RIGIDITY OF GRAPH JOINS AND HENDRICKSON'S CONJECTURE

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Abstract. Whiteley [9] gives a complete characterization of the infinitesimal flexes of complete bipartite frameworks. Our work generalizes a specific infinitesimal flex to include joined graphs, a family of graphs that contain the complete bipartite graphs. We use this characterization to identify new families of counterexamples, including infinite families, in $\mathbb{R}^5$ and above to Hendrickson's conjecture on generic global rigidity.

1. Introduction

A d-dimensional framework of a graph is a mapping from the vertices of the graph to points in Euclidean d-space. A natural question to ask is whether a graph is *locally rigid*, i.e. can we can move the vertices of the framework while preserving distances between adjacent vertices in the graph? Furthermore, when a framework is locally rigid, another question to ask is whether the graph is *globally rigid*, i.e. do the edge lengths uniquely define a framework up to Euclidean motions?

Hendrickson [8] found two necessary conditions for a graph to be generically globally rigid and conjectured that they were also sufficient. Connelly [4] discovered a family of complete bipartite graphs in $\mathbb{R}^3$ and higher that satisfied Hendrickson's two conditions but were not generically globally rigid:

Theorem 1.1 (Connelly [4])**.** *If $a, b \geq d+2$ and $a+b = \binom{d+2}{2}$, then $K_{a,b}$ is generically almost-globally rigid in $\mathbb{R}^d$.*

Work has been done on identifying counterexamples that are subgraphs of Connelly's family in [6]. Our work extends Theorem 1.1 in the opposite direction, exhibiting a family of counterexamples that have Connelly's graphs as subgraphs. Connelly and Whiteley [5] showed that an operation known as coning preserves local and global rigidity. In particular, coning can be used to construct new counterexamples in higher dimensions from known counterexamples. We identify a family of counterexamples that are subgraphs of coned graphs.

Frank and Jiang [6] found a graph in $\mathbb{R}^5$ that could be "attached" to graphs that are sufficiently rigid to form an infinite number of counterexamples to Hendrickson's conjecture. However, one step of the proof was aided by a computer program, so the result could not be generalized using solely their methods. We follow the same direction of their argument, giving a conceptual proof of generic local rigidity for the graph attachment in order to exhibit graph attachments in higher dimensions.

In this paper, we introduce the notion of the *quadric rigidity matrix*, which generalizes one of Whiteley's [9] conditions for infinitesimal rigidity. We use the quadric rigidity matrix to characterize all infinitesimal flexes of joined graphs and for the construction of the aforementioned families of graphs.

Thanks to Dylan Thurston, Andrew Fanoe, and Kiril Ratmanski for their support and suggestions throughout this project. This paper was partially supported by NSF RTG Grant DMS 07-39392.

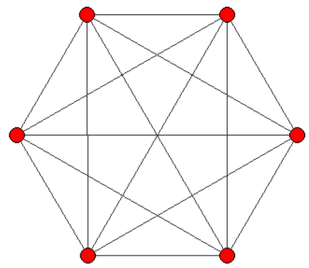

FIGURE 1. A complete graph.

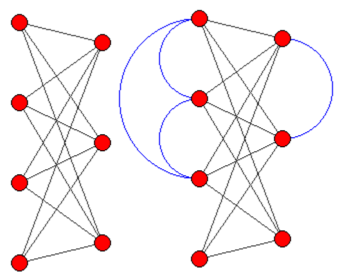

FIGURE 2. The complete bipartite graph $K_{4,3}$ and the joined graph $(K_3 \cup K_1) + (K_2 \cup K_1)$, respectively.

2. Graph Theory Preliminaries

A *graph* $G = (V, E)$ is a 2-tuple consisting of a set $V = \{v_1, v_2, ..., v_{|V|}\}$ of vertices and a set $E \subseteq V \times V$ of edges between the vertices. In this paper, all graphs are assumed to be *simple*: the set of edges has no selfloops, multiple edges between the same two vertices, or any directed edges. We denote an edge connecting vertices v_i and v_j as v_iv_j. Vertices v_i and v_j are *adjacent* if $v_iv_j \in E$. A *subgraph* $G' = (V', E')$ of a graph $G = (V, E)$ is a graph with vertex set $V' \subseteq V$ and edge set $E' \subseteq (V' \times V') \cap E$. That is, a subgraph cannot contain an edge that connects vertices outside of the vertex subset or an edge not in the original graph. A subgraph is called a *factor* if it has the same vertex set. An *induced subgraph* is a subgraph whose edge set is precisely $(V' \times V') \cap E$. The *edge complement* of a graph $G = (V, E)$, denoted $\overline{G} = (V, E')$, where $uv \in E' \Leftrightarrow uv \notin E$.

A graph is connected if, for all pairs of vertices v_i and v_j, there exists a path of vertices starting from v_i and ending at v_j. A graph is *k-(vertex)-connected* if all induced subgraphs with $v - k + 1$ vertices are connected. Equivalently, a graph is k-connected if deleting any subset of $k - 1$ vertices (and edges incident on those vertices) leaves the resulting graph connected.

The *disjoint union* of two graphs G and H, denoted $G \cup H$, is the graph formed by the disjoint union of the vertex sets and edge sets. The *graph join* of graphs G and H, denoted $G + H$, is the graph whose vertex set is the disjoint union $V_G \cup V_H$ and whose edge set is the disjoint union $E_G \cup E_H \cup \{v_gv_h | v_g \in V_G, v_h \in V_H\}$. We call such a graph a *joined graph*, and any edge in $E_G \cup E_H$ is *extraneous*. We will call a joined graph $G + H$ *balanced* if $|V_G|, |V_H| \geq d + 1$.

