ON FUNCTIONAL RELATIONS FOR THE ALTERNATING ANALOGUES OF TORNHEIM'S DOUBLE ZETA FUNCTION

ZHONG-HUA LI

Department of Mathematics, Tongji University, No. 1239 Siping Road, Shanghai 200092, China
Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan
E-mail address: lizhmath@gmail.com

Abstract. We give two functional relations for the alternating analogues of Tornheim's double zeta function. Using the functional relations, we give new proofs of some evaluation formulas found by H. Tsumura for these alternating series.
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1. Introduction

For $s_1, s_2, s_3 \in \mathbb{C}$ with $\Re(s_1 + s_3) > 1$, $\Re(s_2 + s_3) > 1$ and $\Re(s_1 + s_2 + s_3) > 2$, the Tornheim's double zeta function is defined as

$$T(s_1, s_2, s_3) := \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3}}. \tag{1.1}$$

It has two alternating analogues:

$$R(s_1, s_2, s_3) := \sum_{m,n=1}^{\infty} \frac{(-1)^n}{m^{s_1} n^{s_2} (m+n)^{s_3}}, \tag{1.2}$$

and

$$S(s_1, s_2, s_3) := \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{m^{s_1} n^{s_2} (m+n)^{s_3}}. \tag{1.3}$$

Since L. Tornheim [6] introduced the series $T(p, q, r)$ for $p, q, r \in \mathbb{N}$ in 1950's, a lot of results on evaluating the values $T(p, q, r)$ in terms of Riemann zeta values have been found. See for example [1, 3, 4, 5, 6, 7, 10] and the references therein. In [10], H. Tsumura proved a functional relation for Tornheim's double zeta function which represents $T(a, b, s) + (-1)^b T(b, s, a) + (-1)^a T(s, a, b)$ with $a, b \in \mathbb{N}$ and $s \in \mathbb{C}$ via Riemann zeta function. After that, T. Nakamura [4] gave a "simpler" version of this functional relation. And in [2], K. Matsumoto, T. Nakamura, H. Ochiai and H. Tsumura showed that these two functional relations are the same.

The alternating analogues of Tornheim's double zeta series were first introduced by M. V. Subbarao and R. Sitaramachandrarao in [5]. They posed the problem to evaluate $S(r, r, r)$ and $R(r, r, r)$ for $r \in \mathbb{N}$. In a series of papers [7, 8, 9, 11], H. Tsumura obtained some fascinating results on evaluating $S(p, q, r)$ and $R(p, q, r)$.

He gave an evaluation formula for $S(r,r,r)$ for any positive odd integer r in [7], and for $R(r,r,r)$ for any positive odd integer r in [8]. In [9], he obtained the evaluation formula for $S(p,q,r)$ with $p,q,r\in\mathbb{N}$ and $p+q+r$ odd. To evaluate $R(p,q,r)$, H. Tsumura introduced the partial Tornheim's double series defined by

$$\mathfrak{T}_{b_1,b_2}(p,q,r) := \sum_{m,n=0}^{\infty} \frac{1}{(2m+b_1)^p(2n+b_2)^q(2m+2n+b_1+b_2)^r}, \tag{1.4}$$

where $b_1,b_2\in\{1,2\}$. Then in [11, Theorem 4.1], he proved that for any $p,q,r\in\mathbb{N}$ with $r\geqslant 2$ and $p+q+r$ odd, and for $b_1,b_2\in\{1,2\}$, the values $R(p,q,r)$ and $\mathfrak{T}_{b_1,b_2}(p,q,r)$ can be expressed as polynomials in Riemann zeta values $\zeta(j)$ $(2\leqslant j\leqslant p+q+r)$ with rational coefficients.

In this paper, we give two functional relations for $S(s_1,s_2,s_3)$ and $R(s_1,s_2,s_3)$ in Theorem 3.3, from which we obtain new proofs for formulas of $S(p,q,r)$ and $R(p,q,r)$ mentioned in the last paragraph. The method used here is different from that of H. Tsumura [10] and of T. Nakamura [4] for Tornheim's double zeta function case. In fact, it is also valid for proving T. Nakamura's functional relation for Tornheim's double zeta function. We give this new proof in Section 2. Then in Section 3, we prove our main theorem (Theorem 3.3). In Section 4, we give new proofs of H. Tsumura's results mentioned in the last paragraph.

