Level Sets of the Takagi Function: Generic Level Sets

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ABSTRACT

The Takagi function $\tau : [0,1] \to [0,1]$ is a continuous non-differentiable function constructed by Takagi in 1903. This paper studies the level sets $L(y) = \{x : \tau(x) = y\}$ of the Takagi function $\tau(x)$. It shows that for a "generic" full Lebesgue measure set of ordinates y, the level sets are finite sets, and that the expected number of points on such a level is infinite. Complementing this, it shows that the set of ordinates y on which the level set has positive Hausdorff dimension has full Hausdorff dimension 1 (but Lebesgue measure zero). The results are obtained by studying a notion of "local level set" introduced in a previous paper [14], and using a singular measure parameterizing all such sets.

1. Introduction

The Takagi function $\tau(x)$ is a function defined on the unit interval $x \in [0,1]$ which was introduced by Takagi [20] in 1903 as an example of a continuous nondifferentiable function. It can be defined by

$$\tau(x) := \sum_{n=0}^{\infty} \frac{\ll 2^n x \gg}{2^n} \tag{1.1}$$

where $\ll x \gg$ is the distance from x to the nearest integer, although Takagi's original definition was slightly different.

An alternate interpretation of the Takagi function involves the symmetric tent map $T : [0,1] \to [0,1]$, given by

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}, \\ 2-2x & \text{if } \frac{1}{2} \le x \le 1 \end{cases} \tag{1.2}$$

(see [10] for further references). Then we have

$$\tau(x) := \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{1}{2^n} T^{(n)}(x)\right), \tag{1.3}$$

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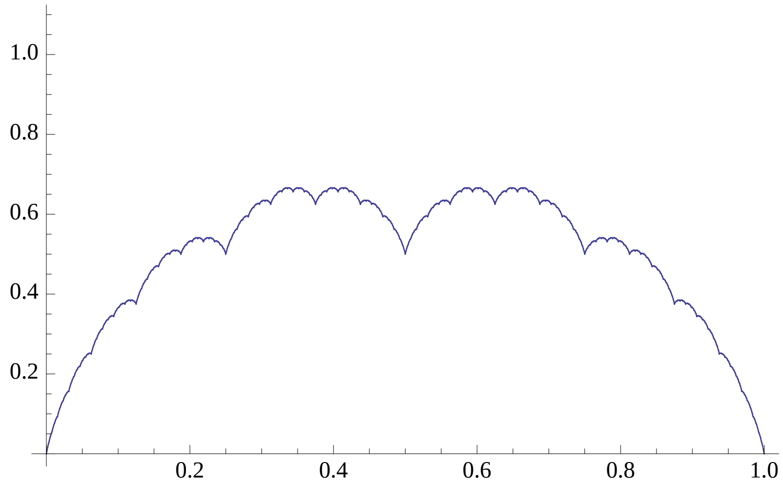

Figure 1: Graph of the Takagi function $\tau(x)$.

where $T^{(n)}(x)$ denotes the n-th iterate of $T(x)$. The Takagi function can be extended to a periodic map on the real line having period 1, and its Fourier series coefficients have a relatively simple form ([12, Sect. 2]). The Takagi function shows up in a surprising number of different places in mathematics, including Bernoulli convolutions ([12, p. 195]), distribution of binary digit sums ([18], [21], [5]) and dynamical systems ([22]).

In this paper we consider certain properties of the graph of the Takagi function

$$\mathcal{G}(\tau) := \{(x, \tau(x)) : \ 0 \le x \le 1\},$$

which is pictured in Figure 1. It is well known that the values of the Takagi function satisfy $0 \le \tau(x) \le \frac{2}{3}$. It is also known that this graph has Hausdorff dimension 1 in $\mathbb{R}^2$; see Mauldin and Williams [17, Theorem 7]. They add the remark that they do not know whether the this graph is σ-finite ([17, p. 800]). Here we study the structure of the level sets of this graph. We make the following definition, which contains a special convention concerning dyadic rationals which simplifies theorem statements.

Definition 1.1. For $0 \le y \le \frac{2}{3}$ the *level set* $L(y)$ at level y is

$$L(y) := \{x : \ \tau(x) = y, \quad 0 \le x \le 1\}.$$

By convention the symbol x specifies a particular binary expansion; so each dyadic rational value $x = \frac{m}{2^n}$ in a level set will appear twice, labeled by each of its two possible binary expansions.

Level sets have a complicated and interesting structure, depending on the value of y. A good deal is known about them. It is known that there are different levels y where the level set $L(y)$ is finite, countably infinite, or uncountably infinite, respectively. Concerning the size of level sets, measured by Hausdorff dimension, in 1984 Baba [3] showed that the level set $L(\frac{2}{3})$ has Hausdorff dimension $\frac{1}{2}$, so is uncountable. The second author recently showed ([16]) that the Hausdorff dimension of any level set is bounded above by 0.668 and conjectured that the example of Baba achieves the largest possible dimension.

This paper is a sequel to the paper [14], which approached the study of level sets of the Takagi function via the notion of "local level set". Local level sets are sets determined locally by combinatorial operations on the binary expansion of a real number x; they are closed sets and each level set decomposes into a disjoint union of local level sets. The structure of local level sets L_x^{loc} is completely analyzable: they are either finite sets or Cantor sets. Information of the Hausdorff dimension of such sets can be deduced from properties of the binary expansion of x. That paper introduced a certain subset $\Omega^L \subset [0,1]$, called the *deficient digit set*, which comprises the left endpoints of all local level sets. It therefore parameterizes all local level sets. We showed [14, Theorem 4.6] that Ω^L is a closed set of measure zero. We also introduced a singular measure $d\mu_S$ which is supported on Ω^L, the *Takagi singular measure*. These were used to prove in [14] that on almost all levels y there are finitely many local level sets; the expected number of local level sets, measured with respect to Lebesgue measure on $y \in [0, \frac{2}{3}]$, was shown to be $\frac{3}{2}$. However we showed there also are a dense set of levels in $[0, \frac{2}{3}]$ which have infinitely many local level sets.

In this paper we will study "generic" level sets in a number of senses. The *ordinate (y-axis) notion of genericity* is to draw an ordinate y at random using Lebesgue measure in $[0, \frac{2}{3}]$, and then to ask what is the nature of the level set $L(y)$. The *abscissa (x-axis) notion of genericity* is to draw a number x at random in $[0,1]$ with respect to Lebesgue measure, and then to ask what is the nature of the level set $L(\tau(x))$. A weaker notion of "generic set " is to be *generic in the Hausdorff dimension sense*, meaning it is a a set of (full) Hausdorff dimension 1.

It is now known that abscissa generic level sets and ordinate generic level sets have a quite different structure. In 2008 Buczolich [4] showed that the generic ordinate level set is a finite set, while in [14, Theorem 1.4] we showed that the "generic" abscissa local level set is an uncountable Cantor set of Hausdorff dimension 0; it immediately follows that a "generic" abscissa level set is uncountable. Our first main result in this paper is a new proof of the finiteness result of Buczolich for generic ordinate level sets, using local level sets and the Takagi singular measure. Our method gives extra information: it shows that the expected number of points in a randomly drawn ordinate level set is infinite. Concerning the generic level sets in the abscissa sense, we formulate a conjecture asserting that such sets will have Hausdorff dimension 0.

Our second main result concerns the set of levels having a "large" level set, namely one of positive Hausdorff dimension. We show that the set of ordinate values labelling these levels is "generic" in the Hausdorff dimension sense, although it has Lebesgue measure 0. In the process of proving this we show that the deficient digit set Ω^L has full Hausdorff dimension 1. Now we state these results in more detail.

1.1. Generic level sets: Results

In Sect. 2 and 3 of this paper we recall preliminary results on the Takagi function and local level sets, following the previous paper [14].

The first notion we consider is that of a "generic" ordinate level set, which is a random $L(y)$ for $y \in [0, \frac{2}{3}]$ drawn uniformly with respect to Lebesgue measure. We obtain the following result.

Theorem 1.2. (Ordinate generic level sets) *(1) For a full Lebesgue measure set of ordinate points $y \in [0, \frac{2}{3}]$ the level set $L(y)$ is a finite set.*

(2) For a random level set $L(y)$ with level y drawn uniformly from $y \in [0, \frac{2}{3}]$, the expected number of elements in $L(y)$ is infinite.

We prove Theorem 1.2 in Sect. 4 and 5. This result is proved using explicit calculations of the Takagi singular measure of various subsets of Ω^L given in the fine decomposition of the deficient digit set Ω^L made in Sect. 4 below. These calculations make use of self-similarity properties of the Takagi singular measure.

As remarked above, part (1) of this result was proved in 2008 by Buczolich [4]. He proves the almost everywhere finiteness of level sets by a method that directly studies the graph of the Takagi function. His proof shows the graph $\mathcal{G}(\tau) = \mathcal{G}_I \bigcup \mathcal{G}_R$ (nonconstructively) partitions into an irregular 1-set $\mathcal{G}_R$ and a regular 1-set $\mathcal{G}_R$, and that the irregular set $\mathcal{G}_I$ has y-projection of Lebesgue measure 0 and x-projection of full measure 1. Here an *irregular* 1-*set* or *purely unrectifiable* 1-*set* is a set in $\mathbb{R}^2$ of Hausdorff dimension 1 that intersects every continuously differentiable curve in a set of $\mathcal{H}^1$-measure zero. By Besicovitch's theorem such a set has 1-dimensional projections of measure 0 in almost all directions, see Falconer [7, Theorem 6.1.3]. A *regular* 1-*set* is a set that can be covered by countably many rectifiable curves. Our proof of Theorem 1.2 uses the Takagi singular measure, whose support lies in Ω^L, and the part of the graph $\mathcal{G}(\tau)$ that lies above Ω^L is covered by a single rectifiable curve, the flattened Takagi function described in Sect. 4. This follows from the BV property of this function [14, Theorem 5.3]). Thus our proof makes use of an explicitly identified set that presumably belongs to the regular part $\mathcal{G}_R$ of Buczolich's partition.

