Condorcet domains of tiling type

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Abstract

We propose a method to construct "large" Condorcet domains by use of so-called rhombus tilings. Then we explain that this method fits to unify several previously known constructions of Condorcet domains. Finally, we discuss some conjectures on the size of such domains.

Keywords: rhombus tiling, weak Bruhat order, pseudo-line arrangement, alternating scheme, Fishburn's conjecture

1 Introduction

In the social choice theory, a *Condorcet domain*, further abbreviated as a CD, is meant to be a set of preferences with the property that, whenever the chosen preferences of all voters belong to this set, the aggregated (social) preference determined by the natural majority rule does not contain cycles. For a state of the art in this field, see, e.g., [12]. A challenging problem in the field is to construct CDs of "large" size. Several interesting methods based on different ideas have been proposed in literature.

One of them is a method of Abello [1] who constructed large CDs by completing a maximal chain in the Bruhat lattice. Chameni-Nembua [2] handled distributive sublattices in the Bruhat lattice. Fishburn [6] used a clever combination of "never conditions" to construct so-called "alternating schemes". Galambos and Reiner [8] proposed an approach using the second Bruhat order. However, each of these methods (which are briefly reviewed in the Appendix to this paper) is rather indirect and it may take some efforts to see that objects generated by the method are good CDs indeed.

In this paper we construct a class of complete (inclusion-wise maximal) CDs by using known planar graphical diagrams called *rhombus tilings*. Our construction and proofs are rather transparent and the CDs constructed admit a good visualization. It should be noted that the obtained CD class is essentially the same as each of three above-mentioned classes (namely, proposed by Abello, by Chameni-Nembua, and by Galambos and Reiner); see Appendix. Our main result (Theorem 4) asserts that any hump-hole domain is a subdomain of a tiling CD. As a consequence, three conjectures posed by Fishburn, by Monjardet, and by Galambos and Reiner turn out to be equivalent. A simple example shows that these conjectures are false.

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2 Linear orders and the Bruhat poset

Let X be a finite set whose elements are thought as *alternatives*. A *linear order* on X is a complete transitive binary relation $<$ on X. It ranges the elements of X, say, $x_1 < \ldots < x_n$, where $n = |X|$. Therefore, we can encode the linear orders on X by words of the form $x_1 \ldots x_n$, regarding x_1 as the least (or worst) alternative, x_2 as the next alternative, and so on; then x_n is the greatest (or best) alternative. The set of linear orders on X is denoted by $\mathcal{L}(X)$. If $Y \subset X$, we have a natural restriction map $\mathcal{L}(X) \to \mathcal{L}(Y)$.

In what follows we identify the ground set X with the set $[n]$ of integers $1, \ldots, n$ (and denote $\mathcal{L}(X)$ as $\mathcal{L}([n])$). The natural linear order $1 < 2 < \ldots < n$ is denoted by α, and the reversed order $1 > 2 > \ldots > n$ is denoted by ω. We use Greek symbols, e.g., σ, for linear orders on $[n]$, and write $i <_\sigma j$ instead of $i\,\sigma j$.

Let $\Omega = \{(i,j),\ i,j \in [n], i < j\}$. A pair $(i,j) \in \Omega$ is called an *inversion* for a linear order σ if $j <_\sigma i$. In other words, the symbol j occurs before i in the order $\sigma = s_1 \ldots s_n$. The set of inversions for σ is denoted by $Inv(\sigma)$. For example, $Inv(\alpha) = \emptyset$ and $Inv(\omega) = \Omega$.

Definition. For linear orders $\sigma, \tau \in \mathcal{L}([n])$, we write $\sigma \ll \tau$ if $Inv(\sigma) \subset Inv(\tau)$. The relation $\ll$ on $\mathcal{L}$ is called the *weak Bruhat order*, and the partially ordered set $(\mathcal{L}, \ll)$ is called the *Bruhat poset*.

Clearly $\ll$ is indeed a partial order, and the linear orders α and ω are the minimal and maximal elements. It is known that the Bruhat poset is a lattice, but we will not use this fact later on. Let us say that a linear order τ *covers* a linear order σ if $Inv(\tau)$ equals $Inv(\sigma)$ plus exactly one inversion. Drawing an arrow from σ to τ if τ covers σ, we obtain the so-called *Bruhat digraph*. The Bruhat poset $(\mathcal{L}, \ll)$ is the transitive closure of this digraph, and the latter is the Hasse diagram of the former. Ignoring the directions of arrows, we obtain the *Bruhat graph* (or the permutohedron) on the set $\mathcal{L}$. For $n = 3$ the Bruhat digraph is drawn in Fig. 1.

