GENERIC CONFORMAL DIMENSION ESTIMATES FOR RANDOM GROUPS

JOHN M. MACKAY

Abstract. We give a lower and an upper bound for the conformal dimension of the boundaries of certain small cancellation groups.

We apply these bounds to the few relator and density models for random groups. This gives generic bounds of the following form, where l is the relator length, going to infinity.

(a) $1 + 1/C < \mathcal{C}\dim(\partial_\infty G) < Cl/\log(l)$, for the few relator model, and

(b) $1 + l/(C\log(l)) < \mathcal{C}\dim(\partial_\infty G) < Cl$, for the density model, at densities $d < 1/24$.

In particular, for the density model at densities $d < 1/24$, as the relator length l goes to infinity, the random groups will pass through infinitely many different quasi-isometry classes.

1. Introduction

1.1. **Overview.** In the study of random groups, one considers typical properties of finitely presented groups. There are several ways to make this idea precise. We will work in two of the most common models for a random group: the few relator model and the density model, both due to Gromov. Our goal is to study the large scale geometry of such groups.

In each of these models, a typical group is (Gromov) hyperbolic [10, 11]. To any hyperbolic group G we can associate a boundary at infinity $\partial_\infty G$, which is a metric space where the metric is canonically defined up to a quasi-symmetric homeomorphism. The boundary captures the quasi-isometry type of the group: two finitely presented hyperbolic groups are quasi-isometric if and only if their boundaries are quasi-symmetric.

A quasi-symmetric invariant of a metric space X is its *conformal dimension*: this is the infimal Hausdorff dimension of all quasi-symmetrically equivalent metric spaces [18, 16]. Consequently, the conformal dimension of $\partial_\infty G$, denoted by $\mathcal{C}\dim(\partial_\infty G)$, is canonically defined and depends only on the quasi-isometry type of the group.

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Bourdon found a family of hyperbolic groups whose boundaries are all homeomorphic to the Menger curve (also called the Menger sponge), but whose conformal dimensions take a dense set of values in $(1, \infty)$ [4]. Therefore, there is an infinite collection of non-quasi-isometric hyperbolic groups all of which have a topological Menger curve as a boundary.

This is of particular interest to us here since, in both models, the boundary of a random group is homeomorphic to the Menger curve.

In this paper we will provide generic estimates for the conformal dimension of the boundary of a hyperbolic group. As a consequence, we show that in the density model, as the lengths of the relators tend to infinity, the conformal dimension of the boundary also tends to infinity, passing through infinitely many different quasi-isometry types.

1.2. **Statement of results.** The simplest model of a random group is given by the few relator model. Throughout this paper we fix a finite generating set S, $|S| = m \geq 2$.

Definition 1.1 (Few relator model)**.** *Fix a finite number of relators $n \geq 1$. Consider all cyclically reduced words of length at most l in $\langle S \rangle$. Consider all presentations $\langle S \mid r_1, \ldots, r_n \rangle$ where $r_1, \ldots, r_n$ are chosen from this set of words uniformly and independently at random.*

A property $\mathcal{P}$ is generic *in the few relator model (for fixed n), if the proportion of all such presentations at length l which satisfy $\mathcal{P}$ goes to 1 as $l \to \infty$. In this case, we say that* a random (few relator) group has property $\mathcal{P}$.

This model was introduced by Gromov [10], who observed that a random few relator group will satisfy the $C'(1/6)$ small cancellation condition, and so be hyperbolic. The algebraic properties of these groups, such as freeness of subgroups and isomorphism type have been studied by Arzhantseva, Ol'shanskii, Kapovich, Schupp, and others [1, 12, 13]. For more discussion, see [17, I.3.c].

The geometry of such groups was considered by Champetier [6]. He used small cancellation techniques to show that generic few relator groups have boundaries homeomorphic to a Menger curve (see Theorem 1.7).

The few relator model can be viewed as the "density 0" case of a more general model, where the number of relators grows as $l \to \infty$.

Definition 1.2 (Density model [11, Chapter 9])**.** *Fix a parameter $d \in (0, 1)$, called the* density*. Consider all cyclically reduced words of length l in $\langle S \rangle$. Consider all presentations which choose as relators $(2m-1)^{dl}$ of these words uniformly and independently at random.*

A property $\mathcal{P}$ holds generically in the density model (at fixed density d), if the proportion of all such presentations at length l which satisfy $\mathcal{P}$ goes to 1 as $l \to \infty$.

Gromov showed that the density model has a phase transition: for densities $d < 1/2$, a random group will be one ended and hyperbolic, but for densities $d > 1/2$, a random group will be trivial or $\mathbb{Z}/2\mathbb{Z}$ ([11, Section 9.B], [17, Theorem 11]).

The boundary of a random group at density $d < 1/2$ is homeomorphic to the Menger curve. At densities $d < 1/24$, this follows from Champetier's Theorem 1.7. A proof that applies to all densities $0 < d < 1/2$ is given in [8].

Since we know that random groups in both the few relator and density model are hyperbolic, it makes sense to ask for estimates of the conformal dimension of their boundaries.

For any hyperbolic group with boundary homeomorphic to the Menger curve, the conformal dimension of the boundary will be strictly greater than one [15], and finite [7]. In this paper we give explicit non-trivial bounds for the conformal dimension of a random group.

Theorem 1.3. *There exists $C > 1$ so that, for fixed $m \geq 2$, $n \geq 1$, the conformal dimension of a random few relator group satisfies*

$$1 + \frac{1}{C\log(2m-1)} \leq \mathcal{C}\mathrm{dim}(\partial_\infty G) \leq C\log(2m-1) \cdot \frac{l}{\log(l)}.$$

Note that the lower bound is independent of l, and the upper bound is sub-linear. The conclusion of this theorem involves n implicitly: let $P(m, n, l)$ be the proportion of all groups with m generators and n cyclically reduced relators of word length at most l which satisfy the above estimate. Then for fixed m, n we have $P(m, n, l) \to 1$ as $l \to \infty$, however the rate of convergence depends on n.

Theorem 1.4. *There exists $C > 1$ so that, for fixed $m \geq 2$, $0 < d < 1/24$, the conformal dimension of a random group at density d satisfies*

$$1 + \frac{d}{C} \cdot \frac{l}{\log(l)} \leq \mathcal{C}\mathrm{dim}(\partial_\infty G) \leq \frac{C}{|\log(d)|} \cdot l.$$

In particular, as $l \to \infty$, generic groups pass through infinitely many different quasi-isometry classes.

Gromov [11, 9.B, p.276, (g)] and Pansu [17, IV.b., p.70] had asked whether the conformal dimension of a random group in the density model can be used to detect the particular density d. Theorem 1.4 gives partial progress towards solving this problem.

There are several natural questions that remain. For example, does the conformal dimension of a random few relator group go to infinity as the relator length goes to infinity? Can one find a function $f(d,l)$ so that the conformal dimension of a random group at density d satisfies $f(d,l) \lesssim \mathcal{C}\dim(\partial_\infty G) \lesssim f(d,l)$? (We write $x \lesssim y$ if $x \leq Cy$, for some suitable constant C.)

For more background on random groups we refer the reader to [9] and [17].

1.3. Outline of proof.

The random groups that we consider are all $C'(1/6)$ small cancellation groups. In the few relator model this is straightforward to prove; a more refined estimate is found in Proposition 2.2, where we show that a generic few relator group will be $C'(\lambda)$ with $\lambda \lesssim \frac{\log(l)}{l}$. In the density model, we have the following result.

Proposition 1.5 ([11, Section 9.B])**.** *For $d > 0$, and $\lambda > 2d$, a random group at density d has the $C'(\lambda)$ metric small cancellation condition.*

For $\lambda > 0$ and $2d > \lambda$, a random group at density d does not have the $C'(\lambda)$ metric small cancellation condition.

In particular, at densities $d < 1/12$, a random group has a $C'(1/6)$ small cancellation presentation.

Recall that a group presentation $\langle S|R\rangle$ is $C'(\lambda)$ if every word u which appears in two distinct ways in (cyclic conjugates of) relators $r_1, r_2 \in R$, or their inverses, satisfies $|u| < \min\{|r_1|, |r_2|\}$.

Specifying a finite generating set S for G allows one to define the Cayley graph $\Gamma = \Gamma(G,S)$. The Cayley graph of a $C'(1/6)$ group is δ-hyperbolic, with δ equal to twice the maximum relator word length (Lemma 3.11).

As a hyperbolic metric space, for any sufficiently small visual parameter $\epsilon > 0$, $\partial_\infty\Gamma$ carries a visual metric comparable to $e^{-\epsilon(\cdot,\cdot)}$, where $(\cdot,\cdot)$ denotes the Gromov product. A simple upper bound on the conformal dimension of $\partial_\infty G$ is given by the Hausdorff dimension of this metric space, which equals $\frac{1}{\epsilon}h(G)$ [7, Corollary 7.6]. Here $h(G)$ is the volume entropy of the group (with respect to S). In an m-generator group, we always have $h(G) \leq \log(2m-1)$. Thus,

$$\mathcal{C}\dim(\partial_\infty G) \leq \frac{1}{\epsilon}h(G) \leq \frac{1}{\epsilon}\log(2m-1). \tag{1.6}$$

To give a good upper bound for the conformal dimension, then, we would like to choose ϵ as large as possible. The standard estimate for an admissible ϵ is $\epsilon \leq \log(2)/(4\delta)$ [5, III.H.3.21]. Thus, generically we have $\mathcal{C}\dim(\partial_\infty G) \lesssim \delta \lesssim l$ (in the few relator model, or the density model with $d < 1/12$).

We cannot find a significantly better estimate for δ, since the relators of size l give bigons with sides separated by a distance of order l. However, work of Bonk and Foertsch [2] lets us find a better estimate for ϵ using the concept of "asymptotic upper curvature" (Section 4).