Recall that in [14, Theorem 1.4] we showed that a "generic" local level set L_x^{loc} obtained by drawing x with the uniform distribution on $[0, 1]$ (Lebesgue measure) is with probability one an uncountable set of Hausdorff dimension 0. This implies that a "generic" level set $L(\tau(x))$ is uncountable with probability one. This contrasts with Theorem 1.2. We may conjecture the following strengthening of that result.

Conjecture 1.3. (Abscissa generic level sets) *For a full Lebesgue measure set of abscissa points $x \in [0, 1]$ the level set $L(\tau(x))$ is a Cantor set of Hausdorff dimension* 0

The abscissa and ordinate "generic" results make differing predictions on the size of a "random" level set, in that sampling a point x on the abscissa favors picking level sets which are "large". These results taken together indicate that the Takagi function must have "infinite slope" over part of its domain. In particular it is not a function of bounded variation.

Our second result reconciles the abscissa and ordinate sampling viewpoints of level sets, by showing that the set of levels y whose level set $L(y)$ has positive Hausdorff dimension itself has full Hausdorff dimension. We let $\dim_H(\Gamma)$ denote the Hausdorff dimension of a set Γ.

Theorem 1.4. (Positive Hausdorff dimension level sets) *Let Γ_H^{ord} be the set of ordinates $y \in [0, \frac{2}{3}]$ such that the Takagi function level set $L(y)$ has positive Hausdorff dimension, i.e.*

$$\Gamma_H^{ord} := \{y : \ \dim_H(L(y)) > 0\}.$$

Then Γ_H^{ord} has full Hausdorff dimension, i.e.

$$\dim_H(\Gamma_H^{ord}) = 1. \tag{1.4}$$

The set Γ_H^{ord} has Lebesgue measure 0 by Theorem 1.2 above. The proof of this result occupies Sect. 6 and 7. We first study in Sect. 6 the set of abscissas in the deficient digit set Ω^L that give rise to a level set of positive Hausdorff dimension, namely

$$\Gamma_H^L := \{x \in \Omega^L : dim_H(L_x^{loc}) > 0\}.$$

We show this set has Hausdorff dimension 1 (Theorem 6.1), obtaining from it the fact that the deficient digit set Ω^L itself has Hausdorff dimension 1. We do this by finding some nice Cantor-like sets Λ_{2r} inside Γ_H^L, depending on an integer parameter r, which have $\dim_H(\Lambda_{2r}) \to 1$ as $r \to \infty$. In Sect. 7 we derive Theorem 1.4 from this fact by showing that the Takagi function $\tau(x)$ restricted to the set Λ_{2r} is a bi-Lipschitz map; such maps preserve Hausdorff dimension.

1.2. Extensions of Results and Related Work

The results we have found suggest that level sets of the Takagi function should exhibit a nontrivial multifractal spectrum. We define the *multifractal spectrum function* for level sets as:

$$f_\tau(\alpha) := \dim_H\{y : \dim_H(L(y) > \alpha). \tag{1.5}$$

The main result of [16] implies that

$$f_\tau(\alpha) = 0 \quad \text{for} \quad \alpha \geq 0.668, \tag{1.6}$$

and a conjecture made in [16] would imply $f_\tau(\alpha) = 0 \quad \text{for} \quad \alpha \geq \frac{1}{2}$. Theorem 1.4 of this paper shows that $f_\tau(0) = 1$. It seems reasonable to expect that $f_\tau(\alpha) > 0$ might hold for $0 \leq \alpha < \frac{1}{2}$. We leave resolving this question for further work. This problem might be approached using local level sets, which should also have a multifractal spectrum for such values varying over $x \in \Omega^L$, possibly identical to that for level sets in general.

There has been much study of the non-differentiable nature of the Takagi function in various directions, see for example Allaart and Kawamura [1], [2], and references therein.

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2. Preliminaries: Properties of the Takagi Function

2.1. Basic identities

We first derive an alternate formula for the Takagi function, expressed directly in terms of the binary expansion of $x = 0.b_1b_2b_3....$ For $0 \leq x \leq 1$ the distance to the nearest integer function is

$$\ll x \gg \; := \begin{cases} x & \text{if} \quad 0 \leq x < \frac{1}{2}, \quad \text{i.e. } b_1 = 0 \\ 1 - x & \text{if} \quad \frac{1}{2} \leq x \leq 1, \quad \text{i.e. } b_1 = 1. \end{cases} \tag{2.1}$$

For $n \geq 0$, we have

$$\ll 2^n x \gg \; = \begin{cases} 0.b_{n+1}b_{n+2}b_{n+3}... & \text{if} \quad b_{n+1} = 0 \\ 0.\bar{b}_{n+1}\bar{b}_{n+2}\bar{b}_{n+3}... & \text{if} \quad b_{n+1} = 1, \end{cases} \tag{2.2}$$

where we use the bar-notation

$$\bar{b} = 1 - b, \quad \text{for} \quad b = 0 \text{ or } 1, \tag{2.3}$$

to mean complementing a bit.

Lemma 2.1. (Takagi function formulas) *For $x = 0.b_1b_2b_3...$ the Takagi function is given by*

$$\tau(x) = \sum_{m=1}^{\infty} \frac{l_m}{2^m}, \tag{2.4}$$

in which $0 \leq l_m = l_m(x) \leq m - 1$ is the integer

$$l_m(x) = \#\{b_i : \; 1 \leq i < m, \quad b_i \neq b_m\} \tag{2.5}$$

In terms of the digit sum function $N_m^1(x) = b_1 + b_2 + ... + b_m$*,*

$$l_m(x) = \begin{cases} N_{m-1}^1(x) & \text{if} \quad b_m = 0, \\ \\ (m-1) - N_{m-1}^1(x) & \text{if} \quad b_m = 1. \end{cases} \tag{2.6}$$

Proof. From the definition

$$\tau(x) = \sum_{n=0}^{\infty} \frac{\ll 2^n x \gg}{2^n}. \tag{2.7}$$

Now (2.2) gives

$$\frac{\ll 2^n x \gg}{2^n} = \begin{cases} \sum_{j=1}^{\infty} \frac{b_{n+j}}{2^{n+j}} & \text{if} \quad b_{n+1} = 0, \\ \\ \sum_{j=1}^{\infty} \frac{\bar{b}_{n+j}}{2^{n+j}} & \text{if} \quad b_{n+1} = 1. \end{cases} \tag{2.8}$$

We substitute this into the formula for $\tau(x)$ and collect all terms having a given denominator $\frac{1}{2^m}$, coming from $m = n + j$ with $1 \le j \le m$. For $m = n + j$ we get a contribution of $\frac{1}{2^m}$ whenever $b_{n+j} := b_m = 1$ and $b_{n+1} = 0$, and whenever $b_{n+j} := b_m = 0$ and $b_{1+n} = 1$, otherwise get 0. Adding up over j, we find the total contribution is $\frac{l_m}{2^m}$ where l_m counts the number of b_j, $1 \le j < m$ having the opposite parity to b_m, which is (2.4). The formulas (2.6) follow by inspection; note that $m - N_m^1(x) = N_m^1(1-x)$ (making an appropriate convention for dyadic rationals). $\square$

We also recall basic functional equations satisfied by the Takagi function [14, Lemma 2.2].

Lemma 2.2. (Takagi functional equations) *The Takagi function satisfies two functional equations, each valid for* $0 \le x \le 1$:

the reflection equation

$$\tau(x) = \tau(1-x), \tag{2.9}$$

and the dyadic self-similarity equation

$$2\tau(\frac{x}{2}) = \tau(x) + x. \tag{2.10}$$

The Takagi function is the unique continuous function on $[0,1]$ *that satisfies both these functional equations.*

We also will use a self -similarity property of the graph of the Takagi function. To describe it we require some functions determined by the binary expansion of x.

Definition 2.3. Let $x \in [0,1]$ have a binary expansion

$$x = \sum_{j=1}^{\infty} \frac{b_j}{2^j} = 0.b_1 b_2 b_3 ..., \tag{2.11}$$

with each $b_j \in \{0,1\}$. For each $j \ge 1$ we define the following integer-valued functions.

(1) The *digit sum function* $N_j^1(x)$ is

$$N_j^1(x) := b_1 + b_2 + \cdots + b_j. \tag{2.12}$$

We also set

$$N_j^0(x) := j - N_j^1(x). \tag{2.13}$$

These functions count the number of 1's (resp. 0's) in the first j binary digits of x.

(2) The *deficient digit function* $D_j(x)$ is given by

$$D_j(x) := N_j^0(x) - N_j^1(x) = j - 2N_j^1(x) = j - 2(b_1 + b_2 + \cdots + b_j). \tag{2.14}$$

The name "deficient digit function" reflects the fact that $D_j(x)$ counts the excess of binary digits $b_k = 0$ over those with $b_k = 1$ in the first j digits, i.e. it is positive if there are more 0's than 1's. We use here the convention that x means the binary expansion, noting that dyadic rationals have two different binary expansions, and the functions $N_j^0(x)$, $N_j^1(x)$, and $D_j(x)$ depend on which binary expansion is used.

We call a dyadic rational $B = \frac{k}{2^n} = 0.b_1b_2...b_n0^\infty$ *balanced* if $D_n(B) = 0$; if this holds then $n = 2m$ is necessarily even. The Takagi function graph has the following local self-similarity property ([14, Lemma 2.5]), which easily follows from Lemmas 2.1 and 2.2.

