Rotation sets of invariant separating continua of annular homeomorphisms

Shigenori Matsumoto

Abstract. Let f be a homeomorphism of the closed annulus A isotopic to the identity, and let $X \subset \mathrm{Int}A$ be an f-invariant continuum which separates A into two domains, the upper domain U_+ and the lower domain U_-. Fixing a lift of f to the universal cover of A, one defines the rotation set $\tilde{\rho}(X)$ of X by means of the invariant probabilities on X. For any rational number $p/q \in \tilde{\rho}(X)$, f is shown to admit a p/q periodic point in X, provided that (1) X consists of nonwandering points or (2) X is an attractor and the frontiers of U_- and U_+ coincides with X. Also the Carathéodory rotation numbers of $U_\pm$ are shown to be in $\tilde{\rho}(X)$ for any separating invariant continuum X.

1. Introduction

Let f be a homeomorphism of the closed annulus $A = S^1 \times [-1, 1]$, isotopic to the identity, i. e. f preserves the orientation and each of the boundary components $\partial_\pm A = S^1 \times \{\pm 1\}$. Suppose there is an f-invariant partition of A; $A = U_- \cup X \cup U_+$, where $U_\pm$ is a connected open set containing the boundary component $\partial_\pm A$ and X is a connected compact set. Let

$$\pi : \tilde{A} = \mathbb{R} \times [-1, 1] \to S^1 \times [-1, 1]$$

be the universal covering map and $T : \tilde{A} \to \tilde{A}$ a generator of the covering transformation group; $T(\xi, \eta) = (\xi + 1, \eta)$. Denote by $p : \tilde{A} \to \mathbb{R}$ the projection onto the first factor.

Fix once and for all a lift $\tilde{f} : \tilde{A} \to \tilde{A}$ of f. Then the function $p \circ \tilde{f} - p$ is T-invariant and can be looked upon as a function on the annulus A. Define the *rotation set* $\tilde{\rho}(X)$ as the set of values $\mu(p \circ \tilde{f} - p)$, where μ ranges over the f-invariant probability measures supported on X. The rotation set is a compact interval (maybe one point) in $\mathbb{R}$, which depends upon the choice of the lift $\tilde{f}$ of f.

The first example of an invariant continuum X such that the frontiers of $U_\pm$ satisfy $\mathrm{Fr}(U_+) = \mathrm{Fr}(U_-) = X$ and that the rotation set $\tilde{\rho}(X)$ is not a singleton is constructed by G. D. Birkhoff in his 1932 year paper [**B**], and is refered to as a *Birkhoff attractor*. It turns out that the Birkhoff attractor is an indecomposable continuum ([**C, L2**]). Furthermore it is shown by P. Le Calvez ([**L1**]) that for any

1991 *Mathematics Subject Classification.* Primary 37E30, secondary 37E45.

Key words and phrases. continuum, rotation set, Carathéodory rotation number, Brouwer line, foliations.

The author is partially supported by Grant-in-Aid for Scientific Research (C) No. 20540096.

rational number p/q between the two Carathéodory rotation numbers (See below for the definition.), there is a point $x \in \pi^{-1}(X)$ such that $\tilde{f}^q(x) = T^p(x)$. However the proof uses strongly the twist map properties of the Birkhoff attractor. This motivates us to consider the same problem for just a homeomorphism without the twist condition.

Theorem 1. *Assume that X is an attractor such that $\mathrm{Fr}(U_+) = \mathrm{Fr}(U_-) = X$. Then for any rational number $p/q \in \tilde{\rho}(X)$, there is a point $x \in \pi^{-1}(X)$ such that $\tilde{f}^q(x) = T^p(x)$.*

Before starting the proof of this theorem in Sect. 3, we need to establish the following result first of all in Sect. 2.

Theorem 2. *Assume X consists of nonwandering points. Then the same conclusion as Theorem 1 holds.*

The topological condition in Theorem 1 and the nonwandering condition in Theorem 2 are necessary as can be seen by Example A of [**W**]. Analogous result has already been obtained for area preserving homeomorphisms (Lemma 5.4, [**FL**]), or more generally for homeomorphisms without wandering points.

