

LINEAR COVER TIME IS EXPONENTIALLY UNLIKELY

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ABSTRACT. We show that the probability that a simple random walk covers a finite, bounded degree graph in linear time is exponentially small.

More precisely, for every D and C , there exists $\alpha = \alpha(D, C) > 0$ such that for any graph G , with n vertices and maximal degree D , the probability that a simple random walk, started anywhere in G , will visit every vertex of G in its first Cn steps is at most $e^{-\alpha n}$.

We conjecture that the same holds for α that does not depend on D , provided that the graph G is simple.

1. INTRODUCTION

Let $G = (V, E)$ be a finite connected graph, let $\{X_t\}_{t=0}^\infty$ be a simple random walk on G started at $X_0 = v$. Let τ_{cov} be the *cover time* of the walk, i.e. the first time t such that for every $v \in G$ there is $s \leq t$ such that $X_s = v$. Our main result is:

Theorem 1.1. *For every D and C , there exists $\alpha = \alpha(D, C) > 0$ such that for any graph G , with n vertices and maximal degree D , and every starting vertex $v \in V$ we have*

$$\mathbf{P}_v(\tau_{\text{cov}} \leq Cn) \leq e^{-\alpha n}.$$

In certain special cases, the result follows from a direct application of Hoeffding's inequality. For example, if the graph is a path of length n then the probability to hit the end of the path within Cn steps is exponentially small. However, this approach fails in general since more typically there is a fixed probability to have hit any specific vertex by time Cn .

A naive approach to this problem would be to consider the Doob martingale of some related random variable. Natural choices include either the cover time itself or the number of uncovered vertices. However, these martingales could have large differences. For example when considering a simple random walk on a complete binary tree of height h , if the walk has already covered half of the tree and is now at the root, the next step

would cause a very large change in the value of either of these martingales.

The proof of Theorem 1.1 relies heavily on the following fact: The expected number of visits to a vertex v before covering $B_v(r)$ (the ball of radius r around v) grows to infinity with r , **even when we allow the walk to behave arbitrarily outside of $B_v(r)$** . To make this more precise, let us make some definitions.

A stochastic process X_t on the vertices of G is said to be a *random walk* if X_{t+1} is a neighbor of X_t , almost surely. For a subset of the vertices $S \subset V$, a random walk is *S -simple* if the distribution of X_{t+1} given the history X_0, \dots, X_t is uniform on the neighbors of X_t whenever $X_t \in S$.

For X a random walk on G and S a subset of vertices let $\tau_{\text{cov}}^*(S)$ be the first time t such that $X_t \notin S$ and for every $v \in S$ there is $s \leq t$ such that $X_s = v$. Let $\ell_t^v = |\{s < t \mid X_s = v\}|$ be the number of visits to v until time t .

Lemma 1.2. *For every D and C , there exists $r = r(D, C)$, such that if G is a graph of maximal degree at most D and v is a vertex of G such that $B_v(r) \neq V$, then any $B_v(r)$ -simple random walk, started outside $B_v(r)$ satisfies*

$$\mathbb{E}(\ell_{\tau_{\text{cov}}^*(B_v(r))}^v) \geq C.$$

The proof of Theorem 1.1 then proceeds by constructing a certain submartingale (which is reminiscent of the Doob martingale), which bounds the cover time from below, has expectation $2Cn$ and has bounded differences. Then by Hoeffding's bounds, the value of this submartingale at time Cn is exponentially unlikely to be less than Cn , which means that the walk hasn't covered the graph by this time.

Lemma 1.2 is of interest in itself. For example, a direct consequence is the well-known fact that the expected cover time of bounded degree graphs grows superlinearly in the number of vertices (see subsection 1.1). A more subtle implication is that for this result to hold one only needs the random walk to be simple in the vicinity of some constant fraction of the vertices. In particular, the cover time of random walk on a bounded degree graph which is simple in all but a sublinear number of vertices is also superlinear. In fact, our main Theorem applies to these kind of random walks as well.