Lemma 2.4. (Takagi exact self-similarity)
Let $B' = 0.b_1b_2\ldots b_{2m}0^\infty = \frac{k}{2^{2m}}$ be a balanced dyadic rational; that is, it has $D_{2m}(B') = 0$, so that $D_j(B') = j - 2m$ for all $j \geq 2m$. Then for $x = B' + \frac{x'}{2^m}$ with any $x' \in [0,1]$, there holds

$$\tau(x) = \tau(B') + \frac{\tau(x')}{2^{2m}}. \tag{2.15}$$

That is, on the dyadic interval $[\frac{k}{2^{2m}}, \frac{k+1}{2^{2m}}]$ the graph of the function $\tau(x)$ is a miniature version of its full graph, vertically shifted by $\tau(B)$ and shrunk by a factor $\frac{1}{2^{2m}}$.

2.2. Local level sets

The notion of local level set L_x^{loc} is attached to the binary expansion of an abscissa point $x \in [0,1]$. We show that certain combinatorial flipping operations applied to the binary expansion of x yield new points x' in the same level set. The totality of points reachable from x by these combinatorial operations will comprise the local level set L_x^{loc} associated to x.

Let a binary expansion of $x \in [0,1]$ be given:

$$x := \sum_{j=1}^{\infty} \frac{b_j}{2^j} = 0.b_1b_2b_3..., \quad \text{each } b_j \in \{0,1\}. \tag{2.16}$$

The *flip operation* (or *complementing operation*) on a single binary digit b is

$$\bar{b} := 1 - b. \tag{2.17}$$

For a given x we call a *balance point* any digit position j at which the deficient digit function $D_j(x)$ defined by (2.14) has a tie-value $D_j(x) = 0$; note that all such j are even. We now associate to x its *balance-set* $Z(x)$,

$$Z(x) := \{c_k : \quad D_{c_k}(x) = 0\}. \tag{2.18}$$

where we define $c_0 = c_0(x) = 0$ and set $c_0(x) < c_1(x) < c_2(x) < ....$ This sequence of tie-values may be finite or infinite. If it is finite, ending in $c_n(x)$, we make the convention to adjoin a final “balance point” $c_{n+1}(x) = +\infty$. We define a *block* to be an indexed set of digits between two consecutive balance points,

$$B_k(x) := \{b_j : \ c_k(x) < j \le c_{k+1}(x)\}, \tag{2.19}$$

where we include the second balance point but not the first. Next, we define an equivalence relation on blocks, written $B_k(x) \sim B_{k'}(x')$, to mean the block endpoints agree ($c_k(x) = c_{k'}(x')$ and $c_{k+1}(x) = c_{k'+1}(x')$) and either $B_k(x) = B_{k'}(x')$ or $B_k(x) = \bar{B}_{k'}(x')$, where the bar operation flips all the digits in the block, i.e.

$$b_j \mapsto \bar{b}_j := 1 - b_j, \qquad c_k < j \le c_{k+1}. \tag{2.20}$$

Finally, we let the equivalence relation $x \sim x'$ mean that these points have identical balance-sets $Z(x) \equiv Z(x')$, and furthermore every block $B_k(x) \sim B_k(x')$ for $k \ge 0$. Note that $x \sim 1-x$; this corresponds to a flipping operation being applied to every binary digit. In [14] we showed that the equivalence relation $x \sim x'$ implies that $\tau(x) = \tau(x')$, so that x and x' are in the same level set of the Takagi function, cf. Theorem 2.6 below.

Definition 2.5. The *local level set* L_x^{loc} associated to x is the set of equivalent points,

$$L_x^{loc} := \{x' : \quad x' \sim x\}. \tag{2.21}$$

We use again the convention that x and x' denote binary expansions, so that dyadic rationals require special treatment.

We recall basic properties of local level sets, as follows ([14, Theorem 3.1, Corollary 3.2]).

Theorem 2.6. *(1) Local level sets L_x^{loc} are closed sets. Two local level sets either coincide or are disjoint.*

(2) Each local level set L_x^{loc} is contained in a level set: $L_x^{loc} \subseteq L(\tau(x))$. That is, if $x_1 \sim x_2$ then $\tau(x_1) = \tau(x_2)$.

(3) Each level set $L(y)$ partitions into local level sets

$$L(y) = \bigcup_{\substack{x \in \Omega^L \\ \tau(x) = y}} L_x^{loc} \tag{2.22}$$

Here Ω^L denotes the collection of leftmost endpoints of all local level sets.

(4) A local level set L_x^{loc} is a finite set if the balance-set $Z(x)$ is finite; otherwise it is a Cantor set (uncountable perfect set).

2.3. Deficient digit set Ω^L

In [14] we studied the set of leftmost endpoints Ω^L of local level sets; this set parametrizes the complete collection of all local level sets. The leftmost endpoints all lie in $[0, \frac{1}{3})$. Here we start with a alternative definition of Ω^L given directly in terms of the binary expansion. Theorem 2.10 below states that this combinatorial definition coincides with the one above, and gives basic properties of this set.

Definition 2.7. The *deficient digit set* Ω^L consists of all points

$$\Omega^L := \{x = \sum_{j=1}^{\infty} \frac{b_j}{2^j} : \quad D_j(x) \geq 0 \text{ for all } \ j \geq 1\}.$$

where the deficient digit function $D_j(x) = N_j^0(x) - N_j^1(x) = j - 2N_j^1(x)$ counts the number of binary digits 0 minus that of binary digits 1 in the first n digits.

The deficient digit set is a Cantor-type set obtained by removing a certain countable collection of open intervals from the unit interval, which we describe using the following definitions.

Definition 2.8. (1) The *breakpoint set* $\mathcal{B}'$ consists of $B_\emptyset' = 0$ together with the collection of all dyadic rationals $B' = \frac{n}{2^{2m}}$ that have binary expansions of the form

$$B' = 0.b_1b_2...b_{2m-1}b_{2m} \quad \text{for some} \ \ m \geq 1,$$

that satisfy the condition

$$D_j(B') \geq 0 \quad \text{for} \ \ 1 \leq j \leq 2m-1, \quad \text{and} \quad N_{2m}(B') = 0, \tag{2.23}$$

This condition implies $b_{2m} = 1$.

(2) The *small breakpoint set* $\mathcal{B}$ is the subset of $\mathcal{B}'$ consisting of $B_\emptyset = 0$ plus all dyadic rationals in $\mathcal{B}'$ that satisfy the extra condition that the last two binary digits $b_{2m-1} = b_{2m} = 1$.

We may rewrite a dyadic rational in the restricted breakpoint set as

$$B = 0.b_1b_2...b_l01^k, \quad \text{with} \ \ k \geq 2, \tag{2.24}$$

and here $2m = k + l + 1$.

In [14] we used the small breakpoint set $\mathcal{B}$ to label the intervals removed from $[0,1]$ to create the deficient digit set Ω^L. Here the breakpoint set $\mathcal{B}'$ will be used in Sect. 4.1 to label a decomposition of Ω^L into finer pieces.

Definition 2.9. For each dyadic rational $B = 0.b_1b_2...b_l01^k$, $k \geq 2$ in the small breakpoint set $\mathcal{B}$ $(B \neq B_\emptyset)$ we associate the open interval

$$I_B := (x(B)^-, x(B)^+) \tag{2.25}$$

having the endpoints

$$\begin{aligned} x(B)^- &:= 0.b_1b_2...b_l01^k(01)^\infty \\ x(B)^+ &:= 0.b_1b_2...b_l10^k(00)^\infty, \end{aligned}$$

necessarily with $k \geq 2$. We also set

$$I_{B_\emptyset} := (x(B_\emptyset)^-, x(B_\emptyset)^+) := (0.(01)^\infty, 1.(00)^\infty) = (\frac{1}{3}, 1). \tag{2.26}$$

The following result gives properties of the deficient digit set Ω^L ([14, Theorem 4.6].)

Theorem 2.10. (Properties of the Deficient Digit Set)

(1) The deficient digit set Ω^L comprises the set of leftmost endpoints of all local level sets. It satisfies $\Omega^L \subset [0, \frac{1}{3}]$.

(2) The deficient digit sum set Ω^L is a closed, perfect set (Cantor set). It is given by

$$\Omega^L = [0,1) \setminus \bigcup_{B \in \mathcal{B}} I_B. \tag{2.27}$$

where the omitted open intervals I_B, for B in the small breakpoint set, have right endpoint a dyadic rational and left endpoint a rational number with denominator $3 \cdot 2^k$ for some $k \geq 1$.

(3) The deficient digit set Ω^L has Lebesgue measure zero.

In [14, Lemma 4.5] it is shown that the value of the endpoints of the removed intervals satisfies

$$x_B^+ - x_B^- = \tau(x(B)^-) - \tau(x(B)^+) = \frac{1}{2^{k+l} \cdot 3}, \tag{2.28}$$

so that linear interpolation of a function across the a removed interval always has slope -1.

3. Takagi Singular Function and Singular Measure

We recall from [14] the definition and properties of the Takagi singular function and Takagi singular measure. We deduce one new result (Theorem 3.3) which states that the Takagi function $\tau(x)$ is nondecreasing on the set $\frac{1}{2}\Omega^L$.

3.1. Flattened Takagi function and Takagi singular function

We define the *flattened Takagi function* $\tau^L(x)$ to agree with the Takagi function on the deficient digit set Ω^L and to be defined by linear interpolation on all the intervals I_B for B in the small breakpoint set $\mathcal{B}$. As mentioned above, it has slope -1 on these intervals. We showed in [14, Theorem 5.3] the following result.

Theorem 3.1. (Flattened Takagi function) *The flattened Takagi function $\tau^L(x)$ is of bounded variation. It has a minimal monotone decomposition (Jordan decomposition) given by*

$$\tau^L(x) = f_u(x) + f_d(x), \tag{3.29}$$

with upward part $f_u(x) = \tau^L(x) + x$ and downward part $f_d(x) = -x$ both being continuous functions. Its total variation $V_0^1(\tau^L) = 2$.