Theorem 4.1. *If $G = \langle S|R\rangle$ is a $C'(\lambda)$ presentation of a group, with $\lambda \leq \frac{1}{6}$, and $|r| \leq M$ for all $r \in R$, then*

$$\mathcal{C}\dim(\partial_\infty(G)) \leq \frac{M}{\log(\frac{1}{\lambda} - 4)} \log(2m-1).$$

Combining this theorem with Propositions 2.2 and 1.5, we obtain the upper bounds in Theorems 1.3 and 1.4.

It is more difficult to obtain lower bounds for the conformal dimension. A key inspiration for our work is the following result of Champetier.

Theorem 1.7 ([6, Theorem 4.18])**.** *Suppose $G = \langle S|R\rangle$ is a $C'(1/12)$ presentation, with $|S| \geq 2$ and $|R| \geq 1$. Suppose further that every reduced word $u \in \langle S\rangle$ of length 12 appears at least once in some cyclic conjugate of some $r^{\pm 1}, r \in R$. Then $\partial_\infty G$ is homeomorphic to the Menger curve.*

Random groups certainly contain every word of length 12 as a subword of some relator. In fact, generic few relator presentations contain every word of length $C\log(l)$ as a subword of some relator (Proposition 2.6), while generic presentations at density d contain every word of length Cl, for $C < d$, as a subword of some relator (Proposition 2.7).

Champetier builds a cone in the Cayley complex of a $C'(1/12)$ group that gives an arc in its boundary. We modify his construction to produce a sub-complex similar to one of Gromov's "round trees" [11, 3], giving a Cantor set of curves in the boundary. Finally, a lemma of Pansu and Bourdon gives the following lower bound for the conformal dimension.

Theorem 5.1. *Suppose $G = \langle S|R\rangle$ is a $C'(1/12)$ presentation, with $|S| \geq 2$ and $|R| \geq 1$, where $|r| \leq M$ for all $r \in R$. Suppose further that for some $M^* \geq 12$, every reduced word $u \in \langle S\rangle$ of length M^* appears at least once in some cyclic conjugate of some relator $r \in R$, or its inverse. Then for some universal constant $C > 0$, we have*

$$\mathcal{C}\dim(\partial_\infty G) \geq 1 + C \cdot \frac{M^*}{\log(M)}.$$

This theorem combines with Propositions 2.6 and 2.7 to complete the proof of Theorems 1.3 and 1.4.

1.4. **Outline of paper.** In Section 2 we consider random groups in both models and their small cancellation properties. Standard results about the geometry of $C'(1/6)$ groups, including hyperbolicity, are given in Section 3.

Asymptotic upper curvature bounds are used in Section 4 to give a generic upper bound for conformal dimension. A round tree subcomplex is built in Section 5, and the proof of Theorem 5.1 is found in Section 6.

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2. Random groups and small cancellation

Our goal in this section is to study subwords of random groups in the few relator model and density model. We find out what lengths subwords should be to be unique in the presentation, or, on the other hand, so that every possible subword of that length appears.

These calculations are fairly routine counting and probability arguments; the main technicality arises from counting cyclically reduced words rather than just reduced words.

First, we recall the definition of the metric small cancellation condition [14].

Definition 2.1. *The presentation $G = \langle S \mid R\rangle$ satisfies the metric small cancellation condition $C'(\lambda)$, for some $0 < \lambda < 1$, if every piece u which is a subword of some cyclic conjugate of $r^{\pm 1}$, $r \in R$, satisfies $|u| < \lambda|r|$.*

Recall that a piece *is a common initial segment of two distinct cyclic conjugates of $r_1, r_2 \in R \cup R^{-1}$, where r_1 may equal r_2.*

2.1. **Small cancellation in the few relator model.** We have $m \geq 2, n \geq 1$ fixed. Our goal in this subsection is to show that generic few relator presentations satisfy strong small cancellation properties.

Proposition 2.2. *There exists $0 < C_0 < \infty$, depending only on m, n, so that generic few relator presentations are $C'(\lambda_0(l))$, where $\lambda_0(l) = C_0 \frac{\log l}{l}$.*

This result is essentially sharp, as shown by Proposition 2.6.

We begin with some preliminary observations. In the following, the notation $A \asymp B$ indicates that $A \lesssim B \lesssim A$.

Let N_l be the number of cyclically reduced words of length l in F_m. It is easy to see that $N_l \asymp (2m-1)^l$, with multiplicative error of $\frac{4}{3}$. More precise estimates are in Subsection 2.2 below. Let $N_{\leq l}$ be the number of cyclic reduced words of length at most l in F_m. Again, $N_{\leq l} \asymp (2m-1)^l$. The number of presentations where all relators have length at most l is $N_{\leq l}^n = (N_{\leq l})^n$.

Let $N_{[0.99l,l]}^n$ be the number of presentations where all n relators have length at least $0.99l$, but no more than l. This is generic, since

$$\frac{N_{\leq l}^n - N_{[0.99l,l]}^n}{N_{\leq l}^n} \leq \frac{n \cdot N_{\leq 0.99l} \cdot N_{\leq l}^{n-1}}{N_{\leq l}^n} \lesssim (2m-1)^{-0.01l},$$

which goes to zero as $l \to \infty$.

So to show that a property is generic, it suffices to show that it is generic within the class of presentations where all relators have lengths between $0.99l$ and l.

Proof of Proposition 2.2. Let $N_{[0.99l,l],\lambda_2}^n$ be the number of presentations with n relations all of lengths between $0.99l$ and l, satisfying the $C'(\lambda_0)$ condition. We wish to find C_0 so that $N_{[0.99l,l]}^n - N_{[0.99l,l],\lambda_0}^n = o(N_{\leq l}^n)$.

It suffices to bound the proportion of presentations $N_{(l_i),\lambda_0}^c / N_{(l_i)}$ which are not $C'(\lambda_0)$, for specified lengths $l_1, \ldots, l_n$ in $[0.99l, l]$. Here $N_{(l_i)}$ is the number of presentations with relators of length $|r_i| = l_i$, $i = 1, \ldots, n$, $N_{(l_i),\lambda_0}$ is the number of those which are $C'(\lambda_0)$, and $N_{(l_i),\lambda_0}^c = N_{(l_i)} - N_{(l_i),\lambda_0}$.

If we fail to be $C'(\lambda_0)$, then there is a word u of length equal to $\lceil 0.99l\lambda_0 \rceil$ which appears in two distinct places in the words $r_1, \ldots, r_n$, or their inverses.

Case 1: The word u appears in two different words.

There are $\binom{2n}{2} \leq 4n^2$ choices for the words $r_i^{\pm 1}$ and $r_{i'}^{\pm 1}$. Given this choice, the number of ways u can appear is bounded from above by the product of the number of choices of (1) the location of u in these words, (2) the word u, (3) the remainder of the words r_i and $r_{i'}$, and (4) the other words. Call these numbers A_1, A_2, A_3 and A_4 respectively. Clearly,

$$A_1 \leq l^2, \quad A_2 \leq \frac{4}{3}(2m-1)^{|u|},$$
$$A_3 \leq (2m-1)^{l_i - |u|} \cdot (2m-1)^{l_{i'} - |u|}, \quad \text{and} \quad A_4 = \prod_{j \neq i,i'} N_{l_j}.$$

Since we have

$$\frac{A_1 A_2 A_3 A_4}{\prod_{j=1,\ldots,n} N_{l_j}} \lesssim \frac{l^2 (2m-1)^{|u|} (2m-1)^{l_i - |u|} (2m-1)^{l_{i'} - |u|}}{N_{l_i} \cdot N_{l_{i'}}}$$
$$\lesssim l^2 (2m-1)^{-|u|},$$

Case 1 occurs with probability P_1 at most $P_1 \lesssim n^2 l^2 (2m-1)^{-|u|}$. Observe that

$$-|u| = -\lceil 0.99 l \lambda_0 \rceil \leq -0.99 l \lambda_0 = -0.99 C_0 \log(l).$$

Since $l^2 = (2m-1)^{2\log(l)/\log(2m-1)}$, provided

$$2 - 0.99 C_0 \log(2m-1) < 0,$$

the probability P_1 will go to zero as l goes to infinity.

Case 2: The word u appears in the same word r_i in two distinct ways.

Let P_2 be the probability this occurs among presentations of lengths (l_i).

Lemma 2.3. *There is a subword v of u, of length at least $0.2C_0\log(l)$, which appears in r_i in two non-intersecting locations as either v or v^{-1}.*

Proof. Consider r_i as a labelling on the oriented circle. Let u_1 and u_2 be the two words on the boundary r_i so that each is labelled by u or u^{-1}.

If the initial segment of u_1 of length $\lceil 0.2 C_0 \log(l) \rceil$ does not intersect u_2, then let v be that subword, and we are done.

Otherwise, without loss of generality, up to relabelling u_1 and u_2, we can assume that the initial letter of u_1 is not in u_2 but that the initial segment of u_1 of length $\lceil 0.2 C_0 \log(l) \rceil$ does meet u_2.

If the word u has opposite orientations in u_1 and u_2, we let v be the initial segment of u_1 of length $\lceil 0.2 C_0 \log(l) \rceil$. Then v^{-1} also appears in the tail segment of u_2, disjoint from v.