The *complete graph* on i vertices, denoted K_i, is defined recursively, where K_1 is a single vertex, and $K_i = K_1 + K_{i-1}$. In other words, a complete graph is a graph in which all vertices are connected to one another. Figure 1 is the graph K_6. The *complete bipartite graph* on a and b vertices, denoted $K_{a,b}$, is $\overline{K_a} + \overline{K_b}$. We can think of a complete bipartite graph as taking two sets of vertices and

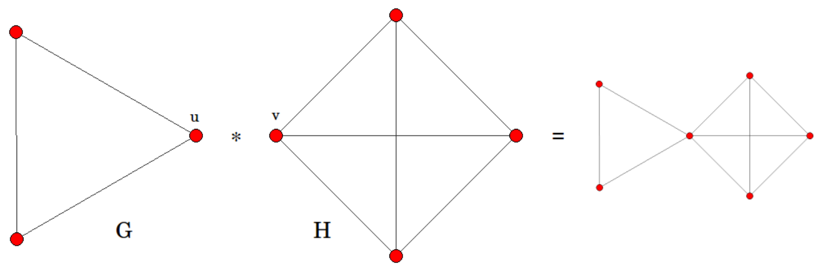

FIGURE 3. The graph amalgamation $(G; u) * (H; v)$.

adding an edge for each possible pair of vertices. $V_{\overline{K_a}}$ and $V_{\overline{K_b}}$ are referred to as the two *bipartite classes*. Figure 2 contains examples of a complete bipartite graph and a joined graph.

A *vertex amalgamation* $(G; u_1, u_2, ..., u_i) * (H; v_1, v_2, ..., v_i)$ is the graph $(G \cup H)/R$, where R is the equivalence relation $\{u_1 = v_1, u_2 = v_2, ..., u_i = v_i\}$. Intuitively, a vertex amalgamation takes vertices of two graphs and pastes them together to get the resulting graph, as in Figure 3.

Two graphs G and H are *isomorphic* if there exists a bijective function $\phi : V_G \to V_H$ such that $uv \in E_G$ if and only if $\phi(u)\phi(v) \in E_H$.

3. FRAMEWORKS

A *d-dimensional framework* is a 2-tuple (G, p) where G is a graph and p is a mapping, known as a *configuration*, that takes elements of V_G to $\mathbb{R}^d$. We can imagine each vertex in G placed into d-dimensional Euclidean space connected by straight rods that represent each edge of G. Two frameworks $(G, p) = (G; p_1, p_2, ..., p_v)$ and $(G, q) = (G; q_1, q_2, ..., q_v)$ are *equivalent* if for all pairs of adjacent vertices v_i and v_j, $||p_i - p_j|| = ||q_i - q_j||$. They are *congruent* if all pairwise distances between points are equal. A *generic configuration* is a mapping in which the coordinates of the vertices are algebraically independent over $\mathbb{Z}$; that is, no non-trivial polynomial with integer coeffients over the coordinates is 0. A *generic framework* is a framework whose configuration is generic.

A framework (G, p) is said to be *globally rigid* if any equivalent framework (G, q) is also congruent. Alternatively, any equivalent configuration can be reached by some Euclidean motion. A framework (G, p) is said to be *locally flexible* if there exists a parametric curve in the space of configurations $C : [0, 1] \to \mathbb{R}^{vd}$ such that $C(0) = p$, $(G, C(1))$ is not congruent to (G, p), and for all $x \in [0, 1]$, $(G, C(x))$ is equivalent to (G, p). Otherwise, it is said to be *locally rigid*. From these definitions, we see that K_n is locally and globally rigid for any framework.

A graph is *generically locally rigid* (GLR) if any generic framework is locally rigid. Similarly, a graph is *generically globally rigid* (GGR) if any generic framework is globally rigid. A graph is *generically redundantly rigid* (GRR) if deleting any edge from the graph leaves a GLR graph.

Proposition 3.1. *If $G = (V, E)$ is a graph that is not generically globally rigid, then any factor G' is also not generically globally rigid.*

Proof. Suppose we had two equivalent, non-congruent frameworks (G, p) and (G, q) for generic p and q. Then (G', p) and (G', q) are equivalent, non-congruent frameworks. $\square$

The following theorems demonstrate that GLR and GGR are properties of the underlying graph, and not the framework.

Theorem 3.2. *If any generic framework of a graph G is locally rigid, then all generic frameworks of G are locally rigid.*

The above result is a corollary of Theorem 4.2.

Theorem 3.3 (Connelly [3], Gortler, Healy, Thurston [7])**.** *If any generic framework of a graph G is globally rigid, then all generic frameworks of G are globally rigid.*

For non-generic frameworks, we have problems like all points lying on a hyperplane that might yield unexpected rigidity properties. Thus, we consider only generic configurations because we can give characterizations of rigidity based on the underlying graph alone. An example of such a characterization comes from Hendrickson [8], who found necessary conditions for a graph to be GGR and conjectured that they were also sufficient.

