Computing the first eigenpair of the p-Laplacian via inverse iteration of sublinear supersolutions

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Abstract

We introduce an iterative method for computing the first eigenpair (λ_p, e_p) for the p-Laplacian operator with homogeneous Dirichlet data as the limit of (μ_q,u_q) as $q \to p^-$, where u_q is the positive solution of the sublinear Lane-Emden equation $-\Delta_p u_q = \mu_q u_q^{q-1}$ with same boundary data. The method is shown to work for any smooth, bounded domain. Solutions to the Lane-Emden problem are obtained through inverse iteration of a supersolution which is derived from the solution to the torsional creep problem. Convergence of u_q to e_p is in the C^1-norm and the rate of convergence of μ_q to λ_p is at least $O(p-q)$.

Keywords: p-Laplacian, first eigenvalue and eigenfunction, inverse iteration, Lane-Emden problem, torsional creep problem.

1 Introduction

In this paper we develop an iterative method to obtain the first eigenpair (λ_p, e_p) of the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda \left|u\right|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_p u := \operatorname{div} \left|\nabla u\right|^{p-2} \nabla u$, $p > 1$, is the p-Laplacian operator and $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, is any smooth, bounded domain. The p-Laplacian equation appears in several mathematical models in fluid dynamics, such as in the modelling of non-Newtonian fluids and glaciology [5, 14, 23, 33], turbulent flows [18], climatology [17] nonlinear diffusion (where it is called the N-diffusion equation; see [34] for the original article and [24] for some current developments), flow through porous media [35], power law materials [6] and in the study of torsional creep [28].

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The first eigenvalue λ_p of (1) is variationally characterized by

$$\lambda_p = \min_{u \in W_0^{1,p}(\Omega)/\{0\}} R\left(u\right) > 0$$

where R is the Rayleigh quotient

$$R\left(u\right) = \frac{\int_\Omega \left|\nabla u\right|^p dx}{\int_\Omega \left|u\right|^p dx}.$$

The first eigenfunction e_p of (1) is characterized by the fact that the minimum of R is attained at e_p, so that

$$\lambda_p = \frac{\int_\Omega \left|\nabla e_p\right|^p dx}{\int_\Omega e_p^p dx}.$$

It is well-known that λ_p is isolated and simple, and that the corresponding eigenfunction $e_p \in C^{1,\alpha}\left(\overline{\Omega}\right)$ can be taken positive. Since R is homogeneous, we may assume $\|e_p\|_\infty = 1$, where $\|\cdot\|_\infty$ stands for the L^∞-norm.

In the one-dimensional case the first eigenpair (λ_p, e_p) is explicitly determined by solving the corresponding ODE boundary value problem. If $\Omega = (a,b)$, then $\lambda_p = \left(\pi_p / \left(b-a\right)\right)^{p-1}$ and $e_p = (p-1)^{-1/p} \sin_p\left(\pi_p\left(x-a\right)/\left(b-a\right)\right)$, where $\pi_p := 2(p-1)^{1/p}\int_0^1 (1-s^p)^{-1/p} ds$ and $\sin_p$ is a $2\pi_p$-periodic function that generalizes the classical sine function (see [11, 32]).

When $p = 2$, we have $\Delta_p = \Delta$, the Laplacian operator, whose first eigenpair (λ_p, e_p) is well-known for domains with simple geometry (that is, domains which admit some kind of symmetry); for more general domains it can be determined by several numerical methods (see [12] and references therein). However, if $p \neq 2$ and $N \geqslant 2$, the first eigenpair is not explicitly known even for simple symmetric domains such as a square or a ball, and there are few available numerical methods to deal directly with the eigenproblem (1) in these domains (see [10], [30] and [37]).

On the other hand, several numerical methods are available to solve homogeneous Dirichlet problems for the (Poisson) p-Laplacian equation in the form

$$-\Delta_p u = f\left(x\right)$$

when f depends only on $x \in \Omega$ (see [2, 8, 9, 19, 21, 36]). This fact motivated the development of our recent inverse iterative method for finding the first eigenpair in [10]. If Ω is a N-dimensional ball, the convergence of the method was established and numerical evidence for its applicability when Ω is a 2-dimensional square were also presented. In the special case of the Laplacian operator, the method was proved to work in general domains and can also be used to obtain other eigenpairs (see [12]). However, since the method was based on the iteration of the nonlinear p-Laplacian equation in (1), the difficulties in dealing with the nonlinearity on the right-hand side of the equation prevented us from showing that the method works in any domain and any $p > 1$.