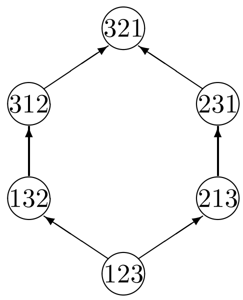

Fig. 1.

3 Condorcet domains

A set $\mathcal{D} \subset \mathcal{L}$ is called *cyclic* if there exist three alternatives i, j, k and three linear orders in $\mathcal{D}$ whose restrictions to $\{i, j, k\}$ have the form either ijk, jki, kij or kji, jik, ikj.

Otherwise $\mathcal{D}$ is called an *acyclic set of linear orders*, or a *Condorcet domain* (CD). Such domains are of interest in the social choice theory (see, e.g., [12]) because if all preferences of the voters form a CD then the naturally aggregated 'social preference' has no cycles (and therefore it is a linear order when the number of voters is odd). Conversely, if $\mathcal{D}$ is cyclic then there exist preference profiles which yield cycles in the 'social preference'.

In what follows we deal only with the domains $\mathcal{D}$ that contain the distinguished orders α and ω. An important problem is constructing 'large' CDs. More precisely, we say that a CD $\mathcal{D}$ is *complete* if it is inclusion-wise maximal, i.e. adding to $\mathcal{D}$ any new linear order would violate the acyclicity.

In the case $n = 3$ there are exactly four complete CDs. These are:

a) the set of four orders 123,132, 312 and 321. These orders are characterized by the property that the alternative 2 is never the worst. If we draw the corresponding utility functions, we observe that each of them has exactly one hump (or "peak"). Due to this, we call such a CD the *hump domain* and denote it as $\mathcal{D}_3(\cap)$.

b) the set of orders 123, 213, 231, 321. In these orders the alternative 2 is never the best. This CD is called the *hole domain* and denoted by $\mathcal{D}_3(\cup)$.

c) the set $\{123, 213, 312, 321\}$. Here the alternative 3 is never the middle. We denote this domain by $\mathcal{D}_3(\rightarrow)$.

d) the set $\mathcal{D}_3(\leftarrow) = \{123, 132, 231, 321\}$. Here the alternative 1 is never the middle.

A *casting* is a mapping c from the set $\binom{[n]}{3}$ of triples ijk ($i < j < k$) to the set $\{\cap, \cup, \rightarrow, \leftarrow\}$. For a casting c, we define $\mathcal{D}(c)$ to be the set of linear orders $\sigma \in \mathcal{L}$ whose restriction to any triple ijk (further denoted as $\sigma|_{ijk}$) belongs to $\mathcal{D}_3(c(ijk))$. The previous observations can be summarized as follows.

Proposition 1. 1) *For any casting c, the domain $\mathcal{D}(c)$ is a Condorcet domain.*

2) *Every Condorcet domain is contained in a set of the form $\mathcal{D}(c)$.*

Note that a casually chosen casting may produce a small CD. As Fishburn writes in [6]: ".. it is far from obvious how the restrictions should be selected jointly to produce a large acyclic set." In Sections 4–6 we describe and examine a simple geometric construction generating a representable class of complete CDs. Some facts given in these Sections are known, possibly being formulated in different terms. Nevertheless, we prefer to give short proofs to have our presentation self-contained.

4 Rhombus tilings

The complete CDs that we are going to introduce one-to-one correspond to certain known geometric arrangements on the plane, called rhombus tilings. We start with recalling this notion; this is dual, via a sort of planar duality, to the notion of pseudo-line arrangement (see, e.g., [5, 7] and see also [4] for some generalizations).

In the upper half-plane $\mathbb{R} \times \mathbb{R}_{>0}$, we fix n vectors $\xi_1, \ldots, \xi_n$ going clockwise around $(0, 0)$. It is convenient to assume that these vectors have the same length. The sum of n segments $[0, \xi_i]$, $i = 1, \ldots, n$, forms a *zonogon*; we denote it by Z_n. In other words, Z_n is the set of points $\sum_i a_i\xi_i$ over all $0 \le a_i \le 1$. It is a center-symmetric $2n$-gon with the *bottom* vertex $b = 0$ and the *top* vertex $t = \xi_1 + \ldots + \xi_n$. A *tile* (more precisely, an *ij-tile* for $i, j \in [n]$) is a rhombus congruent to the sum of two segments $[0, \xi_i]$ and $[0, \xi_j]$.