Theorem 3.4 (Hendrickson [8])**.** *If a non-complete graph G is generically globally rigid in $\mathbb{R}^d$, then it is $(d+1)$-connected and generically redundantly rigid.*

Intuitively, a graph that is GGR requires $(d+1)$-connectivity because if we could disconnect the graph into two components by deleting d vertices, we would be able to reflect one component across the hyperplane defined by the coordinates of those d vertices and obtain an equivalent, but not congruent framework. A graph that is GGR requires redundant rigidity because otherwise, we can delete some non-redundant edge, flex the graph, and replace the edge with the same length to get a non-congruent framework. There exist frameworks in which this is not possible, but Hendrickson argues that they are not generic since they lie on critical points of a manifold.

Connelly [4] and Frank and Jiang [6] found families of counterexamples to Hendrickson's conjecture. Such a counterexample is said to be *generically almost-globally rigid*[1]. We will generalize the results of Connelly [4] and Frank and Jiang [6] in the remainder of this paper.

4. Infinitesimal Flexes and Equilibrium Stresses

Let $f_G : R^{vd} \to R^e$ be the mapping where we take the coordinates of the configuration and output the edge-length squared of each edge. That is, $f_G(p = (p_1, p_2, ..., p_v)) = (..., ||p_i - p_j||^2, ...)$. The *rigidity matrix* of a framework is the Jacobian $df_G(p)$ and has dimensions $e \times vd$. For example, the rigidity matrix for the graph K_3 with coordinates $p_1 = (0,2), p_2 = (2,-2), p_3 = (1,3)$ could be written as

$$2 * \begin{array}{c} \\ v_1v_2 \\ v_1v_3 \\ v_2v_3 \end{array} \begin{array}{c} \begin{array}{cccccc} p_{1,x} & p_{1,y} & p_{2,x} & p_{2,y} & p_{3,x} & p_{3,y} \end{array} \\ \begin{pmatrix} -2 & 4 & 2 & -4 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -5 & -1 & 5 \end{pmatrix} \end{array}.$$

An *infinitesimal motion* is an element of the kernel of $df_G(p)$. Equivalently, an infinitesimal motion $p' = (p'_1, p'_2, ..., p'_v)$ satisfies, for any edge v_iv_j, $(p_i - p_j) \cdot (p'_i - p'_j) = 0$. Infinitesimal motions generalize the notion of Euclidean motions and local flexes. To see this, consider the time derivative of f_G. Since Euclidean motions and local flexes preserve edge lengths, we wish to have that derivative be equal to 0. More explicitly, for any edge v_iv_j,

[1]Frank and Jiang [6] refer to these graphs as *generically partially rigid.*

$$\begin{aligned}\frac{d}{dt}(p_i - p_j)^2 &= \frac{d}{dt}(p_i - p_j) \cdot (p_i - p_j) \\ &= 2(p_i - p_j) \cdot \frac{d}{dt}(p_i - p_j) \\ &= 2[(p_i - p_j) \cdot (p_i' - p_j')],\end{aligned}$$

which is zero if it is an infinitesimal motion. Any infinitesimal motion that is not a Euclidean motion is an *infinitesimal flex.* A graph with no infinitesimal flex is *infinitesimally rigid.* Using the equivalence of the two definitions of an infinitesimal flex, Asimow and Roth [1] proved the following theorems that demonstrate the connection between the local rigidity and the rigidity matrix.

Theorem 4.1 (Asimow and Roth [1])**.** *A framework (G,p) is infinitesimally rigid if and only if* $\operatorname{rank} df_G(p) = vd - \binom{d+1}{2} \Leftrightarrow \dim\ker df_G(p) = \binom{d+1}{2}$.

Since the Euclidean motions are infinitesimal motions and have dimension $\binom{d+1}{2}$, $\binom{d+1}{2}$ is the smallest possible dimension for the kernel, which is the best possible.

Theorem 4.2 (Asimow and Roth [1])**.** *A graph with at least $d+1$ vertices is generically locally rigid if and only if a generic framework of it is infinitesimally rigid.*

Proposition 4.3. *If $G = (V,E)$ is a graph that is generically locally rigid, then adding an edge yields a generically locally rigid graph.*

Proof. Adding a row to the rigidity matrix cannot decrease the dimension of the image, but since the image already has dimension $vd - \binom{d+1}{2}$, the resulting graph's image also has dimension $vd - \binom{d+1}{2}$. □

Proposition 4.4. *Given a graph G that is generically locally rigid in $\mathbb{R}^d$, adding a vertex v to G and at least d edges connected to that vertex yields a generically locally rigid graph.*

Proof. We only need to consider the case where we add d edges, since adding more follows from Proposition 4.3. Consider the rigidity matrix of G. Adding v increases both the column and row size by d. The resulting matrix is block triangular, so consider the submatrix formed by the newly added rows and columns. For generic configurations, the determinant of the submatrix forms an algebraic equation and hence must be non-zero. Thus, the submatrix is of maximal rank and the resulting graph is also GLR. □

An *(equilibrium) stress* is a vector $\omega = (..., \omega_{ij}, ...) \in \mathbb{R}^e$ such that for all vertices $v_i \in V$,

$$\sum_{j | v_i v_j \in E} \omega_{ij}(p_i - p_j) = 0.$$

By multiplying out $(df_G)^T\omega$, we find that this definition is equivalent to saying that $\omega \in \ker(df_G)^T$. We denote the vector space of stresses as $\Omega(G,p)$.

From these definitions, we find that $\dim\ker df_G(p) = vd - e + \dim\Omega(G,p)$. This yields a crucial characterization of redundant edges.