In this work we consider a different inverse iterative approach, also based on the solution of the Poisson p-Laplacian equation, but built around an eigenproblem which has a sublinear nonlinearity on its right-hand side. This type on nonlinearity is more manageable and we are able to prove that the iterative method works for any smooth, bounded domain. It is based on obtaining positive solutions $v_{\mu,q}$ for the Lane-Emden type problem

$$\begin{cases} -\Delta_p v = \mu\, |v|^{q-2}\, v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

After rescaling, μ and $v_{\mu,q}$ produce a family of pairs $\{(\mu_q, u_q)\}_{1<q<p}$ converging to the first eigenpair (λ_p, e_p) when $q \to p^-$, the convergence $u_q \to e_p$ being in $C^1\left(\overline{\Omega}\right)$. We will now describe the method in more detail.

It is well known that for each fixed $\mu > 0$, problem (2) has a unique solution $v_{\mu,q}$, if $1 < q < p$ (see [27]). If $q = p$, we have the p-Laplacian eigenvalue problem. If $q > p$, positive solutions of (2) usually are not unique. A nonuniqueness result for ring-shaped domains is given in [22] when q is close to the Sobolev critical exponent p^* ($p^* = Np/(N-p)$, if $1 < p < N$, and $p^* = \infty$, if $p \geqslant N$). On the other hand, as proved in [1], positive solutions are unique when Ω is a ball, while for general bounded domains the uniqueness of positive solutions that reach the minimum energy (ground states) was established in [20] under the conditions $1 < p < N$ and $1 < q < p^*$.

Now, in order to construct the approximating sequence to the first eigenpair, first choose any $\mu > 0$ and a sequence (q_n), $1 < q_n < p$, such that $q_n \to p^-$. It is important to notice that μ need not to be taken close to λ_p. This point is crucial, since good *a priori* estimates for λ_p are hard to obtain. For each q_n we need to solve the Lane-Emden problem (2) in order to find v_{μ,q_n}, which is a degenerate nonlinear problem almost as hard to solve as the eigenvalue problem for the p-Laplacian (1) itself. In order to obtain the solutions v_{μ,q_n} we first solve the much easier *torsional creep problem*

$$\begin{cases} -\Delta_p \phi = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

For example, if Ω is a ball centered at $x_0 \in \mathbb{R}^N$ with radius $R > 0$, it is easy to verify that the *torsion function* ϕ is the radial function

$$\phi(r) = \frac{p-1}{pN^{\frac{1}{p-1}}}\left(R^{\frac{p}{p-1}} - |r|^{\frac{p}{p-1}}\right), \quad r = |x - x_0| \leqslant R. \tag{4}$$

Then compute $k_p = \|\phi\|_\infty^{1-p}$ and set

$$\phi_0 = \left(\frac{\mu}{k_p}\right)^{\frac{1}{p-q_n}} \frac{\phi}{\|\phi\|_\infty}.$$

ϕ_0 is a supersolution to (2). One immediately sees that the easiest choice is $\mu = k_p$, so that $\phi_0 = \phi/\|\phi\|_\infty$. Now apply inverse iteration to ϕ_0, finding a sequence of iterates (ϕ_m) which satisfy

$$\begin{cases} -\Delta_p \phi_{m+1} = \mu\phi_m^{q_n-1} & \text{in } \Omega, \\ \phi_{m+1} = 0 & \text{on } \partial\Omega. \end{cases}$$

This can be done by a number of numerical methods. Finite volume based methods are presented in [4, 21]; finite element based methods are also available (see [26] and the references therein).

After a preestablished tolerance limit has been reached at some ϕ_m, where m is a function of μ and q_n, set

$$v_{\mu,q_n} = \phi_m$$

and define u_{q_n} and μ_{q_n} as

$$\mu_{q_n} := \frac{\mu}{\|v_{\mu,q_n}\|_\infty^{p-q_n}} \quad \text{and} \quad u_{q_n} := \frac{v_{\mu,q_n}}{\|v_{\mu,q_n}\|_\infty}.$$

In Theorem 7 we show that $\mu_{q_n} \to \lambda_p$ and $u_{q_n} \to e_p$ in $C^1\left(\overline{\Omega}\right)$ when $q_n \to p^-$. Stopping at any point q in the sequence (q_n) will give an approximation for the first eigenpair of the p-Laplacian, as shown in Algorithm 1 below.