Finally, if u has the same orientation in both u_1 and u_2, let w be the initial segment of u_1 disjoint from u_2, of length at most $0.2C_0\log(l)$. Since the words u_1 and u_2 are both copies of u, u is made up of repeated copies of w followed by some tail w'. We write $u = w^{2k}w'$, for some integer k, and word w' of length $|w'| < 2\lceil 0.2 C_0 \log(l) \rceil$, thus $|w^k| \geq 0.2 C_0 \log(l)$, so $v = w^k$ is our required word. (In some of these estimates we assumed that l was sufficiently large.) □

We can now find, analogous to Case 1, that P_2, is bounded from above by the product of the number of choices of i, the locations of v in this word, the word v, the remainder of the word r_i, all divided by N_{l_i}. Therefore

$$P_2 \lesssim \frac{n \cdot l^2 \cdot (2m-1)^{|v|} \cdot (2m-1)^{l_i - 2|v|}}{N_{l_i}} \lesssim l^2 (2m-1)^{-|v|}.$$

Now

$$-|v| \leq -\lceil 0.2 l \lambda_0 \rceil \leq -0.2 C_0 \log(l),$$

and, as before, $l^2 = (2m-1)^{2\log(l)/\log(2m-1)}$, so provided

$$2 - 0.2C_0 \log(2m-1) < 0,$$

the probability P_2 will go to zero as l goes to infinity.

Combining the cases:

We have shown that

$$\frac{N^c_{(l_i),\lambda_0}}{N_{(l_i)}} \leq P_1 + P_2$$

goes to zero as $l \to \infty$, independent of the choice of l_i between $0.99l$ and l, provided that C_0 is sufficiently large. It suffices to take $C_0 = 11/\log(2m-1)$. □

2.2. **Counting cyclically reduced words.** In this subsection we give some lemmas we will use in the remainder of this section. We will need the following lemma which counts the number of ways to fill in a cyclically reduced word.

Lemma 2.4. *We count all reduced words w of length $n+2$ with first and last letter fixed in $\langle s_1, s_2, \ldots, s_m \rangle$.*

There are essentially three different cases. Let p_n, q_n and r_n count the number of reduced words of length $n+2$ of the forms s_1us_1, $s_1us_1^{-1}$ and s_1us_2, respectively. Then, for all $n \geq 1$, we have:

$$\frac{\max\{p_n, q_n, r_n\}}{\min\{p_n, q_n, r_n\}} \leq 1 + \frac{2}{(2m-1)^n}.$$

Proof. Note that $p_1 = 2m-1$, and $q_1 = r_1 = 2m-2$. Clearly,

$$\begin{aligned} p_n &= p_{n-1} + (2m-2)r_{n-1}, \\ q_n &= q_{n-1} + (2m-2)r_{n-1}, \text{ and} \\ r_n &= p_{n-1} + q_{n-1} + (2m-3)r_{n-1}. \end{aligned}$$

One observes that, by induction, when n is odd, $p_n = q_n + 1$ and $r_n = q_n$, while when n is even, $p_n = r_n = q_n + 1$.

A simple recurrence relation calculation gives that

$$q_n = \begin{cases} \frac{1}{2m}\big((2m-1)^{n+1} - 1\big) & \text{if } n \text{ is odd,} \\ \frac{1}{2m}\big((2m-1)^{n+1} - (2m-1)\big) & \text{if } n \text{ is even.} \end{cases}$$

Therefore

$$q_n \geq \frac{2m-1}{2m}\big((2m-1)^n - 1\big) \geq \frac{1}{2}(2m-1)^n,$$

and

$$\frac{\max\{p_n, q_n, r_n\}}{\min\{p_n, q_n, r_n\}} = \frac{q_n+1}{q_n} = 1 + \frac{1}{q_n} \leq 1 + \frac{2}{(2m-1)^n}.$$ □

This proof implies that $N_l = 2mp_{l-1} \asymp (2m-1)^l$.

The following lemma estimates the probability of omitting a specified word.

Lemma 2.5. *Fix a reduced word r_0 of length $g(l) < l/4$, $g(l) > 4$. Let N_{r_0} be the number of all cyclically reduced words of length l which omit r_0. Then the proportion N_{r_0}/N_l is at most*

$$\frac{N_{r_0}}{N_l} \leq \exp\left(\frac{2}{(2m-1)^{(l/2)-1}} - \frac{l}{9g(l)(2m-1)^{g(l)}}\right).$$

Proof. Consider a cyclically reduced relator r_1 of length l which omits r_0. Let $A = \lfloor \frac{l}{2(g(l)+1)} \rfloor$. Let us split up r_1 into an initial letter, then words $u_1, u_2, \ldots, u_A$ of length $g(l)+1$, plus a tail of length t, where t must be between $(l/2)-1$ and $3l/4$. Each word u_i consists of an initial letter, plus a word of length $g(l)$, which is not r_0.

The initial letter of r_1 has $2m$ possibilities. For each $i = 1, \ldots, A$, either the initial letter of u_i matches the inverse of the initial letter of r_0, or it does not. In the former case, the remaining $g(l)$ letters have $(2m-1)^{g(l)}$ possibilities, while in the latter case there are only $(2m-1)^{g(l)}-1$ possibilities, since the word r_0 is excluded. The number of possibilities for the remaining t letters is bounded by $\max\{p_t, q_t, r_t\}$ (as defined in Lemma 2.4). Altogether, we have a bound

$$\begin{aligned}
\frac{N_{r_0}}{N_l} &\leq \frac{2m\left((2m-1)^{g(l)} + (2m-2)\left[(2m-1)^{g(l)}-1\right]\right)^A \max\{p_t, q_t, r_t\}}{2m(2m-1)^{(g(l)+1)A}\min\{p_t, q_t, r_t\}} \\
&\leq \left(\frac{(2m-1)^{g(l)} + (2m-2)\left[(2m-1)^{g(l)}-1\right]}{(2m-1)^{g(l)+1}}\right)^A \left(1 + \frac{2}{(2m-1)^t}\right) \\
&= \left(1 - \frac{(2m-2)}{(2m-1)^{g(l)+1}}\right)^A \left(1 + \frac{2}{(2m-1)^t}\right) \\
&\leq \exp\left(\frac{2}{(2m-1)^t} - A \cdot \frac{(2m-2)}{(2m-1)^{g(l)+1}}\right), \text{using } 1 + x \leq e^x.
\end{aligned}$$

Observe that $A \geq \frac{l}{6g(l)}$, and $\frac{2m-2}{2m-1} \geq \frac{2}{3}$, thus:

$$\frac{N_{r_0}}{N_l} \leq \exp\left(\frac{2}{(2m-1)^{(l/2)-1}} - \frac{l}{9g(l)(2m-1)^{g(l)}}\right). \qquad \square$$

2.3. **Short subwords of generic few relator presentations.**

Proposition 2.6. *There exists a constant C (depending on m) so that a generic few relator presentation with relator lengths at most l contains every word of length $\lceil C \log(l) \rceil$ as a subword of some relator.*

Proof. Let $N_{g(l)}$ be the number of cyclically reduced words in $\langle S \rangle$ of length l which contain every word of length at most $g(l)$. To prove the proposition, it suffices to show that $(N_l - N_{g(l)})/N_l \to 0$ as $l \to \infty$, where $g(l) = \lceil C \log(l) \rceil$.

By Lemma 2.5, the probability of an individual relator omitting a fixed word r_0 of length $g(l)$ is at most

$$\exp\left(1 - \frac{l}{9g(l)(2m-1)^{g(l)}}\right).$$

There are at most $\frac{4}{3}(2m-1)^{g(l)}$ choices for r_0, so the probability of missing some word of length $g(l)$ satisfies

$$\begin{aligned}\frac{N_l - N_{g(l)}}{N_l} &\leq \frac{4}{3}(2m-1)^{g(l)} \cdot \exp\left(1 - \frac{l}{9g(l)(2m-1)^{g(l)}}\right) \\ &\leq 4\exp\left(\log(2m-1)g(l) - \frac{l}{9g(l)(2m-1)^{g(l)}}\right).\end{aligned}$$

Note that since $g(l) = \lceil C\log(l)\rceil$, $(2m-1)^{g(l)}$ behaves like $l^{C\log(2m-1)}$ for large l. Thus, if $C\log(2m-1) < 1$, then $\frac{N_l - N_{g(l)}}{N_l}$ will go to zero as $l \to \infty$. □

2.4. **Short subwords in the density model.** The following proposition is a version of [17, Prop. 9]. Ollivier sketches a proof for $0 < C < d < 1$; for completeness we provide a proof in the following special case.

Proposition 2.7. *For any $0 < C < d < 1/2$, a generic presentation at density d contains every word of length $\lceil Cl \rceil$ as a subword of some relator.*

Proof. This follows a similar proof to Proposition 2.6. There are $(2m-1)^{dl}$ reduced words chosen independently, so the probability that they all omit a particular word r_0 of length $g(l) = \lceil Cl \rceil$ is, by Lemma 2.5, at most

$$\begin{aligned}&\left[\exp\left(\frac{2}{(2m-1)^{(l/2)-1}} - \frac{l}{9g(l)(2m-1)^{g(l)}}\right)\right]^{(2m-1)^{dl}} \\ &\quad = \exp\left(\frac{2(2m-1)^{dl}}{(2m-1)^{(l/2)-1}} - \frac{l(2m-1)^{dl}}{9g(l)(2m-1)^{g(l)}}\right) \\ &\quad \lesssim \exp\left(\frac{-1}{10C}(2m-1)^{(d-C)l-1}\right),\end{aligned}$$

for sufficiently large l.

Again, there are at most $\frac{4}{3}(2m-1)^{g(l)}$ choices for r_0, so the probability that some word of length $g(l) = \lceil Cl \rceil$ is omitted is at most

$$\begin{aligned}
&\lesssim \frac{4}{3}(2m-1)^{g(l)} \cdot \exp\left(\frac{-1}{10C}(2m-1)^{(d-C)l-1}\right) \\
&\lesssim \exp\left(2\log(2m-1)Cl - \frac{1}{10C}(2m-1)^{(d-C)l-1}\right),
\end{aligned}$$

for large l, and this goes to zero as $l \to \infty$. □

3. Cayley graphs of small cancellation groups

In $C'(1/6)$ small cancellation groups, geodesic bigons and triangles are known to have certain special forms [19, 6]. In this section we recall these standard facts, and give some extensions to the case of geodesic n-gons which will be needed in Section 4.