A *rhombus tiling* (or simply a *tiling*) is a subdivision T of the zonogon Z_n into a set of tiles which satisfy the following condition: if two tiles intersect then their intersection consists of a common vertex or a common edge. Figures 2 and 4 illustrate examples of rhombus tilings.

Orienting the edges of T upward, we obtain the structure of a planar digraph G_T on the set of vertices of T. The tiles of T are just the (inner two-dimensional) faces of G_T.

Next we need some more definitions. By a *snake* of a tiling T we mean a directed path in the digraph G_T going from the bottom vertex b to the top vertex t. For $i \in [n]$, the union of i-tiles is called an *i-track*, where an i-tile is a tile having an edge congruent to ξ_i. (The term "track" is borrowed from [9]; other known terms are "de Bruijn line", "dual path", "stripe".) One easily shows that the i-tiles form a sequence in which any two consecutive tiles have a common i-edge, and the first (last) tile contains the i-edge lying on the left (resp. right) boundary of Z_n. Also the following simple property takes place.

Lemma 1. *Every snake intersects an i-track by exactly one i-edge.*

Indeed, removing the i-track Q cuts the zonogon into two parts, upper and lower ones, and all i-edges of Q are directed from the lower part to the upper one. Therefore, any directed path of G_T can intersect Q at most once. This implies that any snake intersects Q exactly once (since it goes from the lower to the upper part of $Z_n - Q$). $\Box$

This lemma shows that any snake contains exactly one i-edge, for each i. So the sequence of "colors" of edges in a snake constitutes a word $\sigma = i_1 \ldots i_n$, which is a linear order on $[n]$. In what follows we do not distinguish between snakes S and their corresponding linear orders σ, denoting the snake as $\mathcal{S}(\sigma)$ and saying that the linear order σ is *compatible* with the tiling T. The set of linear orders compatible with T is denoted by $\Sigma(T)$.

Example 1. When $n = 3$, there are exactly two tilings of the zonogon (hexagon) Z_3, as depicted below:

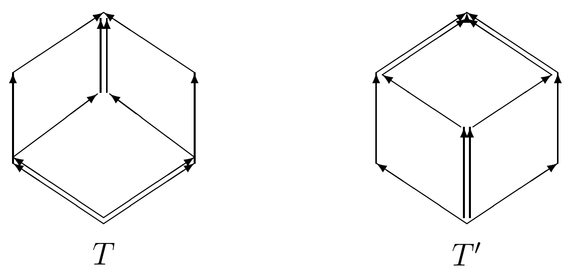

Fig. 2.

The set $\Sigma(T)$ consists of four orders, namely: 123,132,312,321. This is precisely the hump domain $\mathcal{D}(\cap)$. In its turn, the set $\Sigma(T')$ consists of four orders 123,213,231,321, which is just the hole domain $\mathcal{D}(\cup)$.

So, the domains $\Sigma(T)$ and $\Sigma(T')$ are CDs in this example. In Section 6 we explain that a similar property holds for any rhombus tiling.

5 Structure of the poset $\Sigma(T)$

Fix a tiling T of the zonogon Z_n. The snakes of T are partially ordered "from left to right" in a natural way. The minimal element is the leftmost snake $\mathcal{S}(\alpha)$ going along the left boundary of Z_n, and the maximal element is the rightmost snake $\mathcal{S}(\omega)$ going along the right boundary of Z_n. The set $\Sigma(T)$ equipped with this partial order is, obviously, a (distributive) lattice: for two (or more) snakes, their greatest lower bound is the left envelope of the snakes and their least upper bound is the right envelope.

In order to better understand a relationship between the partial order on $\Sigma(T)$ and the weak Bruhat order on $\mathcal{L}$, let us consider the mapping $\psi = \psi_T : Rho(T) \to \Omega$. Here $Rho(T)$ is the set of tiles in T and Ω is the set of pairs (i, j) with $i < j$. This mapping associates to each ij-tile the pair (i, j).

Lemma 2. *The mapping $\psi : Rho(T) \to \Omega$ is a bijection.*

We have to check that for any pair $(i, j) \in \Omega$, there exists exactly one ij-tile in the tiling T. It is clear for pairs of the form (i, n). Indeed, such tiles form the n-track and we can argue as in the proof of Lemma 1. If $j < n$ then the assertion follows by induction applied to the reduced tiling $T|_{[n-1]}$, see Section 6. $\square$

Given a snake $\mathcal{S}(\sigma)$, let $L(\sigma)$ be the set of tiles of the tiling T lying on the left from $\mathcal{S}(\sigma)$. The next assertion gives a visual description of inversions for a linear order $\sigma \in \Sigma(T)$.