An interesting open question is to determine the right quantitative version of 1.2. One can obtain an exponential lower bound for r in terms of C (and fixed D) by considering a simple random walk on a d -dimensional

torus, for $d \geq 3$. The question is whether the power to change the behavior of the walk outside of $B_v(r)$ can reduce significantly the expected number of visits to v before covering.

1.1. Related Works. The cover time of a simple random walk on graphs is a fairly natural concept which has been studied extensively in the past 30 years. Almost all results about the cover time are about its expectation. The most important lower bound is that of Feige [3] who proved that $\mathbb{E}_u(\tau_{\text{cov}}) \geq (1 - o(1))n \log n$ for any simple graph on n vertices and any starting vertex u . This implies that the probability to cover the graph in Cn steps cannot be more than $O(C/\log n)$ uniformly for all vertices.

The only concentration-type result the authors are aware of is that of Aldous [1] who proved that if $\max_{u,v} \mathbb{E}_u(\tau_{\text{hit}}^v) \ll \max_u \mathbb{E}_u(\tau_{\text{cov}})$ (where τ_{hit}^v is the first time the walk visits v) then for any starting vertex u we have $\tau_{\text{cov}}/\mathbb{E}_u(\tau_{\text{cov}}) \rightarrow 1$ in distribution. Notice that our main result applies for any bounded degree graph, even if the cover time is not concentrated around its mean.

The interested reader is referred to [2, 4] for further information about the cover time. More information about the importance of cover times in Computer Science can be found in [5].

2. PROOF OF THE MAIN THEOREM

Given a graph $G = (V, E)$, a vertex $v \in V$ and $r \in \mathbb{N}$ let $A_v(r)$ be the annulus of radius r around v and assume that $A_v(r) \neq \emptyset$. (For the convenience of the reader, we have included a legend of notation on the last page.) Given a walk X_t on G let $\mathcal{F}_t = \sigma(X_0, \dots, X_t)$ and let $\ell^v(r) = \ell_{\tau_{\text{cov}}^*(B_v(r))}^v$ be the number of visits to v before covering and exiting $B_v(r)$ (or ∞ if the walk never covers $B_v(r)$). Define

$$L_t^v(r) = \inf \mathbb{E}(\ell^v(r)(Y) \mid \mathcal{F}_t) \tag{2.1}$$

where the infimum is taken over all $B_v(r)$ -simple random walks Y that agree with X in the first t steps (i.e. $\mathbf{P}(Y_0 = X_0, \dots, Y_t = X_t) = 1$). The stochastic process $\{L_t^v(r) : t \geq 0\}$ is adapted to the filtration \mathcal{F}_t and is somewhat similar to the Doob martingale. However, here we take expectation with respect to a different process than the random walk itself.

The next few Lemmas show that $L_t^v(r)$ is, in fact, a submartingale with bounded differences and that it does not change its value when the walk is outside of $B_v(r)$.

Lemma 2.1. $L_t^v(r)$ is a sub-martingale.

Proof.

$$\begin{aligned} \mathbb{E}(L_{t+1}^v(r) | \mathcal{F}_t) &= d_{X_t}^{-1} \sum_{u \sim X_t} \mathbb{E}(L_{t+1}^v(r) | \mathcal{F}_t, X_{t+1} = u) \\ &= d_{X_t}^{-1} \sum_{u \sim X_t} \inf \mathbb{E}(\ell^v(r)(Y) | \mathcal{F}_t, X_{t+1} = u) \end{aligned}$$

where for each summand the infimum is taken over all $B_v(r)$ -simple random walks which agree with X in the first $t+1$ steps. Given a vector $\{Y^u\}_{u \sim X_t}$ of such random walks we can combine them into a single such random walk Y in the following way: $Y_s = X_s$ for $s \leq t+1$ and $Y_s = Y_s^u$ for $s > t+1$ if $X_{t+1} = u$. Obviously, $\mathbb{E}(L_{t+1}^v(r) | \mathcal{F}_t, X_{t+1} = u)$ is the same under Y^u and under Y . Hence