Throughout this section, $G = \langle S \mid R \rangle$, with $S = \{s_1, \ldots, s_m\}$, and $R = \{r_1, \ldots, r_n\}$.

Definition 3.1. *A* diagram *for a reduced word $w \in G$ is a connected, contractible, finite, pointed, planar 2-complex $\mathcal{D}$ which satisfies the following conditions:*

1. *Each edge of $\mathcal{D}$ is oriented and labelled with an element of S,*
2. *For each face $B \subset \mathcal{D}$, reading the edge labels along its boundary ∂B gives a (cyclic conjugate of) a word $r^{\pm 1}$, $r \in R$.*
3. *The base point lies on the boundary $\partial\mathcal{D}$, and reading the edge labels from this point around $\partial\mathcal{D}$ ('counter-clockwise') gives w.*

We say $\mathcal{D}$ is reduced *if there are never two distinct faces B_1, B_2 which intersect in at least one edge, so that the labellings on ∂B_1 and ∂B_2, read from this edge clockwise and counter-clockwise respectively, agree.*

Lemma 3.2 (Strebel [19])**.** *Suppose $\mathcal{D}$ is a reduced diagram homeomorphic to a disc. For a vertex v, let $d(v)$ denote its degree. For a face B, let $|\partial B|$ denote its degree, let $e(B)$ denote the number of exterior edges of B, and let $i(B)$ denote the number of interior edges. Then*

$$6 = 2\sum_v (3 - d(v)) + \sum_B (6 - 2e(B) - i(B)). \tag{3.3}$$

Proof. Suppose $\mathcal{D}$ has V vertices, E edges and F faces. Then

$$1 + E = V + F = \sum_v 1 + \sum_B 1 \tag{3.4}$$

$$2E = \sum_v d(v) \tag{3.5}$$

$$2E = \left(\sum_B |\partial B| \right) + |\partial \mathcal{D}| = \sum_B \left(2e(B) + i(B) \right). \tag{3.6}$$

Consider $6 \cdot (3.4) - 2 \cdot (3.5) - (3.6)$. □

Definition 3.7. *The Cayley graph $\Gamma(G,S) = \Gamma^1(G,S)$ of a group G with finite generating set S is the graph with vertex set G, and an unoriented edge between $\{g, gs\}$ for all $g \in G$, $s \in S \cup S^{-1}$.*

Suppose P is a geodesic n-gon in the Cayley graph $\Gamma(G,S)$, where $G = \langle S \mid R \rangle$ satisfies $C'(\lambda)$, for some $\lambda \in (0, \frac{1}{6}]$. We want to show that P is slim; that is, any side of P is contained in a suitable neighborhood of the other sides.

As P is a closed loop, van Kampen's lemma states that there is a reduced diagram $\mathcal{D}$ for P. We may assume that the boundary word is cyclically reduced, and that $\mathcal{D}$ is homeomorphic to a disc; this only makes it harder to show that P is slim.

We remove all vertices of degree 2 from $\mathcal{D}$ and relabel edges with the corresponding words in $\langle S \rangle$. So now all vertices have degree at least 3.

In this reduced diagram, there are two kinds of faces that have external edges, those where a endpoint of a side of P lies in the interior of an external edge, and all others. We call the former kind distinguished; there are at most n of them.

When $e(B) = 1$ and B is not distinguished, the external edge with label u is a geodesic in $\Gamma(G,S)$, and so $|u| \leq \frac{1}{2}|\partial B|$. Now each remaining edge of B is internal, and so a piece of G, and so has length less than $\lambda |\partial B|$. Thus

$$\frac{1}{2}|\partial B| \leq \sum \Big\{ |t| : t \text{ internal edge of } B \Big\} < i(B) \lambda |\partial B|,$$

So $i(B) > \frac{1}{2\lambda}$, thus $i(B) \geq \lfloor \frac{1}{2\lambda} + 1 \rfloor =: d_{Ext}(\lambda) \geq 4$.

Note also that each edge of an interior face B ($e(B) = 0$) has length strictly less than $\lambda |\partial B|$, so

$$i(B) > \frac{1}{\lambda} \quad \Rightarrow \quad i(B) \geq \left\lfloor \frac{1}{\lambda} + 1 \right\rfloor =: d_{Int}(\lambda) \geq 7.$$

Thus (3.3) splits into cases as follows.

$$
\begin{aligned}
6 = {} & 2\sum_v (3-d(v)) + \sum_{B,\, e(B)=0} (6-i(B)) + \sum_{\substack{B,\, e(B)=1 \\ \text{not dist.}}} (4-i(B)) + \\
& \sum_{\substack{B,\, e(B)=1 \\ \text{dist.}}} (4-i(B)) + \sum_{B,\, e(B)=k\geq 2} (6-2k-i(B)) \\
\leq {} & -(d_{Int}(\lambda)-6)F_I + 3n,
\end{aligned}
\tag{3.8}
$$

where F_I is the number of interior faces of $\mathcal{D}$. We have shown the following.

Lemma 3.9. *In the above situation,*

$$F_I \leq \frac{3n-6}{d_{Int}(\lambda)-6}.$$

Lemma 3.10. *Suppose $G = \langle S|R\rangle$ is $C'(\frac{1}{6})$, and that the diagram $\mathcal{D}$ has no vertices of degree two. Then any two distinct faces $B, B' \subset \mathcal{D}$ are either disjoint, meet at a single point, or meet along a single edge.*

Also, the boundary of any face B is a simple curve, i.e., the face does not bump into itself.

Proof. If the boundary of a face B is not a simple curve, B encloses a subdiagram $\mathcal{D}'$ in the interior of $\mathcal{D}$, all of whose vertices (except perhaps one) have degree at least three, and all of whose faces have degree at least seven. This contradicts (3.3).

Similarly, if two faces meet at more than a single edge, they enclose a subdiagram $\mathcal{D}'$ in the interior of $\mathcal{D}$, and this has at most two vertices of degree two. This again contradicts (3.3). □

Lemma 3.9 immediately implies that reduced diagrams for geodesic bigons have no internal faces, and that reduced diagrams for geodesic triangles have at most three internal faces. We can make more precise statements in these cases. (See [19, Theorem 35] and [6, Proposition 3.6].)

Lemma 3.11. *Reduced diagrams for geodesic bigons in a $C'(1/6)$ group have a specific form, as illustrated by Figure 1.*

Reduced diagrams for geodesic triangles in a $C'(1/6)$ group have no interior faces. In particular, the Cayley graph is $2M$-hyperbolic, where $M = \max_{r\in R}|R|$.

After removing spurs, the reduced diagram for a geodesic triangles has no more than six connected faces. (If it is $C'(1/8)$, no more than three connected faces.)

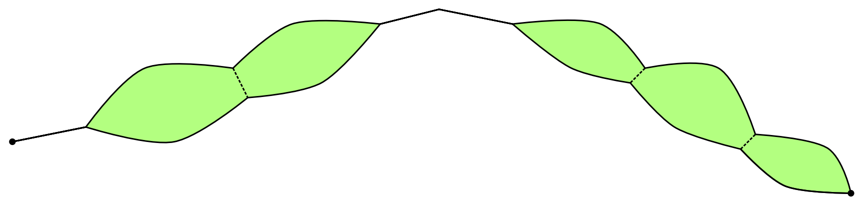

FIGURE 1. A geodesic bigon in a $C'(1/6)$ group

Recall that if a geodesic triangle has sides γ_{12}, γ_{13} and γ_{23} joining vertices P_1, P_2 and P_3, and $\mathcal{D}$ is a reduced diagram for the triangle, then the spur of $\mathcal{D}$ containing P_1 is the maximal subdiagram of $\mathcal{D}$ bounded by γ_{12}, γ_{13} and a vertex or a single internal edge.

Proof. Geodesic bigons are the $n = 2$ case in (3.8). We assume there is more than one face in the reduced diagram, and reduce to the case that the diagram is homeomorphic to a closed disc.

Equation (3.8) immediately implies that $F_I = 0$, all vertices have degree 3, the endpoints of the geodesic lie in two faces with one internal edge, and all other faces have exactly two external and two internal edges. This implies that the bigon has the specified form.

Geodesic triangles are a little more complicated. Without loss of generality, we can assume that the diagram is homeomorphic to a disc. If the endpoint of a geodesic lies in a face with one interior edge, delete this face from the diagram and relocate the endpoint on to the adjacent face. Continue this as far as possible for each endpoint, until we are left with the diagram $\mathcal{D}$.

If what remains has more than one face, then each face containing an endpoint has at least two interior edges. So, by (3.8), with $n = 3$, we have $6 \leq 0 - F_I + 0 + 2 \cdot 3 + 0$, so $F_I = 0$. Moreover, there must be three different faces corresponding to each endpoint, with exactly two interior edges. All other faces with one exterior edge have exactly four interior edges. There are no faces with exactly two interior and two exterior faces, as they would be removed with the spurs. All vertices have degree equal to three.

Thus the dual diagram $\mathcal{D}^*$ has no interior vertices, all faces have degree three, it has three exterior vertices with degree two, and all other (exterior) vertices have degree four. It may be the case that $\mathcal{D}^*$ consists of a single triangular face, and so $\mathcal{D}$ consists of three faces. This is the only possibility when the group is $C'(1/8)$.