Corollary 1. $\psi(L(\sigma)) = Inv(\sigma)$.

Indeed, let (i, j) be an inversion for σ. Then the edge of color i is situated in the snake $\mathcal{S}(\sigma)$ after the edge of color j. Therefore, the i- and j-tracks meet before they reach the snake $\mathcal{S}(\sigma)$, and hence the ij-tile where they meet lies on the left from $\mathcal{S}(\sigma)$. Conversely, if ij-tile lies on the left from the snake $\mathcal{S}(\sigma)$, then the i- and j-tracks meet before $\mathcal{S}(\sigma)$, implying that the j-edge appears in the snake before the i-edge. $\square$

Let us return to the partial order on $\Sigma(T)$. It is clear that a snake $\mathcal{S}(\sigma)$ lies on the left from a snake $\mathcal{S}(\tau)$ if and only if $L(\sigma) \subseteq L(\tau)$, that is (due to Corollary 1), if and only if $\sigma \ll \tau$. So the partial order on $\Sigma(T)$ is induced by the weak Bruhat order on $\mathcal{L}$. In reality, a sharper property takes place: *the covering relation on the poset $\Sigma(T)$ is the same as that on the Bruhat poset.* In other words, we assert that if a snake $\mathcal{S}(\tau)$ lies on the right from $\mathcal{S}(\sigma)$ and there is no snake between them, then these snakes differ by one tile.

Indeed, suppose that these snakes coincide until a vertex v and that the next elements are different: the edge e of $\mathcal{S}(\sigma)$ leaving v has color i, the edge e' of $\mathcal{S}(\tau)$ leaving v has color j, and $i \neq j$. Clearly $i < j$. We claim that the edges e, e' belong to a tile in T. Otherwise T would have an l-edge leaving v such that $i < l < j$, and we could draw an intermediate snake between $\mathcal{S}(\sigma)$ and $\mathcal{S}(\tau)$. Now consider the ij-tile ρ with the bottom at v. The first left edge of ρ (namely, e) belongs to the snake $\mathcal{S}(\sigma)$. One can see that the second left edge of ρ (which has color j) belongs to $\mathcal{S}(\sigma)$ as well. (If $\mathcal{S}(\sigma)$ contains another edge leaving the vertex $v+\xi_i$ then one can produce an intermediate snake between $\mathcal{S}(\sigma)$ and $\mathcal{S}(\tau)$.) For a similar reasons, both right edges of ρ belong to $\mathcal{S}(\tau)$. Thus, our snakes differ only by the tile ρ, as required.

As a consequence, we obtain that *any maximal chain in the poset* $\Sigma(T)$ *is a maximal chain in the Bruhat poset* $(\mathcal{L}, \ll)$.

6 Condorcet domains of tiling type

In this section we show that for any rhombus tiling T, the set $\Sigma(T)$ is a CD. The main role in the proof plays the reduction of a tiling under deleting elements from $[n]$. Let $i \in [n]$. As is said above, the i-track divides the zonogon into two parts: above and below the track. Remove this track from the tiling and move the upper part by the vector $-\xi_i$. As a result, we obtain a rhombus tiling T' of the reduced zonogon $Z' = Z_{n-1}$ determined by the vectors $\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_n$. The tiling T' is called the *reduction* of T by the alternative i and is denoted as $T|_{[n]-i}$.

Under this operation, a snake $\mathcal{S}(\sigma)$ compatible with the tiling T is transformed into a snake (corresponding to the restricted linear order $\sigma|_{[n]-i}$) which is compatible with the reduced tiling $T|_{[n]-i}$. This gives the restriction mapping

$$\Sigma(T) \to \Sigma(T|_{[n]-i}).$$

One can iterate the reduction operation by deleting alternatives in an arbitrary order, so as to reach a subset $X \subset [n]$. This gives the corresponding restriction mapping

$$\Sigma(T) \to \Sigma(T|_X).$$

Theorem 1. *The set* $\Sigma(T)$ *is a complete Condorcet domain.*

Proof. Consider the restriction of linear orders from $\Sigma(T)$ to a triple ijk, where $i < j < k$. By reasonings above, the restricted orders get into the domain $\Sigma(T|_{ijk})$, which is either $\mathcal{D}(\cup)$ or $\mathcal{D}(\cap)$ (defined in Section 2). Therefore, $\Sigma(T)$ is a CD.

