# Stable $\mathbb{A}^1$-homotopy and $R$-equivalence

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**Abstract**

We prove that existence of a $k$-rational point can be detected by the stable $\mathbb{A}^1$-homotopy category of $S^1$-spectra.

## 1 Introduction

Suppose $k$ is a field and $X$ a smooth proper $k$-variety. By the Lang-Nishimura lemma [Nis55], one knows that existence of a $k$-point is a $k$-birational invariant. By a remark of Morel and Voevodsky, one also knows that existence of a $k$-rational point is an unstable $\mathbb{A}^1$-homotopy invariant; see, e.g., [MV99, §3 Remark 2.5], where it is observed that this is a consequence of the fact that the Nisnevich topology is used in the construction of the unstable $\mathbb{A}^1$-homotopy category. The purpose of this note is to, in a sense, combine the two results above and to show that the ability to detect rational points persists in the linearization of $\mathbb{A}^1$-homotopy theory (i.e., the Morel-Voevodsky stable $\mathbb{A}^1$-homotopy theory of $S^1$-spectra) as well as its abelianization (i.e., Morel's $\mathbb{A}^1$-derived category).

Write $\mathbf{SH}^s_k$ for the Morel-Voevodsky stable $\mathbb{A}^1$-homotopy category of $S^1$-spectra (see [Mor05, Definition 4.1.1] for a precise definition). Let $\Sigma^\infty_s X_+$ denote the $\mathbb{A}^1$-localization of the simplicial suspension spectrum of $X$ with a disjoint basepoint attached. The 0-th $S^1$-stable $\mathbb{A}^1$-homotopy sheaf of $X$, denoted $\boldsymbol{\pi}^s_0(X_+)$ is the Nisnevich sheaf associated with the presheaf on $\mathcal{S}m_k$

$$U \mapsto \mathrm{Hom}_{\mathbf{SH}^s_k}(\Sigma^\infty_s U_+, \Sigma^\infty_s X_+).$$

The structure morphism $X \to \operatorname{Spec} k$ induces a morphism of sheaves $\boldsymbol{\pi}^s_0(X_+) \to \boldsymbol{\pi}^s_0 \operatorname{Spec} k_+$. The sheaf $\boldsymbol{\pi}^s_0(X_+)$ is a birational invariant of smooth, proper $k$-varieties. If $X$ has a $k$-rational point, the map $\boldsymbol{\pi}^s_0(X_+) \to \boldsymbol{\pi}^s_0(\operatorname{Spec} k_+)$ is a split epimorphism. We prove a converse to this statement.

**Theorem 1.** *If $X$ is a smooth proper $k$-variety, then the following conditions are equivalent:*

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*i) X has a k-rational point,*

*ii) there is a stable $\mathbb{A}^1$-homotopy class of maps $\Sigma_s^{\infty} \operatorname{Spec} k_+ \to \Sigma_s^{\infty} X_+$ splitting the structure map $\Sigma_s^{\infty} X_+ \to \Sigma_s^{\infty} \operatorname{Spec} k_+$, and*

*iii) the morphism of sheaves $\boldsymbol{\pi}_0^s(X_+) \to \boldsymbol{\pi}_0^s(\operatorname{Spec} k_+)$ is a split epimorphism.*

To put this result in context, we observe how these results combined with those of [AH10] give a framework for comparing rational points and 0-cycles of degree 1.

*Remark* 2. Let $\Sigma_{\mathbb{P}^1}$ denote the operation of smashing with the simplicial suspension spectrum of $(\mathbb{P}^1, \infty)$, and let $\Omega_{\mathbb{P}^1}$ be the adjoint looping functor. If $E$ is any $S^1$-spectrum, there is a map $E \to \Omega_{\mathbb{P}^1}\Sigma_{\mathbb{P}^1} E$. We can iterate this functor to obtain a tower

$$E \to \Omega_{\mathbb{P}^1}\Sigma_{\mathbb{P}^1} E \to \Omega^2_{\mathbb{P}^1}\Sigma^2_{\mathbb{P}^1} E \cdots .$$

The 0-th stable $\mathbb{A}^1$-homotopy sheaf of $E$, denoted $\boldsymbol{\pi}_0^{s\mathbb{A}^1}(E)$, can be computed by means of the formula

$$\boldsymbol{\pi}_0^{s\mathbb{A}^1}(E) = \operatorname{colim}_n \boldsymbol{\pi}_0^s(\Omega^n_{\mathbb{P}^1}\Sigma^n_{\mathbb{P}^1} E).$$