Proposition 4.5 (Frank and Jiang [6])**.** *Removing an edge e of a generically locally rigid graph G preserves local rigidity if and only if for any $r \in \mathbb{R}$, there exists a stress with value r on e.*

Proof. Since both G and $G-\{e\}$ are GLR, the space of flexes is $\binom{d+1}{2}$, so adding edge e to $G-\{e\}$ increases $\dim\Omega(G,p)$. Thus, a stress with non-zero value on e must exist. Conversely, deleting an edge with non-zero stress decreases the space of stresses by at least 1 because scaling that stress creates a one-dimensional subspace, so $\binom{d+1}{2} = \dim\ker df_G(p) \geq \dim\ker df_{G-\{e\}}(p)$. Because the Euclidean motions are infinitesimal flexes of all frameworks and has dimension $\binom{d+1}{2}$, we have equality. □

If we restrict ourselves to only balanced complete bipartite graphs, we obtain a tidy characterization of the stresses and flexes. The following theorem is a formula for the dimension of the stresses.

Theorem 4.6 (Bolker and Roth [2])**.** *Given some balanced complete bipartite graph $K_{a,b}$ where $a+b \leq \binom{d+2}{2}$, the space of stresses $\dim\Omega(K_{a,b})$ for a generic configuration is $(a-d-1)(b-d-1)$.*

Corollary 4.7. *If $K_{a,b}$ is generically locally rigid and $a,b \geq d+2$, then $K_{a,b}$ is generically redundantly rigid.*

Proof. Since $\dim\Omega(K_{a,b}) > 0$, there must exist a stress which is non-zero on some edge. That edge is then redundant by Proposition 4.5, so by symmetry, all the edges of $K_{a,b}$ are redundant. □

Whiteley [9] explicitly describes the infinitesimal flexes that arise from the stresses of a complete bipartite framework. When $v < \binom{d+2}{2}$, there exists at least one *quadric surface* that passes through all v points. A quadric surface is a $(d-1)$-dimensional surface in $\mathbb{R}^d$ whose space is the locus of zeroes of some quadratic polynomial in d variables. That is, a quadric surface can be viewed as the set of all points $p = (p_1, p_2, ..., p_d)$ that satisfy the equation

$$\sum_{i=1}^{d} A_i p_i^2 + \sum_{1\leq j<k\leq d} 2B_{j,k}(p_j p_k) + \sum_{l=1}^{d} C_l p_l + D = 0$$

for some real coefficients A_i, $B_{j,k}$, C_l, D not all zero. A quadric surface can also be defined as the set $\{p \in \mathbb{R}^d \mid (p,1)^T Q(p,1) = 0\}$ for some symmetric $(d+1)\times(d+1)$ matrix. To see that this definition is equivalent to the polynomial equation counterpart, we let $A_i := [Q]_{i,i}$, $B_{j,k} := 2[Q]_{j,k}$, $C_l := 2[Q]_{d+1,l}$, $D := [Q]_{d+1,d+1}$. Expanding out $(p,1)^T Q(p,1)$ yields the polynomial equation definition.

Let the *quadric flex* be the flex Qx_i for all vertices x_i of one bipartite class, and $-Qy_i$ for all vertices y_i of the other bipartite class. Intuitively, this flex pushes one bipartite class into the quadric surface and the other class outwards from the surface, as seen in Figure 4. To see that this is in fact an infinitesimal flex, for any pair of adjacent vertices v_a, v_b,

$$\begin{aligned}(p_a - p_b)\cdot(Qp_a - (-Qp_b)) &= p_a\cdot Qp_a + p_b\cdot Qp_b + p_a\cdot Qp_b - p_b\cdot Qp_a \\ &= p_a\cdot Qp_b - p_b^T Qp_a,\end{aligned}$$

but since Q is symmetric, $p_a\cdot Qp_b = p_a^T Qp_b = p_b^T Qp_a = p_b\cdot Qp_a$, so $(p_a-p_b)\cdot(Qp_a-(-Qp_b)) = 0$.

Proposition 4.8 (Whiteley [9])**.** *For generic d-dimensional frameworks of balanced complete bipartite graphs with fewer than $\binom{d+2}{2}$ vertices, the quadric flexes spans the space of all infinitesimal flexes modulo Euclidean motions.*

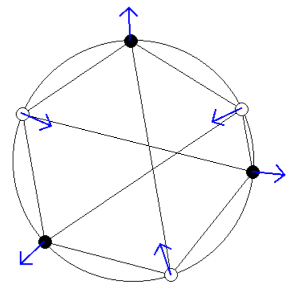

FIGURE 4. $K_{3,3}$ on a circle and its corresponding quadric flex. The bipartite classes are shown in different colors.

Sketch of Proof. The space of flexes is

$$\begin{aligned} vd - e + \dim \Omega(G,p) &= (a+b)d - ab + (a-d-1)(b-d-1) \\ &= -a - b + d^2 + 2d + 1 \\ &= \binom{d+1}{2} + \binom{d+2}{2} - a - b \end{aligned}$$

However, the dimension of the space of quadric surfaces passing through those points is $\binom{d+2}{2} - a - b$, so all the non-trivial infinitesimal flexes can be written as a linear combination of Euclidean motions and quadric flexes. □

From here on, we will only consider balanced joined graphs.

5. Quadric Rigidity Matrix

Specifying the coordinates of a single point p_i forces any quadric surface Q containing p_i to satisfy the linear constraint $p_i^T Q p_i = 0$. We find a similar constraint by adding extraneous edges to vertices within the same bipartite class.

