u-\tilde u)+(\tilde u-u_h), \tag{55}$$

the first term on the right hand side of (55), $u-\tilde u$, is in general smaller, by a factor of size $O(h\log(h))$, than the second one, $\tilde u-u_h$ (Theorem 1). Then, to estimate the error $u-u_h$ we can safely omit the term $u-\tilde u$ in (55). Comparing with the analysis of [33] for a nonstationary linear Stokes model problem the main difference is that the two terms in (55) are taken into account. In Theorem 4 below we show that this kind of technique can also be applied to the nonlinear Navier-Stokes equations. The advantage of this point of view is that hypothesis (49) is not required for the proof of Theorem 4. Let us finally observe that $(\dot u_h,\dot p_h)$ are the MFE approximations to the solution $(\tilde u_t,\tilde p_t)$ of the Stokes problem that we obtain deriving respect to the time variable the Stokes problem (26)-(27). Then, we will denote by $\xi_{\rm vel}((\dot u_h,\dot p_h),f_t,H^j)$, $j=-1,0,1$, any a posteriori error estimator of the error $u_h-\tilde u_t$ in the norm of $H^j(\Omega)^d$ in the approximation to the velocity of the corresponding steady Stokes problem. The proof of the following theorem follows the steps of the proof of [20, Theorem 1].

Theorem 4 *Let (u,p) be the solution of (1)-(2) and let (u_h,p_h) be its MFE Galerkin approximation. Then, the following a posteriori error bound holds for*

$0 \le t \le T$ *and a constant* C *independent of* h.

$$
\begin{aligned}
&\|(u-u_h)(t)\|_0 \le C\|u_0-u_h(0)\|_0 + C\xi_{\rm vel}((u_h(0),p_h(0)),f(0),L^2)\\
&+\xi_{\rm vel}((u_h(t),p_h(t)),f(t),L^2) + Ct^{1/2}\max_{0\le s\le t}\xi_{\rm vel}((u_h,p_h),f,L^2) \qquad (56)\\
&+Ct^{1/2}\max_{0\le s\le t}\xi_{\rm vel}((\dot u_h,\dot p_h),f_s,H^{-1}).
\end{aligned}
$$

Proof Let us denote by $\eta = u-\tilde u$. From (26)-(27) it follows that

$$\eta_t + A\eta + \Pi(F(u,u)-F(u_h,u_h)) = \Pi(\dot u_h - \tilde u_t).$$

Then η satisfies the equation

$$
\begin{aligned}
\eta(t) &= e^{-At}\eta(0) + \int_0^t e^{-A(t-s)}\Pi(F(\tilde u,\tilde u)-F(u,u))\ ds\\
&+\int_0^t e^{-A(t-s)}\Pi(F(u_h,u_h)-F(\tilde u,\tilde u)\ ds) + \int_0^t e^{-A(t-s)}\Pi(\dot u_h-\tilde u_t)\ ds.
\end{aligned}
$$

Taking into account (7) we get

$$
\begin{aligned}
\|\eta(t)\|_0 \le \|\eta(0)\|_0 + C\int_0^t \frac{\|A^{-1/2}\Pi(F(\tilde u,\tilde u)-F(u,u))\|_0}{\sqrt{t-s}}\ ds\\
+C\int_0^t \frac{\|A^{-1/2}\Pi(F(u_h,u_h)-F(\tilde u,\tilde u))\|_0}{\sqrt{t-s}}\ ds + C\int_0^t \frac{\|A^{-1/2}\Pi(\dot u_h-\tilde u_t)\|_0}{\sqrt{t-s}}\ ds.
\end{aligned}
$$

We first observe that for any $v\in L^2(\Omega)^d$ we have $\|A^{-1/2}\Pi v\|_0 \le C\|v\|_{-1}$. Then, taking into account (35) we get

$$
\begin{aligned}
\|A^{-1/2}\Pi(F(\tilde u,\tilde u)-F(u,u))\|_0 &\le C\|\tilde u-u\|_0,\\
\|A^{-1/2}\Pi(F(u_h,u_h)-F(\tilde u,\tilde u))\|_0 &\le C\|u_h-\tilde u\|_0.
\end{aligned}
$$

Let us observe that in order to apply (35) we require u_h to satisfy (32), which holds due to (24), and $\|\tilde u\|_\infty$ and $\|\nabla\tilde u\|_{L^{2d/(d-1)}}$ to be bounded. Using (4) both norms are bounded in terms of $\|\tilde u\|_2$. Applying (19) we get

$$
\begin{aligned}
\|\tilde u\|_2 &\le C\left(\|\dot u_h\|_0 + \|u_h\cdot\nabla u_h\|_0\right)\\
&\le C\left(\|A_h u_h\|_0 + \|\Pi_h F(u_h,u_h)\|_0 + \|\Pi_h f\|_0 + \|u_h\cdot\nabla u_h\|_0\right).
\end{aligned}
$$

Finally, using that $\|A_h u_h\|_0$ is uniformly bounded, see (28), and reasoning as in (29) to bound the second and forth terms above we conclude $\|\tilde u\|_2$ is uniformly bounded. Then, we arrive at

$$
\begin{aligned}
\|\eta(t)\|_0 &\le \|\eta(0)\|_0 + C\int_0^t\frac{\|\eta(s)\|_0}{\sqrt{t-s}}\ ds + C\int_0^t\frac{\|u_h(s)-\tilde u(s)\|_0}{\sqrt{t-s}}\ ds\\
&+C\int_0^t\frac{\|\dot u_h(s)-\tilde u_s(s)\|_0}{\sqrt{t-s}}\ ds.
\end{aligned}
$$

And then

$$\begin{aligned}\|\eta(t)\|_0 &\le \|\eta(0)\|_0 + C\int_0^t \frac{\|\eta(s)\|_0}{\sqrt{t-s}}\,ds + Ct^{1/2}\max_{0\le s\le t}\xi_{\rm vel}((u_h,p_h),f,L^2)\\ &\quad + Ct^{1/2}\max_{0\le s\le t}\xi_{\rm vel}((\dot u_h,\dot p_h),f_s,L^2).\end{aligned}$$