The structure map $X \to \operatorname{Spec} k$ induces a morphism $\boldsymbol{\pi}_0^{s\mathbb{A}^1}(X_+) \to \boldsymbol{\pi}_0^{s\mathbb{A}^1}(\operatorname{Spec} k_+)$. One says that $X$ has a rational point up to stable $\mathbb{A}^1$-homotopy if the latter map is a split epimorphism. By [AH10, Theorem 1], if $k$ is an infinite perfect field having characteristic unequal to 2, we know that a smooth proper $k$-scheme $X$ has a 0-cycle of degree 1 if and only if it has a rational point up to stable $\mathbb{A}^1$-homotopy. Thus, under the stated hypotheses on $k$, the difference between a 0-cycle of degree 1 and $k$-rational point is measured by the difference between $S^1$-stable and $\mathbb{P}^1$-stable $\mathbb{A}^1$-homotopy theory.

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## 2 Proof of Theorem 1

Let us introduce some notation for the rest of the note. Throughout, suppose $k$ is a field. Let $\mathcal{S}m_k$ denote the category of schemes that are separated, smooth, and have finite type over $\operatorname{Spec} k$. By a *space* we will mean a simplicial Nisnevich sheaf of sets on $\mathcal{S}m_k$. We write $\mathbf{SH}_k^s$ for the stable $\mathbb{A}^1$-homotopy category of $S^1$-spectra, e.g., as defined in [Mor05, §5]. We write $\mathbf{SH}_k$ for the stable $\mathbb{A}^1$-homotopy category of $\mathbb{P}^1$-spectra, e.g., as defined in [Jar00]. Recall that a sheaf $\mathcal{F}$ of abelian groups on $\mathcal{S}m_k$ is called *strictly* $\mathbb{A}^1$*-invariant* if for any $U \in \mathcal{S}m_k$ and any $i \geq 0$ the map

$$H^i_{Nis}(U, \mathcal{F}) \to H^i_{Nis}(U \times \mathbb{A}^1, \mathcal{F})$$

induced by pullback along the projection $U \times \mathbb{A}^1 \to U$ is a bijection. For recollections about the $\mathbb{A}^1$-derived category, see [Mor06, §3.2].

### The Hurewicz homomorphism

There is a canonical abelianization functor $\mathbf{SH}^s(k) \to \mathrm{D}_{\mathbb{A}^1}(k)$. The abelianization functor induces a Hurewicz morphism $\pi_0^s(\Sigma^\infty \mathscr{X}_+) \to \mathbf{H}_0^{\mathbb{A}^1}(\mathscr{X})$. The following result is a consequence of the stable $\mathbb{A}^1$-connectivity theorem [Mor05, Theorem 6.1.8].

**Proposition 2.1.** *If $\mathscr{X}$ is a space, the canonical morphism $\pi_0^s(\Sigma^\infty \mathscr{X}_+) \to \mathbf{H}_0^{\mathbb{A}^1}(\mathscr{X})$ is an isomorphism of strictly $\mathbb{A}^1$-invariant sheaves.*

The next result follows immediately from Proposition 2.1 and [Aso10, Theorem 2.2.9].

**Corollary 2.2.** *If $k$ is an infinite field, the sheaf $\boldsymbol{\pi}_0^s(X_+)$ is a birational invariant of smooth and proper $k$-varieties.*

Because of this proposition, we can (and will) replace the 0-th stable $\mathbb{A}^1$-homotopy sheaf by the zeroth $\mathbb{A}^1$-homology sheaf of a space in the sequel.

## 2.1 Strict $\mathbb{A}^1$-invariance and birationality

**Definition 2.3.** Suppose $\mathcal{F}$ is a presheaf of sets on $\mathcal{S}m_k$. We say $\mathcal{F}$ is *birational* if for any open dense immersion $U \to U'$ in $\mathcal{S}m_k$, the map $\mathcal{F}(U') \to \mathcal{F}(U)$ is an isomorphism. We say $\mathcal{F}$ is $\mathbb{A}^1$*-invariant*, if for any smooth scheme $U$ the map $\mathcal{F}(U) \to \mathcal{F}(U \times \mathbb{A}^1)$ is a bijection.