Proposition 5.1. *Given a complete bipartite framework, if x_i and x_j are vertices in the same bipartite class, then adding the edge $x_i x_j$ imposes the linear constraint $p_i^T Q p_j = p_j^T Q p_i = 0$ on the space of quadric surfaces Q whose quadric flex preserves the length of $x_i x_j$.*

Proof. We wish to find a quadric surface Q such that the quadric flex satisfies the infinitesimal flex condition $(p_i - p_j) \cdot (p_i' - p_j') = 0$. Then

$$\begin{aligned} (p_i - p_j) \cdot (p_i' - p_j') &= (p_i - p_j) \cdot (Qp_i - Qp_j) \\ &= p_i \cdot Qp_i + p_j \cdot Qp_j - p_i \cdot Qp_j - p_j \cdot Qp_i \\ &= -p_i \cdot Qp_j - p_j \cdot Qp_i. \end{aligned}$$

However, since Q is symmetric, $p_i \cdot Qp_j = p_i^T Q p_j = p_j^T Q p_i = p_j \cdot Qp_i$, so $p_i^T Q p_j = p_j^T Q p_i = 0$. □

To see that both the constraints from vertices and edges are in fact linear, we look at the polynomial form for a quadric. Suppose we have a configuration that maps vertices x and y to p and q in $\mathbb{R}^d$, respectively. For the vertex constraint of the vertex x, we obtain

$$\sum_{i=1}^{d} A_i p_i^2 + \sum_{1\leq j<k\leq d} 2B_{j,k}(p_j p_k) + \sum_{l=1}^{d} 2C_l p_l + 1 = 0.$$

Similarly, the edge constraint of the edge xy yields

$$\sum_{i=1}^{d} A_i(p_i q_i) + \sum_{1\leq j<k\leq d} B_{j,k}(p_j q_k + p_k q_j) + \sum_{l=1}^{d} C_l(p_l + q_l) + 1 = 0,$$

where A_i, B_i, and C_i are variables representing the coefficients of the quadric polynomial. We define the *(d-dimensional) constraint mapping* $m : \mathbb{R}^{2d} \to \mathbb{R}^{\binom{d+2}{2}}$ where (p, q) is mapped to

$$\begin{aligned}(p_1q_1, p_2q_2, \ldots, p_dq_d,\\ p_1q_2 + p_2q_1, p_1q_3 + p_3q_1, \ldots, p_{d-1}q_d + p_dq_{d-1},\\ p_1 + q_1, p_2 + q_2, \ldots, p_d + q_d, 1).\end{aligned}$$

These are the coefficients of the A_i, B_i, and C_i variables in the edge constraint. For a single point, the constraint mapping is $m(p, p)$. Since the vertex and edge constraints form a system of linear equations, it is natural to define the quadric rigidity matrix.

Definition 5.2. Let $G + H$ be a joined graph with vertex set V and extraneous edge set E'. The *quadric rigidity matrix* of $G + H$ is the $(|V| + |E'|) \times (\binom{d+2}{2})$ matrix whose rows are the constraint mappings $m(v, v)$ for all $v \in V$ and $m(v_i, v_j)$ for all $v_iv_j \in E'$.

Since the quadric flex automatically preserves non-extraneous edge lengths, those edges do not impose any constraint on the quadric rigidity matrix. Suppose we have the joined graph $(K_2 \cup K_1) + \overline{K_3}$ with configuration $p_1 = (4, -5), p_2 = (2, 4), p_3 = (-1, 3), p_4 = (-4, -1), p_5 = (-9, 0), p_6 = (5, 7)$ such that the extraneous edge connects v_1 and v_2. We can write the quadric rigidity matrix of this joined framework as

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_1v_2 \end{array} \begin{array}{c} \begin{array}{cccccc} x^2 & y^2 & xy & x & y & 1 \end{array} \\ \begin{pmatrix} 16 & 25 & -40 & 8 & -10 & 1 \\ 4 & 16 & 16 & 4 & 8 & 1 \\ 1 & 9 & -6 & -2 & 6 & 1 \\ 16 & 1 & 8 & -8 & -2 & 1 \\ 81 & 0 & 0 & -18 & 0 & 1 \\ 25 & 49 & 24 & 10 & 14 & 1 \\ 8 & -20 & 6 & 6 & -1 & 1 \end{pmatrix} \end{array}$$

The following results are crucial for the main results of this paper.

Proposition 5.3. *The quadric rigidity matrix of a joined framework (G, p) has rank $\binom{d+2}{2}$ if and only if (G, p) is infinitesimally rigid.*

Proof. The rank is less than $\binom{d+2}{2}$ if and only if a quadric surface satisfying those constraints exists, which is equivalent to infinitesimal flexibility by Proposition 4.8. $\square$

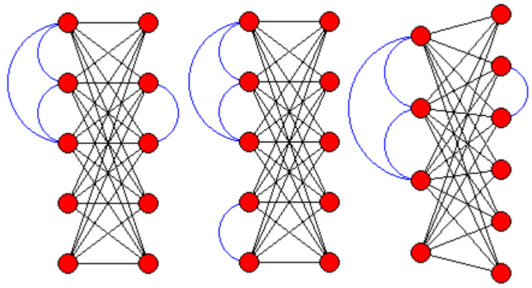

FIGURE 5. Since the quadric rigidity matrix is the same for all three graphs, they are either all rigid, or all flexible in $\mathbb{R}^3$.