The flattened Takagi function is pictured in Figure 2.

The *Takagi singular function* $\tau^S(x)$ is the upward part of the Jordan decomposition of $\tau^L(x)$, e.g.

$$\tau^S(x) := f_u(x) = \tau^L(x) + x.$$

In [14, Theorem 5.5] we established the following properties of this function.

Theorem 3.2. (Takagi singular function properties) *The Takagi singular function $\tau^S(x)$ has the following properties.*

(1) The Takagi singular function $\tau^S(x)$ is a Cantor function. That is, it is a nondecreasing function having $\tau^S(0) = 0, \tau^S(1) = 1$, which has derivative zero at almost all points of $[0,1]$. The support of its points of increase is contained in Ω^L.

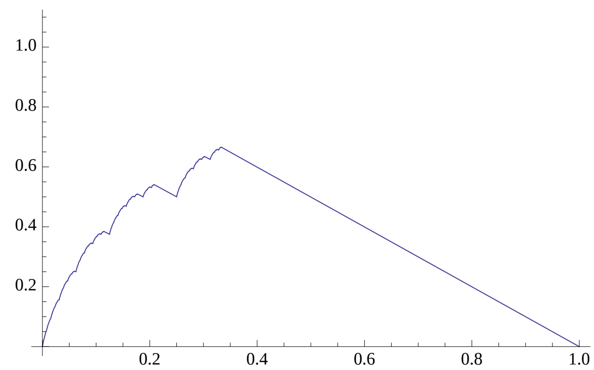

Figure 2: Graph of flattened Takagi Function $\tau^L(x)$.

(2) The Takagi singular function $\tau^S(x)$ is the integral of a nonnegative Radon measure μ_S that is singular continuous with respect to Lebesgue measure, i.e. $\tau^S(x) = \int_0^x d\mu_S$. The measure μ_S has support contained in the deficient digit set Ω^L, so that

$$\int_0^1 d\mu_S = \int_{\Omega^L} d\mu_S = 1. \tag{3.30}$$

Furthermore, for every Borel set K in $[0,1]$,

$$\mu_S(K) = meas(\tau^S(K)), \tag{3.31}$$

where meas denotes Lebesgue measure.

(3) The support of the measure μ_S is exactly the deficient digit set. That is, it is the closure of the set of points of increase of the Takagi singular function.

We call $d\mu_S$ the *Takagi singular measure*. It is a probability measure on $[0,1]$, and $\tau^S(x)$ is its cumulative distribution function. The Takagi singular function is pictured in Figure 3.

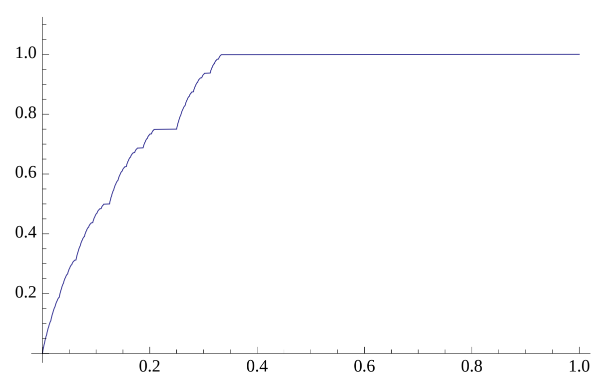

Figure 3: Graph of Takagi singular function $\tau^S(x)$.

We deduce from Theorem 3.2 a nondecreasing property of the Takagi function on $\frac{1}{2}\Omega^L$. This result later will be important in establishing the bi-Lipschitz property in Theorem 7.1.

Theorem 3.3. (Increasing Property of Takagi function on $\frac{1}{2}\Omega^L$) *Let* $\frac{1}{2}\Omega^L := \{\frac{1}{2}x : \ x \in \Omega^L\}$, *which is*

$$\frac{1}{2}\Omega^L = \{x : \ D_j(x) \geq 1 \textit{ for all } \ j \geq 1\}. \tag{3.32}$$

The Takagi function $\tau(x)$ *restricted to the domain* $\frac{1}{2}\Omega^L$ *is a nondecreasing function. That is, if* $x_1, x_2 \in \frac{1}{2}\Omega^L$ *with* $x_1 \geq x_2$*, then* $\tau(x_1) \geq \tau(x_2)$*. Furthermore, it is strictly increasing on* $\frac{1}{2}\Omega^L$*, except at a certain countable set of points* x.

Proof. We have

$$\frac{1}{2}\Omega^L = \{x = 0.0b_1b_2b_3... : \ x' = 0.b_1b_2b_3... \in \Omega^L\}.$$

Now the digit sum function $N_j^1(x) = N_{j-1}^1(x')$ for $j \geq 1$ (here $N_0^1(x') = 0$) so that, for $j \geq 1$,

$$D_j(x) = j - 2N_j^1(x) = 1 + (j-1) - 2N_{j-1}^1(x') = 1 + D_{j-1}(x') \geq 1.$$

This verifies (3.32) and also that $\frac{1}{2}\Omega^L \subset \Omega^L$. Lemma 2.2 gives for $x \in [0, \frac{1}{2}]$,

$$2\tau(x) = \tau(2x) + 2x.$$

This applies if $x \in \frac{1}{2}\Omega^L \subset [0, \frac{1}{6}]$, the point being that $2x \in \Omega^L$ gives $\tau(2x) = \tau^L(2x)$. Thus

$$2\tau(x) = \tau(2x) + 2x = \tau^L(2x) + 2x = \tau^S(2x).$$

By Theorem 3.2 the singular Takagi function is monotone nondecreasing on $[0,1]$, giving the nondecreasing property.

We now show that $\tau(x)$ is strictly increasing on $\frac{1}{2}\Omega^L$, except at certain rational values. The strict increasing property holds for all $x \in \frac{1}{2}\Omega^L$ that are not rational numbers with binary expansions ending in either 0^∞ or $(01)^\infty$, using [14, Theorem 4.8] . We need here the additional fact that the proofs of both parts of Theorem 4.8 show that if $x \in \frac{1}{2}\Omega^L$ then all the approximants x_k constructed belong to $\frac{1}{2}\Omega^L$ as well. The exceptional rational values are not strictly increasing; they group into pairs of tie values $\tau(\frac{1}{2}x(B)^-) = \tau(\frac{1}{2}x(B)^+)$ for $B \in \mathcal{B}$, where $I_B = (x(B)^-, x(B)^+)$ is given in Definition 2.9. $\square$

3.2. Expected number of local level sets

In [14, Theorem 5.8] we proved the following result.

Theorem 3.4. (Expected number of local level sets) *For a full Lebesgue measure set of ordinate points* $y \in [0, \frac{2}{3}]$ *the number* $N^{loc}(y)$ *of local level sets at level* y *is finite. Furthermore*

$$\int_0^{\frac{2}{3}} N^{loc}(y)dy = 1. \tag{3.33}$$

That is, the expected number of local level sets on a randomly drawn ordinate level y *is* $\frac{3}{2}$.

The proof uses the Coarea formula for functions of bounded variation ([6, Sec. 5.5]), applied to the flattened Takagi function τ^L.

4. Structure of Takagi Singular Measure

We now compute values of the Takagi singular measure on various subsets of Ω^L.

4.1. Fine partition of deficient digit set

We partition the set Ω^L of left endpoints of local level sets into finer pieces, as follows:

$$\Omega^L = \Omega^L_\infty \bigcup \Omega^L_{fin} \tag{4.1}$$

in which

$$\Omega^L_\infty := \{x \in \Omega^L : \ D_j(x) = 0 \text{ for infinitely many } j \geq 1\}. \tag{4.2}$$

and Ω^L_{fin} is its complement,

$$\Omega^L_{fin} := \{x \in \Omega^L : \ D_j(x) \geq 1 \text{ for all sufficiently large } j\}. \tag{4.3}$$

The latter set can be further partitioned into subsets labelled by elements of the breakpoint set $\mathcal{B}'$ in Definition 2.8. To each $B' \in \mathcal{B}'$ we associate the set

$$\Omega^L(B') := \{x = B' + \frac{x'}{2^{2m}} : \ x' \in \frac{1}{2}\Omega^L\}. \tag{4.4}$$

In particular for $m = 0$ we have one set $B' = B_0 = 0$ with $\Omega^L(B_0) = \frac{1}{2}\Omega^L$.

Lemma 4.1. (Fine Partition of Deficient Digit Set) *The set Ω^L_{fin} has a partition*

$$\Omega^L_{fin} = \bigcup_{B' \in \mathcal{B}'} \Omega^L(B'), \tag{4.5}$$

with union over the breakpoint set $\mathcal{B}^{'}$. Each set $\Omega^L(B')$ is a closed set.

Proof. Elements $x \in \Omega^L(B')$ have $D_j(x) \geq 0$ for all $j \geq 1$, $D_{2m}(x) = D_{2m}(B') = 0$, and $D_j(x) > 0$ for $j \geq 2m + 1$. In particular $\Omega^L(B') \subset \Omega^L_{fin}$. The sets are disjoint for different B' because the value $2m$ is uniquely determined for each element of the set $\Omega^L(B')$, and this determines the initial digits B' uniquely. Finally we see that each element $x \in \Omega^L_{fin}$ has associated to it a unique maximal value $2m$ of j such that $D_j(x) = 0$, (j is necessarily even) and this assigns it to a particular $\Omega^L(B')$. $\square$

4.2. Singular measure calculations

The Takagi singular measure $d\mu_S$ is not translation-invariant. In Theorem 4.2 below we use its self-similarity properties to compute the μ_S-measure of certain sets inside Ω^L, namely the sets Ω^L_∞ and the sets $\Omega^L(B')$ in the fine partition of Ω^L in Lemma 4.1.