Algorithm 1 Inverse iteration for the first p-Laplacian eigenpair (λ_p, e_p)

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set $\mu$                                                          (an arbitrary positive number)
set $q$                                                            ($q$ should be chosen close to $p$)
solve $-\Delta_p\phi_p = 1$ in $\Omega$, $\phi_p = 0$ on $\partial\Omega$                 (torsion function)
set $\phi_0 = (\mu/k_p)^{\frac{1}{p-q}} \phi_p / \|\phi_p\|_\infty$                       (supersolution)
for $n = 0, 1, 2, \ldots$ do
    solve $-\Delta_p\phi_{m+1} = \mu\phi_m^{q-1}$ in $\Omega$, $\phi_{m+1} = 0$ on $\partial\Omega$   (Inverse iterative sequence)
end for
return $\mu / \|\phi_{m+1}\|_\infty^{p-q}$                                     (first eigenvalue $\lambda_p$)
return $\phi_{m+1} / \|\phi_{m+1}\|_\infty$                                     (first eigenfunction $e_p$)
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The outline of the paper is as follows. In Section 2 we present some preliminary results that will be used in the sequel. The sequence of approximates is built in Section 3 and the proof of its convergence to the first eigenpair is given in Section 4. In Section 5 we present some numerical results for the unit ball of dimensions $N = 2, 3$ and 4. These results compare very well with the ones presented in [10].

The main advantage of the method presented here, besides its applicability to general domains, is that approximations to both λ_p and e_p are obtained with the desired precision by an iteration process which is numerically simple and, in the case of a ball, also explicit.

2 Preliminary results

In this section we state simple versions of some results on the p-Laplacian. We begin with the following comparison principle (see [16] for a more general version).

Lemma 1 *For $i \in \{1, 2\}$, let $h_i \in C\left(\overline{\Omega}\right)$ and $u_i \in W^{1,p}(\Omega)$ be such that $-\Delta_p u_i = h_i$ in Ω. If $h_1 \le h_2$ in Ω and $u_1 \leqslant u_2$ on $\partial\Omega$, then $u_1 \leqslant u_2$ in Ω.*

The following result is a simple version of a general result proved in the classical paper [31] of Lieberman.

Theorem 2 *[31, Thm 1] Suppose that* $u \in W^{1,p}(\Omega)$ *is a weak solution of the Dirichlet problem*

$$\begin{cases} -\Delta_p u = f(x,u) & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega \end{cases}$$

where f *is a continuous function such that*

$$|f(x,\xi)| \leqslant \Lambda \quad \textit{fol all } (x,\xi) \in \Omega \times [-M, M]$$

for positive constants Λ *and* M.

If $\|u\|_\infty \leqslant M$*, then there exists* $0 < \alpha < 1$*, depending only on* Λ*,* p *and* N*, such that* $u \in C^{1,\alpha}\left(\overline{\Omega}\right)$*; moreover we have*

$$\|u\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} \leqslant C,$$

where C *is a positive constant that depends only on* Λ*,* p*,* N *and* M*.*

Thus, denoting by ϕ is the solution of the torsional creep problem (3) in the domain Ω, one can easily verify using (4) and the comparison principle in balls that $0 < \phi \leqslant M$ in Ω for some positive constant M. Hence, Theorem 2 implies that $\phi \in C^{1,\alpha}\left(\overline{\Omega}\right)$ for some $0 < \alpha < 1$.

For the next lemma set

$$k_p := \|\phi\|_\infty^{1-p} > 0. \tag{5}$$

Lemma 3 $k_p \leqslant \lambda_p$.