$$d_{X_t}^{-1} \sum_{u \sim X_t} \inf \mathbb{E}(\ell^v(r)(Y) | \mathcal{F}_t, X_{t+1} = u) \geq \inf \mathbb{E}(\ell^v(r)(Y) | \mathcal{F}_t) \quad (2.2)$$

where the infimum is now taken over all $B_v(r)$ -simple random walks which agree with X in the first $t+1$ steps. (In fact we have equality in equation (2.2), but we don't need this.) In comparison, in the definition of $L_t^v(r)$ we have the same expectation but the infimum is taken over all $B_v(r)$ -simple random walks which agree with X in the first t steps. This latter set contains the former, hence

$$\mathbb{E}(L_{t+1}^v(r) | \mathcal{F}_t) \geq L_t^v(r).$$

■

Lemma 2.2. If $X_t \notin B_v(r)$ and $X_{t+1} \notin B_v(r)$ then $L_{t+1}^v(r) = L_t^v(r)$.

Proof. Since the infimum in the definition of $L_t^v(r)$ includes all the $B_v(r)$ -simple random walks Y where $Y_{t+1} = X_{t+1}$ with probability 1, we see that $L_{t+1}^v(r) \geq L_t^v(r)$. Similarly, if we have $X_{t+2} = X_t$ then $L_{t+2}^v(r) \geq L_{t+1}^v(r)$. However, since $L_t^v(r)$ only depends on X_t and on which vertices were visited in $B_v(r)$ and on ℓ_t^v and none of these changes between time t and $t+2$ if $X_{t+2} = X_t$, we get that $L_t^v(r) = L_{t+2}^v(r) \geq L_{t+1}^v(r) \geq L_t^v(r)$. ■

In fact, when inside $B_v(r)$, this process is a martingale and when traversing an edge outside of $B_v(r)$ its value doesn't change, so the only times when $L_t^v(r)$ exhibits its "sub"-ness is when taking a step from the outside to the inside of $B_v(r)$.

Lemma 2.3. There exists $M = M(D, r)$, such that $|L_{t+1}^v(r) - L_t^v(r)| \leq M$.

Proof. Consider $L_t^v(r) - \ell_t^v$. This is the infimum of the expected number of visits to v between times t and $\tau_{\text{cov}}^*(B_v(r))$ where the infimum is with respect to any $B_v(r)$ -simple random walk that agrees with X in the first t steps. This number is nonnegative and bounded above by the expectation when we take the walk X itself. This is at most $D^{2D^{r+1}} + 2D^{r+1}$, since after every visit to v there is a probability of at least $D^{-2D^{r+1}}$ that X now performs a depth first search of $B_v(r)$, and during such a search the walk may visit v no more than $2D^{r+1}$ times. Since $|\ell_{t+1}^v - \ell_t^v| \leq 1$ we get that

$$|L_{t+1}^v(r) - L_t^v(r)| \leq D^{2D^{r+1}} + 2D^{r+1} + 1.$$

■

Now we can turn to the proof of the main result.

Proof of Theorem 1.1. Given D and C , let $r = r(D, 4C)$, as given by Lemma 1.2. If $G = (V, E)$ is a connected graph with maximal degree at most D , and $n = |V| > D^{r+1}$ then for every $v \in V$ we have $A_v(r) \neq \emptyset$. Hence, we can define $L_t^v(r)$ and we have $L_0^v(r) \geq 4C$ for all $v \in V \setminus B_{X_0}(r)$.

Consider the sum

$$L_t = \sum_{v \in G \setminus B_{X_0}(r)} L_t^v(r).$$

By Lemma 2.1 we know that L_t is a sub-martingale too, since all of the $L_t^v(r)$ are adapted to the same filtration. Combining Lemmas 2.3 and 2.2 shows that $|L_{t+1} - L_t| \leq M$, provided we incorporate a factor D^{r+2} into the constant $M = M(D, r)$ from Lemma 2.3. We now have

$$L_0 \geq 4C|V \setminus B_{X_0}(r)| \geq 3Cn,$$

for sufficiently large n .

We can now apply the Hoeffding-Azuma inequality to get

$$\mathbf{P}(L_t \leq 2Cn) \leq e^{-n^2/2tM}$$

for any t .