Theorem 4.2. (Takagi singular measure: fine partition) *For each balanced dyadic rational $B' = 0.b_1 b_2 ... b_{2m} = \frac{k}{2^{2m}}$ in the deficient digit set Ω^L the fine partition set $\Omega^L(B')$ is a closed set, and its Takagi singular measure is*

$$\mu_S(\Omega^L(B')) := \int_{\Omega^L(B')} d\mu_S = \frac{1}{2^{2m+1}}. \tag{4.6}$$

Proof. We already know that $\mu_S(\Omega^L) = 1$ via Theorem 3.2(2).

Claim 1. *The Takagi singular measure of $\frac{1}{2}\Omega^L$ is given by*

$$\mu_S(\Omega^L(B_0)) := \mu_S(\frac{1}{2}\Omega^L) = \frac{1}{2}\mu_S(\Omega^L) = \frac{1}{2}. \tag{4.7}$$

To prove the claim, we use the self-similarity relation in Lemma 2.2(1),

$$2\tau(\frac{1}{2}x) = x + \tau(x), \qquad \text{for} \quad 0 \le x \le 1.$$

It $x \in \Omega^L$ then $\tau(x) = \tau^L(x)$ so that we obtain

$$2\tau(\frac{1}{2}x) = x + \tau(x) = x + \tau^L(x) = \tau^S(x). \tag{4.8}$$

Thus if $x_1 < x_2$ with both $x_i \in \Omega^L$, then

$$\begin{aligned}
\int_{\frac{1}{2}x_1}^{\frac{1}{2}x_2} \mu_S &= \tau^S(\frac{1}{2}x_2) - \tau^S(\frac{1}{2}x_1) \\
&= \left(\tau(\frac{1}{2}x_2) + \frac{1}{2}x_2\right) - \left(\tau(\frac{1}{2}x_1) + \frac{1}{2}x_1\right) \\
&= \frac{1}{2}\left(\tau^S(x_2) - \tau^S(x_1)\right) + \frac{1}{2}\left(x_2 - x_1\right) \\
&= \frac{1}{2}\int_{x_1}^{x_2} \mu_S + \frac{1}{2}\left(x_2 - x_1\right).
\end{aligned} \tag{4.9}$$

We may rewrite this as

$$|\int_{\frac{1}{2}x_1}^{\frac{1}{2}x_2} \mu_S - \frac{1}{2}\int_{x_1}^{x_2} \mu_S| \le \frac{1}{2} meas([x_1, x_2]) \tag{4.10}$$

where the last term denotes the Lebesgue measure of the interval $[x_1, x_2]$.

Now by (2.27) for each $m \ge 1$ we obtain a covering of Ω^L using

$$\Omega^L \subset \mathcal{P}_{2m} := [0,1) \smallsetminus \bigcup_{\substack{B \in \mathcal{B} \\ |B| \le 2m}} I_B. \tag{4.11}$$

in which we remove only a finite number of the "flattened" open intervals I_B corresponding to those $B \in \mathcal{B}$ (the small breakpoint set) having dyadic length at most $2m$. The set $\mathcal{P}_{2m}$ is a closed set comprised of a finite number of intervals, $[x_j, x_j^{'}]$, say, having both endpoints $x_j, x_j^{'} \in \Omega^L$. Adding up the relations (4.10) over these intervals yields

$$|\int_{\frac{1}{2}\mathcal{P}_{2m}} \mu_S - \frac{1}{2}\int_{\mathcal{P}_{2m}} \mu_S| \le \frac{1}{2} meas(\mathcal{P}_{2m}), \tag{4.12}$$

in which $meas(\mathcal{P}_{2m})$ denotes the Lebesgue measure of the set $\mathcal{P}_{2m}$. Next we note that the $\mathcal{P}_{2m}$ form a nested family $\mathcal{P}_2 \supset \mathcal{P}_4 \supset \mathcal{P}_6 \supset \cdots$ of closed sets, with

$$\Omega^L = \bigcap_{m=1}^{\infty} \mathcal{P}_{2m}.$$

Since these sets are Borel measurable and this Radon measure μ_S is outer regular (using [11, Theorem E, Sect. 52]), we have

$$\lim_{m\to\infty}\int_{\mathcal{P}_{2m}}\mu_S=\int_{\Omega^L}\mu_S,$$

$$\lim_{m\to\infty}\int_{\frac{1}{2}\mathcal{P}_{2m}}\mu_S=\int_{\frac{1}{2}\Omega^L}\mu_S,$$

cf. Evans and Gariepy [6, Theorem 1, p. 2]. Now Theorem 2.10 (2), (3) ([14, Theorem 4.6]) together establish that

$$meas(\mathcal{P}_{2m})\to 0 \quad \text{as} \quad m\to\infty.$$

Thus letting $m\to\infty$ in (4.12) yields

$$\int_{\frac{1}{2}\Omega^L}\mu_S=\frac{1}{2}\int_{\Omega^L}\mu_S,$$

which with $\int_{\Omega^L}\mu_S=1$ proves Claim 1.

Claim 2. *Let $B'=\frac{k}{2^{2m}}\in\mathcal{B}'$ and suppose that $x_i=B'+\frac{x_i'}{2^{2m}}$ for $i=1,2$, with both $x_i'\in\Omega^L$. Then*

$$\int_{x_1}^{x_2}\mu_S=\frac{1}{2^{2m}}\left(\int_{x_1'}^{x_2'}\mu_S+(x_2'-x_1')\right). \tag{4.13}$$

Since B' is a balanced dyadic rational, we may deduce (4.13) using the formula of Lemma 2.4, in analogous fashion to (4.9). This proves Claim 2.

Now we complete the proof. Claim 2 yields the formula

$$|2^{2m}\int_{x_1}^{x_2}\mu_S-\int_{x_1'}^{x_2'}\mu_S|\leq meas([x_1',x_2']). \tag{4.14}$$

For any $B'\in\mathcal{B}'$ we have

$$\Omega^L(B'):=\{x:\ x=B'+\frac{x'}{2^{2m}} \ \text{ with } \ x'\in\frac{1}{2}\Omega^L\}.$$

Now we may cover the set $\frac{1}{2}\Omega^L$ with $\frac{1}{2}\mathcal{P}_{2n}$. We apply the approximation bound (4.14) summed up over all the intervals in $\mathcal{P}_{2n}$, to obtain

$$|2^{2m}\int_{B'+\frac{1}{2^{2m}}(\frac{1}{2}\mathcal{P}_{2n})}\mu_S-\int_{\frac{1}{2}\mathcal{P}_{2n}}\mu_S|\leq\frac{1}{2}meas(\mathcal{P}_{2n}).$$

Letting $n\to\infty$ we deduce, using $meas(\mathcal{P}_{2n})\to 0$, that

$$2^{2m}\int_{B'+\frac{1}{2^{2m}}(\frac{1}{2}\Omega^L)}\mu_S=\int_{\frac{1}{2}\Omega^L}\mu_S=\frac{1}{2}.$$

This yields, since $\Omega^L(B'):=B'+\frac{1}{2^{2m}}(\frac{1}{2}\Omega^L)$, that

$$\int_{\Omega^L(B')}\mu_S=\frac{1}{2^{2m+1}},$$

as asserted. $\Box$.

4.3. Singular measure of Ω_∞^L

The calculations of the last section yield the following consequence.

Theorem 4.3. *Let μ_S denote the Takagi singular measure. The sets Ω_{fin}^L and Ω_∞^L are Borel sets, hence measurable. We have*

$$\mu_S(\Omega_{fin}^L) = 1, \tag{4.15}$$

which shows that

$$\mu_S(\Omega_\infty^L) = 0. \tag{4.16}$$

Consequently the image of this set under the Takagi singular function τ^S satisfies

$$meas(\tau^S(\Omega_\infty^L)) = 0, \tag{4.17}$$

where meas denotes Lebesgue measure.

Proof. Each set $\Omega^L(B')$ is closed, hence their disjoint union Ω_{fin}^L is a Borel set, hence is μ_S-measurable. The set Ω^L is closed, hence $\Omega_\infty^L = \Omega^L \setminus \Omega_{fin}^L$ is also a Borel set, hence is μ_S-measurable, and

$$\mu_S(\Omega_\infty^L) = \mu_S(\Omega^L) - \mu_S(\Omega_{fin}^L).$$

(In fact one can easily show that the closure of Ω_{fin}^L is Ω^L.)

Since $\mu_S(\Omega^L) = 1$ the theorem will follow on showing $\mu_S(\Omega_{fin}^L) = 1$. We have

$$\mu_S(\Omega_{fin}^L) = \sum_{B \in \mathcal{B}'} \mu_S(\Omega^L(B)),$$

where $\mathcal{B}'$ is the breakpoint set. Theorem 4.2 now gives $\mu_S(\Omega^L(B)) = \frac{1}{2^{2m+1}}$, where $B = 0.b_1 \cdots b_{2m} = \frac{k}{2^{2m}}$, with k odd. Recall from [14, Lemma 4.2] that the number of balanced dyadic rationals in Ω^L having the form $\frac{k}{2^{2m}}$ for an odd k is the m-th Catalan number $C_m = \frac{1}{m}\binom{2m}{m}$. Here for $m = 0$ we have $C_0 = 1$ corresponding to the element $B_0 = 0$.