Let $\hat{U}_\pm = U_\pm \cup \partial_\infty U_\pm$ be the Carathéodory compactification of $U_\pm$, where $\partial_\infty U_\pm$ is the space of the prime ends, which is homeomorphic to the circle ([**E, M**]). As is well known, the homeomorphism f restricted to $U_\pm$ extends to a homeomorphism $\hat{f}_\pm : \hat{U}_\pm \to \hat{U}_\pm$. Denoting $I_+ = [0,1]$ and $I_- = [-1,0]$, define a homeomorphism

$$\Psi_\pm : \hat{U}_\pm \to S^1 \times I_\pm$$

such that $\Psi_\pm(\partial_\infty U_\pm) = S^1 \times 0$. By some abuse of notations denote by $\pi : \check{U}_\pm \to \hat{U}_\pm$ the universal covering map. (Thus $\pi^{-1}(U_\pm)$ is considered to be a subspace of both $\tilde{A}$ and $\check{U}_\pm$. This is a natural convention since the universal covering space is defined to be the set of homotopy classes of paths starting at a given base point.) Let $\check{\Psi}_\pm : \check{U}_\pm \to \mathbb{R} \times I_\pm$ be a lift of $\Psi_\pm$, and define $\check{p}_\pm : \check{U}_\pm \to \mathbb{R}$ by $\check{p}_\pm = p \circ \check{\Psi}_\pm$.

Let $\check{f}_\pm : \check{U}_\pm \to \check{U}_\pm$ be the lift of $\hat{f}_\pm$ such that $\check{f}_\pm = \tilde{f}$ on $\pi^{-1}(U_\pm)$. The rotation number of the restriction of $\check{f}_\pm$ to $\pi^{-1}(\partial_\infty U_\pm)$, denoted by $\check{\rho}_\pm$, is called the *Carathéodory rotation number* of $U_\pm$.

The second half of this paper is devoted to the following Theorem

Theorem 3. *The Carathéodory rotation numbers $\check{\rho}_\pm$ belong to $\tilde{\rho}(X)$.*

This statement was already known for area preserving homeomorphisms by Lemma 5.4 of [**FL**].

In [**Ha**] a diffeomorphism of the annulus A admitting a pseudo-circle as a minimal set is given. See also [**He**]. Another nontrivial example of a minimal set, a Warsaw circle, is constructed in [**W**]. More examples with various topological properties are constructed in [**FK**]. In [**MN**] we have shown that the Carathéodory rotation numbers $\check{\rho}_\pm(X)$ of a minimal set X are irrational, but could not prove that they are the same. Theorem 3, together with Theorem 2, yields the following results.

Corollary 4. *If X is a minimal set, then $\tilde{\rho}(X)$ is a singleton consisting of an irrational number $\check{\rho}_- = \check{\rho}_+$.*

2. Proof of Theorem 2

First of all let us state a deep and quite useful theorem of P. Le Calvez ([**L3**]) which plays a key role in the whole paper. A fixed point free and orientation preserving homeomorphism F of the plane $\mathbb{R}^2$ is called a *Brouwer homeomorphism.* A proper oriented simple curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ is called a *Brouwer line* for F if $F(\gamma) \subset R(\gamma)$ and $F^{-1}(\gamma) \subset L(\gamma)$, where $R(\gamma)$ (resp. $L(\gamma)$) is the right (left) side complementary domain of γ, which is decided by the orientation of γ.

Theorem 2.1. *Let F be a Brouwer homeomorphism commuting with the elements of a group Γ which acts on $\mathbb{R}^2$ freely and properly discontinuously. Then there is a Γ-invariant oriented topological foliation of $\mathbb{R}^2$ whose leaves are Brouwer lines of F.*

Let f and $\tilde{f}$ be as in Sect. 1. In order to show Theorem 2, by considering $\tilde{f}^q \circ T^{-p}$ instead of $\tilde{f}$, it suffices to show the following proposition.

Proposition 2.2. *If X consists of nonwandering points and $\tilde{\rho}(X)$ contains 0, then $\tilde{f}$ admits a fixed point in $\pi^{-1}(X)$.*

We shall prove the proposition by the absurdity. So let us assume that the lift $\tilde{f}$ does not have a fixed point in $\pi^{-1}(X)$. Then the distance of a point $x \in \pi^{-1}(X)$ and $\tilde{f}(x)$ has a positive lower bound since it is T-invariant. Therefore there is an open annular neighbourhood V of the attractor X such that $\tilde{f}$ does not admit a fixed point in $\pi^{-1}(V)$.