Substituting $t = 2Cn$ we get

$$\mathbf{P}(L_{2Cn} \leq 2Cn) \leq e^{-n/4CM}. \quad (2.3)$$

Let τ_{cov}^* be the first time $t > \tau_{\text{cov}}$ such that $X_t \notin B_{X_{\tau_{\text{cov}}}}(2r)$. Note that $\tau_{\text{cov}}^* \geq \tau_{\text{cov}}^*(B_v(r))$ for all $v \in V$. Note also that if $t \geq \tau_{\text{cov}}^*(B_v(r))$ then $\ell_t^v \geq$

$\ell_{\tau_{\text{cov}}^*(B_\nu(r))}^\nu = L_t^\nu$ for all $\nu \in V$, and summing this inequality over ν gives

$$\begin{aligned} t &= \sum_{\nu \in V} \ell_t^\nu \\ &\geq \sum_{\nu \in V} L_t^\nu = L_t. \end{aligned}$$

Thus if $L_t > t$ then we must have $\tau_{\text{cov}}^*(B_\nu(r)) > t$ for some $\nu \in V$ and hence $\tau_{\text{cov}}^* > t$ as well. Thus $\mathbf{P}(\tau_{\text{cov}}^* \leq t) \leq \mathbf{P}(L_t \leq t)$. Substituting $t = 2Cn$ gives

$$\begin{aligned} \mathbf{P}(\tau_{\text{cov}}^* \leq 2Cn) &\leq \mathbf{P}(L_{2Cn} \leq 2Cn) \\ &\leq e^{-n/4CM}, \end{aligned}$$

by equation (2.3). Finally we note that $\mathbf{P}(\tau_{\text{cov}}^* - \tau_{\text{cov}} \geq t)$ decays exponentially fast, at a rate depending only on D and r , regardless of the history until time τ_{cov} . Hence,

$$\mathbf{P}(\tau_{\text{cov}} < Cn) \leq \mathbf{P}(\tau_{\text{cov}}^* \leq 2Cn) + \mathbf{P}(\tau_{\text{cov}}^* - \tau_{\text{cov}} \geq Cn) \leq e^{-\alpha n},$$

for a constant α that depends only on D and r which in turn depends only on D and C . ■

3. PROOF OF THE MAIN LEMMA

Define

$$\phi(r) = \min \mathbb{E}(\ell^\nu(r)) / d_\nu$$

where the minimum is take over all connected graphs $G = (V, E)$ of maximal degree at most D and vertices $\nu \in V$ such that $A_\nu(r) \neq \emptyset$ and over all $B_\nu(R)$ -simple random walks started outside of $B_\nu(r)$. Then one may restate the main Lemma as

$$\lim_{r \rightarrow \infty} \phi(r) = \infty.$$

We will prove this fact by induction on the value of $\phi(r)$. More precisely, we will show that if $\phi(r) = K$ then there is some $R > r$ such that $\phi(R) \geq K + e^{-3Kd_\nu-4}$. Obviously, this is enough, as iterations of $K \mapsto K + e^{-3Kd_\nu-4}$ tend to infinity.

For a set of vertices $S \subset V$ write $B_S(r) = \cup_{\nu \in S} B_\nu(r)$. The following is a weaker, but more general version of Theorem 1.1, showing that the weighted sum of the number of visits to a set S of vertices is unlikely to be small for $B_S(r)$ -simple random walk.

Lemma 3.1. *Fix $r \in \mathbb{N}$ and let $K = \phi(r)$. For any $\varepsilon > 0$ there is some $a = a(r, \varepsilon) > 0$ such that if $G = (V, E)$ is a connected graph of maximal degree at most D and $\{a_v\}_{v \in V}$ a probability distribution on some $S \subset V$ with $\max_{v \in S} a_v \leq a$ then for any $B_S(r)$ -simple random walk started outside $B_S(r)$ we have*

$$\mathbf{P}\left(\sum_{v \in S} a_v \ell^v(r) < K \sum_{v \in S} a_v d_v - \varepsilon\right) < \varepsilon.$$