A standard application of a generalized Gronwall lemma [26] gives

$$\begin{aligned}\|\eta(t)\|_0 &\le C\|\eta(0)\|_0 + Ct^{1/2}\max_{0\le s\le t}\xi_{\rm vel}((u_h,p_h),f,L^2)\\ &\quad + Ct^{1/2}\max_{0\le s\le t}\xi_{\rm vel}((\dot u_h,\dot p_h),f_s,L^2).\end{aligned}$$

Now, using decomposition (55) we conclude the proof. □

We observe that using the same proof, a similar bound for the $H^1(\Omega)^d$ norm of the error can be obtained changing only $\xi_{\rm vel}((u_h,p_h),f,L^2)$ by $\xi_{\rm vel}((u_h,p_h),f,H^1)$ and $\xi_{\rm vel}((\dot u_h,\dot p_h),f_t,H^{-1})$ by $\xi_{\rm vel}((\dot u_h,\dot p_h),f_t,L^2)$. Let us also remark that Theorem 4 allows to a posteriori obtain upper error bounds for the error in the approximation to the nonlinear Navier-Stokes equations using only upper error bounds for some Stokes problems depending only on the data and the computed approximation. However, the estimation of the error at a time t requires the estimation of the error of a family of Stokes problems with right hand side depending on τ, for all $\tau\in[0,t]$.

We now propose a simple procedure to estimate the error which is based on computing a MFE approximation to the solution $(\tilde u(t^*),\tilde p(t^*))$ of (26)-(27) on a MFE space with better approximation capabilities than $(X_{h,r},Q_{h,r-1})$ in which the Galerkin approximation (u_h,p_h) is defined. This procedure was applied to the p-version of the finite-element method for evolutionary convection-reaction-diffusion equations in [19]. The main idea here is to use a second approximation of different accuracy than that of the Galerkin approximation of (u,p) and whose computational cost hardly adds to that of the Galerkin approximation itself.

Let us fix any time $t^*\in(0,T]$ and let approximate the solution $(\tilde u,\tilde p)$ of the Stokes problem (26)-(27) by solving the following discrete Stokes problem: find $(\tilde u_h(t^*),\tilde p_h(t^*))\in\widetilde X\times\widetilde Q$ satisfying

$$\begin{aligned}\left(\nabla\tilde u_h(t^*),\nabla\tilde\phi\right)+\left(\nabla\tilde p_h(t^*),\tilde\phi\right) &= \left(f,\tilde\phi\right)-\left(F(u_h(t^*),u_h(t^*)),\tilde\phi\right) \quad (57)\\ &\quad -\left(\dot u_h(t^*),\tilde\phi\right)\quad\forall\tilde\phi\in\widetilde X,\\ \left(\nabla\cdot\tilde u_h(t^*),\tilde\psi\right) &= 0\quad\forall\tilde\psi\in\widetilde Q, \quad (58)\end{aligned}$$

where $(\widetilde X,\widetilde Q)$ is either:

(a) The same-order MFE over a finer grid. That is, for $h'<h$, we choose $(\widetilde X,\widetilde Q)=(X_{h',r},Q_{h',r-1})$.

(b) A higher-order MFE over the same grid. In this case we choose $(\widetilde{X}, \widetilde{Q}) = (X_{h,r+1}, Q_{h,r})$.

We now study the errors $u - \tilde{u}_h$ and $p - \tilde{p}_h$.

Theorem 5 *Let (u, p) be the solution of (1)–(2) and for $r = 2, 3, 4$, and let (10)–(11) hold with $k = r + 2$ Then, there exists a positive constant C such that the postprocessed MFE approximation to u, $\tilde{u}_h$ satisfies the following bounds for $r = 2, 3, 4$ and $t \in (0, T]$:*

(i) if the postprocessing element is $(\widetilde{X}, \widetilde{Q}) = (X_{h',r}, Q_{h',r-1})$, then

$$\|u(t) - \tilde{u}_h(t)\|_j \leq \frac{C}{t^{(r-2)/2}}(h')^{r-j} + \frac{C}{t^{(r-1)/2}} h^{r+1-j} |\log(h)|^{r'}, \quad j = 0, 1, \tag{59}$$

$$\|p(t) - \tilde{p}_h(t)\|_{L^2/\mathbb{R}} \leq \frac{C}{t^{(r-2)/2}}(h')^{r-1} + \frac{C}{t^{(r-1)/2}} h^{r} |\log(h)|^{r'}, \tag{60}$$

(ii) if the postprocessing element is $(\widetilde{X}, \widetilde{Q}) = (X_{h,r+1}, Q_{h,r})$, then

$$\|u(t) - \tilde{u}_h(t)\|_j \leq \frac{C}{t^{(r-1)/2}} h^{r+1-j} |\log(h)|^{r'}, \qquad j = 0, 1, \tag{61}$$

$$\|p(t) - \tilde{p}_h(t)\|_{L^2/\mathbb{R}} \leq \frac{C}{t^{(r-1)/2}} h^{r} |\log(h)|^{r'}. \tag{62}$$

For $r = 2$ only the case $j = 1$ in (59) and (61) holds. In (59)–(62), $r' = 2$ when $r = 2$ and $r' = 1$ otherwise.

Proof The cases $r = 3, 4$ have been proven in Theorems 5.2 and 5.3 in [16]. Following the same arguments, we now prove the results corresponding to $r = 2$ and $(\widetilde{X}, \widetilde{Q}) = (X_{h',r}, Q_{h',r-1})$, the case $(\widetilde{X}, \widetilde{Q}) = (X_{h,r+1}, Q_{h,r})$ being similar, yet easier. We decompose the error $u - \tilde{u}_h = (u - s_{h'}) + (s_{h'} - \tilde{u}_h)$, where $(s_{h'}, q_{h'}) \in X_{h',2} \times Q_{h',1}$ is the solution of

$$(\nabla s_{h'}, \nabla \phi_{h'}) - (q_{h'}, \nabla \cdot \phi_{h'}) = (f - F(u, u) - u_t, \phi_{h'}) \ \forall \phi_{h'} \in X_{h',2}, \tag{63}$$