The overall strategy of the proof is to modify the homeomorphism f to a new one g without creating new fixed points such that the restrictions of $\tilde{g}$ to the lifts of the both boundary circles $\pi^{-1}(\partial_\pm A)$ are nontrivial rigid translations by the same translation number. Then by glueing the two boundary circles we obtain a torus T^2 and a homeomorphism on T^2. Now we can apply Theorem 2.1 to the lift of the homeomorphism to the universal covering space. This yields a topological foliation on T^2, which has long been well understood. Analyzing the foliation, we shall show Proposition 2.2. We first prepare a lemma which is necessary for the desired modification.

Lemma 2.3. *Assume $\tilde{f}$ does not admit a fixed point in the lift $\pi^{-1}(V)$ of an annular neighbourhood V of X. Then the Carathéodory rotation number $\check{\rho}_\pm$ is nonzero.*

Proof: Consider the mapping $\tilde{f} - \mathrm{Id}$ defined on $\pi^{-1}(V)$. Since it is T-invariant, it yields a mapping from V, still denoted by

$$\tilde{f} - \mathrm{Id} : V \to \mathbb{R}^2 \setminus \{0\}.$$

Clearly for any positively oriented essential simple closed curve γ in V, the degree of the map

$$\tilde{f} - \mathrm{Id} : \gamma \to \mathbb{R}^2 \setminus \{0\}$$

must be the same. If the curve γ is contained in $U_\pm$, then the degree can be studied by considering the map $\check{f}_\pm$ defined on the lift $\check{U}_\pm$ of the Carathéodory compactification $\hat{U}_\pm$. If the Carathéodory rotation number $\check{\rho}_\pm$ is nonzero, the degree is clearly 0.

To analyze the case $\check{\rho}_\pm = 0$, we need the following form of the Cartwright-Littlewood theorem [**CL**].

Theorem 2.4. *If $\check\rho_+ = 0$ and if* $\mathrm{Fix}(\tilde f) \cap \pi^{-1}(X) = \emptyset$, *then the map $\hat f_+$ on $\partial_\infty U_+$ is Morse Smale and the attractors (resp. repellors) of $\hat f_+|_{\partial_\infty U_+}$ are attractors (resp. repellors) of the whole map $\hat f_+$.*

This is slightly stronger than the usual version in which it is assumed that $\mathrm{Fix}(f) \cap X = \emptyset$. However the proof works under the assumption of Theorem 2.4. See e. g. Sect. 3 of [**MN**].

Let us complete the proof of Lemma 2.3. Theorem 2.4 enables us to compute the degree of the curve δ in $U_\pm$ when $\check\rho_\pm = 0$. The degree is n if $\delta \subset U_-$ and $-n$ if $\delta \subset U_+$, where n is the number of the attractors. Since the degree must be the same in U_- and U_+, the conclusion follows. □

We shall make the following assumption.

Assumption 2.5. The Carathéodory rotation number $\check\rho_-$ is negative.

Now let us start the modification of f. Condition (4) below will be used in Sect. 4.

Lemma 2.6. *Under the assumption of Lemma 2.3 and Assumption 2.5, there exists a homeomorphism g of A such that*
(1) $g = f$ in some neighbourhood of X,
(2) $\tilde g$ does not admit a fixed point in $\tilde A$, where $\tilde g$ is the lift of g such that $\tilde g = \tilde f$ on $\pi^{-1}(X)$,
(3) $\tilde g$ is a negative rigid translation by the same translation number on $\pi^{-1}(\partial_\pm A)$, and
(4) $\check p_- \circ \check g_- - \check p_- \le -c$ on $\hat U_-$ for some positive number c.