**Lemma 2.4.** *Suppose $\mathcal{F}$ is a birational and $\mathbb{A}^1$-invariant presheaf. The free abelian group $\mathbb{Z}(\mathcal{F})$ is also birational, both $\mathcal{F}$ and $\mathbb{Z}(\mathcal{F})$ are sheaves, $\mathbb{Z}(\mathcal{F})$ is Nisnevich flasque, and $\mathbb{Z}(\mathcal{F})$ is strictly $\mathbb{A}^1$-invariant.*

*Proof.* To check that $\mathcal{F}$ is a Nisnevich sheaf, we just have to check that $\mathcal{F}$ takes an elementary distinguished square

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow \pi \\ U & \longrightarrow & X, \end{array}$$

(where $\pi$ is étale, and the map $\pi^{-1}(X \backslash U) \to X \backslash U$ is an isomorphism) to a cartesian square. Since $\mathcal{F}$ is birational, both the bottom and top maps are isomorphisms. By definition $\mathbb{Z}(\mathcal{F})$ is also birational, and hence $\mathbb{Z}(\mathcal{F})$ is also a Nisnevich sheaf.

Since $\mathcal{F}$ is $\mathbb{A}^1$-invariant, it follows immediately that $\mathbb{Z}(\mathcal{F})$ is also $\mathbb{A}^1$-invariant. We will now show that $\mathbb{Z}(\mathcal{F})$ is flasque. To see this, recall that the Nisnevich cohomology can be computed by means of Čech cochains. Therefore, suppose $X$ is an irreducible smooth scheme, $u : U \to X$ is a Nisnevich cover of $X$. By lifting the generic point $\eta$ of $X$, we can find a component of $U$ that is birational to $X$. Since each map $U^{\times n+1} \to U^{\times n}$ is also a Nisnevich cover, it follows that $\mathbb{Z}(\mathcal{F})(U^{\times n}) \to \mathbb{Z}(\mathcal{F})(U^{\times n+1})$ is injective and thus all higher Nisnevich cohomology of $\mathbb{Z}(\mathcal{F})$ vanishes. □

**Corollary 2.5.** *If $\mathcal{F}$ is a birational and $\mathbb{A}^1$-invariant sheaf of sets, the canonical map $\mathcal{F} \to \mathbb{Z}[\mathcal{F}]$ induces an isomorphism $\mathbf{H}_0^{\mathbb{A}^1}(\mathcal{F}) \to \mathbb{Z}[\mathcal{F}]$.*

*Proof.* By definition $\mathbf{H}_0^{\mathbb{A}^1}(\mathcal{F}) = H_0(L_{\mathbb{A}^1}^{ab}\mathbb{Z}[\mathcal{F}])$. However, since $\mathbb{Z}[\mathcal{F}]$ is Nisnevich flasque, it follows that $\mathbb{Z}[\mathcal{F}]$ is $\mathbb{A}^1$-local, i.e., $L_{\mathbb{A}^1}^{ab}(\mathbb{Z}[\mathcal{F}]) = \mathbb{Z}[\mathcal{F}]$. □

*Example* 2.6. Suppose $X$ is an $\mathbb{A}^1$-rigid smooth proper $k$-scheme. Given an open dense immersion $U \to U'$, the map $X(U') \to X(U)$ is an isomorphism; indeed any such map is uniquely determined by where it sends the generic point of each component. As a consequence $\mathbb{Z}(X)$ is a strictly $\mathbb{A}^1$-invariant sheaf. Because $\mathbb{Z}(X)$ is $\mathbb{A}^1$-local, we see that $C_*^{\mathbb{A}^1}(X) = \mathbb{Z}(X)$, and thus that $\mathbf{H}_0^{\mathbb{A}^1}(X) = \mathbb{Z}(X)$. Thus, $X$, $\mathbb{Z}(X)$, and $\mathbf{H}_0^{\mathbb{A}^1}(X)$ are all birational sheaves. By Lemma 2.4 they are all strictly $\mathbb{A}^1$-invariant. As a consequence, we deduce that if $k$ is infinite and $X'$ is any smooth proper variety that is stably $k$-birationally equivalent to a smooth proper $\mathbb{A}^1$-rigid variety $X$, then $\mathbf{H}_0^{\mathbb{A}^1}(X') = \mathbf{H}_0^{\mathbb{A}^1}(X)$.

## 2.2 Birational connected components and the main result

Suppose $X$ is a smooth proper variety over a field $k$. If $L/k$ is a separable, finitely generated extension, recall that two $L$-points in $X$ are $R$-equivalent if they can be connected by the images of a chain of morphisms from $\mathbb{P}^1_L$ [Man86]. There is a birational sheaf related to $R$-equivalence classes of points in $X$.