To check the completeness of this domain, let us try to add to it a new linear order ρ. Let $\mathcal{S}(\rho)$ be the snake for ρ drawn in the zonogon. Then $\mathcal{S}(\rho)$ is not compatible with the tiling T. Let e be the first edge of the snake $\mathcal{S}(\rho)$ that is not an edge of T. There are three possible cases, as depicted in Figure 3.

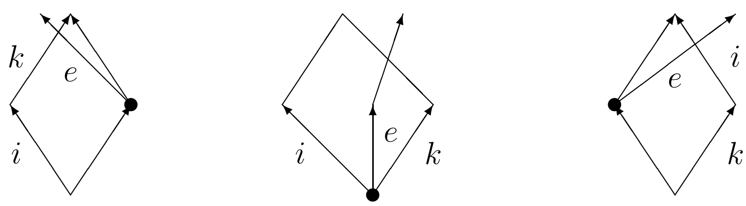

Fig. 3

Consider the middle case. Let the edge e be parallel to a vector ξ_j, and let the tile covering e be the ik-tile; it is clear that $i < j < k$. On the other hand, in the linear order ρ the alternative j occurs earlier than both i and k. Two subcases are possible: either $j <_\rho i <_\rho k$ or $j <_\rho k <_\rho i$. In the first subcase, add to ρ two linear orders from the domain $\Sigma(T)$, namely: $i <' k <' j$ (realized by a snake going through the left side of the ik-tile), and the linear order ω, yielding $k <_\omega j <_\omega i$. As a result, we obtain a cyclic triple. In the second subcase, we act symmetrically, by adding to ρ a linear order $k <'' i <'' j$ (realized by a snake going through the right side of the ik-tile) and the linear order α (yielding $i <_\alpha j <_\alpha k$), which again gives a cyclic triple.

Two other cases are examined in a similar way. $\square$

We refer to a domain of the form $\Sigma(T)$ as a *Condorcet domain of tiling type*, or a *tiling CD*.

7 Main result

A domain $\mathcal{D}$ in $\mathcal{L}$ is called a *hump-hole domain* if, for any triple ijk, either the hump condition $\mathcal{D}(\cap)$ or the hole condition $\mathcal{D}(\cup)$ is satisfied. As is seen from the proof of Theorem 1,

$(*)$ *any tiling CD is a hump-hole domain.*

We claim that the converse is also true.

Theorem 2. *Every hump-hole domain is contained in a Condorcet domain of tiling type.*

We need some preparations before proving this theorem.

Let σ be a linear order on $[n]$. A subset $X \subset [n]$ is an *ideal* of σ if $x \in X$ and $y <_\sigma x$ imply $y \in X$. In other words, if we represent σ as a word $i_1 \ldots i_n$, then an ideal of σ corresponds to an initial segment of this word. Denote by $Id(\sigma)$ the set of ideals of σ (including the empty set); so it is a set-system of cardinality $n+1$. For example, $Id(\alpha)$ consists of the intervals $[0], [1], \ldots, [n-1], [n]$.

Let $\mathcal{D}$ be a subset of $\mathcal{L}$. We associate to $\mathcal{D}$ the following set-system

$$Id(\mathcal{D}) = \cup_{\sigma \in \mathcal{D}} Id(\sigma).$$

Example 2. Let $\mathcal{D}$ be the hump domain for $n = 3$; it consists of the four orders 123, 132, 312, and 321. Then $Id(\mathcal{D})$ consists of the seven sets $\emptyset$, 1,3,12,13,23, and 123=[3], that is, of all subsets of [3] except for $\{2\}$ (since 2 is never the worst).

Similarly, if $\mathcal{D}$ is the hole domain, then $Id(\mathcal{D})$ consists of all subsets of [3] except for $\{1, 3\}$.

Consider a tiling T. We associate to each of its vertices v the subset $sp(v)$ of $[n]$ as follows. Let $\mathcal{S}(\sigma)$ be a snake passing v. Then $sp(v)$ is the ideal of the order σ corresponding to the part of $\mathcal{S}(\sigma)$ from the beginning to v. (One can see that $sp(v)$ does not depend on the choice of a snake σ passing v.) Equivalently, the set $sp(v)$ consists of all alternative which are 'not better than v'. One more equivalent definition is that $sp(v)$ consists of the elements $i \in [n]$ such that the i-track goes below the vertex v. The collection of sets $sp(v)$ over the set of vertices v of T, is denoted by $Sp(T)$ and called the *spectrum* of T. One can see that a linear order σ belongs to $\Sigma(T)$ if and only if the inclusion $Id(\sigma) \subset Sp(T)$ holds.