2. A functional relation for Tornheim's double zeta function

In [10, Theorem 4.5], H. Tsumura proved the following functional relation for Tornheim's double zeta function:

$$\begin{aligned} &T(a,b,s)+(-1)^bT(b,a,s)+(-1)^aT(s,a,b) \\ &= 2\sum_{\substack{j=0\\ j\equiv a(2)}}^{a}(2^{1-a+j}-1)\zeta(a-j)\sum_{l=0}^{j/2}\frac{(\pi i)^{2l}}{(2l)!}\binom{b-1+j-2l}{j-2l}\zeta(b+j+s-2l) \\ &\quad -4\sum_{\substack{j=0\\ j\equiv a(2)}}^{a}(2^{1-a+j}-1)\zeta(a-j)\sum_{l=0}^{(j-1)/2}\frac{(\pi i)^{2l}}{(2l+1)!}\sum_{\substack{k=0\\ k\equiv b(2)}}^{b}\zeta(b-k) \\ &\quad\times\binom{k-1+j-2l}{j-2l-1}\zeta(k+j+s-2l), \end{aligned} \tag{2.1}$$

where (2) means mod 2, $a,b\in\mathbb{N}\cup\{0\}$, $b\geqslant 2$, $s\in\mathbb{C}$, except for the singular points of both sides. In [4], T. Nakamura gave a "simpler" version, which can be restated as the following theorem.

Theorem 2.1 ([4, Theorem 1.2])**.** *For all $a,b\in\mathbb{N}$ and $s\in\mathbb{C}$ except for the singular points, we have*

$$T(a,b,s)+(-1)^bT(b,a,s)+(-1)^aT(s,a,b)=2N(a,b,s)+2N(b,a,s), \tag{2.2}$$

where

$$N(a,b,s):=\sum_{j=0}^{a/2}\binom{a+b-2j-1}{b-1}\zeta(2j)\zeta(a+b+s-2j).$$

In [2], K. Matsumoto T. Nakamura, H. Ochiai and H. Tsumura showed that the right-hand sides of (2.1) and (2.2) are the same. In this section, we first restate their proof with a different method to obtain the key formulas used in the proof. Then we give a new proof of the functional relation (2.2).

Recall that the Bernoulli polynomials $\{B_n(x)\}$ and Bernoulli numbers $\{B_n\}$ are defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad \frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},$$

respectively. It is known that $B_n(0)=B_n(1)=B_n$ for any $n\neq 1$ and $B_1(0)=B_1=-B_1(1)=-\frac{1}{2}$. We recall the formulas

$$\zeta(2n)=-\frac{(2\pi i)^{2n}}{2(2n)!}B_{2n}, \tag{2.3}$$

$$B_n(1/2)=(2^{1-n}-1)B_n, \tag{2.4}$$

$$B_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k(x)y^{n-k}. \tag{2.5}$$

From the translation formula (2.5), we immediately get the following lemma.

Lemma 2.2. *For any nonnegative integer n, we have*

$$\sum_{k=0}^{n}\binom{2n}{2k}B_{2k}(x)y^{2n-2k}=\frac{1}{2}(B_{2n}(x+y)+B_{2n}(x-y)), \tag{2.6}$$

and

$$\sum_{k=0}^{n}\binom{2n+1}{2k}B_{2k}(x)y^{2n+1-2k}=\frac{1}{2}(B_{2n+1}(x+y)-B_{2n+1}(x-y)). \tag{2.7}$$

Using the above lemma, we get the following key formulas for proving the fact that the right-hand side of (2.1) equals that of (2.2).

Lemma 2.3. *For any nonnegative integer n, we have*

$$\sum_{k=0}^{n}(2^{1-2k}-1)\zeta(2k)\frac{(\pi i)^{2n-2k}}{(2n-2k)!}=\zeta(2n), \tag{2.8}$$

and

$$\sum_{k=0}^{n}(2^{1-2k}-1)\zeta(2k)\frac{(\pi i)^{2n-2k}}{(2n-2k+1)!}=-\frac{1}{2}\delta_{n,0}, \tag{2.9}$$

where δ_{ij} is the Kronecker's delta symbol.