$$(\nabla \cdot s_{h'}, \psi_{h'}) = 0 \quad \forall \psi_{h'} \in Q_{h',1}, \tag{64}$$

that is, $s_{h'}$ is the Stokes projection of u onto $V_{h'}$. Since in view of (17)–(18) we have

$$\|u - s_{h'}\|_1 + \|p - q_{h'}\|_{L^2/\mathbb{R}} \leq C M_2 h',$$

we only have to estimate $s_{h'} - \tilde{u}_h$ and $q_{h'} - \tilde{p}_h$. To do this, we subtract (57) from (63), and take inner product with $\tilde{e}_h = s_{h'} - \tilde{u}_h$ to get

$$\|\nabla \tilde{e}_h\|_0^2 \leq \left(\|u_t - \dot{u}_h\|_{-1} + \|F(u_h, u_h) - F(u, u)\|_{-1}\right) \|\tilde{e}_h\|_1 .$$

Now applying Lemma 4, (35) and (24) the proof of (59) is finished.

To prove (60), again we subtract (57) from (63), rearrange terms and apply the inf-sup condition (5) to get

$$\beta \|q_{h'} - \tilde{p}_h\|_{L^2/\mathbb{R}} \leq \|\nabla \tilde{e}_h\|_0 + \|u_t - \dot{u}_h\|_{-1} + \|F(u_h, u_h) - F(u, u)\|_{-1}$$

and the proof is finished with the same arguments used to prove (59). □

To estimate the error in $(u_h(t^*), p_h(t^*))$ we propose to take the difference between the postprocessed and the Galerkin approximations:

$$\tilde\eta_{h,\mathrm{vel}}(t^*) = \tilde u_h(t^*) - u_h(t^*), \quad \tilde\eta_{h,\mathrm{pres}}(t^*) = \tilde p_h(t^*) - p_h(t^*).$$

In the following theorem we prove that this error estimator is efficient and asymptotically exact both in the $L^2(\Omega)^d$ and $H^1(\Omega)^d$ norms and it has the advantage of providing an improved approximation when added to the Galerkin MFE approximation.

Theorem 6 *Let (u,p) be the solution of (1)-(2) and fix any positive time $t^* > 0$. Assume that condition (49) is satisfied. Then, there exist positive constants h_0, $\gamma_0 < 1$, and C_1, C_2, C_3 and C_4 such that, for $h < h_0$ and $0 < \gamma < \gamma_0$, the error estimators $\tilde\eta_{h,\mathrm{vel}}(t^*)$ $\tilde\eta_{h,\mathrm{pres}}(t^*)$ satisfy the following bounds when $(\widetilde X, \widetilde Q) = (X_{h',r}, Q_{h',r-1})$ and $h' < \gamma h$:*

$$C_1 \le \frac{\|\tilde\eta_{h,\mathrm{vel}}(t^*)\|_j}{\|(u-u_h)(t^*)\|_j} \le C_2, \ j = 0,1, \quad C_3 \le \frac{\|\tilde\eta_{h,\mathrm{pres}}(t^*)\|_{L^2/\mathbb{R}}}{\|(p-p_h)(t^*)\|_{L^2/\mathbb{R}}} \le C_4. \quad (65)$$

Furthermore, if $(\widetilde X, \widetilde Q) = (X_{h',r}, Q_{h',r-1})$, with $h' = h^{1+\epsilon}$, $\epsilon > 0$, or $(\widetilde X, \widetilde Q) = (X_{h,r+1}, Q_{h,r})$ then

$$\lim_{h\to 0} \frac{\|\tilde\eta_{h,\mathrm{vel}}(t^*)\|_j}{\|(u-u_h)(t^*)\|_j} = 1, \ j = 0,1, \quad \lim_{h\to 0} \frac{\|\tilde\eta_{h,\mathrm{pres}}(t^*)\|_{L^2/\mathbb{R}}}{\|(p-p_h)(t^*)\|_{L^2/\mathbb{R}}} = 1. \quad (66)$$

For the mini element, the case $j = 0$ in (65) and (66) must be excluded.

Proof We will prove the estimates for the velocity in the case $r = 3, 4$, since the estimates for the pressure and the case $r = 2$ are obtained by similar arguments but with obvious changes. Let us observe that for $j = 0, 1$

$$\begin{aligned}\|u(t^*) - u_h(t^*)\|_j &\le \|\tilde\eta_{h,\mathrm{vel}}(t^*)\|_j + \|\tilde u_h(t^*) - u(t^*)\|_j \\ &\le \|\tilde\eta_{h,\mathrm{vel}}(t^*)\|_j + \frac{C}{(t^*)^{(r-2)/2}}(h')^{r-j} \\ &\quad + \frac{C}{(t^*)^{(r-1)/2}} h^{r+1-j} |\log(h)|.\end{aligned}$$

On the other hand

$$\begin{aligned}\|\tilde\eta_{h,\mathrm{vel}}(t^*)\|_j &\le \|u(t^*) - u_h(t^*)\|_j + \|\tilde u_h(t^*) - u(t^*)\|_j \\ &\le \|u(t^*) - u_h(t^*)\|_j + \frac{C}{(t^*)^{(r-2)/2}}(h')^{r-j} \\ &\quad + \frac{C}{(t^*)^{(r-1)/2}} h^{r+1-j} |\log(h)|.\end{aligned}$$

Using (49) we get

$$\left| \frac{\|\tilde\eta_{h,\mathrm{vel}}(t^*)\|_j}{\|(u-u_h)(t^*)\|_j} - 1 \right| \le \frac{C}{C_r} \left((t^*)^{-(r-2)/2} \left(\frac{h'}{h}\right)^{r-j} + (t^*)^{-(r-1)/2} |\log(h)| h \right). \quad (67)$$

Taking $h' \leq \gamma h$ and h and γ sufficiently small, the bound (65) is readily obtained. The proof of (66) follows straightforwardly from (67), since in the case when $(\widetilde{X}, \widetilde{Q}) = (X_{h',r}, Q_{h',r-1})$ with $h' = h^{1+\epsilon}$, $\epsilon > 0$, the term $(h'/h)^{r-j} \to 0$ when h tends to zero, and in the case when $(\widetilde{X}, \widetilde{Q}) = (X_{h,r+1}, Q_{h,r})$ the term containing the parameter h' is not present. $\square$