**Theorem 2.7.** *If $X$ is a smooth proper $k$-variety, there is a birational and $\mathbb{A}^1$-invariant sheaf $\pi_0^{b\mathbb{A}^1}(X)$ together with a canonical map $X \to \pi_0^{b\mathbb{A}^1}(X)$ functorial for morphisms of proper varieties such that for any separable finitely generated extension $L/k$ the induced map $X(L) \to \pi_0^{b\mathbb{A}^1}(X)$ factors through an isomorphism $X(L)/R \to \pi_0^{b\mathbb{A}^1}(X)(L)$.*

*Proof.* Everything except the statement of functoriality is included in [AM09, Theorem 6.2.1]. Since $\pi_0^{b\mathbb{A}^1}(X)$ is a birational and $\mathbb{A}^1$-invariant sheaf, to construct a morphism $\pi_0^{b\mathbb{A}^1}(Y) \to \pi_0^{b\mathbb{A}^1}(X)$, it suffices to observe that by the definition of $R$-equivalence a morphism $f : Y \to X$ induces morphisms $X(L)/R \to Y(L)/R$ for every finitely generated separable extension $L/k$. □

If $X$ is a smooth proper variety, we can consider the sheaf $\mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X))$. By Lemma 2.4, it follows that $\mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X))$ is a strictly $\mathbb{A}^1$-invariant sheaf, and Corollary 2.5 gives rise to a canonical identification $\mathbf{H}_0^{\mathbb{A}^1}(\pi_0^{b\mathbb{A}^1}(X)) \xrightarrow{\sim} \mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X))$. As a consequence of Theorem 2.7 we deduce the existence of a canonical morphism

$$\psi_X : \mathbf{H}_0^{\mathbb{A}^1}(X) \to \mathbb{Z}[\pi_0^{b\mathbb{A}^1}(X)].$$

Because $\mathbb{Z}[\pi_0^{b\mathbb{A}^1}(X)]$ is a strictly $\mathbb{A}^1$-invariant sheaf, existence of this morphism also follows immediately from [Aso10, Lemma 2.2.3].

*Remark* 2.8. It seems reasonable to expect that the morphism $\psi_X : \mathbf{H}_0^{\mathbb{A}^1}(X) \to \mathbb{Z}[\pi_0^{b\mathbb{A}^1}(X)]$ is an isomorphism. Since our goal is to get as quickly as possible to the connection with rational points we did not pursue this further.

**Corollary 2.9.** *If $X$ is a smooth proper $k$-variety, then the set $X(k)$ is non-empty if and only if the map $\mathbf{H}_0^{\mathbb{A}^1}(X) \to \mathbb{Z}$ induced by the structure map is a split surjection.*

*Proof.* If $X(k)$ is non-empty, then we get a morphism $\mathbb{Z} = \mathbf{H}_0^{\mathbb{A}^1}(\operatorname{Spec} k) \to \mathbf{H}_0^{\mathbb{A}^1}(X)$ that splits the map induced by the structure morphism. Conversely, note that the map $\mathbf{H}_0^{\mathbb{A}^1}(X) \to \mathbb{Z}[\pi_0^{b\mathbb{A}^1}(X)]$ is functorial in $X$, and thus the morphism $\mathbf{H}_0^{\mathbb{A}^1}(X) \to \mathbb{Z}$ factors through the morphism $\psi_X$. A splitting $\mathbb{Z} \to \mathbf{H}_0^{\mathbb{A}^1}(X)$ therefore gives rise to a non-trivial morphism $\mathbb{Z} \to \mathbb{Z}[\pi_0^{b\mathbb{A}^1}(X)]$, i.e., an element of $\mathbb{Z}[\pi_0^{b\mathbb{A}^1}(X)](k)$. The group $\mathbb{Z}[\pi_0^{b\mathbb{A}^1}(X)](k)$ is by definition the free abelian group on the set $X(k)/R$ (see Theorem 2.7). Since the group $\mathbb{Z}[\pi_0^{b\mathbb{A}^1}(X)](k)$ is a non-trivial free abelian group, we deduce that $X(k)/R$ has at least 1 element, and therefore $X(k)$ is non-empty. □

*Proof of Theorem 1.* Combine Corollary 2.9 and Proposition 2.1. □

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