Proof. (2.8) follows from (2.3), (2.4) and (2.6). And (2.9) follows from (2.3), (2.4) and (2.7). □

We come to prove the fact that the right-hand sides of (2.1) and (2.2) are the same. The right-hand side of (2.1) is $2R_1+2R_2$, where

$$R_1=\sum_{\substack{j=0\\ j\equiv a(2)}}^{a}(2^{1-a+j}-1)\zeta(a-j)\sum_{l=0}^{j/2}\frac{(\pi i)^{2l}}{(2l)!}\binom{b-1+j-2l}{j-2l}\zeta(b+j+s-2l),$$

and

$$\begin{aligned}R_2 =& -2 \sum_{\substack{j=0\\ j\equiv a(2)}}^{a} (2^{1-a+j}-1)\zeta(a-j) \sum_{l=0}^{(j-1)/2} \frac{(\pi i)^{2l}}{(2l+1)!} \sum_{\substack{p=0\\ p\equiv b(2)}}^{b} \zeta(b-p)\\ &\times \binom{p-1+j-2l}{j-2l-1}\zeta(p+j+s-2l).\end{aligned}$$

We show that $R_1 = N(a,b,s)$ and $R_2 = N(b,a,s)$. For R_1, let $a-j=2k$ and $k+l=n$, we get

$$R_1 = \sum_{k=0}^{a/2}(2^{1-2k}-1)\zeta(2k)\sum_{n=k}^{a/2}\frac{(\pi i)^{2n-2k}}{(2n-2k)!}\binom{a+b-2n-1}{a-2n}\zeta(a+b+s-2n).$$

Changing the order of n and k, we get

$$R_1 = \sum_{n=0}^{a/2}\left(\sum_{k=0}^{n}(2^{1-2k}-1)\zeta(2k)\frac{(\pi i)^{2n-2k}}{(2n-2k)!}\right)\binom{a+b-2n-1}{a-2n}\zeta(a+b+s-2n),$$

which is $N(a,b,s)$ by (2.8). Similarly for R_2, we have

$$\begin{aligned}R_2 =& -2\sum_{k=0}^{a/2}(2^{1-2k}-1)\zeta(2k)\sum_{n=k}^{(a-1)/2}\frac{(\pi i)^{2n-2k}}{(2n-2k+1)!}\sum_{m=0}^{b/2}\zeta(2m)\\ &\times\binom{a+b-2m-2n-1}{a-2n-1}\zeta(a+b+s-2m-2n)\\ =& -2\sum_{n=0}^{(a-1)/2}\left(\sum_{k=0}^{n}(2^{1-2k}-1)\zeta(2k)\frac{(\pi i)^{2n-2k}}{(2n-2k+1)!}\right)\sum_{m=0}^{b/2}\zeta(2m)\\ &\times\binom{a+b-2m-2n-1}{a-2n-1}\zeta(a+b+s-2m-2n),\end{aligned}$$

which is $N(b,a,s)$ by (2.9). Hence the right-hand sides of (2.1) and (2.2) are the same.

In the rest of this section, we give a new proof of the functional relation (2.2). We set

$$F(a,b,s) := T(a,b,s) + (-1)^b T(b,s,a) + (-1)^a T(s,a,b).$$

It is well-known that the Tornheim's double zeta function satisfies the following recursive formula

$$T(s_1,s_2,s_3) = T(s_1-1,s_2,s_3+1) + T(s_1,s_2-1,s_3+1).$$

We find that

$$\begin{aligned}F(a,b,s) =&(T(a-1,b,s+1)+T(a,b-1,s+1))\\ &+(-1)^b(T(b,s+1,a-1)-T(b-1,s+1,a))\\ &+(-1)^a(-T(s+1,a-1,b)+T(s+1,a,b-1)),\end{aligned}$$

which is just

$$F(a,b,s) = F(a-1,b,s+1) + F(a,b-1,s+1). \tag{2.10}$$

As stated in [1], we have the following general lemma.

Lemma 2.4. *Let $X(s_1, s_2, s_3)$ be a function satisfying the recursive relation*

$$X(a, b, s) = X(a-1, b, s+1) + X(a, b-1, s+1).$$

Then for any $a, b \in \mathbb{N} \cup \{0\}$ and $s \in \mathbb{C}$, we have

$$\begin{aligned} X(a,b,s) = &\sum_{j=1}^{a} \binom{a+b-j-1}{b-1} X(j, 0, a+b+s-j) \\ &+ \sum_{j=1}^{b} \binom{a+b-j-1}{a-1} X(0, j, a+b+s-j). \end{aligned} \tag{2.11}$$

One can prove this lemma by induction on $a+b$.