Proposition 5.4. *Let $G_1 = G + H$ and $G_2 = G' + H'$ be two balanced joined graphs where $G \cup H$ is isomorphic to $G' \cup H'$. That is, the resulting graphs from deleting all non-extraneous edges are isomorphic. Then $G+H$ is generically locally rigid if and only if $G'+H'$ is generically locally rigid.*

Proof. G_1 and G_2 have the same quadric rigidity matrix, since the matrix is only dependent on the extraneous edges and the vertices, and not on the edges between G and H. □

Figure 5 shows three graphs that satisfy the conditions in Proposition 5.4. In particular, all three graphs are generically locally rigid because ten generic points do not lie on a quadric surface in $\mathbb{R}^3$.

The quadric rigidity matrix presents a faster method of deciding local rigidity for balanced joined graphs. The dimensions of the quadric rigidity matrix is strictly smaller than that of the rigidity matrix. Let e' be the number of extraneous edges. Then, the number of rows in the quadric rigidity matrix is $|V_G| + |V_H| + e'$, which is less than $|V_G||V_H| + e'$, the number of rows in the rigidity matrix. The number of columns in the quadric rigidity matrix is $\binom{d+2}{2} < 2(d+1)d \leq vd$, so the quadric rigidity matrix is overall smaller. If we fix the dimension parameter, the complexity of determining local rigidity is reduced from $O(v^2 e)$ to $O(v + e')$.

Recognizing a balanced joined graph takes exponential time by the naive algorithm of checking all balanced partitions of the vertices. However, there is an $O(|V|^2)$ algorithm. The complete bipartite graph $K_{a,b}$has edge complement is the graph $K_a \cup K_b$, which has two connected components. A *connected component* is an equivalence class of the relation "u is connected to v." For a balanced joined graph, its edge complement has at least two connected components, which can be partitioned into two sets of size at least $d+1$. Our algorithm uses dynamic programming and runs as follows for a graph G:

(1) If $|V_G| < 2d+2$, reject.
(2) Compute the edge-complement of G.
(3) Using a depth-first search, find the vertex-sets $V_1, V_2 ..., V_n$ of the connected components of $\overline{G}$.
(4) Initialize a string array A indexed from 1 to $|V|$.
(5) For each vertex-set V_i, do the following. Set A$[|V_i|]$:= "V_i". For each j such that A$[j]$ is nonempty, set A$[|V_i| + j]$:= A$[j]$ + "V_i".
(6) If there is an i such that $d+1 \leq i \leq |V_G| - (d+1)$ and A$[i]$ is not an empty string, return A$[i]$.

Each step takes at most $O(|V|^2)$ time, so the overall algorithm runs in $O(|V|^2)$ time. At the end of step 5, A$[i]$ is nonempty if and only if there exists a partition of the connected components such

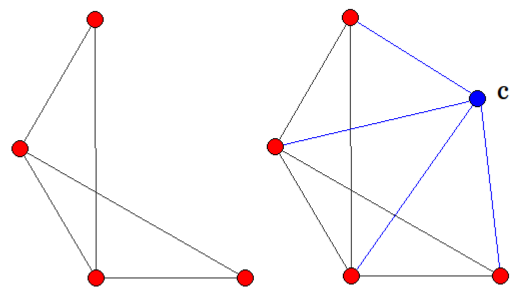

Figure 6. A graph and its coning, respectively.

that the size of one subset is i. Step 6 ensures that i is chosen such that the size of both subsets is at least $d+1$. Note that if we did not require the joined graph to be balanced, we would only need to test connectivity on $\overline{G}$.

6. Partial Coning of Connelly's Graphs

Connelly [4] provided the first known counterexamples to Hendrickson's conjecture by Theorem 1.1. Let the *coning* of a graph G be the graph $G' = G + K_1$. That is, we add a new vertex and connect it to every other vertex. Connelly and Whiteley [5] demonstrate that the coning operation preserves all the forms of rigidity.

Theorem 6.1 (Connelly and Whiteley [5])**.** *The cone of a graph G is generically [locally, redundantly, globally] rigid in $\mathbb{R}^d$ if and only if G is generically [locally, redundantly, globally] rigid in $\mathbb{R}^{d-1}$.*

Proposition 6.2. *The cone of a graph G is $(k+1)$-connected if G is k-connected.*

Proof. In the cone of G, there are two different ways to delete k vertices. If the cone vertex is deleted, then the result follows immediately from the k-connectivity of G. If the cone vertex is not deleted, then the resulting graph is still connected because the cone vertex is adjacent to all other vertices. □

In particular, the cone of a graph in Theorem 1.1 is also GAGR in the next-highest dimension. It turns out that for those graphs in $\mathbb{R}^5$ and above, only a *partial coning* is necessary. A partial coning is where the cone vertex is joined to only a subset $V' \subsetneq V_G$. We provide a specific type of partial coning that yields a family of GAGR graphs. A partial coning of $K_{9,6}$ as shown in Figure 7 is the smallest graph in this family.

Theorem 6.3. *If $a > b \geq d+1$ and $a+b = \binom{d+1}{2}+1$, then the partial coning $(K_{1,d+1} \cup \overline{K_{(a-d-2)}}) + \overline{K_b}$ is generically almost-globally rigid in $\mathbb{R}^d$ for $d > 3$.*

Proof. First, $(K_{1,d+1} \cup \overline{K_{(a-d-2)}}) + \overline{K_b}$ is in fact a partial coning of a GAGR complete bipartite graph. All extraneous edges are connected to the same vertex. Removal of that vertex leaves $K_{a-1,b}$. Since $a > b \geq d+1$, $a-1 \geq d+1$, so we have a GAGR graph in $\mathbb{R}^{d-1}$.