Proof. Let e_p be the first eigenfuncion associated with λ_p satisfying $\|e_p\|_\infty = 1$ in Ω. Since

$$\begin{cases} -\Delta_p e_p = \lambda_p e_p^{p-1} \leqslant \lambda_p = -\Delta_p \left(\lambda_p^{\frac{1}{p-1}} \phi\right) & \text{in } \Omega, \\ e_p = 0 = \lambda_p^{\frac{1}{p-1}} \phi & \text{on } \partial\Omega, \end{cases}$$

it follows from the comparison principle that

$$0 < e_p \leqslant \lambda_p^{\frac{1}{p-1}} \phi \quad \text{in } \Omega.$$

Hence,

$$1 = \|e_p\|_\infty \leqslant \lambda_p^{\frac{1}{p-1}} \|\phi\|_\infty,$$

from what follows our claim. ■

Remark 4 *It follows from Picone's identity (see [3]) that, in fact, the inequality is strict, that is,* $k_p < \lambda_p$ *(for details, see [15, Lemma 8.1]).*

The following result is well-known and follows from Theorem 2.

Theorem 5 *Let* $-\Delta_p^{-1} : C^1\left(\overline{\Omega}\right) \to W_0^{1,p}(\Omega)$ *be the operator defined as follows: for each* $v \in C^1\left(\overline{\Omega}\right)$ *let* $-\Delta_p^{-1}v := u \in W_0^{1,p}(\Omega)$ *be the unique solution of the Dirichlet problem*

$$\begin{cases} -\Delta_p u = v & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $-\Delta_p^{-1}$ *is continuous and compact. Moreover,* $-\Delta_p^{-1}v \in C^{1,\alpha}\left(\overline{\Omega}\right)$ *for each* $v \in C^1\left(\overline{\Omega}\right)$.

In the remainder of the paper (λ_p, e_p) denotes the first eigenpair of (1), ϕ denotes the torsion function of Ω and $k_p := \|\phi\|_\infty^{1-p}$.

3 Construction of the sequence of approximates

As mentioned before, if $q < p$, then for each $\mu > 0$ the Lane-Emden problem

$$\begin{cases} -\Delta_p v = \mu\,|v|^{q-2}\,v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

has a unique positive solution $v_{\mu,q}$, which can be obtained via standard variational, and therefore *non-constructive*, arguments. The existence and uniqueness of solutions of (6) in the case $1 < q < p$ implies that the map $\mu \mapsto v_{\mu,q}$ is well-defined and monotone, in the sense that $\mu_1 < \mu_2$ implies $v_{\mu_1,q} < v_{\mu_2,q}$ in Ω, since $v_{\mu_1,q} = (\mu_1/\mu_2)^{1/(p-q)}\,v_{\mu_2,q}$ for any μ_1, $\mu_2 > 0$.

The basis of our constructive method is given by

Theorem 6 *Suppose* $1 < q < p$. *For each* $\mu > 0$ *the unique positive solution* $v_{\mu,q} \in C^{1,\alpha}\left(\overline{\Omega}\right) \cap W_0^{1,p}(\Omega)$ *of (6) satisfies*

$$0 < \left(\frac{\mu}{\lambda_p}\right)^{\frac{1}{p-q}} e_p \leqslant v_{\mu,q} \leqslant \left(\frac{\mu}{k_p}\right)^{\frac{1}{p-q}} \frac{\phi}{\|\phi\|_\infty} \quad \text{in } \Omega. \tag{7}$$

Moreover, $v_{\mu,q}$ *is the limit, in the* $C^1\left(\overline{\Omega}\right)$ *norm, of the sequence* $\{v_n\} \subset C^{1,\alpha}\left(\overline{\Omega}\right) \cap W_0^{1,p}(\Omega)$ *iteratively defined by*

$$v_0 := \left(\frac{\mu}{k_p}\right)^{\frac{1}{p-q}} \frac{\phi}{\|\phi\|_\infty} \tag{8}$$

and, for $n \geq 1$:

$$\begin{cases} -\Delta_p v_{n+1} = \mu v_n^{q-1} & \text{in } \Omega, \\ v_{n+1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

Proof. Define $\underline{v}_{\mu,q} := m e_p$ and $\overline{v}_{\mu,q} := \dfrac{M\phi}{\|\phi\|_\infty}$ where

$$m := \left(\frac{\mu}{\lambda_p}\right)^{\frac{1}{p-q}} \quad \text{and} \quad M := \left(\frac{\mu}{k_p}\right)^{\frac{1}{p-q}}.$$