Now we apply Lemma 2.4 to $F(a, b, s)$. We first compute $F(j, 0, a+b+s-j)$ and $F(0, j, a+b+s-j)$. It is easy to see $F(j, 0, a+b+s-j) = F(0, j, a+b+s-j)$. And we have

$$\begin{aligned} &F(j, 0, a+b+s-j) \\ =&T(j, 0, a+b+s-j) + T(0, a+b+s-j, j) + (-1)^j T(a+b+s-j, j, 0) \\ =&(1+(-1)^j)\zeta(j)\zeta(a+b+s-j) - \zeta(a+b+s). \end{aligned}$$

Then we get

$$\begin{aligned} &\sum_{j=1}^{a} \binom{a+b-j-1}{b-1} F(j, 0, a+b+s-j) \\ =&2\sum_{j=0}^{a/2} \binom{a+b-2j-1}{b-1} \zeta(2j)\zeta(a+b+s-2j) - \sum_{j=1}^{a} \binom{a+b-j-1}{b-1} \zeta(a+b+s) \\ =&2\sum_{j=0}^{a/2} \binom{a+b-2j-1}{b-1} \zeta(2j)\zeta(a+b+s-2j) + 2\binom{a+b-1}{b} \zeta(0)\zeta(a+b+s), \end{aligned}$$

and similarly

$$\begin{aligned} &\sum_{j=1}^{b} \binom{a+b-j-1}{a-1} X(0, j, a+b+s-j) \\ =&2\sum_{j=0}^{b/2} \binom{a+b-2j-1}{a-1} \zeta(2j)\zeta(a+b+s-2j) + 2\binom{a+b-1}{a} \zeta(0)\zeta(a+b+s). \end{aligned}$$

Here we use the fact that $\zeta(0) = -\frac{1}{2}$. Combining these two equations and Lemma 2.4, we finish the proof of Theorem 2.1.

3. Functional relations for $S(s_1, s_2, s_3)$ and $R(s_1, s_2, s_3)$

As in [12], we define

$$\zeta(\bar{s}_1, \bar{s}_2) := \sum_{m>n>0} \frac{(-1)^{m+n}}{m^{s_1} n^{s_2}},$$

$$\zeta(\bar{s}_1, s_2) := \sum_{m>n>0} \frac{(-1)^{m}}{m^{s_1} n^{s_2}},$$

$$\zeta(s_1, \bar{s}_2) := \sum_{m>n>0} \frac{(-1)^{n}}{m^{s_1} n^{s_2}},$$

and

$$\zeta(\bar{s}) := \sum_{m=1}^{\infty} \frac{(-1)^m}{m^s} = (2^{1-s} - 1)\zeta(s).$$

It is easy to see that

$$\zeta(\bar{s})\zeta(t) = \zeta(\bar{s}, t) + \zeta(t, \bar{s}) + \zeta(\overline{s+t}), \tag{3.1}$$

$$\zeta(\bar{s})\zeta(\bar{t}) = \zeta(\bar{s}, \bar{t}) + \zeta(\bar{t}, \bar{s}) + \zeta(s+t). \tag{3.2}$$

For $a, b \in \mathbb{N}$ and $s \in \mathbb{C}$, we define

$$F_1(a, b, s) := S(a, b, s) + (-1)^b R(b, s, a) + (-1)^a R(a, s, b),$$

$$F_2(a, b, s) := R(a, b, s) + (-1)^b R(s, b, a) + (-1)^a S(a, s, b).$$

Similar to Tornheim's double zeta function, we have the recursive relations:

$$S(s_1, s_2, s_3) = S(s_1 - 1, s_2, s_3 + 1) + S(s_1, s_2 - 1, s_3 + 1),$$

$$R(s_1, s_2, s_3) = R(s_1 - 1, s_2, s_3 + 1) + R(s_1, s_2 - 1, s_3 + 1).$$

Then we get the following lemma.

Lemma 3.1. *We have the following recursive relations:*

$$F_1(a, b, s) = F_1(a - 1, b, s + 1) + F_1(a, b - 1, s + 1), \tag{3.3}$$

$$F_2(a, b, s) = F_2(a - 1, b, s + 1) + F_2(a, b - 1, s + 1). \tag{3.4}$$

Before applying Lemma 2.4 to F_1 and F_2, we make some preparations.