Proof: The modification in U_- will be done in the following way. We identify $\hat U_-$ with $S^1 \times [-1, 0]$ by the homeomorphism Ψ_- and the universal covering space $\check U_-$ with $\mathbb{R} \times [-1, 0]$. Thus $\check p_-$ is just the projection onto the first factor; $\check p_-(\xi, \eta) = \xi$. Now Assumption 2.5 implies that the lift

$$\check f_- : \mathbb{R} \times [-1, 0] \to \mathbb{R} \times [-1, 0]$$

of $\hat f_-$ satisfies that $\check p_- \circ \check f_-(\xi, 0) < \xi - 2c$ for some $c > 0$. Therefore changing the coordinates of $[-1, 0]$ if necessary, one may assume that $\check p_- \circ \check f_-(\xi, \eta) \le \xi - c$ if $(\xi, \eta) \in \mathbb{R} \times [-1/2, 0]$. Define a homeomorphism h of $S^1 \times [-1, 0]$ by

$$h(\xi, \eta) = (\xi + \varphi(\eta) \mathrm{mod} 1, \eta),$$

where $\varphi : [-1, 0] \to (-\infty, 0]$ is a continuous function such that $\varphi([-1/2, 0]) = 0$ and

$$\varphi(\eta) \le -\sup\{(\check p_- \circ \check f_- - \check p_-)(\xi, \eta) \mid \xi \in S^1\} - c.$$

Define $g = f \circ h$. Then its lift $\check g_-$ satisfies

$$\check p_- \circ \check g_- - \check p_- \le -c$$

on $\check U_- = \mathbb{R} \times [-1, 0]$. Clearly condition (3) for $\pi^{-1}(\partial_- A)$ can be established by a further obvious modification.

Now to modify f in U_+, we do the same thing as in U_-. If the Carathéodory rotation number $\check\rho_+$ is negative, then with an auxiliary modification we are done. If it is positive insert a time one map of the Reeb flow. □

The rest of this section is devoted to the proof that the rotation set $\tilde\rho(X)$ for g in Lemma 2.6 does not contain 0. Consider the torus T^2 which is obtained from A by glueing the two boundary curves $\partial_- A$ and $\partial_+ A$. Then the condition (3) above shows that g induces a homeomorphism of T^2, again denoted by g. The universal cover of T^2 is $\mathbb{R}^2$ and $\tilde{A} = \mathbb{R} \times [-1, 1]$ is a subset of $\mathbb{R}^2$. The lift $\tilde{g} : \tilde{A} \to \tilde{A}$ can be extended uniquely to a lift $\tilde{g} : \mathbb{R}^2 \to \mathbb{R}^2$ of $g : T^2 \to T^2$. The covering transformation group Γ is isomorphic to $\mathbb{Z}^2$, generated by the horizontal translation T and the vertical translation by 2, denoted by S. Since $\tilde{g}$ is a Brouwer homeomorphism, there is a Γ-invariant oriented foliation on $\mathbb{R}^2$ whose leaves are Brouwer lines for $\tilde{g}$. This yields an oritented foliation $\mathcal{F}$ on the torus T^2. The proof is divided into several cases.

Case 1. *The foliation $\mathcal{F}$ does not admit a compact leaf.* Then $\mathcal{F}$ is conjugate either to a linear foliation or to a Denjoy foliation, both of irrational slope. The lift of $\mathcal{F}$ to the open annulus $\mathbb{R}^2/\langle T \rangle$ is conjugate to a foliation by vertical lines. There is defined a projection from $\mathbb{R}^2/\langle T \rangle$ to S^1 along the leaves of the foliation. This lifts to a projection $q : \mathbb{R}^2 \to \mathbb{R}$. Clearly q restricted to $\tilde{A}$ is within a bounded error of the first factor projection $p : \tilde{A} \to \mathbb{R}$ that we have used for the definition of the rotation set $\tilde{\rho}(X)$. The asymptotic property of the rotation number;

$$\mu(p \circ \tilde{g} - p) = \frac{1}{n}\mu(p \circ \tilde{g}^n - p),$$

shows that one can use the projection q instead of p in the definition of $\tilde{\rho}(X)$.

Assume the foliation is oriented upward. Then the Brouwer property of $\mathcal{F}$, the compactness of X and the equivariance of q shows that there is $a > 0$ such that $q \circ \tilde{g} - q \geq a$ on X, showing that $\tilde{\rho}(X)$ is contained in $[a, \infty)$.