We have

$$-\Delta_p \underline{v}_{\mu,q} \leqslant \mu \underline{v}_{\mu,q}^{q-1} \quad \text{and} \quad -\Delta_p \overline{v}_{\mu,q} \geqslant \mu \overline{v}_{\mu,q}^{q-1} \quad \text{in } \Omega. \tag{10}$$

Indeed, in Ω in we have

$$-\Delta_p \underline{v}_{\mu,q} = \lambda_p \underline{v}_{\mu,q}^{p-1} = \lambda_p \underline{v}_{\mu,q}^{p-q} \underline{v}_{\mu,q}^{q-1} = \lambda_p \left(m e_p \right)^{p-q} \underline{v}_{\mu,q}^{q-1} \leqslant \lambda_p m^{p-q} \underline{v}_{\mu,q}^{q-1} = \mu \underline{v}_{\mu,q}^{q-1}$$

and

$$-\Delta_p \overline{v}_{\mu,q} = k_p M^{p-1} = k_p M^{p-q} M^{q-1} \geqslant k_p M^{p-q} \left(\frac{M\phi}{\|\phi\|_\infty} \right)^{q-1} = \mu \overline{v}_{\mu,q}^{q-1}.$$

Since $\underline{v}_{\mu,q} = 0 = \overline{v}_{\mu,q}$ on Ω the inequalities in (10) mean that $\underline{v}_{\mu,q}$ and $\overline{v}_{\mu,q}$ are, respectively, sub- and supersolutions for (6).

Moreover, $\underline{v}_{\mu,q}$ and $\overline{v}_{\mu,q}$ are ordered, that is $\underline{v}_{\mu,q} \leqslant \overline{v}_{\mu,q}$ in Ω. For, since $k_p \leqslant \lambda_p$, we have

$$\begin{aligned} \lambda_p m^{p-1} &= \lambda_p \left(\frac{\mu}{\lambda_p} \right)^{\frac{p-1}{p-q}} \\ &= \mu^{\frac{p-1}{p-q}} \left(\frac{1}{\lambda_p} \right)^{\frac{q-1}{p-q}} \leqslant \mu^{\frac{p-1}{p-q}} \left(\frac{1}{k_p} \right)^{\frac{q-1}{p-q}} = k_p \left(\frac{\mu}{k_p} \right)^{\frac{p-1}{p-q}} = k_p M^{p-1}, \end{aligned}$$

whence

$$-\Delta_p \underline{v}_{\mu,q} = \lambda_p \underline{v}_{\mu,q}^{p-1} \leqslant \lambda_p m^{p-1} \leq k_p M^{p-1} = -\Delta_p \overline{v}_{\mu,q}$$

in Ω. Thus, since $\underline{v}_{\mu,q} = \overline{v}_{\mu,q} = 0$ on $\partial\Omega$, we obtain $\underline{v}_{\mu,q} \leqslant \overline{v}_{\mu,q}$ in Ω by applying the comparison principle.

Since $u \mapsto \mu u^{q-1}$ is increasing and $\underline{v}_{\mu,q} \leqslant \overline{v}_{\mu,q}$ in Ω, the comparison principle also implies that the sequence $\{v_n\}$ defined by the iteration process (9) starting with the supersolution $\overline{v}_{\mu,q}$ satisfies

$$\underline{v}_{\mu,q} \leqslant v_{n+1} \leqslant v_n \leqslant \overline{v}_{\mu,q} \quad \text{in } \Omega.$$

Hence, v_n converges to a function $v_{\mu,q}$ a.e. in Ω. Since $\|v_n\|_\infty \leqslant \|\overline{v}_{\mu,q}\|_\infty = M$, it follows from Theorem 2 that $\{v_n\} \subset C^{1,\alpha}\left(\overline{\Omega}\right)$ for some $0 < \alpha < 1$ (which does not depend on n) and that

$$\|v_n\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} \leqslant C$$

for some positive constant C which is independent of n.

Thus, from Arzela-Ascoli theorem we conclude that $v_n \to v$ in the C^1 norm.