Proof of Theorem 2. Let $\mathcal{D}$ be a hump-hole domain. Our aim is to show the existence of a tiling T such that $Id(\mathcal{D}) \subset Sp(T)$. We will use a criterion due to Leclerc and Zelevinsky [11] (see also [3, Sec. 5.3]), on a system of subsets of $[n]$ that can be extended to the spectrum $Sp(T)$ of a tiling T. It is based on the following notion. Two subsets A, B of $[n]$ are said to be *separated* (more precisely, *strongly separated*, in terminology of [11]) from each other if the convex hulls of $A \setminus B$ and $B \setminus A$ (as the corresponding intervals in $\mathbb{R}$) do not intersect. For example, the sets $\{1,2\}$ and $\{2,4\}$ are separated, whereas $\{1,3\}$ and $\{2\}$ are not. In particular, A and B are separated if one includes the other. A collection of sets is called *separated* if any two of its elements are separated.

Theorem 3 [11]. *The spectrum $Sp(T)$ of any rhombus tiling T is separated. Conversely, if $\mathcal{X}$ is a separated system, then there exists a tiling T such that $\mathcal{X} \subset Sp(T)$.*

Due to this theorem, it suffices to show that for every hump-hole domain $\mathcal{D}$, the system $Id(\mathcal{D})$ is separated. Suppose this is not so for some $\mathcal{D}$. Then there exist two sets $A, B \in Id(\mathcal{D})$ and a triple $i < j < k$ in $[n]$ such that A contains j but none of i, k, whereas B contains i, k but not j. We can restrict the members of $\mathcal{D}$ to the set $\{i, j, k\}$, or assume that $n = 3$. Then $Id(\mathcal{D}|i, j, k)$ contains both sets $\{j\}$ and $\{i, k\}$. Thus, we are neither in the hump domain nor in the hole domain case, as we have seen in Example 2.
□

Now we combine Theorem 2 and a slight modification of property $(*)$, yielding the main assertion in this paper. Let us say that a domain $\mathcal{D}$ is *semi-connected* if the linear orders α and ω can be connected in the Bruhat graph by a path in which all vertices belong to $\mathcal{D}$.

Theorem 4. 1) *Every domain of tiling type is semi-connected.*

2) *Every semi-connected Condorcet domain is a hump-hole domain.*

3) *Every hump-hole domain is contained in a domain of tiling type.*

Proof of Theorem 4.

Any domain of the form $\Sigma(T)$ is semi-connected since it contains a maximal chain of the Bruhat poset, yielding the first claim.

It is easy to see that the semi-connectedness is stable under reductions. Because of this, we can restrict ourselves to the case $n = 3$. In this case there exist exactly four CDs. Two of them, where one of the alternatives 1 and 3 is never the middle, are not semi-connected. The other two domains are semi-connected; they are just hump and hole domains. This implies the second claim.

The third claim is just Theorem 2. □

As a consequence, we obtain that the CDs constructed by Abello[1], Galambos and Reiner [8], and Chameni-Nembua [2] (see the Appendix for a brief outline), as well as maximal hump-hole domains, are CDs of tiling type. Moreover, all these classes of CDs are equal.

8 On Fishburn's conjecture

Fishburn [6] constructed Condorcet domains by the following method. Given a set of linear orders and a triple $i < j < k$, the 'never condition' $jN1$ means the requirement that, in the restriction of each linear order to the set $\{i, j, k\}$, the alternative j is never the worst. One can see that this is exactly the case of 'hump condition'. Similarly, the 'never condition' $jN3$ (saying that "the alternative j is never the best") is equivalent to the 'hole condition'.

Fishburn's alternating scheme is defined by the following combination of hump and hole conditions. For each triple $i < j < k$, we impose the hump condition when j is even, and impose the hole condition when j is odd. The set of linear orders obeying these conditions constitutes the *Fishburn domain* and we denote its cardinality by $\Phi(n)$.

By Theorem 2, the Fishburn domain $\mathcal{D}$ is contained in a CD of tiling type. Also it is a complete CD, as is shown in [8]. So $\mathcal{D}$ is exactly a tiling CD. The corresponding tiling for $n = 8$ is drawn in Fig. 4.

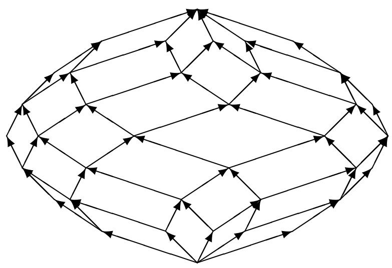

Fig. 4

Fishburn conjectured that *the size of any hump-hole CD does not exceed* $\Phi(n)$.