Proof. For any submartingale L_t one can construct a martingale M_t such that

- (1) $M_0 = L_0$
- (2) $M_t \leq L_t$
- (3) If $L_{t+1} = L_t$ then $M_{t+1} = M_t$
- (4) If the differences of L_t are bounded by L then the differences of M_t are bounded by $2L$

Now, apply this to $L_t^v(r)$ to get the martingales M_t^v , and let

$$M^v = \lim_{t \rightarrow \infty} M_t^v \leq \lim_{t \rightarrow \infty} L_t^v(r) = \ell^v(r).$$

It now follows that M^v and M^u are uncorrelated when the distance between v and u is more than $2r$. This is since $M^v = \sum_{t=0}^{\infty} M_{t+1}^v(r) - M_t^v(r)$ and we have $(M_{t+1}^v - M_t^v)(M_{t+1}^u - M_t^u) = 0$ by Lemma 2.2 and property 3 above and for $s \neq t$ we have $\mathbb{E}((M_{t+1}^v - M_t^v)(M_{s+1}^u - M_s^u)) = 0$ because these are martingales. Also, the variance of each M^v is bounded by the second moment of $\ell^v(r)$ which is bounded by some function of D and r only, since $\ell^v(r)$ has exponential tails with parameter depending only on D and R (see Lemma 2.3). Let $N = N(D, r)$ be such a bound for $\text{Var}(M^v)$.

Now let

$$M = \sum_{v \in S} a_v M^v \leq \sum_{v \in S} a_v \ell^v(r).$$

We bound $\text{Var}(M)$ by

$$\begin{aligned} \text{Var}(M) &= \sum_{v \in S} \sum_{u \in S} a_v a_u \text{Cov}(M^v, M^u) \\ &\leq \sum_{v \in S} \sum_{u \in B_v(r)} a_v a_u \text{Cov}(M^v, M^u) \\ &\leq \sum_{v \in S} a_v \sum_{u \in B_v(r)} a_u \sqrt{\text{Var}(M^v) \text{Var}(M^u)} \\ &\leq \sum_{v \in S} a_v a D^{r+1} N \\ &\leq a D^{r+1} N. \end{aligned}$$

and the Lemma holds by choosing a small enough and applying Chebyshev's inequality \blacksquare

Let $\tau_v(r)$ be the positive hitting time of $A_v(r)$ and for $w \in B_v(r)$ let $a_w^v(r) = \mathbb{E}_w(\ell_{\tau_v(r)}^v)$, where the expectation is with respect to a simple random walk. Obviously, this expectation is the same for any $B_v(r)$ -simple random walk.

Lemma 3.2. *Let $G = (V, E)$ be a finite graph, $v \in V$ a vertex and $r \in \mathbb{N}$ such that $A_v(r) \neq \emptyset$. Then*

$$\sum_{w \in A_v(r)} d_w a_w^v(r) = d_v.$$

Proof. $a_w^v(r)$ is equal to the sum of the probabilities of all paths which start at w and end at v and do not return to $A_v(r)$. For each of these paths, the probability that a simple random walk would traverse it is exactly d_v/d_w times the probability of traversing it in the reverse direction. Hence,

$$\sum_{w \in A_v(r)} d_w a_w^v(r) = d_v \sum_{w \in A_v(r)} \mathbf{P}_v(X_{\tau_v(r)} = w) = d_v$$

where the last equality follows since the walk hits exactly one vertex of $A_v(r)$. \blacksquare

Let $m^v(r) = \max_{w \in A_v(r)} a_w^v(r)$.

Lemma 3.3. *Given a graph $G = (V, E)$ with maximal degree at most D and a vertex $v \in V$ and $r \in \mathbb{N}$ such that $B_v(r) \neq V$, there is $r' \leq r$ such that*

$$m^v(r') \leq \sqrt{\frac{d_v a_v^v(r+1)}{r}}$$

for any $B_v(r)$ -simple random walk.

Proof. As in Lemma 3.2 we have

$$d_w a_w^v(r') = d_v \mathbf{P}_v(X_{\tau_v(r')} = w)$$

for every $w \in A_v(r')$ when $r' \leq r$.

