Existence of Spherically Symmetric Initial Data with Zero Energy, Unit Mass, and Virial less than - 1/2 for the Relativistic Vlasov-Poisson Equation with Attractive Coupling

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Abstract

In a recent paper, Kiessling and Tahvildar-Zadeh proved that any spherically symmetric classical solution of the attractive relativistic Vlasov-Poisson equation launched by sufficiently regular initial data with zero total energy and virial less than or equal to -1/2 will blow up in finite time. They left open whether such data exist. Interestingly, the simplest conceivable ansatz, a ball of material centered at the origin with momenta directed inward, must have virial strictly larger than -1/2! This has raised the question whether any such data exist at all. In this brief note, we settle this question by constructing an entire class of these initial data. The ansatz we employ is a core-halo structure, i.e. a dense inner core surrounded by a thin shell of material further out.

1 Introduction

The relativistic Vlasov-Poisson (rVP) system is given by

$$\text{rVP}^{\pm}: \begin{cases} \left(\partial_t + \frac{p}{\sqrt{1+|p|^2}}\cdot\nabla_q \pm \nabla_q\varphi_t(q)\cdot\nabla_p\right) f_t(p,q) = 0 \\ \triangle_q\varphi_t(q) = 4\pi \int f_t(p,q)\, d^3p \\ \varphi_t(q) \asymp -|q|^{-1} \text{ as } |q|\to\infty; \end{cases}$$

rVP$^+$ models a system with repulsive interaction while rVP$^-$ models a system with attractive interaction. One of the earliest papers to appear on the subject is [GS85] wherein Glassey and Schaeffer show that global classical solutions to rVP$^\pm$ will exist for initial data that are spherically symmetric, compactly supported in momentum space, vanish on characteristics with vanishing angular momentum, and have $\mathfrak{L}^\infty$-norm below a critical constant $\mathcal{C}_\infty^\pm$ with $\mathcal{C}_\infty^+ = \infty$ and $\mathcal{C}_\infty^- < \infty$. More recently, Hadžić and Rein ([HR07]) showed the non-linear stability of a wide class of steady-state solutions of rVP$^-$ against certain allowable perturbations utilizing energy-Casimir functionals. Shortly thereafter, Lemou, Méhats, and Raphaël ([LMR08, LMR09]) investigated non-linear stability versus the formation of singularities in rVP$^-$ through concentration compactness techniques. Kiessling and Tahvildar-Zadeh ([KTZ08]) have extended the theorem of Glassey and Schaeffer for rVP$^-$ by proving global existence of classical solutions for initial data that are spherically symmetric, compactly supported in momentum space, vanish on characteristics with vanishing angular momentum, and have $\mathfrak{L}^\beta$-norm below a critical constant $\mathcal{C}_\beta^-$ with $\mathcal{C}_\beta^- < \infty$ and identically zero for $\beta < 3/2$. The sharp value of this constant was subsequently determined for all values of β in [You10].

Glassey and Shaeffer also investigated what may happen when rVP$^-$ is launched by initial data with $\|f\|_\infty > \mathcal{C}_\infty^-$. They proved that negative energy data lead to "blow-up" (i.e. system collapse) in finite time. In [LMR08], Lemou, Méhats, and Raphaël proved that collapse for systems launched by initial data with negative total energy approaches self-similar collapse. Around the same time, Kiessling and Tahvildar-Zadeh proved that any spherically symmetric classical solution of rVP$^-$ launched by initial data $f_0 \in \mathfrak{P}_3 \cap \mathfrak{C}^1$ [†] with *zero total energy* and total virial less than or equal to

[†]By $\mathfrak{P}_3 \cap \mathfrak{C}^1$ we mean the set of probability measures on $\mathbb{R}^6$ whose first three moments

-1/2 will blow up in finite time (Theorem 6.1 of [KTZ08]).

Now, the authors of [KTZ08] did not give any specific examples, and recently the question was raised whether such data actually exist [Cal]. We here show that a plethora of such functions can be constructed from a not entirely obvious ansatz.

The plan of this brief note is as follows. To begin, we establish the basic formulae needed to carry out our subsequent investigations. We then show that the simplest ansatz (a ball of material centered at the origin) cannot have the desired properties. Finally, we employ a core-halo structure to construct a class of initial data satisfying the requirements in [KTZ08]. In addition, we show that zero-energy initial data can have arbitrarily negative virial.