Lemma 3.2. *We have*

$$\begin{aligned} F_1(j, 0, a+b+s-j) &= F_1(0, j, a+b+s-j) \\ &= (1 + (-1)^j)\zeta(j)\zeta(\overline{a+b+s-j}) - \zeta(\overline{a+b+s}), \end{aligned} \tag{3.5}$$

$$F_2(j, 0, a+b+s-j) = (1 + (-1)^j)\zeta(\bar{j})\zeta(\overline{a+b+s-j}) - \zeta(a+b+s), \tag{3.6}$$

$$F_2(0, j, a+b+s-j) = (1 + (-1)^j)\zeta(\bar{j})\zeta(a+b+s-j) - \zeta(\overline{a+b+s}). \tag{3.7}$$

Proof. (3.5) and (3.6) follow from (3.1), and (3.7) follows from (3.2). □

By Lemma 2.4 and the above two lemmas, we immediately get the main result of this section.

Theorem 3.3. *For all $a, b \in \mathbb{N}$ and $s \in \mathbb{C}$ except for the singular points, we have*

$$S(a, b, s) + (-1)^b R(b, s, a) + (-1)^a R(a, s, b) = 2N_1(a, b, s) + 2N_1(b, a, s), \tag{3.8}$$

and

$$R(a, b, s) + (-1)^b R(s, b, a) + (-1)^a S(a, s, b) = 2N_2(a, b, s) + 2N_3(b, a, s), \tag{3.9}$$

where

$$N_1(a,b,s) := \sum_{j=0}^{a/2} \binom{a+b-2j-1}{b-1} (2^{2j+1-a-b-s}-1)\zeta(2j)\zeta(a+b+s-2j),$$

$$N_2(a,b,s) := \sum_{j=0}^{a/2} \binom{a+b-2j-1}{b-1} (2^{1-2j}-1)(2^{2j+1-a-b-s}-1)\zeta(2j)\zeta(a+b+s-2j),$$

$$N_3(a,b,s) := \sum_{j=0}^{a/2} \binom{a+b-2j-1}{b-1} (2^{1-2j}-1)\zeta(2j)\zeta(a+b+s-2j).$$

Note that

$$N_1(a,b,s) + N_2(a,b,s) + N_3(a,b,s) = (2^{2-a-b-s}-1)N(a,b,s),$$

where $N(a,b,s)$ is defined in Section 2.

4. Applications of functional relations

In [4, Section 3], T. Nakamura used the functional relation (2.2) to give new proofs of some formulas for the special values of $T(p,q,r)$ with $p,q,r \in \mathbb{N}$. For example, we have the evaluation formula of $T(p,q,r)$ when $p+q+r$ is odd as in [1, 4].

Proposition 4.1. *For $p,q,r \in \mathbb{N}$ with $p+q+r$ odd, we have*

$$T(p,q,r) = (-1)^p N(p,r,q) + (-1)^p N(r,p,q) + (-1)^q N(q,r,p) + (-1)^q N(r,q,p).$$

In this section, we use the functional relations (3.8) and (3.9) to deduce some formulas for the special values of $S(p,q,r)$, $R(p,q,r)$ and $\mathfrak{T}_{b_1,b_2}(p,q,r)$ with $p,q,r \in \mathbb{N}$ and $b_1, b_2 \in \{1,2\}$.

Let $a=b=s=r \in \mathbb{N}$ in (3.8) and (3.9), we get

$$S(r,r,r) + 2(-1)^r R(r,r,r) = 4N_1(r,r,r), \tag{4.1}$$

$$(1+(-1)^r)R(r,r,r) + (-1)^r S(r,r,r) = 2N_2(r,r,r) + 2N_3(r,r,r). \tag{4.2}$$

Let $r=2p$ be even in (4.1) and (4.2), we get a formula which was mentioned in [7, Eq. (4.2)].

Proposition 4.2. *For any $p \in \mathbb{N}$, we have*

$$\begin{aligned} &S(2p,2p,2p) + 2R(2p,2p,2p) \\ =&4\sum_{j=0}^{p} \binom{4p-2j-1}{2p-1} (2^{2j+1-6p}-1)\zeta(2j)\zeta(6p-2j) \\ =&2\sum_{j=0}^{p} \binom{4p-2j-1}{2p-1} (2^{2-6p}-2^{2j+1-6p})\zeta(2j)\zeta(6p-2j). \end{aligned}$$

The above formulas give some relations for Riemann zeta values. For example, taking $p=1$, we get the relation $7\zeta(6) = 4\zeta(2)\zeta(4)$.