One may bound $a_v^v(r)$ by considering, for every $r' \leq r$ all the paths which start at v , hit $A_v(r')$ at some specified vertex w and then hit v again before returning to $A_v(r')$. For distinct r' 's or distinct w in the same $A_v(r')$

these are disjoint sets. This summation yields

$$\begin{aligned}
 a_v^v(r+1) &\geq \sum_{r' \leq r} \sum_{w \in A_v(r')} \mathbf{P}_v(X_{\tau_v(r')} = w) a_w^v(r) \\
 &= \sum_{r' \leq r} \sum_{w \in A_v(r')} (a_w^v(r'))^2 d_w / d_v \\
 &\geq \sum_{r' \leq r} (m^v(r'))^2 / d_v = r(m^v(r))^2 / d_v
 \end{aligned}$$

where the middle equality follows by reversibility. \blacksquare

We will also need the following useful Lemma.

Lemma 3.4. *Let x_i be a stochastic process on $\{0, 1\}$, adapted to the filtration \mathcal{F}_i and let $p_i = \mathbb{E}(x_i | \mathcal{F}_{i-1})$. If $p_i \leq \frac{1}{2}$ a.s. for all i , and τ is a stopping time such that $\sum_{i=1}^{\tau} p_i \leq K$ a.s. then*

$$\mathbf{P}(\forall_{i \leq \tau} x_i = 0) \geq e^{-3K}.$$

Proof. Define $M_i = \prod_{j \leq i} (1 - p_j)^{-1}$ if $\forall_{j \leq i} x_j = 0$ and $M_i = 0$ otherwise. It is easily checked that M_i is a martingale adapted to \mathcal{F}_i and $M_0 = 1$. Since $p_j \leq \frac{1}{2}$ for all j we have $1 - p_j \geq e^{-3p_j}$ so $\prod_{j \leq i} (1 - p_j)^{-1} \leq e^{3 \sum_{j \leq i} p_j}$. Since $\sum_{i=1}^{\tau} p_i \leq K$, by the optional stopping Theorem we have $\mathbf{P}(\forall_{i \leq \tau} x_i = 0) \geq e^{-3K}$. \blacksquare

Now we are ready to prove the main Lemma. Very roughly, we show that for some radius R' , by the time we cover $A_v(R')$, we visit v almost Kd_v times in expectation and there is a non-negligible probability that we haven't visited v at all, in which case we will visit v at least once before covering, thus increasing the expected number of visits to v before covering by this probability.

Proof of Lemma 1.2. Let r be such that $K = \phi(r)$. Fix some ε to be chosen later and let $a = a(r, \varepsilon)$ from Lemma 3.1. Let $R = D(K + e^{-3Kd_v-4}) / a^2$. We claim that $\phi(R+r) \geq K + e^{-3Kd_v-4}$. This is enough to show that $\lim_{r \rightarrow \infty} \phi(r) = \infty$.

Let $G = (V, E)$ be a graph with maximal degree at most D and let $v \in V$ a vertex such that $A_v(R) \neq \emptyset$. We want to show that for any $B_v(R)$ -simple random walk started outside $B_v(R)$ we have $\mathbb{E}(\ell^v(R)) \geq K + e^{-3Kd_v-4}$. If $a_w^v(R) \geq K + e^{-3Kd_v-4}$ then we are done (recall that $a_w^v(R)$ is the expected number of visits to v before hitting $A_v(R)$ for a simple random walk started at w). Hence, from now on we assume that

$$a_w^v(R) \geq K + e^{-3Kd_v-4}. \quad (3.1)$$

In this case, by Lemma 3.3 there is $R' \leq R$ such that for all $w \in A_\nu(R')$ we have

$$a_w^\nu(R') \leq \sqrt{\frac{d_\nu a_\nu^\nu(R)}{R}} \leq a.$$

Let t_i enumerate the times the walk is in $A^\nu(R')$ and define

$$b_i = \sum_{j=0}^i a_{X_{t_j}}^\nu(R')$$

and

$$c_i = \ell_{t_{i+1}}^\nu.$$

Claim 3.5. $c_i - b_i$ is a martingale (adapted to the filtration $\mathcal{F}_{t_{i+1}}$).