The Catalan numbers are well known to have the generating function

$$\sum_{j=0}^{\infty} C_m z^{2m} = \frac{1 - \sqrt{1 - 4z^2}}{2z^2}. \tag{4.18}$$

In consequence, taking $z = \frac{1}{2}$, we obtain $\sum_{j=0}^{\infty} \frac{C_m}{2^{2m}} = 2$. Therefore we obtain, using Theorem 4.2, that

$$\mu_S(\Omega_{fin}^L) = \sum_{m=0}^{\infty} C_m \frac{1}{2^{2m+1}} = \frac{1}{2}\left(\sum_{m=0}^{\infty} \frac{C_m}{2^{2m}}\right) = 1,$$

which proves (4.15). Now (4.16) follows, and (4.17) follows from Theorem 3.2 on taking $K = \Omega_\infty^L$ in (3.31). $\square$

5. Cardinality of Global Level Sets

In this section we prove Theorem 1.2, which asserts that for a full measure set of ordinates y the level set $L(y)$ is a finite set, and that the expected number of elements in this set, with respect to Lebesgue measure on $0 \leq y \leq \frac{2}{3}$, is infinite.

Proof of Theorem 1.2. (1) Let Γ_∞^{ord} be the set of infinite levels, i.e.

$$\Gamma_\infty^{ord} := \{y : \quad L(y) \quad \text{is an infinite set}\}. \tag{5.1}$$

To show a full measure set of ordinates having finite level sets, we show that the complementary set Γ_∞^{ord} has Lebesgue measure 0. We have

$$\Gamma_\infty^{ord} \subset \tau(\Omega_\infty^L) \bigcup \Lambda_\infty^{loc},$$

in which $\tau(\Omega_\infty^L) := \{y = \tau(x) : \ x \in \Omega_\infty^L\}$ detects all levels that contain at least one infinite local level set, and

$$\Lambda_\infty^{loc} := \{y : L(y) \ \text{ contains infinitely many different local level sets}\}. \tag{5.2}$$

Theorem 6.2 (1) of our previous paper [14] shows that Λ_∞^{loc} has Lebesgue measure 0. Thus it suffices to prove that $\tau(\Omega_\infty^L)$ has Lebesgue measure 0.

We have the identity

$$\tau^S(x) = \tau^L(x) + x = \tau(x) + x, \quad \text{for} \ \ x \in \Omega^L,$$

with the first equality holding for general x and the second equality only for $x \in \Omega^L$. Now consider $\tau^S(x)$ restricted to $x \in \Omega^L(B)$ for a particular $B \in \mathcal{B}'$, the breakpoint set. We write $B = 0.b_1b_2\cdots b_{2m} = \frac{k}{2^{2m}}$ where k is necessarily odd. Then $x \in \Omega^L(B)$ if and only if

$$x = B + \frac{\frac{1}{2}x'}{2^{2m}}, \qquad \text{with } \frac{1}{2}x' \in \frac{1}{2}\Omega^L.$$

Lemma 2.4 then gives

$$\tau(x) = \tau(B) + \frac{1}{2^{2m}}\tau(\frac{1}{2}x'), \ \ \text{with} \ \ x' \in \Omega^L.$$

We recall from Lemma 2.2 that $2\tau(\frac{1}{2}(x)) = \tau(x) + x$ if $x \in \Omega^L$, whence

$$2^{2m+1}\left(\tau(x) - \tau(B)\right) = \tau(x') + x' = \tau^L(x') + x' = \tau^S(x'), \quad \text{for } x' \in \Omega^L. \tag{5.3}$$

Now the linear map

$$y \mapsto y' := 2^{2m+1}\left(y - \tau(B)\right)$$

sends the interval $[\tau(B), \tau(B) + \frac{1}{2^{2m+1}}]$ onto $[0,1]$ and it follows from the above that it sends $\tau(\Omega^L(B)) \subset [\tau(B), \tau(B) + \frac{1}{2^{2m+1}}]$ onto the range $\tau^S(\Omega^L) = [0,1]$. We conclude from the linearity of the map that

$$\tau(\Omega^L(B)) = [\tau(B), \tau(B) + \frac{1}{2^{2m+1}}].$$

Furthermore the subset of $\tau(\Omega^L(B))$ that maps to $\tau^S(\Omega_\infty^L)$ necessarily has Lebesgue measure 0, because the map is linear and $meas(\tau^S(\Omega_\infty^L)) = 0$ by Theorem 4.3. We conclude that the total Lebesgue measure (allowing overlaps) covered by elements

$$meas\left(\{y = \tau(x) : \ x \in \Omega^L(B), \ x \notin \Omega_\infty^L\}\right) = meas\left(\tau(\Omega^L(B))\right) = \frac{1}{2^{2m+1}}.$$

Adding up these contributions, the summation in Theorem 4.3 gives that the total Lebesgue measure in $y \in [0, \frac{2}{3}]$ covered by images of these sets (counting overlaps with multiplicity) is

$$\sum_{B \in \mathcal{B}'} meas(\tau(\Omega^L(B))) = \sum_{B \in \mathcal{B}'} \mu_S(\Omega^L(B)) = 1.$$

(The images have some overlap, allowing their total measure to exceed the length of the interval $[0, \frac{2}{3}]$.) Viewing these points $x \in \Omega^L(B)$ as labelling left endpoints of local level sets, this says that a lower bound of the total number of local level set endpoints Lebesgue-integrated over $0 \le y \le \frac{2}{3}$, counted with multiplicity, is 1. Here we did not count any local level set endpoints in $\tau(\Omega^L_\infty) := \tau\left(\Omega^L \smallsetminus \Omega^L_{fin}\right)$, coming from the image of Ω^L_∞. Now Theorem 3.4 says that

$$\int_0^{\frac{2}{3}} N^{loc}(y) dy = 1.$$

where $N^{loc}(y)$ counts the number of all local level set endpoints. We have already accounted for the full mass of this integral above, and any omitted contribution to $N^{loc}(y)$ coming from $\tau(\Omega^L_\infty) := \tau\left(\Omega^L \smallsetminus \Omega^L_{fin}\right)$ necessarily contributes an additional nonnegative amount. Thus we may conclude that

$$meas(\tau(\Omega^L_\infty) = 0.$$

as asserted.

(2) We aim to show that the expected size of a global level set is infinite, i.e. to show that

$$\int_0^{\frac{2}{3}} |L(y)| dy = +\infty,$$

where $|L(y)|$ counts the number of elements in $L(y)$. By the discussion above we have

$$\int_0^{\frac{2}{3}} |L(y)| dy = \int_0^1 |L(\tau(x))| \mu_S(x) \ge \sum_{B \in \mathcal{B}'} \frac{1}{2^{2m+1}} 2^{r(B)}, \tag{5.4}$$

in which

$$r(B) := |\{1 \le j < \infty : \ N_j(B) = 0\}|.$$

Here each $r(B)$ is finite and bounded above by m if $B = 0.b_1 b_2 \cdots b_{2m}$. We rewrite this as

$$\int_0^{\frac{2}{3}} |L(y)| dy = \sum_{m=0}^{\infty} \frac{L_m}{2^{2m+1}}, \tag{5.5}$$

in which

$$L_m := \sum_{B \in \mathcal{B}', |B| = 2m} 2^{r(B)}.$$

Now we observe that L_m, the total number of binary sequences of length $2m$ having $N_{2m}(B) = 0$, has a combinatorial interpretation as counting the number the two-dimensional lattice paths of length $2m$ starting at the origin $(0, 0)$, taking steps either $(1, 1)$ or $(1, -1)$, and ending at $(2m, 0)$. These paths groups into collections of paths of size 2^r under the "flipping" (reflection) operation, with each group containing exactly one path in $\mathcal{B}'$. (See the discussion and proof in Feller [9, Theorem 4, p. 90] and also [14, Lemma 4.2].) It follows that

$$L_m = \binom{2m}{m} \sim \frac{1}{\sqrt{\pi m}} 2^{2m}.$$

Thus the terms in the series on the right side of (5.5) decay like $\Omega(\frac{1}{\sqrt{m}})$, so the series (5.5) diverges, giving the result. $\square$.

6. Level Sets of Positive Hausdorff Dimension: Abscissa View

We study level sets having positive Hausdorff dimension. In [14, Sect. 3.3]) we classified those local level sets containing a rational number x that are of positive Hausdorff dimension: this result gives explicitly determinable rational ordinates y having this property. Here we show that the set Γ_H^L of abscissa points in Ω^L that give local level sets having positive Hausdorff dimension has full Hausdorff dimension 1.

Theorem 6.1. (Local level sets of positive Hausdorff dimension) *Let Λ_H^L denote the set of $x \in \Omega^L$ such that the Hausdorff dimension of L_x^{loc} is positive, i.e.*

$$\Lambda_H^L := \{x \in \Omega^L : \quad dim_H(L_x^{loc}) > 0\}. \tag{6.1}$$

This set has full Hausdorff dimension, i.e.

$$dim_H(\Lambda_H^L) = 1. \tag{6.2}$$

In particular, the deficient digit set Ω^L has Hausdorff dimension 1.

Proof. It clearly suffices to prove the first assertion. We set

$$\Gamma_H^L := \{x \in \Omega^L : \ \dim_H(\tau(x)) > 0\} \subset \Omega^L,$$

and aim to prove $\dim_H(\Gamma_H^L) = 1$. For integer $r \geq 1$ let Γ_{2r} consist of all abscissas $x \in [0, 1]$ that satisfy:

(i) $D_j(x) > 0$ for $j \not\equiv 0 \pmod{2r}$.

(ii) $D_{2kr}(x) = 0$ for $k = 1, 2, 3, ...$

These conditions are equivalent to requiring $\Gamma_{2r} \subset \Omega^L$, and that all $x \in \Gamma_{2r}$ have the same balance-set $Z(x) = \{2kr : k \geq 0\} := 2r\mathbb{N}$. We will show later in (7.4) that membership in Γ_{2r} implies that

$$\tau(x_1) \neq \tau(x_2) \quad \text{for distinct } x_1, x_2 \in \Gamma_{2r}$$

so that each point in Γ_{2r} picks out a different level set.