4 A posteriori error estimations. Fully discrete case

In practice, it is not possible to compute the MFE approximation exactly, and, instead, some time-stepping procedure must be used to approximate the solution of (22)-(23). Hence, for some time levels $0 = t_0 < t_1 < \cdots < t_N = T$, approximations $U_h^n \approx u_h(t_n)$ and $P_h^n \approx p_h(t_n)$ are obtained. In this section we assume that the approximations are obtained with the backward Euler method or the two-step BDF which we now describe. For simplicity, we consider only constant stepsizes, that is, for $N \geq 2$ integer, we fix $k = T/N$, and we denote $t_n = nk$, $n = 0, 1, \dots, N$. For a sequence $(y^n)_{n=0}^N$ we denote

$$Dy^n = y^n - y^{n-1}, \quad n = 1, 2 \dots, N.$$

Given $U_h^0 = u_h(0)$, a sequence (U_h^n, P_h^n) of approximations to $(u_h(t_n), p_h(t_n))$, $n = 1, \dots N$, is obtained by means of the following recurrence relation:

$$\begin{aligned} (d_t U_h^n, \phi_h) + (\nabla U_h^n, \nabla \phi_h) & \qquad\qquad\qquad\qquad\qquad\qquad (68) \\ + b\,(U_h^n, U_h^n, \phi_h) - (P_h^n, \nabla \cdot \phi_h) &= (f, \phi_h) \quad \forall \phi_h \in X_{h,r}, \\ (\nabla \cdot U_h^n, \psi_h) = 0, \quad \forall\, \psi_h \in Q_{h,r-1}, & \qquad\qquad\qquad\qquad\qquad\qquad (69) \end{aligned}$$

where $d_t = k^{-1}D$ in the case of the backward Euler method and $d_t = k^{-1}(D + \frac{1}{2}D^2)$ for the two-step BDF. In this last case, a second starting value U_h^1 is needed. Here, we will always assume that U_h^1 is obtained by one step of the backward Euler method. Also, for both the backward Euler and the two-step BDF, we assume that $U_h^0 = u_h(0)$, which is usually the case in practical situations.

We now define the time-discrete postprocessed approximation. Given an approximation $d_t^* U_h^n$ to $\dot{u}_h(t_n)$, the time-discrete postprocessed velocity and pressure $(\widetilde{U}^n, \widetilde{P}^n)$ are defined as the solution of the following Stokes problem:

$$\left(\nabla \tilde{U}^n, \nabla \phi\right) + \left(\nabla \tilde{P}^n, \phi\right) = (f, \phi) - b\,(U_h^n, U_h^n, \phi) - (d_t^* U_h^n, \phi)\,,\ \forall\, \phi \in H_0^1(\Omega)^d, \tag{70}$$

$$\left(\nabla \cdot \tilde{U}^n, \psi\right) = 0,\ \forall\, \psi \in L^2(\Omega)/\mathbb{R}. \tag{71}$$

For reasons already analyzed in [15] and [17] we define

$$d_t^* U_h^n = \Pi_h f - A_h U_h^n - \Pi_h F\,(U_h^n, U_h^n) \tag{72}$$

as an adequate approximation to the time derivative $\dot{u}_h(t_n)$.

For the analysis of the errors $u(t)-\tilde{U}^n$ and $p(t)-\tilde{P}^n$ we follow [17], where the MFE approximations to the Stokes problem (70)–(71) are analyzed. We start by decomposing the errors $u(t)-\tilde{U}^n$ and $p(t)-\tilde{P}^n$ as follows,

$$\begin{aligned} u(t_n)-\tilde{U}^n &= (u(t)-\tilde{u}(t_n))+\tilde{e}^n, && (73)\\ p(t_n)-\tilde{P}^n &= (p(t_n)-\tilde{p}(t_n))+\tilde{\pi}^n, && (74)\end{aligned}$$

where $\tilde{e}^n=\tilde{u}(t_n)-\tilde{U}^n$ and $\tilde{\pi}^n=\tilde{p}(t_n)-\tilde{P}^n$ are the temporal errors of the time-discrete postprocessed velocity and pressure $(\tilde{U}^n,\tilde{P}^n)$. The first terms on the right-hand sides of (73)–(74) are the errors of the postprocessed approximation that were studied in the previous section.

Let us denote by $e_h^n=u_h(t_n)-U_h^n$, the temporal error of the MFE approximation to the velocity, and by $\pi_h^n=p_h(t_n)-P_h^n$, the temporal error of the MFE approximation to the pressure. In the present section we bound $(\tilde{e}^n-e_h^n)$ and $(\tilde{\pi}^n-\pi_h^n)$ in terms of e_h^n.

The error bounds in the following lemma are similar to those of [17, Proposition 3.1] where error estimates for MFE approximations of the Stokes problem (70)–(71) are obtained.

Lemma 5 *There exists a positive constant $C=C(\max_{0\le t\le T}\|A_hu_h(t)\|_0)$ such that*

$$\begin{aligned} \|\tilde{e}^n-e_h^n\|_j &\le Ch^{2-j}\big(\|e_h^n\|_1+\|e_h^n\|_1^3+\|A_he_h^n\|_0\big),\ j=0,1,\quad 1\le n\le N, && (75)\\ \|\tilde{\pi}^n-\pi_h^n\|_{L^2/\mathbb{R}} &\le Ch\big(\|e_h^n\|_1+\|e_h^n\|_1^3+\|A_he_h^n\|_0\big),\quad 1\le n\le N. && (76)\end{aligned}$$