Case 2.1. *The foliation $\mathcal{F}$ admits a compact leaf L of nonzero slope and does not admit a Reeb component.* In this case the argument of Case 1 applies.

Case 2.2. *The foliation $\mathcal{F}$ admits a Reeb component R of nonzero slope.* The Brouwer property of leaves implies that $g(R) \subset \mathrm{Int}(R)$ or $g^{-1}(R) \subset \mathrm{Int}(R)$. On the other hand X must intersect the boundary of R since the slope of R is nonzero, contradicting the assumption that any point of X is nonwandering.

Case 2.3. *The foliation $\mathcal{F}$ admits a compact leaf of slope 0.* First notice that X cannot intersect a compact leaf since any point on the closed leaf of the lifted foliation $\tilde{\mathcal{F}}$ of $\mathcal{F}$ to the covering space $\mathbb{R}^2/\langle T \rangle$ of T^2 is a wandering point of g. Thus X is contained in the interior of a foliated I bundle or a Reeb component, say R. Then one can define a projection $q : \pi^{-1}(\mathrm{Int}(R)) \to \mathbb{R}$ just as in Case 1, and show that the rotation set does not contain 0. This completes the proof of Theorem 2.

3. Proof of Theorem 1

The proof is again by the absurdity. So assume there is no fixed point in $\pi^{-1}(V)$, where V is an open annular neighbourhood of the attractor X. One may further assume that $f(\overline{V}) \subset V$ and $\bigcap_{i \geq 0} f^i(V) = X$. Here we modify f to a new homeomorphism g in a bit different way from the previous setion, using the attracting property of X. As can easily be shown there exists a homeomorphsim g such that

(1) $g = f$ on V,
(2) $\bigcap_{i \geq 0} g^{-i}(A \setminus V) = \partial_- A \cup \partial_+ A$, and
(3) $\tilde{g}$ restricted to $\pi^{-1}(\partial_\pm A)$ is a nontrivial translation by the same translation number, where $\tilde{g}$ is the lift of g such that $\tilde{g} = \tilde{f}$ on $\pi^{-1}(X)$.

Our goal is to show that the rotation set $\tilde{\rho}(X)$ of g does not contain 0. We can also apply Theorem 2.1 to get a foliation $\mathcal{F}$ on T^2.

CASE 1 *The foliation $\mathcal{F}$ does not admit a compact leaf.*

CASE 2.1 *The foliation $\mathcal{F}$ admits a compact leaf of nonzero slope and does not admit a Reeb component.*

In these cases the argument in Sect. 2 works since it does not use the nonwandering condition.

CASE 2.2 *The foliation $\mathcal{F}$ admits a Reeb component R of nonzero slope.*

The boundary $\partial_\pm A$ are identified in T^2 to form a simple closed curve ∂A. The curve ∂A is a repellor of g consisting of nonwandering points. Hence it cannot intersect a boundary leaf of R. But this is impossible since ∂A is of slope 0.

CASE 2.3 *The foliation $\mathcal{F}$ admits a compact leaf of slope zero.*

Consider the covering space $\mathbb{R}^2/\langle T\rangle$, and the lift $\tilde{\mathcal{F}}$ of $\mathcal{F}$. The argument of the previous section shows that $\partial_- A$ is contained in the interior of a Reeb component or a foliated I bundle R and therefore $\partial_+ A$ in SR, where S is the vertical translation by 2. (In fact we can show easily that R is a repelling Reeb component. But we do not use it.) Using the attracting property of X, one can construct a foliation $\mathcal{H}$ on $\mathrm{Int}(A) \setminus X$ consisting of curves

$$\lambda : \mathbb{R} \to \mathrm{Int}(A) \setminus X$$

such that $g\lambda(t) = \lambda(t+1)$.

If we show that X cannot intersect a closed leaf L of $\tilde{\mathcal{F}}$, the rest of the argument is the same as before. So let us assume there is a point $x \in X \cap L$. To fix the idea assume L is oriented from left to right. Then $g^{-1}(x)$ lies on the upper side of L. Since $X = \mathrm{Fr}(\mathrm{U}_-)$, there is a leaf λ of $\mathcal{H}$ such that $\lambda(t) \to \partial_- A$ as $t \to -\infty$ which intersects L. Any point of L is wandering and thus $\partial_- A$ does not intersect L, and clearly $\partial_- A$ lies on the lower side of L. Let t_0 be the minimal value such that $\lambda(t_0)$ lies on L. But then $\lambda(t_0 - 1) = f^{-1}\lambda(t_0)$ lies on the upper side. A contradiction. The opposite case where L is oriented reversely can be dealt with similarly by considering a leaf of $\mathcal{H}$ lying in U_+. This completes the proof of Theorem 1.