Otherwise, the dual of the dual diagram $\mathcal{D}^{**}$ has no faces, three degree one vertices (corresponding to the endpoints), and all other

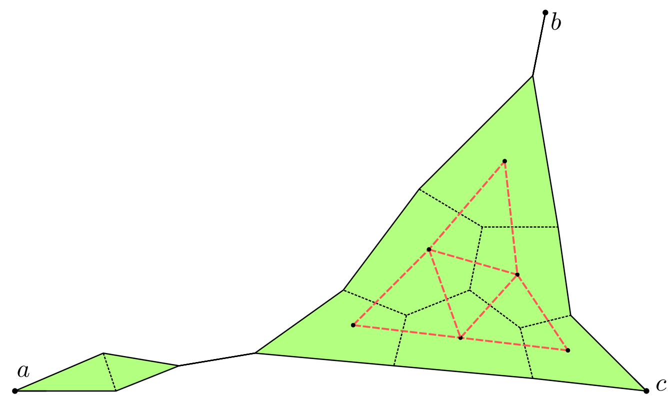

FIGURE 2. A geodesic triangle in a $C'(1/6)$ group

vertices of degree three. Thus this diagram has exactly four vertices in total. This implies that (the relevant parts of) $\mathcal{D}^*$ and $\mathcal{D}$ have the forms shown in Figure 2. □

4. Asymptotic curvature bounds and an upper bound for conformal dimension

4.1. **Outline.** If G is hyperbolic, geodesic triangles in $\Gamma(G,S)$ are uniformly slim. Consequently, geodesic n-gons will be $(C\log(n))$-slim, for some C independent of n. Bonk and Foertsch [2] investigated this further and linked the behavior of geodesic n-gons to the optimal visual parameter ϵ for visual metrics on the boundary of G. In this section we will use these ideas to prove the following theorem.

Theorem 4.1. *If $G = \langle S|R\rangle$ is a $C'(\lambda)$ presentation of a group, with $\lambda \leq \frac{1}{6}$, and $|r| \leq M$ for all $r \in R$, then*

$$\mathcal{C}\dim(\partial_\infty(G)) \leq \frac{M}{\log(\frac{1}{\lambda}-4)}\log(2|S|-1).$$

We recall one of the equivalent definitions of asymptotic upper curvature, and the result which we will need.

Definition 4.2 (Bonk and Foertsch)**.** *A geodesic metric space X has* an asymptotic upper curvature bound κ*, written* $\mathrm{AC}_u(\kappa)$*, for $\kappa \in [-\infty, 0)$, if there exists some C so that every geodesic $(n+1)$-gon in X, $n \in \mathbb{N}$, $n \geq 2$, is $\left(\frac{1}{\sqrt{-\kappa}}\log(n)+C\right)$-slim.*

(Recall that a geodesic $(n+1)$-gon is Δ-slim if every side is in the union of the Δ-neighborhoods of the other n sides.)

Theorem 4.3 ([2, Theorem 1.5])**.** *If a geodesic metric space X is $\mathrm{AC}_u(\kappa)$, for some $\kappa \in [-\infty, 0)$, then for every $0 < \epsilon < \sqrt{-\kappa}$ there is a visual metric on $\partial_\infty X$ with parameter ϵ.*

This result, and the bound in (1.6), reduce the proof of Theorem 4.1 to the following statement.

Theorem 4.4. *If $G = \langle S | R \rangle$ is a $C'(\lambda)$ presentation of a group, with $\lambda \leq \frac{1}{6}$, and $|r| \leq M$ for all $r \in R$, then the Cayley graph $\Gamma(G, S)$ is $AC_u(\kappa)$ with $\kappa = -\frac{1}{M^2} \log^2 \left(\frac{1}{\lambda} - 4\right)$.*

In the following subsection we prove Theorem 4.4.

4.2. **Slim n-gons.** To prove Theorem 4.4, we show that while a reduced diagram for a geodesic n-gon may have interior faces, they cannot be too far from the boundary of the diagram.

Proposition 4.5. *Let $\mathcal{P}$ be a geodesic n-gon in the Cayley graph of a $C'(\lambda)$ group $G = \langle S | R \rangle$, $\lambda \leq 1/6$, and let $\mathcal{D}$ be a reduced diagram for $\mathcal{P}$.*

Then there exists some constant $C = C(\lambda)$ so that, for any $x \in \partial \mathcal{D} = \mathcal{P}$, there is a chain of at most $k+1$ faces joining x to another side of $\mathcal{P}$, with

$$0 \leq k \leq \frac{\log(n)}{\log(\frac{1}{\lambda} - 4)} + C.$$

This means that there are faces $B_0, B_1, \ldots, B_k$ so that $x \in \partial B_0$, $B_j \cap B_{j+1} \neq \emptyset$ for $0 \leq j < k$, and B_k meets another side of $\mathcal{P}$.

Proof. The point x lies in the boundary of some face $B_0 \subset \mathcal{P}$. We may assume that $e(B_0) = 1$ and B_0 is not distinguished, else the single chain B_0 suffices.

By Lemma 3.10, B_0 is homeomorphic to a closed disc. The exterior edge of B_0, denoted by γ_0, is a geodesic in $\Gamma(G, S)$, and $i(B_0) \geq \lfloor \frac{1}{2\lambda} + 1 \rfloor \geq 4$.

Suppose $\mathcal{D}_i$ is a subdiagram of $\mathcal{D}$ homeomorphic to a closed disc. We can combine faces and delete vertices in $\mathcal{D}$ so that $\mathcal{D}_i$ consists of a single face, and all vertices still have degree at least three. Let $i(\mathcal{D}_i)$ denote the number of interior edges of this disc.

Let $\mathcal{D}_0 = B_0$. Then $\mathcal{D}_0$ has one exterior edge, a geodesic, and $i(\mathcal{D}_0) = i(B_0) \geq \lfloor \frac{1}{2\lambda} + 1 \rfloor \geq 4$.

The following lemma shows that number of faces in star neighborhoods of B_0 in $\mathcal{D}$ grows exponentially, until another side of $\mathcal{P}$ is found. An example of this is shown in Figure 3.

Recall that the star neighborhood of a subcomplex $\mathcal{D}' \subset \mathcal{D}$ is the union of all closed cells in $\mathcal{D}$ meeting $\mathcal{D}'$, and is denoted by $\mathrm{St}(\mathcal{D}')$.

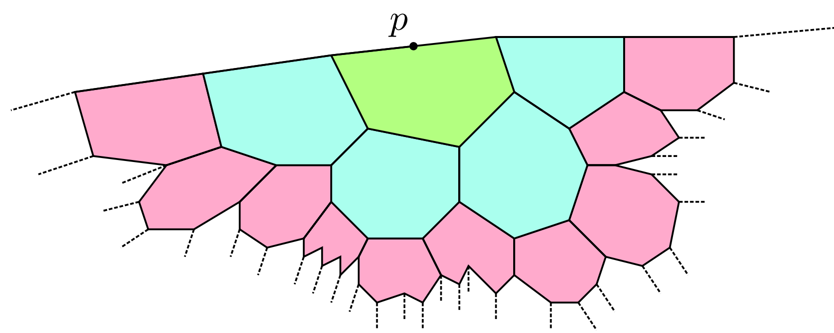

FIGURE 3. Star neighborhoods of B_0

Lemma 4.6. *Suppose a subdiagram $\mathcal{D}_i$ is homeomorphic to a closed disc, with one exterior geodesic edge and $i(\mathcal{D}_i) \geq 4$ interior edges. Let $\mathcal{D}_{i+1} = \mathrm{St}(\mathcal{D}_i)$ be the star neighborhood of $\mathcal{D}_i$ in $\mathcal{D}$. Then either $\mathcal{D}_{i+1}$ meets another side of $\mathcal{P}$, or it is homeomorphic to a closed disc with one exterior geodesic edge and*

$$i(\mathcal{D}_{i+1}) \geq i(\mathcal{D}_i)(d_{Int}(\lambda) - 4) + 2(d_{Ext}(\lambda) - d_{Int}(\lambda)) + 3$$

interior edges.

Proof. $\mathcal{D}_i$ has exactly one exterior geodesic edge. We assume that $\mathrm{St}(\mathcal{D}_i)$ does not meet another side of $\mathcal{P}$, so we can extend this edge to edges on faces B_- and B_+ adjacent to $\mathcal{D}_i$.

Taking a small neighborhood U of $\mathcal{D}_i$ in $\mathrm{St}(\mathcal{D}_i)$, there is a natural way to order the faces in $\mathrm{St}(\mathcal{D}_i) \setminus \mathcal{D}_i$ from $B_- = B_1, B_2, \ldots$ to $B_l = B_+$, so that B_p meets B_q along an edge in $U \setminus \mathcal{D}_i$ if and only if $|p - q| = 1$.

Claim 1: If $p \neq q$, $B_p \neq B_q$.

Suppose we have $B_p = B_q$, $p \neq q$. We can assume that $q \geq p+2$, since Lemma 3.10 states that faces don't bump into themselves. Therefore $\mathcal{D}_i$ and $B_p = B_q$ enclose a subdiagram $\mathcal{D}'$ entirely in the interior of $\mathcal{D}$, that is non-empty since it contains B_{p+1}. Every vertex of $\mathcal{D}'$ except perhaps two has degree at least three, and so, by the same argument as Lemma 3.10, we have a contradiction.

Claim 2: A geodesic edge γ cannot meet a face B in a disconnected set, i.e., it meets B on an edge, at a vertex or not at all.

Suppose otherwise. Then γ and B enclose a non-empty subdiagram $\mathcal{D}'$, containing B. Consider the equation (3.3). Following the calculation of (3.8), we see that

$$6 \leq (6 - 2e(B) - i(B)) \leq 3,$$

a contradiction.

Claim 3: A geodesic edge γ cannot meet two intersecting faces B, B' in a disconnected set.

Because of Claim 2, it must meet B and B', and enclose a non-empty diagram $\mathcal{D}'$, containing both B and B'. If B and B' meet along an edge, (3.3) gives

$$6 \leq (6-2e(B)-i(B))+(6-2e(B')-i(B')) \leq (6-2-2)+(6-2-2) = 4.$$

On the other hand, if B and B' meet only at a vertex v, we have

$$6 \leq (3-d(v))+(6-2e(B)-i(B))+(6-2e(B')-i(B')) \leq -1+3+3 = 5.$$

In both cases we obtain a contradiction.