Proof Let us denote by $l=g+(d_t^*U_h^n-\dot{u}_h(t_n))$ where $g=F(U_h^n,U_h^n)-F(u_h(t_n),u_h(t_n))$. Subtracting (70)–(71) from (26)–(27) we have that the temporal errors $(\tilde{e}^n,\tilde{\pi}^n)$ of the time-discrete postprocessed velocity and pressure are the solution of the following Stokes problem

$$\begin{aligned} (\nabla\tilde{e}^n,\nabla\phi)+(\nabla\tilde{\pi}^n,\phi) &= (l,\phi),\quad \forall\phi\in H_0^1(\Omega)^d, && (77)\\ (\nabla\cdot\tilde{e}^n,\psi) &= 0,\quad \forall\psi\in L^2(\Omega)/\mathbb{R}. && (78)\end{aligned}$$

On the other hand, subtracting (68)-(69) from (22)-(23) and taking into account that, thanks to definition (72) $d_tU_h^n=d_t^*U_h^n$, we get that the temporal errors (e_h^n,π_h^n) of the fully discrete MFE approximation satisfy

$$\begin{aligned} (\nabla e_h^n,\nabla\phi_h)+(\nabla\pi_h^n,\phi_h)=(l,\phi_h),\quad \forall\phi_h\in X_{h,r},\\ (\nabla\cdot e_h^n,\psi_h)=0,\quad \forall\psi_h\in Q_{h,r-1},\end{aligned}$$

and thus (e_h^n,π_h^n) is the MFE approximation to the solution $(\tilde{e}^n,\tilde{\pi}^n)$ of (77)–(78). Using then (21) we get

$$\|\tilde{e}^n-e_h^n\|_j\le Ch^{2-j}\|l\|_0.$$

For the pressure we apply (18) and (20) to obtain

$$\|\tilde{\pi}^n-\pi_h^n\|_{L^2/\mathbb{R}}\le Ch\|\tilde{\pi}^n\|_{H^1/\mathbb{R}}\le Ch\|l\|_0.$$

Then, to conclude, it only remains to bound $\|l\|_0$. From the definition of $d_t^* U_h^n$ it is easy to see that

$$d_t^* U_h^n - \dot{u}_h(t_n) = A_h e_h^n - \Pi_h \left(F\left(U_h^n, U_h^n\right) - F(u_h(t_n), u_h(t_n)) \right),$$

so that

$$\|d_t^* U_h^n - \dot{u}_h(t_n)\|_0 \leq \|A_h e_h^n\|_0 + \|g\|_0.$$

Now, by writing g as

$$g = F(e_h^n, u_h(t_n)) + F(u_h(t_n), e_h^n) - F(e_h^n, e_h^n),$$

and using (29)–(30) we get

$$\|g\|_0 \leq \left(\|A_h u_h(t_n)\|_0 \|e_h^n\|_1 + \|e_h^n\|_1^{3/2} \|A_h e_h^n\|_0^{1/2} \right),$$

from which we finally conclude (75) and (76). □

Let us consider the quantities $\widetilde{U}^n - U_h^n$ and $\widetilde{P}^n - P_h^n$ as a posteriori indicators of the error in the fully discrete approximations to the velocity and pressure respectively. Then, we obtain the following result:

Theorem 7 *Let (u,p) be the solution of (1)–(2) and let (9) hold. Assume that the fully discrete MFE approximations (U_h^n, P_h^n), $n = 0 \ldots, N = T/k$ are obtained by the backward Euler method or the two-step BDF (68)–(69), and let $(\widetilde{U}^n, \widetilde{P}^n)$ be the solution of (70)–(71). Then, for $n = 1, \ldots, N$,*

$$\|\widetilde{U}^n - U_h^n\|_j \leq \|\tilde{u}(t_n) - u_h(t_n)\|_j + C'_{l_0} h^{2-j} \frac{k^{l_0}}{t_n^{l_0}}, \qquad j = 0, 1, \quad (79)$$

$$\|\widetilde{P}^n - P_h^n\|_{L^2/\mathbb{R}} \leq \|\tilde{p}(t_n) - p_h(t_n)\|_{L^2/\mathbb{R}} + C'_{l_0} h \frac{k^{l_0}}{t_n^{l_0}}, \quad (80)$$

where C'_{l_0} is the constant in (82)–(83), $l_0 = 1$ for the backward Euler method and $l_0 = 2$ for the two-step BDF.

Proof In [17, Theorems 5.4 and 5.7] we prove that if (9) and the case $l = 2$ in (17) hold, the errors e_h^n of these two time integration procedures satisfy for k small enough that

$$\|e_h^n\|_0 + t_n \|A_h e_h^n\|_0 \leq \mathcal{C}_{l_0} \frac{k^{l_0}}{t_n^{l_0 - 1}}, \quad 1 \leq n \leq N, \quad (81)$$

for a certain constants $\mathcal{C}_1$ and C_2, where $l_0 = 1$ for the backward Euler method and $l_0 = 2$ for the two-step BDF. Since $\|A_h^{1/2} e_h^n\|_0 \leq \|e_h^n\|_0^{1/2} \|A_h e_h^n\|_0^{1/2}$, and then $\|e_h^n\|_1 \leq C \|e_h^n\|_0^{1/2} \|A_h e_h^n\|_0^{1/2}$, from (81) and (75)-(76) we finally reach that for k small enough

$$\|\tilde{e}^n - e_h^n\|_j \leq C'_{l_0} h^{2-j} \frac{k^{l_0}}{t_n^{l_0}}, \; j = 0, 1, \; 1 \leq n \leq N, \quad (82)$$

$$\|\tilde{\pi}^n - \pi_h^n\|_{L^2/\mathbb{R}} \leq C'_{l_0} h \frac{k^{l_0}}{t_n^{l_0}}, \; 1 \leq n \leq N, \quad (83)$$

where C'_{l_0} is a positive constant.