Now, the continuity of the operator $-\Delta_p^{-1} : C^1\left(\overline{\Omega}\right) \to W_0^{1,p}\left(\Omega\right)$ permits passing to the limit in (9), which yields that $v_{\mu,q} \in C^1\left(\overline{\Omega}\right) \cap W_0^{1,p}\left(\Omega\right)$ is a solution of (6) satisfying

$$0 < \underline{v}_{\mu,q} \leqslant v_{\mu,q} \leqslant \overline{v}_{\mu,q} \quad \text{in } \Omega,$$

proving (7). The regularity $v_{\mu,q} \in C^{1,\alpha}\left(\overline{\Omega}\right)$ follows from Theorem 2. ∎

This iteration process also is known as inverse iteration since $v_{n+1} = -\Delta_p^{-1}(\mu v_n^{q-1})$. It is essentially the sub- and supersolution method starting with the supersolution $\overline{v}_{\mu,q}$; the solution $v_{\mu,q}$ that it produces is characterized as the maximal solution between $\underline{v}_{\mu,q}$ and $\overline{v}_{\mu,q}$.

If one starts the iteration with the subsolution then one obtains an increasing sequence converging to the minimal solution between $\underline{v}_{\mu,q}$ and $\overline{v}_{\mu,q}$. Because of the uniqueness this minimal solution coincides with $v_{\mu,q}$. However, in order to compute the minimal solution from this iteration process, it is necessary to know *a priori* a subsolution, which is exactly one of the unknowns that we wish to find by applying the method.

On the other hand the supersolution $\overline{v}_{\mu,q}$ is easily obtainable since it involves the solution of the simpler problem (3).

For example, if $\Omega = B_R(x_0)$, the ball centered at $x_0 \in \mathbb{R}^N$ with radius $R > 0$, we obtain from (4) that

$$k_p = \|\phi\|_\infty^{1-p} = N\left(\frac{p}{p-1}\right)^{p-1}$$

and

$$\overline{v}_{\mu,q}(r) = \left(\frac{\mu}{k_p}\right)^{\frac{1}{p-q}}\left(1 - |r|^{\frac{p}{p-1}}\right) = \mu^{\frac{1}{p-q}}\left(\frac{p-1}{pN^{\frac{1}{p-1}}}\right)^{\frac{p-1}{p-q}}\left(1 - |r|^{\frac{p}{p-1}}\right)$$

where $r = |x - x_0|$.

In this case it is easy to verify that the sequence v_n converging to $v_{\mu,q}$ is given recursively by the formula

$$v_{n+1}(r) = \int_r^R \left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} \mu v_n(s)^{q-1} ds\right)^{\frac{1}{p-1}} d\theta$$

where $v_0(r) = \overline{v}_{\mu,q}(r)$.

In our method, in order to compute the first eigenpair (λ_p, e_p), we chose any positive value $\mu > 0$ and any sequence $q_n \to p^-$. Then, for each q_n, we apply the inverse iteration of Theorem 6 starting with the supersolution

$$\overline{v}_{\mu,q_n} = \left(\frac{\mu}{k_p}\right)^{\frac{1}{p-q_n}} \frac{\phi}{\|\phi\|_\infty}$$

to obtain approximations for the function v_{μ,q_n}. Hence,

$$\frac{\mu}{\|v_{\mu,q_n}\|_\infty^{p-q_n}} \to \lambda_p \quad \text{and} \quad \frac{v_{\mu,q_n}}{\|v_{\mu,q_n}\|_\infty} \to e_p \text{ (in the } C^1 \text{ norm)}$$

a result that we prove in the next section.

4 Convergence of the method

Theorem 7 *For $\mu > 0$ and for each $1 < q < p$ set*

$$u_q := \frac{v_{\mu,q}}{\|v_{\mu,q}\|_\infty}, \tag{11}$$

where $v_{\mu,q} \in C^{1,\alpha}\left(\overline{\Omega}\right)$ *is the unique positive solution of (6), and*

$$\mu_q := \frac{\mu}{\|v_{\mu,q}\|_\infty^{p-q}}. \tag{12}$$

Then $\mu_q \to \lambda_p$ *and* $u_q \to e_p$ *in* $C^1\left(\overline{\Omega}\right)$ *as* $q \to p^-$.