Proof. $b_{i+1} - b_i = a_{X_{t_i}}^\nu(r) = \mathbb{E}(c_{i+1} - c_i \mid \mathcal{F}_{t_{i+1}})$. ■

In words, we partition the walk into excursions, each of which start and ends at $A_\nu(R')$, and for each excursion we count the number of visits to ν and subtract the expectation.

Let I be the first index such that either $b_I \geq Kd_\nu - \varepsilon$ or $\tau_{\text{cov}}^*(B_\nu(R+r)) \leq t_I$. Obviously, this is a stopping time and also $b_I \leq Kd_\nu + 1$ since $a_w^\nu(R') \leq 1$, for all $w \in A_\nu(R')$.

Claim 3.6.

$$\mathbf{P}(c_I = 0) \geq e^{-3(Kd_\nu+1)}.$$

Proof. The probability to hit ν between t_i and t_{i+1} is at most $a_{X_{t_i}}^\nu(R')$ and $\sum_{i=0}^I a_{X_{t_i}}^\nu(R') = b_I \leq Kd_\nu + 1$. The claim now follows by Lemma 3.4. ■

Claim 3.7.

$$\mathbf{P}(\tau_{\text{cov}}^*(B_\nu(R+r)) \leq t_I) \leq \varepsilon$$

Proof. Obviously, $\tau_{\text{cov}}^*(B_\nu(R+r)) \geq \tau_{\text{cov}}^*(B_S(r))$ where $S = A_\nu(R')$. Let $a_w = a_w^\nu(R') \leq a$ by our assumption on R' . Hence, by Lemma 3.1 and the choice of a we have

$$\mathbf{P}\left(\sum_{w \in S} a_w \ell^w(r) < K \sum_{w \in S} a_w d_w - \varepsilon\right) < \varepsilon$$

which implies

$$\mathbf{P}(b_{\tau_{\text{cov}}^*(B_\nu(R+r))} < Kd_\nu - \varepsilon) < \varepsilon$$

and the claim follows by the definition of I . ■

Claim 3.8.

$$\mathbb{E}(c_I) \geq (Kd_\nu - \varepsilon)(1 - \varepsilon).$$

Proof. $\mathbb{E}(c_I) = \mathbb{E}(b_I)$ by Lemma 3.5 and since I is a stopping time.

$$\mathbb{E}(b_I) \geq (Kd_v - \varepsilon)\mathbf{P}(b_I \geq Kd_v - \varepsilon) \geq (Kd_v - \varepsilon)(1 - \varepsilon)$$

by claim 3.7 and the definition of I . ■

Summing it all up, the expected number of visits to v before $\tau_{\text{cov}}^*(B_v(R+r))$ is at least the expected number of these visits which occur before t_I plus the probability that v has not been visited at all by time t_I (in which case we need to visit it at least once). Lemma 3.8 and Lemma 3.6 bound these from below yielding

$$\begin{aligned} \mathbb{E}(\ell^v(R+r)) &\geq \mathbb{E}(c_I) + \mathbf{P}(c_I = 0) \\ &\geq (Kd_v - \varepsilon)(1 - \varepsilon) + e^{-3(Kd_v+1)} \\ &\geq Kd_v + e^{-3Kd_v-4} \end{aligned}$$

for ε small enough. ■

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Legend:

$B_v(r)$ - the ball of radius r around v

$A_v(r) = B_v(r) \setminus B_v(r-1)$ - the annulus of radius r around v

$\tau_v(r)$ the hitting time of $A_v(r)$

ℓ_t^v the number of visits to v before time t

$\ell^v(r)$ the number of visits to v before time $\tau_v(r)$

$\tau_{\text{cov}}(S)$ the cover time of S

τ_{cov} the cover time of the graph

$\tau_{\text{cov}}^*(S)$ the time to cover and exit S

τ_{cov}^* the time to cover the graph and exit $B_{X_{\tau_{\text{cov}}}}(2r)$

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