Let us decompose the estimators as follows:

$$\begin{aligned} \widetilde{U}^n - U_h^n &= \left(\widetilde{U}^n - \tilde{u}(t_n)\right) + (\tilde{u}(t_n) - u_h(t_n)) + (u_h(t_n) - U_h^n) \\ &= (\tilde{u}(t_n) - u_h(t_n)) + (e_h^n - \tilde{e}^n), \end{aligned} \quad (84)$$

$$\begin{aligned} \widetilde{P}^n - P_h^n &= \left(\widetilde{P}^n - \tilde{p}(t_n)\right) + (\tilde{p}(t_n) - p_h(t_n)) + (p_h(t_n) - P_h^n) \\ &= (\tilde{p}(t_n) - p_h(t_n)) + (\pi_h^n - \tilde{\pi}^n), \end{aligned} \quad (85)$$

which implies

$$\begin{aligned} \|\widetilde{U}^n - U_h^n\|_j &\leq \|\tilde{u}(t_n) - u_h(t_n)\|_j + \|e_h^n - \tilde{e}^n\|_j, \qquad j = 1, 2, \\ \|\widetilde{P}^n - P_h^n\|_{L^2/\mathbb{R}} &\leq \|\tilde{p}(t_n) - p_h(t_n)\|_{L^2/\mathbb{R}} + \|\pi_h^n - \tilde{\pi}^n\|_{L^2/\mathbb{R}}. \end{aligned}$$

Thus, in view of (82)–(83) we obtain (79) and (80) □

Let us comment on the practical implications of this theorem. Observe that from (84) and (85) the fully discrete estimators $\widetilde{U}^n - U_h^n$ and $\widetilde{P}^n - P_h^n$ can be both decomposed as the sum of two terms. The first one is the semi-discrete a posteriori error estimator we have studied in the previous section (see Remark 1) and which we showed it is an asymptotically exact estimator of the spatial error of U_h^n and P_h^n respectively. On the other hand, as shown in (82)–(83), the size of the second term is in asymptotically smaller than the temporal error of U_h^n and P_h^n respectively. We conclude that, as long as the spatial an temporal errors are not too unbalanced (i.e., they are not of very different sizes), the first term in (84) and (85) is dominant and then the quantities $\widetilde{U}^n - U_h^n$ and $\widetilde{P}^n - P_h^n$ are a posteriori error estimators of the spatial error of the fully discrete approximations to the velocity and pressure respectively. The control of the temporal error can be then accomplished by standard and well-stablished techniques in the field of numerical integration of ordinary differential equations.

Now, we remark that $(\widetilde{U}^n, \widetilde{P}^n)$ are obviously not computable. However, we observe that the fully discrete approximation (U_h^n, P_h^n) of the evolutionary Navier-Stokes equation is also the approximation to the Stokes problem (70)-(71) whose solution is $(\widetilde{U}^n, \widetilde{P}^n)$. Then, one can use any of the available error estimators for a steady Stokes problem to estimate the quantities $\|\widetilde{U}^n - U_h^n\|_j$ and $\|\widetilde{P}^n - P_h^n\|_{L^2/\mathbb{R}}$, which, as we have already proved, are error indicators of the spatial errors of the fully discrete approximations to the velocity and pressure, respectively.

To conclude, we show a procedure to get computable estimates of the error in the fully discrete approximations. We define the fully discrete postprocessed approximation $(\widetilde{U}_h^n, \widetilde{P}_h^n)$ as the solution of the following Stokes problem (see [17]):

$$\left(\nabla\widetilde{U}_h^n, \nabla\tilde{\phi}\right) + \left(\nabla\widetilde{P}_h^n, \tilde{\phi}\right) = \left(f, \tilde{\phi}\right) - b\left(U_h^n, U_h^n, \tilde{\phi}\right) - \left(d_t^* U_h^n, \tilde{\phi}\right) \quad \forall\, \tilde{\phi} \in \widetilde{X}, \quad (86)$$

$$\left(\nabla \cdot \widetilde{U}_h^n, \tilde{\psi}\right) = 0 \quad \forall\, \tilde{\psi} \in \widetilde{Q}, \quad (87)$$

where $(\widetilde X,\widetilde Q)$ is as in (57)-(58).

Let us denote by $\tilde e_h^n=\tilde u_h(t_n)-\widetilde U_h^n$ and $\tilde\pi_h^n=\tilde p_h(t_n)-\widetilde P_h^n$ the temporal errors of the fully discrete postprocessed approximation $(\tilde U_h^n,\tilde P_h^n)$ (observe that the semi-discrete postprocessed approximation $(\tilde u_h,\tilde p_h)$ is defined in (57)-(58)). Let us denote, as before, by e_h^n the temporal error of the MFE approximation to the velocity. Then, we have the following bounds.

Lemma 6 *There exists a positive constant $C=C(\max_{0\le t\le T}\|A_hu_h(t)\|_0)$ such that for $1\le n\le N$ the following bounds hold*

$$\|\tilde e_h^n-e_h^n\|_j \le Ch^{2-j}\big(\|e_h^n\|_1+\|e_h^n\|_1^3+\|A_he_h^n\|_0\big),\ j=0,1 \quad (88)$$

$$\|\tilde\pi_h^n-\pi_h^n\|_{L^2(\Omega)/\mathbb{R}} \le Ch\big(\|e_h^n\|_1+\|e_h^n\|_1^3+\|A_he_h^n\|_0\big). \quad (89)$$

Proof The bound (88) is proved in [17, Proposition 3.1]. To prove (89) we decompose

$$\|\tilde\pi_h^n-\pi_h^n\|_{L^2(\Omega)/\mathbb{R}}\le\|\tilde\pi_h^n-\tilde\pi^n\|_{L^2(\Omega)/\mathbb{R}}+\|\tilde\pi^n-\pi_h^n\|_{L^2(\Omega)/\mathbb{R}}.$$

The second term above is bounded in (76) of Lemma 5. For the first we observe that $\tilde\pi_h^n$ is the MFE approximation in $\widetilde Q$ to the pressure $\tilde\pi^n$ in (77)-(78) so that the same reasoning used in the proof of Lemma 5 allow us to obtain

$$\|\tilde\pi_h^n-\tilde\pi^n\|_{L^2(\Omega)/\mathbb{R}}\le Ch\|\tilde\pi^n\|_{H^1/\mathbb{R}}\le Ch\big(\|e_h^n\|_1+\|e_h^n\|_1^3+\|A_he_h^n\|_0\big).$$

□

Using (81) as before, we get the analogous to (82) and (83), i.e., for k small enough the following bound holds

$$\|\tilde e_h^n-e_h^n\|_j \le C'_{l_0}h^{2-j}\frac{k^{l_0}}{t_n^{l_0}},\ j=0,1,\ 1\le n\le N, \quad (90)$$

$$\|\tilde\pi_h^n-\pi_h^n\|_{L^2/\mathbb{R}} \le C'_{l_0}h\frac{k^{l_0}}{t_n^{l_0}},\ 1\le n\le N. \quad (91)$$

where C'_{l_0} is a positive constant.