Remark 3.1. The case where X is an attractor from the side of U_- and a repellor from U_+ cannot be handled by the above argument. But the author does not know such an example with the property $\mathrm{Fr}(U_-) = \mathrm{Fr}(U_+) = X$ and with nontrivial rotation set.

4. Proof of Theorem 3

Assume in way of contradiction that

$$\check{\rho}_- < 0 < \inf \tilde{\rho}(X).$$

Notice that since $\inf \tilde{\rho}(X) > 0$, there is no fixed point of $\tilde{f}$ in $\pi^{-1}(X)$, and therefore we obtain the modification g of f just as in Lemma 2.6.

Lemma 4.1. *For any $C > 0$ there is $n > 0$ such that $p \circ \tilde{g}^n - p \geq C$ on X.*

PROOF: If not, there would be sequences $x_i \in X$ and $n_i \to \infty$ such that

$$(p \circ \tilde{g}^{n_i} - p)(x_i) = \sum_{j=0}^{n_i - 1} (p \circ \tilde{g} - p)(g^j(x_i)) < C,$$

and the accumulation point μ of the sequence of averages of Dirac masses $\frac{1}{n_i}\sum_{j=0}^{n_i-1} g_*^j\delta_{x_i}$ would satisfy $\mu(p\circ\tilde g - p)\le 0$, contradicting the assumption $\inf\tilde\rho(X)>0$. □

Let $\mathcal F$ be the foliation on the 2-torus T^2 that we have constructed in Sect. 2. The argument here again follows the classification of the foliation $\mathcal F$ in Sect. 2 and 3.

CASE 1 and CASE 2.1: Just as in Sect. 2, let $q:\mathbb R^2\to\mathbb R$ be the lift of the projection along the leaves of the foliation $\tilde{\mathcal F}$, the lift of $\mathcal F$ to $\mathbb R^2/\langle T\rangle$. Lemma 4.1 shows that $q\circ\tilde g^n(x)\to\infty$ $(n\to\infty)$ for $x\in\pi^{-1}(X)$, since q is within bounded error of the canonical projection p in $\pi^{-1}(X)$. Thus the foliation $\tilde{\mathcal F}$ is oriented upward. But this shows that $q\circ\tilde g^n(x)\to\infty$ $(n\to\infty)$ for any point $x\in\pi^{-1}(\partial_-A)$, since ∂_-A is compact. On the other hand by condition (3) of Lemma 2.6, $\tilde g$ is a negative translation on $\pi^{-1}(\partial_-A)$. A contradiction.

CASE 2.2: In this case the bondary $\partial_-A\subset\mathbb R^2/\langle T\rangle$ cannot intersect a Reeb component since it consists of nonwandering points. So this case cannot happen.

CASE 2.3 *The foliation $\tilde{\mathcal F}$ admits a compact leaf.* The foliation $\tilde{\mathcal F}$ yields a partition $\mathcal P$ of $\mathbb R/\langle T\rangle$ into compact leaves, interiors of Reeb components and foliated I-bundles. The set $\mathcal P$ is totally ordered by the height. The minimal element which intersect X cannot be a compact leaf by the Brouwer line property. Let R be the closure of the minimal element. Thus R is either a Reeb component or a foliated I-bundle such that $\mathrm{Int}(R)\cap X\neq\emptyset$ and $\partial_-R\cap X=\emptyset$, where ∂_-R is the lower boundary curve of R.

Assume, to fix the idea, that ∂_-R is oriented from the right to the left. Thus the homeomorphism g carries ∂_-R into the upper complement of ∂_-R.