Proof. Since $\|u_q\|_\infty = 1$ and

$$-\Delta_p u_q = \frac{\mu}{\|v_{\mu,q}\|_\infty^{p-1}} v_{\mu,q}^{q-1} = \frac{\mu}{\|v_{\mu,q}\|_\infty^{p-q}} u_q^{q-1} = \mu_q u_q^{q-1},$$

we have that u_q is the unique solution of the problem

$$\begin{cases} -\Delta_p u_q = \mu_q u_q^{q-1} & \text{in } \Omega, \\ u_q = 0 & \text{on } \partial\Omega. \end{cases} \tag{13}$$

As a consequence of (7) we have

$$\frac{\mu}{\lambda_p} \leqslant \|v_{\mu,q}\|_\infty^{p-q} \leqslant \frac{\mu}{k_p},$$

$$0 < \left(\frac{k_p}{\lambda_p}\right)^{\frac{1}{p-q}} e_p \leqslant u_q \leqslant \left(\frac{\lambda_p}{k_p}\right)^{\frac{1}{p-q}} \frac{\phi}{\|\phi\|_\infty} \quad \text{in } \Omega \tag{14}$$

and

$$k_p \leqslant \mu_q \leqslant \lambda_p. \tag{15}$$

Since

$$0 \leqslant \mu_q u_q^{q-1} \leqslant \lambda_p,$$

it follows from Theorem 2 the existence of constants $0 < \alpha < 1$ and $C > 0$ independent of q such that $u_q \in C^{1,\alpha}\left(\overline{\Omega}\right)$ and

$$\|u_q\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} \leqslant C \quad \text{for all } 1 < q < p.$$

Using the compactness of the immersion $C^{1,\alpha}\left(\overline{\Omega}\right) \hookrightarrow C^1\left(\overline{\Omega}\right)$, letting $q_n \to p$ we get, up to a subsequence, $\mu_{q_n} \to \lambda \in [k_p, \lambda_p]$ and $u_{q_n} \to u$ in $C^1\left(\overline{\Omega}\right)$. Taking the limit in (13), we conclude from Theorem 5 that u must satisfy

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\|u\|_\infty = 1$, whence $\lambda = \lambda_p$ and $u = e_p$ because λ is an eigenvalue and $u \neq 0$ is a corresponding eigenfuntion that does not change the signal in Ω (note from (14) that $u > 0$ in Ω). Since these limits are always the same, that is, do not depend on particular subsequences, we are done. ■

Next we prove an error estimate in the approximation of λ_p by μ_q or, alternatively, by the scaled quotient

$$\Lambda_q := \mu \frac{\|v_{\mu,q}\|_q^q}{\|v_{\mu,q}\|_p^p},$$

where $\|\cdot\|_r$ denotes the norm of the $L^r(\Omega)$, that is, $\|w\|_r = \left(\int_\Omega |w|^r dx\right)^{\frac{1}{r}}$.

The upper bound Λ_q together with the lower bound μ_q allows one to better control the accuracy of the approximation to λ_p.

Theorem 8 *There holds:*

(i) $\lambda_p \leqslant \Lambda_q$.

(ii) $\Lambda_q \to \lambda_p$ *as* $q \to p^-$.

(iii) There exists a positive constant K *which does not depend on* q *such that*

$$0 \leqslant \max\{(\lambda_p - \mu_q), (\Lambda_q - \lambda_p)\} \leqslant K(p-q) \tag{16}$$

for all q *sufficiently close to* p, $q < p$.

Proof. (i) follows directly from the variational characterization of λ_p and (2), since

$$\lambda_p \leqslant \frac{\|\nabla v_{\mu,q}\|_p^p}{\|v_{\mu,q}\|_p^p} = \frac{\mu\|v_{\mu,q}\|_q^q}{\|v_{\mu,q}\|_p^p} = \Lambda_q.$$

In order to prove (ii) we note from Theorem 7 that

$$\lim_{q\to p^-} \|u_q\|_q^q = \lim_{q\to p^-} \|u_q\|_p^p = \|e_p\|_p^p, \tag{17}$$

since u_q converges uniformly to e_p when $q \to p^-$. Thus, since

$$\Lambda_q = \mu\frac{\|v_{\mu,q}\|_q^q}{\|v_{\mu,q}\|_p^p} = \frac{\mu}{\|v_{\mu,q}\|_\infty^{p-q}}\frac{\|u_q\|_q^q}{\|u_q\|_p^p} = \mu_q\frac{\|u_q\|_q^q}{\|u_q\|_p^p}, \tag{18}$$

we obtain

$$\lim_{q\to p^-}\Lambda_q = \left(\lim_{q\to p^-}\mu_q\right)\left(\lim_{q\to p^-}\frac{\|u_q\|_q^q}{\|u_q\|_p^p}\right) = \lambda_p.$$