Similarly to (84)–(85) we write $\widetilde U_h^n-U_h^n=\big(\tilde u_h(t_n)-u_h(t_n)\big)+\big(e_h^n-\tilde e_h^n\big)$ and $\widetilde P_h^n-P_h^n=\big(\tilde p_h(t_n)-p_h(t_n)\big)+\big(\pi_h^n-\tilde\pi_h^n\big)$, so that in view of (90)–(91) we have the following result.

Theorem 8 *Let (u,p) be the solution of (1)–(2) and let (9) hold. Assume that the fully discrete MFE approximations (U_h^n,P_h^n), $n=0\dots,N=T/k$ are obtained by the backward Euler method or the two-step BDF (68)–(69), and let $(\widetilde U_h^n,\widetilde P_h^n)$ be the solution of (86)–(87). Then, for $n=1,\dots,N$,*

$$\|\widetilde U_h^n-U_h^n\|_j \le \|\tilde u_h(t_n)-u_h(t_n)\|_j+C'_{l_0}h^{2-j}\frac{k^{l_0}}{t_n^{l_0}},\qquad j=0,1,\quad (92)$$

$$\|\widetilde P_h^n-P_h^n\|_{L^2/\mathbb{R}} \le \|\tilde p_h(t_n)-p_h(t_n)\|_{L^2/\mathbb{R}}+C'_{l_0}h\frac{k^{l_0}}{t_n^{l_0}}, \quad (93)$$

where C'_{l_0} is the constant in (90)–(91), $l_0 = 1$ for the backward Euler method and $l_0 = 2$ for the two-step BDF.

The practical implications of this result are similar to those of Theorem 7, that is, the first term on the right-hand side of (92) is an error indicator of the spatial error (see Theorem 6) while the second one is asymptotically smaller than the temporal error. As a consequence, the quantity $(\widetilde{U}_h^n - U_h^n)$ is a computable estimator of the spatial error of the fully discrete velocity U_h^n whenever the temporal and spatial errors of U_h^n are more or less of the same size. As before, similar arguments apply for the pressure. We remark that having balanced spatial and temporal errors in the fully discrete approximation is the more common case in practical computations since one usually looks for a final solution with small total error.

As in the semi-discrete case, the advantage of these error estimators is that they produce enhanced (in space) approximations when they are added to the Galerkin MFE approximations.

5 Numerical experiments

We consider the equations

$$\begin{aligned} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p &= f, \\ \operatorname{div}(u) &= 0, \end{aligned} \tag{94}$$

in the domain $\Omega = [0,1]\times[0,1]$ subject to homogeneous Dirichlet boundary conditions. For the numerical experiments of this section we approximate the equations using the mini-element [10] over a regular triangulation of Ω induced by the set of nodes $(i/N, j/N)$, $0 \le i, j \le N$, where $N = 1/h$ is an integer. For the time integration we use the two-step BDF method with fixed time step. For the first step we apply the backward Euler method. In the first numerical experiment we study the semi-discrete in space case. To this end in the numerical experiments we integrate in time with a time-step small enough in order to have negligible temporal errors. We take the forcing term $f(t,x)$ such that the solution of (94) with $\nu = 0.05$ is

$$\begin{aligned} u^1(x,y,t) &= 2\pi\varphi(t)\sin^2(\pi x)\sin(\pi y)\cos(\pi y), \\ u^2(x,y,t) &= -2\pi\varphi(t)\sin^2(\pi y)\sin(\pi x)\cos(\pi x), \\ p(x,y,t) &= 20\varphi(t)x^2 y. \end{aligned} \tag{95}$$

We chose $\varphi(t) = t$ in the first numerical experiment.

When using the mini-element it has been observed and reported in the literature (see for instance [40], [41], [7] [34], [37] and [38]) that the linear part of the approximation to the velocity, u_h^l, is a better approximation to the solution u than u_h itself. The bubble part of the approximation is only introduced for stability reasons and does not improve the approximation to the velocity and pressure terms. For this reason in the numerical experiments of this section we only consider the errors in the linear approximation to the velocity. Also, following [5], we postprocess only the linear approximation to the velocity, i.e., we solve the Stokes problem (57)-(58) with u_h^l and $\dot{u}_h^l$ on the right-hand-side instead of u_h and $\dot{u}_h$. The finite element space at the postprocessed step is the same mini-element defined over a refined mesh of size h'. We show the Galerkin errors and the a posteriori error estimates obtained at time $t^* = 0.5$ by taking the difference between the postprocessed and the standard approximations to the velocity and the pressure. In Figure 1, we have represented the errors in the

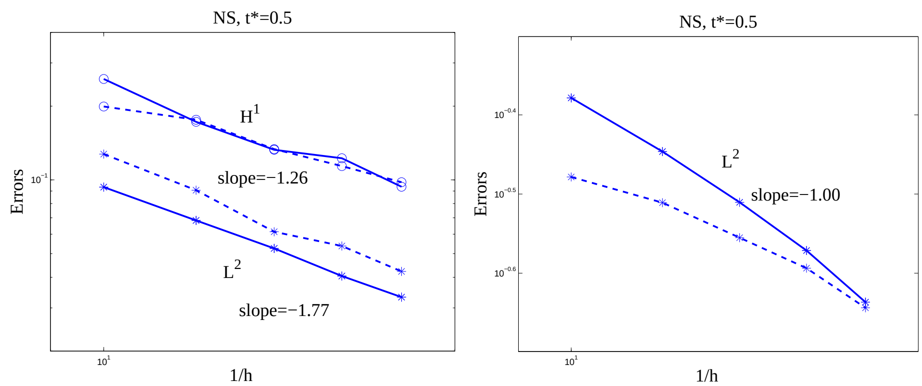

Figure 1: Errors (solid lines) and estimations (dashed lines) in L^2 (asterisks) and H^1 (circles) for $h = 1/10$, $1/12$, $1/14$, $1/16$ and $1/18$ and $h' = 1/24$, $1/30$, $1/34$, $1/38$ and $1/40$ respectively. On the left, error estimations for the first component of the velocity. On the right, error estimations for the pressure.