CASE 2.3.1 *R is a Reeb component.* First notice that the interior leaves of R are oriented upwards by the assumption $\inf\tilde\rho(X)>0$ and the fact that $g(X\cap R)\subset X\cap R$. Choose a simple arc

$$\alpha:[0,1]\to\pi^{-1}(R)$$

such that $\alpha(0)\in\pi^{-1}(\partial_-R)$, $\alpha(1)=\tilde g(\alpha(0))$, and $\alpha((0,1))\subset\mathrm{Int}(\pi^{-1}(R))\setminus\tilde g(\pi^{-1}(R))$. Notice that α is contained in $\pi^{-1}(U_-)$.

Concatenating nonnegative iterates of α, we obtain a simple path $\gamma:[0,\infty)\to\pi^{-1}(R\cap U_-)$ such that $\tilde g\circ\gamma(t)=\gamma(t+1)$ for any $t\ge 0$. Let $q:\pi^{-1}(\mathrm{Int}(R))\to\mathbb R$ be the lift of the projection along the leaves. Since $\gamma([1,\infty))$ is contained in the lift of a compact subset $\tilde g(R)\subset\mathrm{Int}(R)$, the map q is within bounded error of p on $\gamma([1,\infty))$. Therefore by Lemma 4.1 we have $q\circ\gamma(t)\to\infty$ as $t\to\infty$. On the other hand by condition (4) of Lemma 2.6, we have $\check p\circ\gamma(t)\to-\infty$ as $t\to\infty$. In particular the curve γ is proper both in $\pi^{-1}(R)$ and in $\check U_-$. By joining the point $\gamma(0)$ to an appropriate point in $\pi^{-1}(\partial_-A)$, we obtain a simple curve δ in $\pi^{-1}(U_-)$ starting at a point on $\pi^{-1}(\partial_-A)$ which extends γ.

Let x be a point in $\pi^{-1}(\partial_-A)$ left to the initial point of δ. Notice that there is a point of $\pi^{-1}(X)$ on the left of δ since a high iterate of T^{-1} carries a point in $\pi^{-1}(X)$ to the left of δ. (There may also be a point of $\pi^{-1}(X)$ on the right of δ.) Then there is a simple path $\beta:[0,\infty)\to\pi^{-1}(U_-)$ such that $\beta(0)=x$, $\lim_{t\to\infty}\beta(t)\in\pi^{-1}(X)$, and β is disjoint from δ. The path β, extendable in $\pi^{-1}(A)$ is also extendable in $\check U_-$, the lift of the Carathéodory compactification. (See e. g. Lemma 2.5 of [**MN**].) This implies that β defines a simple path in $\check U_-$ joining x to a prime end in $\pi^{-1}(\partial_\infty U_-)$ without intersecting δ, which is impossible since $\pi^{-1}(\partial_\infty U_-)$ is contained in the right side of the proper path δ because $\check{p}_-\delta(t) \to -\infty$, while x is on the left side. A contradiction.

Case 2.3.2 *R is a foliated I-bundle.* Thus the upper boundary curve $\partial_+ R$ of R is also oriented from the right to the left, and its image by g lies on the upper complement of R. The interior leaves of R are oriented upward.

Recall that the boundary component $\partial_- A$ consisting of nonwandering points cannot intersect a compact leaf. Notice that $\partial_- A$ lies in a Reeb component or a foliated I-bundle whose interior leaves are oriented downward since $p\tilde{g}^n(x) \to -\infty$ as $t \to \infty$ for $x \in \pi^{-1}(\partial_- A)$. Let C be the annulus in $\mathbb{R}^2/\langle T\rangle$ bounded by $\partial_- A$ and $\partial_+ R$, the upper boundary curve of R.

Case 2.3.2.1 *The intersection $X \cap C$ has a component which separates $\partial_- A$ from $\partial_+ A$.* One can derive a contradiction by the same argument as in Case 2.3.1, since the path δ cannot evade R.

Case 2.3.2.2 *There is a simple path in U_- joining a point in $\partial_- A$ with a point in $\partial_+ R$.* Notice first of all that $g^{-1}(C) \subset C$. Let $\mathcal{Y}$ be the family of the connected components of $\pi^{-1}(X \cap C)$. Then any element $Y \in \mathcal{Y}$ is compact, and intersects $\pi^{-1}(\partial_+ R)$ since otherwise Y would be a connected component of $\pi^{-1}(X)$ itself.