Galambos and Reiner [8] proposed the following weakening of Fishburn's conjecture (an equivalent conjecture in terms of pseudo-line arrangements was formulated by Knuth [10]):

Galambos-Reiner's conjecture: The size of any GR-domain does not exceed $\Phi(n)$.

Monjardet [12] calls a CD *connected* if it induces a connected subgraph of the Bruhat graph. His conjecture there sounds as follows: *the size of any connected CD does not exceed* $\Phi(n)$.

Due to our main result, the conjectures by Fishburn, by Galambos and Reiner, and by Monjardet are equivalent and they assert that $\gamma_n = \Phi(n)$, where γ_n is the maximum possible size of a tiling CD (for a given n). However, such an equality is false in general.

This is a consequence of some lower bound on γ_n given by Ondjey Bilka, as an anonymous referee of the original version of this paper kindly pointed out to us (though not providing us with details). A simple proof subsequently found by authors is as follows.

Let T and T' be rhombus tilings of zonogons Z_n and $Z_{n'}$, respectively. We will identify the set $[n']$ with the subset $\{n+1,\dots,n+n'\}$ in $[n+n']$. If we merge the top vertex of T with the bottom vertex of T' (putting T' over T), we obtain a partial tiling of the zonogon $Z_{n+n'}$, as illustrated in Fig. 5, where $n=4$ and $n'=3$.

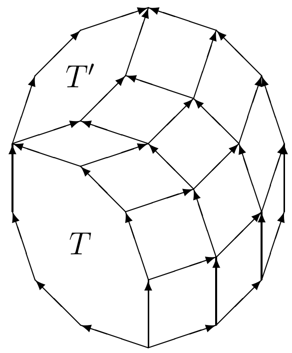

Fig. 5

This partial tiling can be extended (by a unique way) to a complete rhombus tiling $\widehat{T}$ of the whole zonogon $Z_{n+n'}$. If σ is a snake of T and σ' is a snake of T', then the concatenated path $\sigma\sigma'$ is a snake of the tiling $\widehat{T}$. Thus, we obtain the injective map

$$\Sigma(T)\times\Sigma(T')\to\Sigma(\widehat{T}),$$

which gives the inequality $\gamma_n\gamma_{n'}\le\gamma_{n+n'}$.

Now let T and T' be the Fishburn tilings for $n=n'=21$. From the formula for $\Phi(n)$ given in [8] one can compute that $\Phi(21)=4443896$ and $\Phi(42)=19.156.227.207.750$. Then $\Phi(21)^2=19.748.211.658.816>\Phi(42)$. Thus, $\Phi(42)<\gamma(42)$, disproving Fishburn's conjecture.

9 Some reformulations

It is easy to see that any linear order can be realized as a snake in some rhombus tiling. However, this need not hold for a pair of linear orders. For example, the linear orders 213 and 312 (which together with 123 and 321 form the CD $\mathcal{D}_3(\leftarrow)$) cannot appear in the same tiling.

Let us say that two linear orders σ and τ are *strongly consistent* if there exists a tiling T such that $\sigma,\tau\in\Sigma(T)$. For example, σ and τ are strongly consistent if $\sigma\ll\tau$. Using observations and result from previous sections, one can demonstrate some useful equivalence relations.

Proposition 2. *Let σ and τ be linear orders in $[n]$. The following properties are equivalent:*

(i) linear orders σ and τ are strongly consistent;

(ii) the set-system $Id(\sigma) \cup Id(\tau)$ is separated;

(iii) for each triple $i < j < k$, the restrictions of σ and τ to this triple are simultaneously either humps or holes;

(iv) $Id(\sigma) \cup Id(\tau) = Id(\sigma \vee \tau) \cup Id(\sigma \wedge \tau)$;

(iv′) $Id(\sigma) \cup Id(\tau) \subset Id(\sigma \vee \tau) \cup Id(\sigma \wedge \tau)$.

Proof. Properties (i) and (ii) are equivalent by Theorem 3.

Properties (i) and (iii) are equivalent by Theorem 2.

To see that (i) implies (iv), observe that if σ and τ occur in a tiling T, then $\mathcal{S}(\sigma \vee \tau)$ and $\mathcal{S}(\sigma \wedge \tau)$ are the left and right envelopes of the snakes for σ and τ, respectively. Therefore, any vertex of the snake $\mathcal{S}(\sigma \vee \tau)$ is a vertex of $\mathcal{S}(\sigma)$ or $\mathcal{S}(\tau)$. Conversely, each vertex of $\mathcal{S}(\sigma)$ is a vertex of $\mathcal{S}(\sigma \vee \tau)$ or $\mathcal{S}(\sigma \wedge \tau)$.