2 Basic Formulae

We first establish the various formulae we shall need for our particular class of functions. Consider a generic separation-of-variables ansatz:

$$f(p,q) \quad = \quad \mathcal{C}\eta(|q|)\Phi(|p|)\mathscr{L}(\cos(\theta_{p,q})), \tag{1}$$

where $\theta_{p,q}$ is the angle between q and p (both considered as vectors in $\mathbb{R}^3$) and η, Φ, and $\mathscr{L}$ are non-negative. The coefficient $\mathcal{C}$ will be chosen to normalize the total mass to 1.

The total mass is given by

$$\begin{aligned}\iint f(p,q)\ d^3p\ d^3q =& \mathcal{C}\int_0^\infty \eta(|q|)|q|^2 d|q| \int_0^\infty \Phi(|p|)|p|^2 d|p| \iint_{\mathbb{S}^2\times\mathbb{S}^2} \mathscr{L}(\cos(\theta_{p,q}))d\Omega_p d\Omega_q \\ =& \, 8\pi^2\mathcal{C}\int_0^\infty \eta(|q|)|q|^2 d|q| \int_0^\infty \Phi(|p|)|p|^2 d|p| \int_{-1}^1 \mathscr{L}(x)dx.\end{aligned}$$

Using the notation

$$\|g\|_1 \equiv \int_0^\infty |g(r)|dr,$$

and recalling that all functions under consideration are non-negative we see that taking

$$\mathcal{C}^{-1} = 8\pi^2 \|\eta|q|^2\|_1 \|\Phi|p|^2\|_1 \int_{-1}^1 \mathscr{L}(x)dx$$

are finite, that are absolutely continuous with respect to Lebesgue measure, and whose Radon-Nikodym derivative is $\mathfrak{C}^1$.

will ensure that f has total mass 1.

Recall that the energy and virial functionals are defined by

$$\begin{aligned}\mathcal{E}(f) \equiv & \iint \sqrt{1+|p|^2} f(p,q)\, d^3p\, d^3q \\ & - \frac{1}{2} \iiiint \frac{f(p',q')f(p,q)}{|q-q'|}\, d^3p'\, d^3p\, d^3q'\, d^3q \\ \mathcal{V}(f) \equiv & \iint q\cdot p\, f(p,q)\, d^3p\, d^3q.\end{aligned}$$

We begin by computing the virial:

$$\begin{aligned}\mathcal{V}(f) &= \iint q\cdot p f(p,q)\, d^3p\, d^3q \\ &= \mathcal{C}\int_0^\infty \eta(|q|)|q|^3 d|q| \int_0^\infty \Phi(|p|)|p|^3 d|p| \iint_{\mathbb{S}^2\times\mathbb{S}^2} \cos(\theta_{p,q})\mathscr{L}(\cos(\theta_{p,q}))d\Omega_p d\Omega_q \\ &= 8\pi^2\mathcal{C}\|\eta|q|^3\|_1\|\Phi|p|^3\|_1 \int_{-1}^1 x\mathscr{L}(x)dx \\ &= \frac{\|\eta|q|^3\|_1}{\|\eta|q|^2\|_1}\frac{\|\Phi|p|^3\|_1}{\|\Phi|p|^2\|_1}\frac{\int_{-1}^1 x\mathscr{L}(x)dx}{\int_{-1}^1 \mathscr{L}(x)dx}. \qquad (2)\end{aligned}$$

Of course, in order to have a negative virial we must require

$$\int_{-1}^1 x\mathscr{L}(x)dx < 0.$$

For the energy, we compute the "kinetic" and potential energy contributions separately (the portion of the energy we label as "kinetic" also contains the rest mass energy). For the kinetic energy, we have

$$\begin{aligned}KE(f) &\equiv \iint \sqrt{1+|p|^2} f(p,q)\, d^3p\, d^3q \\ &= \mathcal{C}\int_0^\infty \Phi(|p|)\sqrt{1+|p|^2}|p|^2 d|p| \int_0^\infty \eta(|q|)|q|^2 d|q| \\ &\qquad \cdot \iint_{\mathbb{S}^2\times\mathbb{S}^2} \mathscr{L}(\cos(\theta_{p,q}))d\Omega_p d\Omega_q \\ &= \frac{\|\Phi\sqrt{1+|p|^2}|p|^2\|_1}{\|\Phi|p|^2\|_1}. \qquad (3)\end{aligned}$$

We begin computing the potential energy by first computing the spatial distribution ρ associated to f:

$$\begin{aligned}
\rho(q) &\equiv \int f(p,q)\, d^3p \\
&= \mathcal{C}\eta(|q|)\int_0^\infty \Phi(|p|)|p|^2 d|p| \int_{\mathbb{S}^2} \mathscr{L}(\cos(\theta_{p,q}))d\Omega_p \\
&= \frac{\eta(|q|)}{4\pi\|\eta|q|^2\|_1}
\end{aligned}$$