Claim 4: The geodesic edge γ_0 of $\mathcal{P}$ which meets $\mathcal{D}_i$ only meets $B_1 \cup \cdots \cup B_l$ along a single edge of B_1 and a single edge of B_l.

This follows from claims 2 and 3, viewing $\mathcal{D}_i$ as a single face.

Now, $l \geq i(\mathcal{D}_i)$. We calculate the number of interior edges of $\mathcal{D}_{i+1}$. The faces B_1 and B_l have one external and at least $d_{Ext}(\lambda) \geq 4$ internal edges, at least $d_{Ext}(\lambda) - 2$ of which will contribute to $i(\mathcal{D}_{i+1})$.

The faces $B_2, \ldots, B_{l-1}$ have at least $d_{Int}(\lambda) \geq 7$ internal edges, at least $d_{Int}(\lambda) - 3$ of which will contribute to $i(\mathcal{D}_{i+1})$.

Finally, we may have to combine $(l-1)$ of these edges together when we make $\mathcal{D}_{i+1}$ into a single face and remove vertices of degree two. So

$$\begin{aligned} i(\mathcal{D}_{i+1}) &\geq 2(d_{Ext}(\lambda) - 2) + (l-2)(d_{Int}(\lambda) - 3) - (l-1) \\ &= 2(d_{Ext}(\lambda) - 2) + l(d_{Int}(\lambda) - 4) - 2d_{Int}(\lambda) + 7 \\ &\geq i(\mathcal{D}_i)(d_{Int}(\lambda) - 4) + 2(d_{Ext}(\lambda) - d_{Int}(\lambda)) + 3. \qquad \square \end{aligned}$$

We continue the proof of Proposition 4.5.

Note that $d_{Ext}(\lambda) - d_{Int}(\lambda) > \frac{1}{2\lambda} - (\frac{1}{\lambda} + 1) = \frac{-1}{2\lambda} - 1$, so

$$\begin{aligned} i(\mathcal{D}_{i+1}) &\geq (\tfrac{1}{\lambda} - 4)i(\mathcal{D}_i) - (\tfrac{1}{\lambda} - 1) = (\tfrac{1}{\lambda} - 4)\Big(i(\mathcal{D}_i) - \frac{\frac{1}{\lambda} - 1}{\frac{1}{\lambda} - 4}\Big) \\ &\geq (\tfrac{1}{\lambda} - 4)(i(\mathcal{D}_i) - \tfrac{5}{2}). \end{aligned} \tag{4.7}$$

Since $i(\mathcal{D}_i) \geq 9$ for $i = 2$, by induction $i(\mathcal{D}_i) \geq 9$ for all $i \geq 2$.

We now return to bounding the number of faces in a chain joining B_0 to one of the other sides of $\mathcal{P}$.

If there is no $(k+1)$-chain of faces, then every face in $\mathcal{D}_k \setminus \mathcal{D}_{k-1}$, with the exception of two, is an interior face of $\mathcal{D}$. On the one hand, by (4.7) $i(\mathcal{D}_k)$ grows exponentially fast:

$$\begin{aligned} F_I &\geq i(\mathcal{D}_k) \\ &\geq (\tfrac{1}{\lambda}-4)^{k-2} i(\mathcal{D}_2) - \tfrac{5}{2}(\tfrac{1}{\lambda}-4) - \tfrac{5}{2}(\tfrac{1}{\lambda}-4)^2 - \cdots \tfrac{5}{2}(\tfrac{1}{\lambda}-4)^{k-2} \\ &\geq (\tfrac{1}{\lambda}-4)^{k-2}\cdot 9 - \tfrac{5}{2}(\tfrac{1}{\lambda}-4)\left(\frac{(\frac{1}{\lambda}-4)^{k-2}-1}{(\frac{1}{\lambda}-4)-1}\right) \\ &\geq 9(\tfrac{1}{\lambda}-4)^{k-2} - 5(\tfrac{1}{\lambda}-4)^{k-2} = 4(\tfrac{1}{\lambda}-4)^{k-2}. \end{aligned}$$

On the other hand, by Lemma 3.9,

$$F_I \leq \frac{3n-6}{d_{Int}(\lambda)-1} \leq 3n,$$

thus for some $C = C(\lambda)$, we have

$$k \log(\tfrac{1}{\lambda}-4) \leq \log(n) + C. \qquad \square$$

Proof of Theorem 4.4. Proposition 4.5 shows that geodesic n-gon $\mathcal{P}$ is Δ-slim, with

$$\Delta = \frac{M}{\log(\frac{1}{\lambda}-4)} \cdot \log(n) + C,$$

where $M = \max\{|r| \mid r \in R\}$ is the maximum diameter of a face, and C is independent of n. So the Cayley graph $\Gamma(G,S)$ is $AC_u(\kappa)$ with

$$\kappa = -\frac{1}{M^2}\log^2(\tfrac{1}{\lambda}-4). \qquad \square$$

5. Building a round tree in the Cayley complex

Our final goal is to prove the following result.

Theorem 5.1. *Suppose $G = \langle S|R\rangle$ is a $C'(1/12)$ presentation, with $|S| \geq 2$ and $|R| \geq 1$, where $|r| \leq M$ for all $r \in R$. Suppose further that for some $M^* \geq 12$, every word $u \in \langle S\rangle$ of length M^* appears at least once in some cyclic conjugate of some $r^{\pm 1}, r \in R$. Then for some universal constant $C > 0$, we have*

$$\mathcal{C}\dim(\partial_\infty G) \geq 1 + C \cdot \frac{M^*}{\log(M)}.$$

In this section, we will build a round tree in the Cayley complex $\Gamma^2 = \Gamma^2(G,S,R)$. The branching of this tree is controlled by the size of M^* relative to M, and in Section 6 we use a lemma of Bourdon to give the lower bound for the conformal dimension of the boundary.

Definition 5.2. *The Cayley complex $\Gamma^2(G,S,R)$ of a finitely presented group $G = \langle S|R\rangle$ is the universal cover of the complex X, where X has a bouquet of $|S|$ oriented circles as a 1-skeleton, each labelled with a generator from S, and there are $|R|$ discs glued in with boundary labels from the corresponding relators in R.*

Note that the 1-skeleton of $\Gamma^2(G,S,R)$ is the Cayley graph $\Gamma^1(G,S)$.

5.1. **Preliminary lemmas.** As we are modifying Champetier's technique, we need two of his lemmas. Proofs are given for completeness.

Lemma 5.3 (Champetier [6, Lemma 4.19])**.** *Consider a $C'(1/6)$ presentation of a group $G = \langle S|R\rangle$ with Cayley graph $\Gamma = \Gamma(G,S)$.*

For every point $a \in \Gamma$, there are at most two distinct $s \in S \cup S^{-1}$ that satisfy $d(1,as) \leq d(1,a)$. In other words, any geodesic from 1 to a can be extended to any of the neighbors of a, with at most two exceptions.

Of course, for $a \in G, a \neq 1$, there exists $s \in S \cup S^{-1}$ so that $d(1,as) = d(1,a) - 1$. We denote a geodesic between p and q by $[p,q]$.

Proof. Let $\gamma_1 = [a,1]$, and let $b \in \gamma_1$ satisfy $d(a,b) = 1$. Suppose there is some $c \in \Gamma$, $c \neq b$, so that $d(a,c) = 1$ and $d(1,c) \leq d(1,a)$. Let $\gamma_2 = [c,1]$. Note that a,b do not lie in γ_2.

Let $\mathcal{D}$ be a reduced diagram for the geodesic triangle $\gamma_1, [a,c], \gamma_2$. By Lemma 3.11, and its proof, this diagram has a face labelled r_1 with at most one interior edge, containing a,b,c in its boundary. Let $w_i = \gamma_i \cap \partial r_1$, for $i = 1,2$. Since w_i are geodesics, and the interior edge has length at most $|r_1|/6$, we have $|w_i| \geq |r_1|/2 - |r_1|/6 - 1 > |r_1|/6$, for $i = 1,2$.

Suppose now that there is another point c' satisfying the same conditions as c. Then, as before one builds a geodesic triangle from a,b,c', and finds a relator r_2 so that the initial segment of γ_1 which overlaps r_2 has length at least $|r_2|/6$. Thus, by the $C'(1/6)$ condition, r_1 and r_2 are the same relator, and so $c = c'$. □

Lemma 5.4 (Champetier [6, Lemma 4.20])**.** *Consider a $C'(1/12)$ presentation of a group $G = \langle S|R\rangle$ with Cayley graph $\Gamma = \Gamma(G,S)$.*

For every $u' \in \Gamma$, there is at most one $u \in \Gamma$ so that $d(1,u) = d(1,u') + d(u',u) = d(1,u') + 3$, and so that a geodesic $\gamma_u = [1,u]$ starts with a subword of a relator $r_1 \in R$ of length greater than $|r_1|/6 + 3$.

Proof. Suppose $u \in \Gamma$ is such a point, and γ_u is such a geodesic.

Case 1: $u' \notin \gamma_u$.

Then the two geodesics γ_u and $\gamma_{u'} = [u,u'] \cup [u',1]$ form a geodesic bigon that splits at a vertex of $[u,u']$, and so by Lemma 3.11 there is a relator $r_2 \in R$ whose boundary meets both γ_u and $\gamma_{u'}$ from the point they split in a segment of length at least $|r_2|/6$. Thus r_1 and r_2 are the same relators, and so the only possibility is that γ_u and $\gamma_{u'}$ split at u and each begin with $|r_1|/6+3$ of r_1 and its inverse respectively.

Case 2: $u' \in \gamma_u$.