Now we prove error estimate (16). It follows from (i) and (15) that

$$\mu_q \leqslant \lambda_p \leqslant \Lambda_q.$$

Hence,

$$0 \leqslant \max\{(\lambda_p - \mu_q), (\Lambda_q - \lambda_p)\} \leqslant \Lambda_q - \mu_q.$$

Thus, in order to prove (iii) we need only to bound $\Lambda_q - \mu_q$. It follows from (18) that

$$\Lambda_q - \mu_q = \mu_q\left(\frac{\|u_q\|_q^q}{\|u_q\|_p^p} - 1\right) = \mu_q\frac{\int_\Omega\left(u_q^q - u_q^p\right)dx}{\int_\Omega u_q^p dx}.$$

Therefore,

$$\begin{aligned}
\Lambda_q - \mu_q &\leqslant \lambda_p \frac{\int_\Omega \left(u_q^q - u_q^p\right) dx}{\int_\Omega u_q^p dx} \\
&\leqslant \frac{\lambda_p}{\int_\Omega u_q^p dx} \int_\Omega \left[\max_{0\leqslant t\leqslant 1} \left(t^q - t^p\right)\right] dx \\
&= \frac{\lambda_p \left|\Omega\right|}{\int_\Omega u_q^p dx} \left(\frac{q}{p}\right)^{\frac{q}{p-q}} \frac{p-q}{p} \\
&\leqslant \frac{\lambda_p \left|\Omega\right|}{\int_\Omega u_q^p dx} \left(p-q\right).
\end{aligned}$$

Taking into account (17), there exists $R > 0$ such that $\int_\Omega u_q^p dx \geqslant R$ for all q near to p^-. Thus,

$$0 \leqslant \mu \frac{\left\|v_{\mu,q}\right\|_q^q}{\left\|v_{\mu,q}\right\|_p^p} - \mu_q \leqslant \frac{\lambda_p \left|\Omega\right|}{R} \left(p-q\right) = K \left(p-q\right).$$

■

5 Some numerical results

In this section we present some numerical results in the unit ball of dimensions $N = 2, 3, 4$. The table of numerical approximations for the first eigenvalue below was obtained choosing $\mu = k_p$ and taking $q = p - 0.01$. The results compare very well with the ones presented in [10] up to the second decimal digit.

Table 1: First eigenvalue for p-Laplacian in the unit ball.

p	$N = 2$	$N = 3$	$N = 4$
1.1	2.5666	3.86653	5.17607
1.2	2.9601	4.50265	6.0797
1.3	3.3182	5.10982	6.97306
1.4	3.6637	5.71889	7.89478
1.5	4.0053	6.3419	8.86046
1.6	4.3477	6.98495	9.87865
1.7	4.6932	7.65165	10.955
1.8	5.0434	8.34438	12.094
1.9	5.3993	9.06487	13.2991
2.0	5.76161	9.81443	14.5735
2.1	6.13078	10.5942	15.9202
2.2	6.50713	11.405	17.3421
2.3	6.89092	12.2478	18.8418
2.4	7.28234	13.1232	20.422
2.5	7.68152	14.0319	22.0854
2.6	8.08856	14.9747	23.8345
2.7	8.50354	15.9521	25.672
2.8	8.92654	16.9646	27.6004
2.9	9.35759	18.013	29.6225
3.0	9.79673	19.0977	31.7409
3.1	10.244	20.2194	33.9581
3.2	10.6994	21.3785	36.2769
3.3	11.163	22.5755	38.6999
3.4	11.6347	23.8111	41.2298
3.5	12.1146	25.0856	43.8694
3.6	12.6027	26.3997	46.6213
3.7	13.099	27.7539	49.4884
3.8	13.6034	29.1486	52.4734
3.9	14.1161	30.5844	55.5792
4.0	14.6369	32.0618	58.8085

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