Claim 1. *All members of Γ_{2r} have local level sets of positive Hausdorff dimension, satisfying*

$$\dim_H(L(\tau(x))) \geq \frac{\log 2}{\log(2^{2r})} = \frac{1}{2r}.$$

Claim 1 will follow from the spacing of the balance points being an arithmetic progression. This makes each local level set L_x^{loc} a Cantor-like set, which has a standard tree construction covered by 2^k intervals of length 2^{-2rk}, so that $\dim H(L_x^{loc}) = \frac{1}{2r}$. Note that the particular subintervals are chosen differently at each step so this is generally not a self-similar construction, but the Hausdorff dimension lower bound proof for the Cantor set given in Falconer [8, Sect. 2.3] remains valid here, establishing Claim 1.

Claim 1 shows that $\Gamma_{2r} \subset \Gamma_H^L$, so that

$$\bigcup_{r=1}^{\infty} \Gamma_{2r} \subset \Gamma_H^L.$$

To complete the proof it suffices to show the sets Γ_{2r} each have positive Hausdorff dimension, which approaches 1 as $r \to \infty$.

Claim 2. *For all large enough r, the set Γ_{2r} has Hausdorff dimension greater than $1 - \frac{2\log r}{r}$.*

Claim 2 follows by observing that Γ_{2r} is itself a self-similar Cantor set in which each block of $2r$ symbols is drawn from the set

$$X_{2r} := \{x = \frac{m}{2^{2r}} = 0.b_1 b_2 ... b_{2r} : \quad D_j(x) > 0 \text{ for } 1 \leq x < 2r, \quad D_{2r}(x) = 0, \}$$

whose Hausdorff dimension is computable by the method of Falconer [8, Sect. 2.3]. It is well known that

$$|X_{2r}| = C_r = \frac{1}{r+1}\binom{2r}{r},$$

is a Catalan number. Thus we obtain

$$\dim_H(\Gamma_{2r}) = \frac{\log C_r}{\log 2^{2r}} = \frac{\log C_r}{2r \log 2}.$$

However it is well known that $C_r = \frac{2^{2r}}{\pi r^{\frac{3}{2}}} (1 + o(1))$, as the integer $r \to \infty$. We conclude that for large enough r there holds

$$\dim_H(\Gamma_{2r}) > 1 - \frac{2\log r}{r}$$

Claim 2 now follows, so the theorem is proved. $\square$.

7. Level Sets of Positive Hausdorff Dimension: Ordinate View

Our object is to prove Theorem 1.4, which asserts that the set of ordinate levels y having $\dim_H(L(y)) > 0$ has Hausdorff dimension 1. We use the result on abscissas proved in the last section (Theorem 6.1), together with a property showing that the Takagi function restricted to certain small domains in $[0, 1]$ is quite well behaved, i.e. it is bi-Lipschitz map. This allows the transfer of Hausdorff dimension lower bounds from the abscissa case treated in Sect. 6.

7.1. Bi-Lipschitz property of Takagi function on Γ_{2r}

We use the result on abscissas proved in Theorem 6.1 together with the following result.

Theorem 7.1. *Let $X(x) = \{j : D_j(x) = 0\}$ be the balance-set of $x \in [0,1]$. For $r \geq 1$ the Takagi function $\tau(x)$ restricted to the (compact) domain*

$$\Gamma_{2r} := \{x \in \Omega^L : \ Z(x) = 2r\mathbb{N}\},$$

is a bi-Lipschitz map.

Proof. If $1 \leq r \leq 2$, then Γ_{2r} contains one element and the statement is trivial. Assume the integer $r \geq 3$, in which case Γ_{2r} is a Cantor set. Using Lemma 2.1 we have

$$\tau(x) = \sum_{m=0}^{\infty} \frac{l_m}{2^m}$$

with integers $0 \leq l_m \leq m$, given by

$$l_m = l_m(x) := \#\{b_i : 1 \leq i < m, b_i \neq b_m\}.$$

Now for all $x \in \Gamma_{2r}$ we have at balance points $c_k = 2kr$ that $l_{2kr+1} = kr$. Now suppose $x_1, x_2 \in \Gamma_{2r}$ with $x_1 = 0.b_1b_2b_3...$, $x_2 = 0.b_1^{'}b_2^{'}b_3^{'}...$ agree at their first $2kr$ bits, and disagree somewhere between the $(2kr+2)$-th bit and the $(2k+1)r$-th bit.

Now set $l_m = l_m(x_1)$ and $l_m^{'} = l_m(x_2)$. Note that the value $l_m - l_m^{'}$ can change by at most 2 as m increases to $m+1$. Also note that for $x_1, x_2 \in \Gamma_{2r}$ we have $|l_m - l_m^{'}| = 0$ whenever $m = 2jr + i$ for $i = -1, 0, 1, 2$, the equalities coming from the definition of Γ_{2r} and the assumption that $r > 1$, which moreover imply that the digits in x_1 and x_2 for $m = 2jr - 1$ and $m = 2jr$ must all be 1, and the digits for $m = 2jr + 1$ and $m = 2jr + 2$ must all be 0.

We now show that

$$2^{-2kr-1} \geq |x_1 - x_2| \geq 2^{-(2k+2)r+1}. \tag{7.1}$$

Here the upper bound in (7.1) holds by an absolute value estimate, the maximum occurring if the two numbers disagree at all binary digits at and after the $2kr+2$-nd digit. The lower bound holds because they must have a disagreeing digit in the block from the $(2kr+1)$-th digit and the $(2k+2)r$-th digit, however they also have a matching digit in the $(2k+2)r-1$-st place, which prevents too much cancellation.

We next deduce that $\tau(x)$ is a Lipschitz function on this domain Γ_{2r} with Lipschitz constant $2p2^r$, via:

$$\begin{aligned}
|\tau(x_1) - \tau(x_2)| &= |\sum_{m=1}^{\infty} \frac{l_m - l_m^{'}}{2^m}| \\
&\leq \sum_{m=1}^{\infty} \frac{|l_m - l_m^{'}|}{2^m} \\
&\leq \sum_{m=2kr+1}^{\infty} \frac{2r}{2^m} \\
&= 2r2^{-2kr} \leq r2^{2r}|x_1 - x_2|.
\end{aligned} \tag{7.2}$$

The inequality on the second to last line follows from the remarks on $|l_m - l_m^{'}|$ above, and that on the last line uses (7.1).

To show the other direction of the bi-Lipschitz property, we will establish the lower bound

$$|\tau(x_1) - \tau(x_2)| \geq \frac{1}{2^{2r-1}}|x_1 - x_2|, \quad \text{for } x_1, x_2 \in \Gamma_{2r}. \tag{7.3}$$

This would follow from the assertion that, for $x_1, x_2 \in \Gamma_{2r}$ disagreeing first somewhere between the $(2kr+2)$-th bit and the $(2k+1)r$-th bit,

$$|\tau(x_1) - \tau(x_2)| \geq \frac{1}{2^{(2k+2)r}}, \tag{7.4}$$

by using the upper bound in (7.1). To show the assertion (7.4) , we start from

$$|\tau(x_1) - \tau(x_2)| \geq \left| \sum_{j=1}^{2r} \frac{l_{2kr+j} - l_{2kr+j}^{'}}{2^{2kr+j}} \right| - \left(\sum_{m=(2k+2)r+1}^{\infty} \frac{|l_m - l_m^{'}|}{2^m} \right). \tag{7.5}$$

First, we upper bound second term on the right in (7.5) by

$$\sum_{m=(2k+2)r+1}^{\infty} \frac{|l_m - l_m'|}{2^m} \leq \sum_{j=1}^{\infty} \frac{2j}{2^{(2k+2)r+2+j}} = \frac{1}{2^{(2k+2)r}}. \tag{7.6}$$

Here we used the fact that the difference $|l_m - l_m^{'}| = 0$ for $m = (2k+2)r + i$ with $i = 1, 2$, and the difference can increase by at most 2 at each step thereafter. To complete the proof it suffices to lower bound the first term on the right in (7.5) by

$$\left| \sum_{j=1}^{2r} \frac{l_{2kr+j} - l_{2kr+j}^{'}}{2^{2kr+j}} \right| \geq \frac{1}{2^{(2k+2)r-1}}. \tag{7.7}$$

Indeed, combining in (7.5) this estimate with the second term estimate (7.6), will yield (7.4), which yields (7.3). Thus it remains to establish the lower bound (7.7), which is complicated.

To proceed, we construct from x_1, x_2 the dyadic rationals

$$\bar{x}_1 = 0.b_{2kr+1}b_{2kr+2}...b_{(2k+2)r-1} = \frac{m_1}{2^{2r-1}},$$

$$\bar{x}_2 = 0.b_{2kr+1}^{'}b_{2kr+2}^{'}...b_{(2k+2)r-1}^{'} = \frac{m_2}{2^{2r-1}},$$

Claim 1. *We have $\bar{x}_1, \bar{x}_2 \in \Sigma_{2r-1}$ where*

$$\Sigma_{2r-1} := \{\bar{x} = \frac{m}{2^{2r-1}} : \quad \bar{x} \in \frac{1}{2}\Omega^L, \; \textit{with} \;\; N_{2r-1}(\bar{x}) = 1\}. \tag{7.8}$$

For x_1, x_2 there holds

$$\sum_{j=1}^{2r} \frac{l_{2kr+j} - l_{2kr+j}^{'}}{2^{2kr+j}} = \frac{1}{2^{2kr}} \left(\tau(\bar{x}_1) - \tau(\bar{x}_2) \right). \tag{7.9}$$

The membership of $\bar{x}_1, \bar{x}_2$ in Σ_{2r-1} follows since $x_1, x_2 \in \Omega^L$ both have consecutive balance points at $2kr$ and $(2k+2)r$. (This implies that $b_{2kr+1} = b'_{2kr+1} = b_{2kr+2} = b'_{2kr+2} = 0$ and that $D_{(2k+2)r-1}(x_j) = D_{2r-1}(\bar{x}_j) = 1$ for $j = 1, 2$.)