$(K_{1,d+1} \cup \overline{K_{(a-d-2)}}) + \overline{K_b}$ is not GGR, since it is a subgraph of a complete cone of a GAGR graph.

The only vertices we need to consider for $(d+1)$-connectivity are the vertices $u_1, u_2, \ldots, u_{a-d+2}$ not connected to the cone vertex c. Since $b \geq d+2$, we cannot delete all the vertices of the second

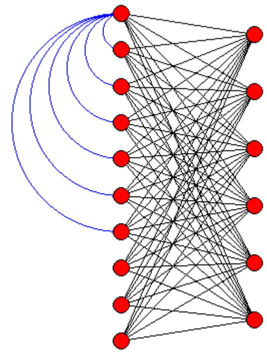

FIGURE 7. A partial coning reimagined as a joined graph. This graph is GAGR in $\mathbb{R}^5$ by Theorem 6.3.

bipartite class, leaving at least one vertex v intact. Then the path $u_i - v - c$ connects vertex u_i to the rest of the graph.

When $a = d + 2$, we have the coned graph of a GLR graph, which is itself GLR. By Proposition 5.4, $(K_{1,d+1} \cup \overline{K_{(a-d-2)}}) + \overline{K_b}$ is GLR since it has the same quadric rigidity matrix. Since there is a linear dependency in the quadric rigidity matrix created by adding the $(d+1)$-th edge, that edge is redundant. By symmetry, all the extraneous edges are redundant. By Corollary 4.7, the bipartite edges are redundant because $a, b \geq d + 2$. Therefore, the entire graph is GRR. $\square$

This idea can be generalized for multiple partial cones of the same graph. However, this requires more technical methods that we will encounter in the next section. We conclude this section with an extension that covers weaker partial conings and other classes of GAGR graphs.

Proposition 6.4. *Let G and G' be generically almost-globally rigid graphs in $\mathbb{R}^d$ such that G' is a factor of G. Then any graph G'' such that $G' \subseteq G'' \subseteq G$ is also generically almost-globally rigid in $\mathbb{R}^d$.*

Proof. $G' \subset G''$ implies $(d + 1)$-connectivity and the other conditions follow immediately from Propositions 3.1 and 4.3. $\square$

7. Graph Attachments in $\mathbb{R}^n$

Frank and Jiang [6] found a graph that could be "attached" to other graphs in $\mathbb{R}^5$ to create GAGR graphs. We generalize the result to higher dimensions. For some $x_1, x_2, x_3, x_4 \in \mathbb{N}$, we can construct the graphs $G_i := \overline{K_{x_i}}$ for $i \in \{1, 2, 3, 4\}$. The *4-chain* C_{x_1,x_2,x_3,x_4} is the graph with vertex set $V = \bigcup_{i=1}^4 V_{G_i}$ and edge set $E = \bigcup_{i=1}^3 E_{G_i+G_{i+1}}$. Formally, we may think of a 4-chain as $G_1+G_2+G_3+G_4$. Frank and Jiang demonstrated that the 4-chain $C_{2,3,5,4}$ could be attached to certain graphs in $\mathbb{R}_5$ to yield GAGR graphs. Attaching a 4-chain to an arbitrary graph G, denoted $C_{w,x,y,z} \lhd G$, is the result of the vertex amalgamation

$$(C_{w,x,y,z}; w_1, w_2, ..., w_{x_1}, z_1, z_2, ..., z_{x_4}) * (G; v_1, v_2, ..., v_{x_1+x_4})$$

for some vertices $v_1, v_2, ..., v_{x_1+x_4}$ in V_G. The vertex amalgamation attaches the vertices of the two ends of the chain to some vertices in G. As demonstrated in the proof, the choice of vertices is irrelevant. We can now state the result of Frank and Jiang.

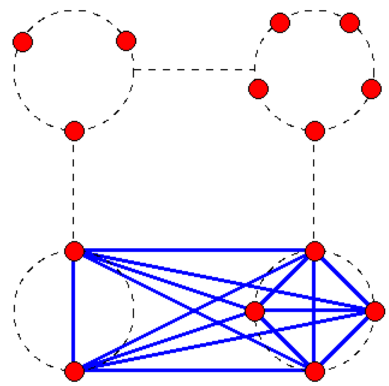

FIGURE 8. The graph attachment $C_{2,3,5,4} \lhd K_6$. A circle of vertices represents each independent set of the 4-chain, and a dashed line between independent sets represents a graph join. K_6 is highlighted by the bolded edges.

Theorem 7.1 (Frank and Jiang [6]). *Let G be a generically redundantly rigid, 6-connected graph. Then $C_{2,3,5,4} \lhd G$ is generically almost-globally rigid in $\mathbb{R}^5$.*

We present the following generalization for higher dimensions.

Theorem 7.2. *Let G be a generically redundantly rigid, $(d+1)$-connected graph. Then*

$$C_{2,i,2d-2-i,d-1} \lhd G,$$

where $2 < i < d-1$, is generically almost-globally rigid in $\mathbb{R}^d$.