Choose a simple curve $\gamma : [0,1] \to \pi^{-1}(C)$ such that
(1) $\gamma(0) \in \pi^{-1}(\partial_- A)$,
(2) $\gamma(1) \in \pi^{-1}(X \cap C)$, and
(3) $\gamma([0,1)) \subset \pi^{-1}(U_- \cap C)$.

Let Y be an element of $\mathcal{Y}$ which contains $\gamma(1)$. Then there are two unbounded connected components of the complement $\pi^{-1}(C) \setminus (Y \cup \gamma)$, one $L(Y \cup \gamma)$ on the left, and the other $R(Y \cup \gamma)$ on the right.

Notice that for any $n > 0$, $\tilde{g}^{-n}\gamma$ is a path in C, and that $p\tilde{g}^{-n}(\gamma(1)) \to -\infty$ and $p\tilde{g}^{-n}(\gamma(0)) \to \infty$ as $n \to \infty$. That is, for any large n, $\tilde{g}^{-n}(\gamma(1)) \in L(Y \cup \gamma)$ and $\tilde{g}^{-n}(\gamma(0)) \in R(Y \cup \gamma)$, showing that $\tilde{g}^{-n}(\gamma)$ intersects γ. On the other hand in $\check{U}_-$, γ defines a curve from a point in $\pi^{-1}(\partial_- A)$ to a prime end in $\pi^{-1}(\partial_\infty U_-)$. But by condition (4) of Lemma 2.6, γ cannot intersect $\tilde{g}^{-n}(\gamma)$ for any large n. A contradiction.

References

[B] G. D. Birkhoff, *Sur quelques courbes fermées remarquables,* Bull. Soc. Math. France **60**(1932) 1-26; also in *Collected Mathematical Papers of G. D. Birkhof,* vol. II, pp. 444-461

[C] M. Charpentier, *Sur quelques propriétés des courbes de M. Birkhoff,* Bull. Soc. Math. France **62**(1934) 193-224.

[CL] M. L. Cartwright and J. E. Littlewood, *Some fixed point theorems,* Ann. Math. **54**(1951) 1-37.

[E] D. B. A. Epstein, *Prime ends,* Proc. London Math. Soc. **42**(1981) 385–414.

[FK] B. Fayad and A. Katok, *Constructions in elliptic dynamics,* Ergod. Th. Dyn. Sys. **24**, Volume dedicated to the memory of Michael Herman, (2004) 1477-1520.

[FL] J. Franks and P. Le Calvez, *Regions of instability for non-twist maps,* Ergod. Th. Dyn. Sys. **23**(2003), 111–141.

[Ha] M. Handel, *A pathological area preserving C^∞ diffeomorphism of the plane,* Proc. A. M. S. **86**(1982) 163–168.

[He] M. R. Herman, *Construction of some curious diffeomorphisms of the Riemann sphere,* J. London Math. Soc. **34**(1986) 375–384.

[L1] P. Le Calvez, *Existence d'orbits quasi-periodiques dans les attracteurs de Birkhoff,* Commun. Math. Phys. **106**(1986) 383-39.

[L2] P. Le Calvez, *Propriétés des attracteurs de Birkhoff,* Ergod. Th. Dyn. Sys. **8**(1987) 241-310

[L3] P. Le Calvez, *Une version feuilletée équivariante du théorème de translation de Brouwer,* Publ. Math. I. H. E. S. **102**(2005) 1–98.

[M] J. Mather, *Topological proofs of some purely topological consequences of Carathéodory's theory of prime ends,* In: Th. M. Rassias, G. M. Rassias, eds., Selected Studies, North-Holland, (1982) 225–255.

[MN] S. Matsumoto and H. Nakayama, *Continua as minimal sets of homeomorphisms of S^2,* Preprints in Arxiv.

[W] R. B. Walker, *Periodicity and decomposability of basin boundaries with irrational maps on prime ends,* Trans. A. M. S. **324**(1991), 303–317.

Department of Mathematics, College of Science and Technology, Nihon University, 1-8-14 Kanda, Surugadai, Chiyoda-ku, Tokyo, 101-8308 Japan

E-mail address: matsumo@math.cst.nihon-u.ac.jp