To show (7.9) holds, we first write

$$\tau(\bar{x}_1) = \sum_{j=1}^{\infty} \frac{\bar{l}_j}{2^j}, \qquad \tau(\bar{x}_2) = \sum_{j=1}^{\infty} \frac{\bar{l}'_j}{2^j}.$$

Now $\bar{l}_j = \bar{l}'_j$ for $j \geq 2r$ because $\bar{x}_1$ and $\bar{x}_2$ each have r zeros and $r-1$ 1's in their first $2r-1$ digits, and all digits thereafter agree. Thus we get

$$\tau(\bar{x}_1) - \tau(\bar{x}_2) = \sum_{j=1}^{2r-1} \frac{\bar{l}_j - \bar{l}'_j}{2^j}.$$

However for $1 \leq j \leq 2r-1$ we have

$$l_{2kr+j} = \bar{l}_{j+2r}, \quad l'_{2kr+j} = \bar{l}'_{j+2r},$$

while for $j = 2r$ we have

$$l_{(2k+2)r} - l'_{(2k+2)r} = \bar{l}_{2r} - \bar{l}'_{2r} = 0$$

since $l_{(2k+2)r} = l'_{(2k+2)r} = (k+1)r$. Substituting these expressions into both sides of (7.9) establishes (7.9) and Claim 1.

Claim 2. *Suppose $r \geq 2$. If $\bar{x}_1, \bar{x}_2 \in \Sigma_{2r-1}$ with $\bar{x}_1 > \bar{x}_2$, then*

$$\tau(\bar{x}_1) - \tau(\bar{x}_2) \geq \frac{1}{2^{2r-1}}. \tag{7.10}$$

To prove Claim 2, since $\bar{x}_1, \bar{x}_2 \in \frac{1}{2}\Omega^L$, Theorem 3.3 shows $\bar{x}_1 > \bar{x}_2$ implies $\tau(\bar{x}_1) \geq \tau(\bar{x}_2)$. This monotonicity property shows that it suffices to prove the inequality (7.10) for consecutive members of the set Σ_{2r-1} (ordering them by size). We assert that two consecutive members necessarily have the form

$$\begin{aligned} \bar{x}_1 &:= 0.b_1 b_2 ... b_{s-1} 1 0^{t+1} 1^{k+t-1} \\ \bar{x}_2 &:= 0.b_1 b_2 ... b_{s-1} 0 1^k (01)^t, \end{aligned}$$

where $s + k + 2t = 2r - 1$, with $k \geq 2, t \geq 0$. Here we let the s-th digit be the first place of disagreement of the two expansions, it must have $s \geq 3$ (they both start with 00) and it must switch a 1 in x_1 to a 0 in x_2. Now, since $\bar{x}_1$ are $\bar{x}_2$ are adjacent elements in Σ_{2p-1}, treating their first $s-1$ binary digits as fixed, we have that $\bar{x}_1$ is the smallest such element of Σ_{2r-1} beginning with a 1 in the s-th position, and $\bar{x}_2$ is the largest such element having a 0 in the s-th position. This requires that the remaining binary expansion of $\bar{x}_2$ have a string of consecutive 1's, say k of them, until $D_{s+k}(\bar{x}_2) = 1$, followed by alternating (01)'s, say $t \geq 0$ of them; we then must have $s + k + 2t = 2r - 1$. and $D_{2r-1}(\bar{x}_2) = 1$. We necessarily have $k \geq 2$, otherwise there is no element $\bar{x}_1$ in Σ_{2r-1} of the given form. The minimal $\bar{x}_1$ must have all its zeros in leading position, from position $s+1$ onward, and there will be $t+1$ leading digits 0, and the remaining $k+t-1$ digits 1. This proves the assertion.

In order that $D_{s+k}(\bar{x}_2) = 1$, we must have $u+1$ digits 0 and u digits 1 in $\bar{x}_2$ to this point, which implies that $s+k = 2u+1$, and also that the first $s-1$ digits of $\bar{x}_1$ and $\bar{x}_2$ necessarily consist of u digits 0 and $u-k$ digits 1. Furthermore the $2r-1$ digits of both numbers will consist of exactly $u+t+1$ digits 0 and $u+t$ digits 1.

We now have

$$\tau(\bar{x}_1) - \tau(\bar{x}_2) = \sum_{j=1}^{2r-1} \frac{\bar{l}_j - \bar{l}'_j}{2^j},$$

because $D_{2r-1}(\bar{x}_1) = D_{2r-1}(\bar{x}_2)$ so that $\bar{l}_j = \bar{l}'_j$ on all subsequent digits. We also have $\bar{l}_j = \bar{l}'_j$ for $1 \le j \le s-1$ and also for $j = 2r-1$. We next compute,

$$\begin{aligned} \bar{l}_s - \bar{l}'_s &= u - (u-k) = k. \\ \bar{l}_{s+j} - \bar{l}'_{s+j} &= (u-k+1) - (u+1) = -k, \quad 1 \le j \le \min(k, t+1) \end{aligned}$$

We also have, for $j \ge t+2$, that

$$\bar{l}_{s+j} - \bar{l}'_{s+j} \ge 0, \tag{7.11}$$

because in this range $\bar{l}_{s+j} = u+t+1 \ge \max_k(\bar{l}_k, \bar{l}'_k)$. We now have two cases. First, if $t+1 \le k$ we obtain, using (7.11)

$$\begin{aligned} \tau(\bar{x}_1) - \tau(\bar{x}_2) &\ge \frac{k}{2^s} - \sum_{j=1}^{t+1} \frac{k}{2^{s+j}} \\ &= \frac{k}{2^{s+t+1}} \ge \frac{1}{2^{2r-1}}. \end{aligned}$$

Secondly, if $k < t+1$, then we have

$$\begin{aligned} \tau(\bar{x}_1) - \tau(\bar{x}_2) &\ge \frac{k}{2^s} - \sum_{j=1}^{k} \frac{k}{2^{s+j}} + \sum_{j=k+1}^{t+1} \frac{\bar{l}_{s+j} - \bar{l}'_{s+j}}{2^{s+j}} \\ &= \frac{k}{2^{s+k}} + \sum_{j=k+1}^{t+1} \frac{\bar{l}_{s+j} - \bar{l}'_{s+j}}{2^{s+j}}. \end{aligned}$$

In the final sum we have, for $i \ge 1$ and $1 \le i \le \frac{t+1-k}{2}$, that

$$\begin{aligned} \bar{l}_{s+k+2i-1} - \bar{l}'_{s+k+2i-1} &= (u-k+1) - (u+i-1) = -k-i+2 \\ \bar{l}_{s+k+2i} - \bar{l}'_{s+k+2i} &= (u-k+1) - (u+i+1) = -k-i \end{aligned}$$

Thus we obtain, for $L = \lfloor \frac{t+1-k}{2} \rfloor$,

$$\begin{aligned} \tau(\bar{x}_1) - \tau(\bar{x}_2) &= \frac{k}{2^s} - \left(\sum_{j=1}^{k} \frac{k}{2^{s+j}} \right) + \left(\sum_{i=1}^{L} \frac{-k-i+2}{2^{s+k+2i-1}} + \frac{-k-i}{2^{s+k+2i}} \right) \\ &\ge \frac{k}{2^{s+t+1}} + \frac{1}{2^{s+k}} \left(\sum_{i=1}^{\infty} \frac{-i+2}{2^{2i-1}} + \frac{-i}{2^{2i}} \right) \\ &= \frac{k}{2^{s+t+1}} \ge \frac{1}{2^{2r-1}}. \end{aligned}$$

Here we used the identity

$$\left(\sum_{i=1}^{\infty}\frac{2-i}{2^{2i-1}}+\frac{-i}{2^{2i}}\right)=0.$$

This proves Claim 2.

Finally, Claims 1 and 2 together establish (7.7), which completes the proof. $\square$

Remark. The Takagi function $\tau(x)$ is not a Lipschitz map on its full domain $[0,1]$, nor is it a Lipschitz function even when restricted to the domain Ω^L. This is because it has arbitrarily steep slopes on Ω^L, as is implicit in the singular function property.

7.2. Hausdorff dimension of Γ_H^{ord}

To conclude the paper we prove Theorem 1.4.

Proof of Theorem 1.4. Let

$$\Gamma_H^{ord} := \{y: \ 0 \le y \le \frac{2}{3} \text{ with } \dim_H L(y) > 0.\}$$

It is well known that bi-Lipschitz maps preserve Hausdorff dimension. By Theorem 7.1 the bi-Lipschitz property holds for the Takagi function τ restricted to the compact domain Γ_{2r}. The range of this map is

$$\tilde{\Gamma}_{2r} := \{y : y = \tau(x), \ x \in \Gamma_{2r}\},$$

which therefore satisfies

$$\dim_H(\tilde{\Gamma}_{2r}) = \dim_H(\Gamma_{2r}) \ge 1 - \frac{2\log r}{2r}.$$

for large enough r, as shown in Claim 2 of the proof of Theorem 6.1.

We also have the inclusion

$$\tilde{\Gamma}_{2r} \subset \Gamma_H^{ord},$$

because for each $x \in \Gamma_{2r}$ the local level set $L_x^{loc} \subset L(\tau(x))$ has positive Hausdorff dimension, which is at least $\frac{1}{2r}$, by Claim 1 of the proof of Theorem 6.1. This shows that, for all large enough p,

$$\dim_H(\Gamma_H^{ord}) \ge 1 - \frac{2\log r}{r}.$$

Letting $r \to \infty$ gives $\dim_H(\Gamma_H^{ord}) \ge 1$, which gives the equality. $\square$

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