first component of the velocity of the Galerkin approximation in the L^2 and H^1 norms and the errors for the pressure in the L^2 norm using solid lines. We have used dashed lines to represent the error estimations. The results for the second component of the velocity are completely analogous and they are not reported here. The L^2 errors of the pressure, on the right of Figure 1, are approximately twice as those of the H^1 errors of the velocity, on the left of Figure 1, in this example. We can observe that with the procedure we propose in this paper we get very accurate estimations of the errors, specially in the H^1 norm of the velocity. The difference between the behavior of the error estimations in the L^2 and H^1 norms of the velocity are due to the fact that for first order approximations the postprocessed procedure increases the rate of convergence of the standard method only in the H^1 norm for the velocity and the L^2 norm for the pressure. However, since the postprocessed method produces smaller errors than the Galerkin method also in the L^2 norm it can also be used to estimate the errors in this norm, as it can be checked in the experiment. On the right of Figure 1 we can clearly observe the asymptotically exact behavior of the estimator in the L^2 errors in the pressure in agreement with (66) of Theorem 6.

Let us denote by

$$\theta_{\rm vel} = \frac{\tilde u_h^1(t^*) - u_h^1(t^*)}{u^1(t^*) - u_h^1(t^*)}, \quad \theta_{\rm pre} = \frac{\tilde p_h(t^*) - p(t^*)}{p(t^*) - p_h(t^*)},$$

the efficiency indexes for the first component of the velocity and for the pressure. In Table 1 we have represented the values of the L^2 and H^1 norms of the velocity index and the $L^2/\mathbb{R}$ norm of the pressure index for the experiments in Figure 1. We deduce again from the values of the efficiency indexes that the a posteriori

h	$\|\theta_{\rm vel}\|_0$	$\|\theta_{\rm vel}\|_1$	$\|\theta_{\rm pre}\|_{L^2/\mathbb{R}}$
1/10	1.3640	0.7721	1.2588
1/12	1.3280	1.0197	1.1602
1/14	1.1695	1.0068	1.1084
1/16	1.3259	0.9290	1.0526
1/18	1.2741	1.0438	1.0167

Table 1: Efficiency indexes

error estimates are very accurate, all the values are remarkably close to 1, which is the optimal value for the efficiency index. More precisely, we can observe that the values of the efficiency index in the L^2 norm for the velocity in this experiment belong to the interval $[1.1695, 1.3640]$. The values in the H^1 norm for the velocity lie on the interval $[0.7721, 1.0438]$ and, finally, the values for the pressure are in the interval $[1.0167, 1.2588]$.

To conclude, we show a numerical experiment to check the behavior of the estimators in the fully discrete case. We choose the forcing term f such that the solution of (94) is (95) with $\varphi(t) = \sin((2\pi + \pi/2)t)$. The value of $\nu = 0.05$ and the final time $t^* = 0.5$ are the same as before. In Figure 2, on the left, we have represented the errors obtained using the implicit Euler method as a time integrator for different values of the fixed time step k ranging from $k = 1/10$ to $k = 1/160$ halving each time the value of k. For the spacial discretization we use the mini-element with always the same value of $h = 1/18$. We use solid lines for the errors in the Galerkin method and dashed lines for the estimations, as before. The L^2 norm errors are marked with asterisks while the H^1 norm errors are marked with circles. We estimate the errors using the postprocessed method computed with the same mini-element over a refined mesh of size $h' = 1/40$. We observe that the Galerkin errors decrease as k decreases until a value that corresponds to the spatial error of the approximation. On the contrary, the error estimations lie on an almost horizontal line, both for the velocity in the L^2 and H^1 norms and for the pressure. This means, as we stated in Section 4.2, that the error estimations we propose are a measure of the spatial errors, even when the errors in the Galerkin method are polluted by errors coming from the temporal discretization. In this experiment the error estimations are very accurate for the spatial errors of the velocity in the H^1 norm and for the errors in the pressure. As commented above, the fact that postprocessing linear

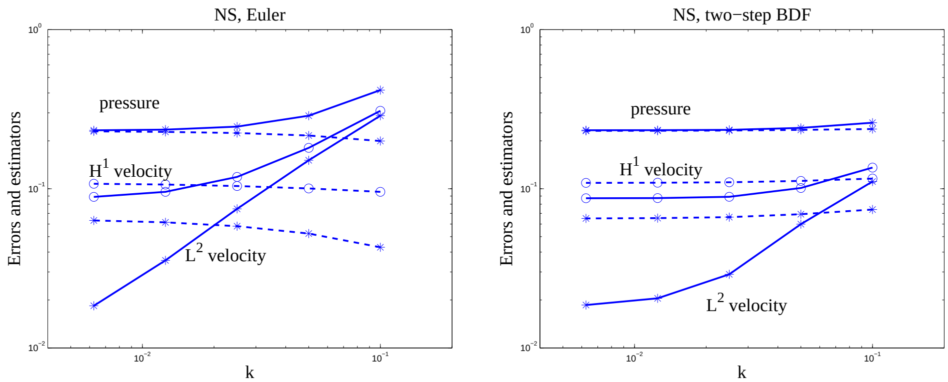

Figure 2: Errors (solid lines) and estimations (dashed lines) in L^2 (asterisks) and H^1 (circles) for $h = 1/18$. On the left: Euler; on the right: two-step BDF for $k = 1/10$ to $k = 1/160$.

elements does not increase the convergence rate in the L^2 norm is reflected in the precision of the error estimations in the L^2 norm. On the right of Figure 2 we have represented the errors obtained when we integrate in time with the two-step BDF and fixed time step. The only remarkable difference is that, as we expected from the second order rate of convergence of the method in time, the temporal errors are smaller for the same values of the fixed time step k. Again, the estimations lie on a horizontal line being essentially the same as in the experiment on the left.

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