Let $r = 2p+1$ be odd in (4.2), we get the evaluation formula of $S(2p+1, 2p+1, 2p+1)$ as in [7, 2].

Proposition 4.3. *For any $p \in \mathbb{N} \cup \{0\}$, we have*

$$\begin{aligned}&S(2p+1,2p+1,2p+1)\\=&2^{-6p}\sum_{j=0}^{p}\binom{4p+1-2j}{2p}(2^{2j-1}-1)\zeta(2j)\zeta(6p+3-2j).\end{aligned}$$

Let $r = 2p+1$ be odd in (4.1). Using the above formula for $S(2p+1, 2p+1, 2p+1)$, we get the evaluation formula of $R(2p+1, 2p+1, 2p+1)$ as in [8, 2].

Proposition 4.4. *For any $p \in \mathbb{N} \cup \{0\}$, we have*

$$\begin{aligned}&R(2p+1,2p+1,2p+1)\\=&2^{-6p-1}\sum_{j=0}^{p}\binom{4p+1-2j}{2p}(2^{6p+2}-2^{2j-1}-1)\zeta(2j)\zeta(6p+3-2j).\end{aligned}$$

Let $a = p$, $b = q$ and $s = r$ in (3.8), we get

$$S(p,q,r)+(-1)^qR(q,r,p)+(-1)^pR(p,r,q)=2N_1(p,q,r)+2N_1(q,p,r). \tag{4.3}$$

Let $a = p$, $b = r$ and $s = q$ in (3.9), we get

$$R(p,r,q)+(-1)^rR(q,r,p)+(-1)^pS(p,q,r)=2N_2(p,r,q)+2N_3(r,p,q),$$

which is

$$(-1)^pR(p,r,q)+(-1)^{p+r}R(q,r,p)+S(p,q,r)=2(-1)^p(N_2(p,r,q)+N_3(r,p,q)). \tag{4.4}$$

The difference of (4.3) and (4.4) gives

$$\begin{aligned}&((-1)^q-(-1)^{p+r})R(q,r,p)\\=&2N_1(p,q,r)+2N_1(q,p,r)-2(-1)^p(N_2(p,r,q)+N_3(r,p,q)),\end{aligned}$$

which deduces the evaluation formula of $R(p, q, r)$ when $p + q + r$ is odd as in [11].

Proposition 4.5. *For $p, q, r \in \mathbb{N}$ with $p + q + r$ odd, we have*

$$R(p,q,r)=(-1)^pN_1(r,p,q)+(-1)^pN_1(p,r,q)+(-1)^qN_2(r,q,p)+(-1)^qN_3(q,r,p).$$

Explicitly, we have

$$\begin{aligned}&R(p,q,r)\\=&(-1)^p\sum_{j=0}^{p/2}\binom{p+r-2j-1}{r-1}(2^{2j+1-p-q-r}-1)\zeta(2j)\zeta(p+q+r-2j)\\&+(-1)^q\sum_{j=0}^{q/2}\binom{q+r-2j-1}{r-1}(2^{1-2j}-1)\zeta(2j)\zeta(p+q+r-2j)\\&+(-1)^p\sum_{j=0}^{r/2}\binom{p+r-2j-1}{p-1}(2^{2j+1-p-q-r}-1)\zeta(2j)\zeta(p+q+r-2j)\\&+(-1)^q\sum_{j=0}^{r/2}\binom{q+r-2j-1}{q-1}(2^{1-2j}-1)(2^{2j+1-p-q-r}-1)\zeta(2j)\zeta(p+q+r-2j).\end{aligned}$$

With the help of the above proposition and (4.3), we get the evaluation formula of $S(p, q, r)$ when $p + q + r$ is odd as in [9].