Suppose we have two such points u, v with corresponding geodesics γ_u, γ_v, and relators r_1, r_2. By Case 1, we can assume that these geodesics both pass through u', and so ∂r_1 and ∂r_2 will meet along a subword w of $\gamma_u \cap \gamma_v$ that includes u', before γ_u, γ_v split at some point p around a relator r_3. As γ_u, γ_v are both geodesics, after p they have to include at least $5|r_3|/12$ of the relator r_3.

If $|w| < |r_i|/12$ for both $i = 1, 2$, then r_1 and r_2 will both meet r_3 along at least $|r_i|/12$ of r_i, and so r_1, r_2, r_3 are all the same relator, and thus $d(u, 1) < d(u', 1)$, a contradiction. Thus $|w| \geq |r_i|/12$ for $i = 1$ or $i = 2$, and so r_1 and r_2 are the same relator, and $u = v$ as desired. □

5.2. **Building a round tree.** Suppose Y is a two dimensional complex with a negatively curved metric, and there is an S^1 action on Y that has a unique fixed point. If, additionally, there is a tree embedded in Y that meets every S^1 orbit in a single point, then we say Y is a *round tree* [11, 7.C$_3$].

Our goal in this section is to build a 2-complex A which is topologically embedded in the Cayley complex Γ^2, and whose one skeleton is a quasi-convex subset of the Cayley graph Γ^1. The complex A will be quasi-isometric to a sector of a round tree; we abuse terminology and simply refer to A as a round tree.

The ideas in this section are inspired by the arguments of Champetier [6] and Bourdon [3]. However, unlike Champetier, we build more than just a single (or finite number) of arcs in the boundary. Unlike Bourdon, we do not have a particular nice hyperbolic building to work in.

We build the round tree inductively. The round tree at step n is denoted by A_n. Its branching is controlled by the index set $T = \{1, \ldots, 3 \cdot 2^{K-3}\}$, where $K = \lfloor M^*/2 - 3 \rfloor$.

Each complex A_n is a union of complexes $A_{\mathbf{a}_n}$ indexed by $\mathbf{a}_n \in T^n$, homeomorphic to a closed disc, which can each be thought of as a triangular region with left edge a geodesic $L_{\mathbf{a}_n}$ from 1, right edge a geodesic $R_{\mathbf{a}_n}$ from 1, and outer edge a path $E_{\mathbf{a}_n}$, where $\mathbf{a}_n \in T^n$. The left tree is $L_n = \bigcup L_{\mathbf{a}_n}$, and the right tree is $R_n = \bigcup R_{\mathbf{a}_n}$, where the unions are over all $\mathbf{a}_n$ as above.

5.2.1. *Initial step.* Let $L_\emptyset = [1, s_1], R_\emptyset = [1, s_2]$ be two distinct edges from the identity in Γ. Combined, $L_\emptyset$ and $R_\emptyset$ give the reduced word $w = s_1^{-1}s_2$ of length 2. Choose some relator $r \in R$ which contains w as a subword, and let A_0 be the face corresponding to r in Γ^2 which contains w as a sub-word in its boundary. Let $E_\emptyset$ be the path of length $|r| - 2$ joining s_1 to s_2 along ∂A_0.

5.2.2. *Inductive step.* Assume we have built A_n. Let us fix $\mathbf{a}_n = (a_1, \ldots, a_n) \in T^n$, and use the notation $E = E_{\mathbf{a}_n} \subset \partial A_{\mathbf{a}_n}$ for the peripheral path joining the endpoints of $L = L_{\mathbf{a}_n}$ and $R = R_{\mathbf{a}_n}$.

Consider the function $d(1, \cdot)$ along E. By induction, this distance is always at least n, and strict local minima are separated by a path of length at least 50. At points $p \in E$ that are not strict local minima for $d(1, \cdot)$, there is at least one generator $s \in S$ that leaves E and extends the distance to the identity by one, i.e., $d(1, ps) = d(1, p) + 1$, by Lemma 5.3.

We can split the path E into segments of length 6 centered on local minima, and of length 3 or 4 in-between. For each endpoint z of the segments we have an edge that leaves E and extends the distance to the identity by one. This can be further extended two more steps to give four points at a distance $d(1, z) + 3$ from the identity. Lemma 5.4 rules out at most one of these, but we can still use three of them. We then extend from each point $K - 3$ times using Lemma 5.3 to give $|T| = 3 \cdot 2^{K-3}$ distinct points at a distance $d(1, z) + K$ from the identity.

Now for each $a_{n+1} \in T$, we have a corresponding geodesic of length K leaving the endpoints of each segment in E. Adjacent paths, and the segment between them, concatenate to give a path of length at most $K + 6 + K \leq M^*$, so there is some relator having this word as a subpath. Add these faces to $A_{\mathbf{a}_n}$ to define $A_{\mathbf{a}_{n+1}}$, and assign the left, right and outer edges to $L_{\mathbf{a}_{n+1}}$, $R_{\mathbf{a}_{n+1}}$, and $E_{\mathbf{a}_{n+1}}$ respectively, where $\mathbf{a}_{n+1} = (a_1, \ldots, a_n, a_{n+1})$.

We show that $E_{\mathbf{a}_{n+1}}$ does not get closer than $n + 1$ to the identity in Γ. To be precise:

Lemma 5.5. *Suppose $u', v' \in E_{\mathbf{a}_n}$ are consecutive endpoints of segments, $u, v \in E_{\mathbf{a}_{n+1}}$ are the corresponding points in $E_{\mathbf{a}_{n+1}}$ (after simplifying the path), and $\gamma_{uv} \subset E_{\mathbf{a}_{n+1}}$ the path joining them, coming from a relator $r_1 \in R$. We show that any geodesic from $p \in \gamma_{uv}$ to the identity must pass through u or v, and include the corresponding sub-path of γ_{uv}.*

Proof. Suppose some geodesic γ_{1p} joins p to the identity without passing through u or v. We can assume that the edge of γ_{1p} adjacent to p is not in γ_{uv}. Let γ_{1v} be the geodesic path joining v to 1 (through v').

Consider the closed path formed by $\gamma_{1p}, \gamma_{1v}, [p,v]_{\gamma_{uv}}$, and the associated reduced diagram. One can glue on a face to this diagram, labelled with the relator r_1, along $[p,v]_{\gamma_{uv}}$ and part of γ_{1v}. Notice that in this diagram, the r_1 face and the face containing 1 are the only two with exterior edges that are not geodesics. Therefore, by the argument of Lemma 3.11, they each have one interior edge, and the diagram is in the form of a standard bigon, as described in Lemma 3.11.

The face adjacent to r_1 is labelled by a relator r_2, which contains $[p,v]_{\gamma_{uv}}$ in its boundary, and also the edge of γ_{1p} adjacent to p, unlike r_1. So the relators r_1, r_2 must be distinct. Thus their overlap is at most $|r_2|/12$, and includes $[v',v]_{\gamma_{1v}} \cup [p,v]_{\gamma_{uv}}$. Since γ_{1p} and γ_{1v} are both geodesics, $[1,v']_{\gamma_{1v}}$ must contain at least $|r_2|/2 - 2|r_2|/12 = |r_2|/3$ of the relator r_2, which contradicts the choice of the paths $[v',v]$. $\square$

5.2.3. *Properties of A.* We have built an infinite polygonal complex $A = \bigcup_{n\in\mathbb{N}} A_n$. It is the union of planar complexes $A_{\mathbf{a}} \subset A$ indexed by $\mathbf{a} = (a_1, a_2, \ldots) \in T^{\mathbb{N}}$, given by $A_{\mathbf{a}} = \bigcup_{n\in\mathbb{N}} A_{(a_1,\ldots,a_n)}$.

Each $A_{\mathbf{a}}$ will carry a CAT(-1) metric, however A may not since the links of the vertices v' as above have simple closed paths of length two (created by the $|T|$ different faces all joined along their edges at v'). Before we consider different metrics on A, we need to understand how it sits inside Γ^2.

The complex A was built abstractly, but with an obvious natural polygonal immersion $i : A \to \Gamma^2$. Denote the 1-skeleton of A by A^1.

Lemma 5.6. *The map $i : A \to \Gamma^2$ is a topological embedding.*

More precisely, for every $p \in A$, every geodesic joining $i(p)$ to $i(1) = 1$ in Γ^1 is the image under i of a geodesic joining p to 1 in A^1.

Proof. By construction, there is at least one geodesic γ joining $i(p)$ to 1 in A^1. Suppose there is some geodesic $\gamma' \subset \Gamma^1$ joining $i(p)$ to 1, whose first edge is not in $i(\mathrm{St}(p))$. Then there is some relation $r_1 \in R$ so that the first $5|r_1|/12$ of γ after $i(p)$ is a subword of r_1.

The geodesic γ is made up of segments in the boundary of relators in A, and special length three extensions that, by Lemma 5.4, do not have any geodesic to the identity which begins with a subword of length $|r|/6 + 3$ of any relator $r \in R$.

Thus no such length three subword appears in the first $5|r_1|/12 - (|r_1|/6 + 3) = |r_1|/4 - 3$ vertices of γ. Therefore, $|r_1|/4 - 3 \geq |r_1|/12$

of r_1 bounds a relator in A, so r_1 is in A, contradicting the hypothesis that γ' left $i(\mathrm{St}(p))$. ☐

Lemma 5.7. *Consider A^1 and Γ^1 with their path metrics d_A and d_Γ. Then the map $i : A^1 \to \Gamma^1$ is a quasi-isometric embedding.*

In other words, (A^1, d_A) is quasi-isometric to (A^1, d_Γ), where d_Γ is the pullback $d_\Gamma(x, y) = d_\Gamma(i(x), i(y))$.

Proof. Take any $x, y \in A^1$. Since i is a topological embedding, clearly $d_A(x, y) \geq d_\Gamma(x, y)$.