In terms of this function, the potential energy contribution is now given by

$$PE(f) \equiv -\frac{1}{2}\iint \frac{\rho(q)\rho(q')}{|q-q'|}\, d^3q'\, d^3q.$$

We calculate this integral directly, and after a little work find

$$PE(f) = -\frac{1}{\|\eta|q|^2\|_1^2}\int_0^\infty \eta(|q|)|q|\left(\int_0^{|q|}\eta(|q'|)|q'|^2 d|q'|\right)d|q|. \qquad (4)$$

3 Balls have Virial greater than $-1/2$

At this point, we can show that the simplest ansatz cannot work. Relaxing the differentiability requirement for a moment, we make the specific choices

$$\begin{aligned}
\eta(|q|) &= \chi_{[0,R]}(|q|), \\
\Phi(|p|) &= \chi_{[0,P]}(|p|), \\
\mathscr{L}(x) &= \chi_{[-1,a]}(x)
\end{aligned}$$

where $R > 0, P > 0, -1 < a < 1$, and χ_I is the characteristic function of the interval I. We have that

$$\begin{aligned}
KE(f) &= \frac{3}{8}\left(\frac{\sqrt{1+P^2}}{P^2} + 2\sqrt{1+P^2} - \frac{\ln(P+\sqrt{1+P^2})}{P^3}\right), \\
PE(f) &= -\frac{3}{5R}.
\end{aligned}$$

The zero-energy requirement allows us to solve for R in terms of P. Calculating the virial and plugging in the formula for $R = R(P)$ gives $\mathcal{V}(f)$ in terms of the parameters P and a. Simple asymptotics shows that for this ansatz

$$\mathcal{V}(f) > -\frac{9}{20}$$

for any choice of parameters. Smoothing out the boundaries of these step functions (taking care to keep the total energy zero) shows that this ansatz is untenable. Of course, this type of initial condition may well lead to collapse after finite time, but this cannot be verified by our virial condition.

4 The Core-Halo Ansatz

Having established formulae for the various quantities of interest and having ruled out the simplest possible ansatz we now proceed to try a core-halo scheme:

$$\begin{aligned} \eta(|q|) &= \chi_{[0,R_1]}(|q|) + \alpha\chi_{[R_2,R_3]}(|q|), && (5)\\ \Phi(|p|) &= \chi_{[0,P]}(|p|), && (6)\\ \mathscr{L}(x) &= \chi_{[-1,a]}(x) && (7) \end{aligned}$$

where $0 < R_1 \leq R_2 \leq R_3, 0 < P$, and $-1 < a \leq 1$. Again, we have relaxed the differentiability requirement to make the computations tractable.

We first consider the specific choices

$$R_1 = \frac{1}{5}, R_2 = 1, R_3 = 2, \text{ and } P = 1.$$

The zero energy condition forces (thanks to Maple)

$$\alpha = \frac{1}{125}\frac{35\ln(1+\sqrt{2}) + 30 - 105\sqrt{2} + 2\sqrt{6480\sqrt{2} - 1655 - 2160\ln(1+\sqrt{2})}}{735\sqrt{2} - 188 - 245\ln(1+\sqrt{2})}$$

(a small but positive number). Again by Maple, choosing $-1 < a \leq -4/5$ gives a virial which is less than $-1/2$. Since our ansatz is compactly supported, we have the requisite number of moments. To complete the argument, we note that we can smooth out the various step functions in such a way that the resulting integrals are as close to the values obtained above as we like. Since α was chosen to force the zero-energy condition and the necessary value was strictly positive, it follows that by smoothing the test functions appropriately we can still choose an $\alpha > 0$ to keep the total energy zero.

Thus, we have constructed an initial datum satisfying all the requirements of Theorem 6.1 in [KTZ08].

In addition, we give an argument showing that virial can be made arbitrarily negative. We make the following choices in (5):

$$\begin{aligned} R_1 &= P^{-2}, \\ R_2 &= P, \\ R_3 &= P^2. \end{aligned}$$

Now, the zero energy condition allows us to solve for $\alpha = \alpha(P)$. Using Maple, we see that as a function of P, α is positive for sufficiently large P and is proportional to $P^{-23/2}$ as $P \to \infty$. Plugging these choices into the formula for the virial and looking at the asymptotics for large P shows that the virial is proportional to $-(1-a)P^3$. Smoothing these functions out as above shows that the virial is unbounded below.

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