Proposition 4.6. *For $p,q,r\in\mathbb{N}$ with $p+q+r$ odd, we have*

$$S(p,q,r)=(-1)^pN_2(p,r,q)+(-1)^qN_2(q,r,p)+(-1)^pN_3(r,p,q)+(-1)^qN_3(r,q,p).$$

More precisely, we have

$$\begin{aligned}
&S(p,q,r)\\
=&(-1)^p\sum_{j=0}^{p/2}\binom{p+r-2j-1}{r-1}(2^{1-2j}-1)(2^{2j+1-p-q-r}-1)\zeta(2j)\zeta(p+q+r-2j)\\
&+(-1)^q\sum_{j=0}^{q/2}\binom{q+r-2j-1}{r-1}(2^{1-2j}-1)(2^{2j+1-p-q-r}-1)\zeta(2j)\zeta(p+q+r-2j)\\
&+(-1)^p\sum_{j=0}^{r/2}\binom{p+r-2j-1}{p-1}(2^{1-2j}-1)\zeta(2j)\zeta(p+q+r-2j)\\
&+(-1)^q\sum_{j=0}^{r/2}\binom{q+r-2j-1}{q-1}(2^{1-2j}-1)\zeta(2j)\zeta(p+q+r-2j).
\end{aligned}$$

The evaluation formula for $S(p,q,r)$ with $p+q+r$ odd given by H. Tsumura in [9] reads

$$\begin{aligned}
&S(p,q,r)=(-1)^pN_2(p,r,q)+(-1)^qN_2(q,r,p)\\
&-2(-1)^p\sum_{j=0}^{(r-1)/2}\zeta(\overline{2j})\sum_{\rho=0}^{p/2}\zeta(\overline{2\rho})\sum_{\mu=0}^{(p-2\rho-1)/2}\binom{p+r-2j-2\rho-2\mu-1}{p-2\rho-2\mu-1}\\
&\times\zeta(p+q+r-2j-2\rho-2\mu)\frac{(\pi i)^{2\mu}}{(2\mu+1)!}\\
&-2(-1)^q\sum_{j=0}^{(r-1)/2}\zeta(\overline{2j})\sum_{\rho=0}^{q/2}\zeta(\overline{2\rho})\sum_{\mu=0}^{(q-2\rho-1)/2}\binom{q+r-2j-2\rho-2\mu-1}{q-2\rho-2\mu-1}\\
&\times\zeta(p+q+r-2j-2\rho-2\mu)\frac{(\pi i)^{2\mu}}{(2\mu+1)!}.
\end{aligned}$$

The third term of the right-hand side of the above equation equals

$$\begin{aligned}
&-2(-1)^p\sum_{j=0}^{(r-1)/2}\zeta(\overline{2j})\sum_{\rho=0}^{p/2}\zeta(\overline{2\rho})\sum_{n=\rho}^{(p-1)/2}\binom{p+r-2j-2n-1}{p-2n-1}\\
&\times\zeta(p+q+r-2j-2n)\frac{(\pi i)^{2n-2\rho}}{(2n-2\rho+1)!}.
\end{aligned}$$

Changing the order of ρ and n, we see that the above formula equals

$$\begin{aligned}
&-2(-1)^p\sum_{j=0}^{(r-1)/2}\zeta(\overline{2j})\sum_{n=0}^{(p-1)/2}\left(\sum_{\rho=0}^{n}\zeta(\overline{2\rho})\frac{(\pi i)^{2n-2\rho}}{(2n-2\rho+1)!}\right)\\
&\times\binom{p+r-2j-2n-1}{p-2n-1}\zeta(p+q+r-2j-2n),
\end{aligned}$$

and using (2.9), we find that it becomes

$$(-1)^p \sum_{j=0}^{(r-1)/2} \binom{p+r-2j-1}{p-1} \zeta(\overline{2j})\zeta(p+q+r-2j).$$

Hence the formula of H. Tsumura is nothing but

$$\begin{aligned} S(p,q,r) =&(-1)^p N_2(p,r,q) + (-1)^q N_2(q,r,p) \\ &+ (-1)^p \sum_{j=0}^{(r-1)/2} \binom{p+r-2j-1}{p-1} \zeta(\overline{2j})\zeta(p+q+r-2j) \\ &+ (-1)^q \sum_{j=0}^{(r-1)/2} \binom{q+r-2j-1}{q-1} \zeta(\overline{2j})\zeta(p+q+r-2j). \end{aligned}$$

Now it is easy to see that the formula of H. Tsumura for $S(p,q,r)$ is the same as that given in Proposition 4.6.