Consider the geodesic triangle between 1, x and y with edges γ_{1x}, γ_{1y} and γ_{xy}. In light of Lemma 3.11, consider the structure of a reduced diagram $\mathcal{D}$ for this triangle.

The geodesics γ_{xy} and γ_{1x} form a spur starting at x that ends at an interior edge of $\mathcal{D}$ joining $p \in \gamma_{xy}$ to $p' \in \gamma_{1x}$.

Likewise, γ_{xy} and γ_{1y} form a spur starting at y that ends at an interior edge of $\mathcal{D}$ joining $q \in \gamma_{xy}$ to $q' \in \gamma_{1y}$. Also, γ_{1x} and γ_{1y} form a spur starting at 1 that ends at an interior edge of $\mathcal{D}$ joining $p'' \in \gamma_{1x}$ to $q'' \in \gamma_{1y}$.

Take a face labelled with some relator r_1 in any one of these spurs. The exterior edges of this face are geodesics, and the interior edges have length at most $|r_1|/12$. Thus the two exterior edges of the face have lengths between $|r_1|/2$ and $|r_1|/2 - 2|r_1|/12 = |r_1|/3$. So the same argument as in Lemma 5.6 shows that the relators in these spurs lie in A.

Since we chose p, p', p'', q, q', q'' to make the spurs as long as possible, the analysis of Lemma 3.11 shows that $[p, q]_{\gamma_{xy}}$ is adjacent to at most three faces in $\mathcal{D}$, thus $d_\Gamma(p, q) \leq 3M/2$.

Similarly, $d_\Gamma(p', p'') = d_A(p', p'') \leq 3M/2$ and $d_\Gamma(q', q'') = d_A(q', q'') \leq 3M/2$. Since p'', q'' lie on the boundary of a single face in A, part of the spur containing 1, $d_A(p'', q'') \leq M/12$.

Likewise, $d_\Gamma(p, p'), d_\Gamma(q, q') \leq M/12$, so

$$\begin{aligned} d_A(x, p') &= d_\Gamma(x, p') \leq d_\Gamma(x, p) + M/12, \quad \text{and} \\ d_A(y, q') &= d_\Gamma(y, q') \leq d_\Gamma(y, q) + M/12. \end{aligned}$$

Combining all these results, we see that

$$\begin{aligned} d_A(x, y) &\leq d_A(x, p') + d_A(p', p'') + d_A(p'', q'') + d_A(q'', q') + d_A(q', y) \\ &\leq \left(d_\Gamma(x, p) + \frac{M}{12} \right) + \frac{3M}{2} + \frac{M}{12} + \frac{3M}{2} + \left(d_\Gamma(q, y) + \frac{M}{12} \right) \\ &\leq d_\Gamma(x, y) + \frac{13M}{4}. \end{aligned}$$

☐

6. A lower bound for conformal dimension

In this section, we will build a model space X quasi-isometric to A^1, and show that $\mathcal{C}\dim(\partial_\infty X)$ has the desired lower bound. Since we have a quasi-symmetric inclusion of $\partial_\infty A$ into $\partial_\infty \Gamma = \partial_\infty G$, this will complete the proof of Theorem 5.1.

Let X be the graph with a vertex for each face in A, and an edge between two vertices if the boundaries of the corresponding faces have non-empty intersection.

Lemma 6.1. $(A^1, d_A) \overset{q.i.}{\simeq} (X, d_X)$

Proof. Let $f : X \to A^1$ be a map that sends each vertex $x \in X$ to some vertex in A on the edge of the corresponding face. Clearly, every point in A^1 is within a d_A-distance of $M/2$ from some point in $f(X)$.

If $d_X(x, y) = 1$ for $x, y \in X$, then $d_A(f(x), f(y)) \leq M$, where M is the maximum perimeter of a face. Thus for any $x, y \in X$, $d_A(f(x), f(y)) \leq M d_X(x, y)$.

Each edge in a geodesic $[f(x), f(y)] \subset A^1$ is the edge of some face in A, and adjacent edges will give intersecting faces (by definition). Adding the faces for x and y to this chain, shows that $d_X(x, y) \leq d_A(f(x), f(y)) + 2$. □

We recall the relevant lemma of Pansu and Bourdon.

Lemma 6.2 ([3, Lemma 1.6])**.** *Suppose Z is a compact metric space containing a family of curves $\mathcal{C} = \{\gamma_i : i \in I\}$, with diameters uniformly bounded away from zero.*

Suppose further that there is a probability measure μ on $\mathcal{C}$ and constants $C > 0$, $\sigma > 0$ such that for all balls $B(z, r)$ in Z

$$\mu(\{\gamma \in \mathcal{C} | \gamma \cap B(z, r) \neq \emptyset\}) \leq C r^\sigma.$$

Then the conformal dimension of Z is at least $1 + \frac{\sigma}{\tau - \sigma}$, where τ is the packing dimension of Z, and in fact $\tau - \sigma \geq 1$.

We need to estimate σ and τ for $Z = \partial_\infty X$.

By Lemma 5.7, any geodesic in (A^1, d_A) is within a uniformly bounded Hausdorff distance from a geodesic with the same endpoints in Γ^1. Thus (A^1, d_A) is also Gromov hyperbolic. Since X is quasi-isometric to A^1, it too is Gromov hyperbolic. The boundary $\partial_\infty X$ of X carries a visual metric ρ with parameter ϵ, for some $\epsilon > 0$.

In other words, for all points $u, v \in \partial_\infty X$, connected by a bi-infinite geodesic $\gamma_{uv} \subset X$,

$$\rho(u, v) \asymp e^{-\epsilon(u \cdot v)},$$

where $(u \cdot v) = d(1, \gamma_{uv})$, and $\asymp$ indicates a multiplicative error of $C_\rho \geq 1$.

Let $\mathcal{C} = \{\partial_\infty A_{\mathbf{a}} : \mathbf{a} \in T^{\mathbb{N}}\}$. There is a natural probability measure μ on $T^{\mathbb{N}}$ so that, for fixed $b_1, \ldots, b_n \in T$, $n \in \mathbb{N}$,

$$\mu(\{(a_1, a_2, \ldots) \in T^{\mathbb{N}} : a_i = b_i, 1 \leq i \leq n\}) = |T|^{-n}.$$

Lemma 6.3. *For this choice of $\mathcal{C}$, ρ, μ, we can take $\sigma = (\log |T|)/\epsilon$.*

Proof. Fix some $z \in \partial_\infty X$. Then z lies in the boundary of some $A_{\mathbf{a}}$, $\mathbf{a} = (a_1, a_2, \ldots) \in T^{\mathbb{N}}$.

Suppose $w \in B(z, r) \subset \partial_\infty X$. Then w lies in the boundary of some $A_{\mathbf{b}}$, $\mathbf{b} = (b_1, b_2, \ldots) \in T^{\mathbb{N}}$, and

$$\frac{1}{C_\rho} e^{-\epsilon(z \cdot w)} \leq \rho(z, w) \leq r,$$

so

$$(z \cdot w) \geq (-1/\epsilon) \log(C_\rho r).$$

Suppose $a_n \neq b_n$, and n is the smallest such n. Then deleting all vertices in X at distance n from the root face will disconnect the boundaries of $A_{\mathbf{a}}$ and $A_{\mathbf{b}}$, and so γ_{zw} must pass within n of the root face. Thus $(z \cdot w) \leq n$, so if $w \in B(z, r)$, $a_m = b_m$ for all

$$m < (-1/\epsilon) \log(C_\rho r).$$

Therefore,

$$\mu(B(z, r)) \leq |T|^{(1/\epsilon)\log(C_\rho r)+1} \leq |T|^{1+\log(C_\rho)/\epsilon} \cdot r^{(\log |T|)/\epsilon}. \qquad \square$$

It remains to bound τ. When we built A_{n+1} from A_n, we added $|T|$ faces to each segment along E_n. So each face in A_n that bordered E_n could have at most $M|T|$ faces joined on to it. This gives a way to label every point in $\partial_\infty X$ by an element of

$$W = \{1, 2, \ldots, M|T|\}^{\mathbb{N}},$$

and we denote that labelling by $f : \partial_\infty X \to W$, which is an injection.

Put the metric ρ_W on W, where

$$\rho_W((a_1, a_2, \ldots), (b_1, b_2, \ldots)) = \exp(-\epsilon \min\{n : a_n \neq b_n\}).$$

Then $f^{-1} : f(\partial_\infty X) \to \partial_\infty X$ is a Lipschitz bijection, so

$$\tau = \dim_{\mathcal{P}}(\partial_\infty X) \leq \dim_{\mathcal{P}}(f(\partial_\infty X)) \leq \dim_{\mathcal{P}}(W) = \log(M|T|)/\epsilon.$$

Thus, by Lemma 6.2,

$$\begin{aligned}\mathcal{C}\dim(\partial_\infty X) &\geq 1 + \frac{\sigma}{\tau - \sigma} \geq 1 + \frac{\log(|T|)/\epsilon}{\log(M|T|)/\epsilon - \log(|T|)/\epsilon} \\ &= 1 + \frac{\log|T|}{\log(M)}.\end{aligned}$$

Since $|T| = 3 \cdot 2^{K-3}$, and $K = \lfloor M^*/2 - 3 \rfloor$, and $M^* \geq 12$,

$$\log|T| = \log(3/8) + K\log(2) \geq CM^*,$$

for $C = 1/100$, say.

Finally,

$$\partial_\infty X \overset{\text{q.s.}}{\simeq} \partial_\infty A \overset{\text{q.s.}}{\subset} \partial_\infty \Gamma = \partial_\infty G,$$

so we have

$$\mathcal{C}\dim(\partial_\infty G) \geq \mathcal{C}\dim(\partial_\infty X) \geq 1 + C \cdot \frac{M^*}{\log(M)}.$$

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Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana IL.

E-mail address: `jmmackay@illinois.edu`