Obviously, (iv) imply (iv′). Let us prove that (iv′) implies (ii). Since $\sigma \wedge \tau \ll \sigma \vee \tau$, the linear orders $\sigma \wedge \tau$ and $\sigma \vee \tau$ are strongly consistent. By the equivalence of (i) and (ii), $Id(\sigma \vee \tau) \cup Id(\sigma \wedge \tau)$ is a separated system. Since $Id(\sigma) \cup Id(\tau) \subseteq Id(\sigma \vee \tau) \cup Id(\sigma \wedge \tau)$, the set-system $Id(\sigma) \cup Id(\tau)$ is separated as well. □

Appendix

Here we briefly outline approaches of Abello [1], Galambos and Reiner [8], and Chameni-Nembua [2], and an interrelation between them and our approach.

Abello

Let $\mathcal{D}$ be a CD. Then there exists a casting c such that $\mathcal{D} \subset \mathcal{D}(c)$ (see Proposition 1). Abello applies this fact to a maximal chain $\mathcal{C}$ in the Bruhat lattice (it had been known that any chain is a CD). In this case the casting c is unique (and is a hump-hole casting), so the domain $\mathcal{C}(c)$ (denoted by $\widehat{\mathcal{C}}$) is also a CD. We call such a CD by *A-domain*. Abello shows that an A-domain is a complete CD.

Different chains can give the same A-domain. Maximal chains $\mathcal{C}$ and $\mathcal{C}'$ are called *equivalent* if the A-domains $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{C}'}$ coincide. In the conclusion of his article Abello gives another characterization of this equivalence. A maximal chain in the Bruhat lattice can be thought as a reduced decomposition (in a product of adjacent transpositions s_i, $i = 1, ..., n-1$) of the inverse permutation ω. Namely, chains are equivalent if one reduced decomposition can be obtained from the other by a sequence of transformations when a decomposition of the form $...s_i s_j...$ (with $|i - j| > 1$) changes to a decomposition of the form $...s_j s_i...$. This characterization played the role of the starting point for Galambos and Reiner approach.

Galambos and Reiner

Let $\mathbf{C}$ be an equivalence class of maximal chains. (In reality, Galambos and Reiner define the equivalence in a somewhat different way; see Definition 2.5 in [8].) Define $\mathcal{D}(\mathbf{C}) := \cup_{\mathcal{C}\in\mathbf{C}}\mathcal{C}$; in their terminology, this domain consists of "permutations visited by an equivalence class of maximal reduced decompositions"). We call such domains by *GR-domains*. It is easy to see (and Galambos and Reiner explicitly mention it) that GR-domains are exactly A-domains. Nevertheless, they give explicit proofs, in Theorems 1 and 2 of [8], that GR-domains are complete CDs.

To give more enlightening representation for these equivalence classes of maximal reduced decompositions, Galambos and Reiner use the so-called arrangements of pseudo-lines. Permutations (or linear orders) from the domain $\mathcal{D}(\mathbf{C})$ are realized in these terms as cutpaths (viz. directed cuts) of such an arrangement. Although they do not prove explicitly that the set of cutpaths of an arrangement forms a complete CD, it can be done rather easily. (We just have done this in Section 6 working in dual terms of rhombus tilings.) One can see from these arguments that GR-domains (as well as A-domains) are nothing but CDs of tiling type.

We prefer to use in this paper the language of rhombus tiling, rather then pseudo-line arrangements, because of their better visualization and simplicity to handle. In all other respects, these approaches are equivalent.

Chameni-Nembua

One more approach was proposed by Chameni-Nembua. A sublattice $\mathcal{L}$ in the Bruhat lattice is called *covering* if the cover relation in this sublattice is induced by the cover relation in the Bruhat lattice.

Chameni-Nembua shows that a distributive covering sublattice in the Bruhat lattice is a CD. Suppose now that $\mathcal{L}$ is a maximal distributive covering sublattice. One can easily see that it contains α and ω and, hence, it contains a maximal chain. Therefore it is a subset of a unique tiling CD. On the other hand, since the tiling CD is a distributive covering sublattice (see Section 4), we can conclude that $\mathcal{L}$ is the whole tiling CD.

Thus, Chameni-Nembua approach gives the same CDs as the rhombus tilings.

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