It is obvious that

$$\begin{aligned} &\mathfrak{T}_{1,2}(p,q,r) = \mathfrak{T}_{2,1}(q,p,r), \quad \mathfrak{T}_{2,2}(p,q,r) = 2^{-p-q-r}T(p,q,r), \\ &R(p,q,r) = -\mathfrak{T}_{1,1}(p,q,r) + \mathfrak{T}_{1,2}(p,q,r) - \mathfrak{T}_{2,1}(p,q,r) + \mathfrak{T}_{2,2}(p,q,r), \\ &S(p,q,r) = \mathfrak{T}_{1,1}(p,q,r) - \mathfrak{T}_{1,2}(p,q,r) - \mathfrak{T}_{2,1}(p,q,r) + \mathfrak{T}_{2,2}(p,q,r). \end{aligned}$$

Thus we get

$$\begin{aligned} \mathfrak{T}_{2,1}(p,q,r) &= -\frac{1}{2}(R(p,q,r) + S(p,q,r)) + \mathfrak{T}_{2,2}(p,q,r), \\ \mathfrak{T}_{1,1}(p,q,r) &= -\frac{1}{2}(R(p,q,r) + R(q,p,r)) + \mathfrak{T}_{2,2}(p,q,r). \end{aligned}$$

Then we obtain the evaluations of $\mathfrak{T}_{b_1,b_2}(p,q,r)$ when $p+q+r$ is odd as in [8, 11].

Proposition 4.7. *For $p,q,r \in \mathbb{N}$ with $p+q+r$ odd, we have*

$$\begin{aligned} \mathfrak{T}_{1,1}(p,q,r) =& -\frac{1}{2}\{(-1)^p N_1(r,p,q) + (-1)^p N_1(p,r,q) + (-1)^p N_2(r,p,q) \\ &+ (-1)^p N_3(p,r,q) + (-1)^q N_1(r,q,p) + (-1)^q N_1(q,r,p) \\ &+ (-1)^q N_2(r,q,p) + (-1)^q N_3(q,r,p)\} + \mathfrak{T}_{2,2}(p,q,r), \end{aligned}$$

$$\begin{aligned} \mathfrak{T}_{1,2}(p,q,r) =&\mathfrak{T}_{2,1}(q,p,r) \\ =& -\frac{1}{2}\{(-1)^p N_2(p,r,q) + (-1)^p N_2(r,p,q) + (-1)^p N_3(p,r,q) \\ &+ (-1)^p N_3(r,p,q) + (-1)^q N_1(r,q,p) + (-1)^q N_1(q,r,p) \\ &+ (-1)^q N_2(q,r,p) + (-1)^q N_3(r,q,p)\} + \mathfrak{T}_{2,2}(p,q,r), \end{aligned}$$

and

$$\begin{aligned} \mathfrak{T}_{2,2}(p,q,r) =&2^{-p-q-r}((-1)^p N(p,r,q) + (-1)^p N(r,p,q) \\ &+ (-1)^q N(q,r,p) + (-1)^q N(r,q,p)). \end{aligned}$$

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References

[1] Huard, J. G., Williams, K. S., Zhang, N. Y.: On Tornheim's double series, Acta Arith. 75(2), 105-117 (1996)

[2] Matsumoto, K., Nakamura, T., Ochiai, H., Tsumura, H.: On value-relations, functional relations and singularities of Mordell-Tornheim and related triple zeta-functions, Acta Arith. 132(2), 99-125 (2008)

[3] Mordell, L.J.: On the evaluation of some multiple series, J. London Math. Soc. 33, 368-371 (1958)

[4] Nakamura, T.: A functional relation for the Tornheim double zeta function, Acta Arith. 125(3), 257-263 (2006)

[5] Subbarao, M. V., Sitaramachandrarao, R.: On some infinite series of L. J. Mordell and their analogues, Pacific J. Math. 119, 245-255 (1985)

[6] Tornheim, L.: Harmonic double series, Amer. J. Math. 72, 303-314 (1950)

[7] Tsumura, H.: On some combinatorial relations for Tornheim's double series, Acta Arith. 105(3), 239-252 (2002)

[8] Tsumura, H.: On alternating analogues of Tornheim's double series, Proc. Amer. Math. Soc. 131, 3633-3641 (2003)

[9] Tsumura, H.: Evaluation formulas for Tornheim's type of alternating double series, Math. Comp. 73, 251-258 (2004)

[10] Tsumura, H.: On functional relations between the Mordell-Tornheim double zeta functions and the Riemann zeta function, Math. Proc. Camb. Phil. Soc. 142, 395-405 (2007)

[11] Tsumura, H.: On alternating analogues of Tornheim's double series II, Ramanujan J. 18, 81-90 (2009)

[12] Zhao, J.: A note on colored Tornheim's double series, perprint, 2009. arxiv: 0907.5106.