For simplicity, let $\mathcal{C}(i,d) := C_{2,i,2d-2-i,d-1}$. Then $\mathcal{C}(3,5)$ is $C_{2,3,5,4}$. The one part of their proof that does not immediately generalize in higher dimensions involves demonstrating that $C_{2,3,5,4} \lhd K_6$ is GLR, in which they provide only a computer-aided proof.

The graph $\mathcal{C}(3,5) \lhd K_6$ can be rewritten as the joined graph $(K_2 \cup \overline{K_5}) + (K_4 \cup \overline{K_3})$. In general, $\mathcal{C}(i,d) \lhd K_{d+1}$ is the joined graph $(K_2 \cup \overline{K_{(2d-2-i)}}) + (K_{d-1} \cup \overline{K_i})$. Reinterpreting the attachment as a joined graph allows us to apply the quadric rigidity matrix.

Lemma 7.3. *For $d \geq 3$, the joined graph $(K_2 \cup \overline{K_{(2d-4)}}) + (K_{d-1} \cup \overline{K_2})$ is generically locally rigid.*

Proof. For simplicity, let $H_d = (K_2 \cup \overline{K_{(2d-4)}}) + (K_{d-1} \cup \overline{K_2})$. H_d has $3d-1$ vertices and $1 + \binom{d-1}{2}$ extraneous edges, so there are $\binom{d+2}{2}$ rows in the quadric rigidity matrix. It suffices to show that the quadric rigidity matrix is of maximal rank.

In $\mathbb{R}^2$, H_2 is achieved by applying Proposition 4.4 on K_2, so it is generically locally rigid. Thus, it has no quadric flex, so its quadric rigidity matrix is of maximal rank. Although H_2 does not have $d+1$ vertices in each bipartite class, we can begin induction from the $d = 2$ case because of its lack of quadric flexes.

Assume H_{d-1} is generically locally rigid in $\mathbb{R}^{d-1}$. Then consider the graph H_d in $\mathbb{R}^d$. We can achieve this graph from the graph H_{d-1} by adding one vertex in the first bipartite class, adding two vertices in the second bipartite class, and adding $d-2$ extraneous edges to one of the two vertices. In terms of the quadric rigidity matrix, we take a $\binom{d+1}{2} \times \binom{d+1}{2}$ matrix and expand to a $\binom{d+2}{2} \times \binom{d+2}{2}$ matrix.

We add $d+1$ columns, namely the d quadratic terms, denoted $Q_1, Q_2, ..., Q_d$ (where Q_i corresponds to the product of the i-th coordinate and the d-th coordinate), and 1 linear term, denoted L_d. We

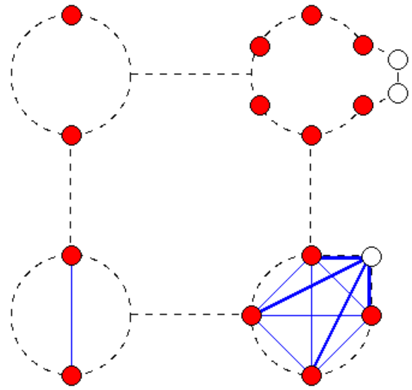

FIGURE 9. The induction step as applied to H_5 to get H_6. The new vertices are shown in a different color and the new extraneous edges are bolded.

also add $d+1$ rows, formed by adding the three new vertices and $d-2$ extraneous edges. We only need to show that there exists some framework whose quadric rigidity matrix has maximal rank[2].

Select a generic framework for H_{d-1} in $\mathbb{R}^{d-1}$. We include that framework into $\mathbb{R}^d$ such that a vertex $v = (v_1, v_2, ..., v_{d-1})$ is mapped to $(v_1, v_2, ..., v_{d-1}, 0)$. We add three new vertices with coordinates

$$\begin{aligned} a &= (0, 0, ..., 0, a_{d-1}, a_d) \\ b &= (0, 0, ..., 0, b_{d-1}, b_d) \\ c &= (0, 0, ..., 0, c_{d-1}, c_d) \end{aligned}$$

and edges $e_1, e_2, ..., e_{d-2}$ all connected to c. Since the original vertices and extraneous edges have 0 in the last coordinate, their values in Q_i and L_d are necessarily 0. It suffices to show that the matrix formed by the new rows and columns are of maximal rank, because the matrix is block triangular.

Since we chose all but the last two coordinates to be 0 for vertices a, b, and c, their constraint mappings must have 0's in the $Q_2, Q_2, ..., Q_{d-1}$ columns. Once again, this is a block triangular matrix, so we need to show that the edges are independent in those $d-2$ columns, and then the vertices in the remaining 3 columns.

The edges connect vertex c to a vertex in the original graph, so the submatrix formed by the $Q_2, Q_3, ..., Q_{d-1}$ columns and the $e_1, e_2, ..., e_{d-2}$ rows are coordinates from the original framework all multiplied by c_d. Since the determinant is an algebraic equation on the coordinates, it must be non-zero since we selected a generic framework in $\mathbb{R}^{d-1}$. We conclude that the submatrix is of maximal rank as long as $c_d \neq 0$.

We are left with the submatrix formed by the Q_1, Q_d, L_d columns and the a, b, c rows. More explicitly, the matrix

$$\begin{pmatrix} a_d^2 & 2a_{d-1}a_d & 2a_d \\ b_d^2 & 2b_{d-1}b_d & 2b_d \\ c_d^2 & 2c_{d-1}c_d & 2c_d \end{pmatrix}.$$

[2]While the proof uses a framework that would have additional infinitesimal flexes (see Whiteley [9]), we are only interested in showing that the graph has no quadric flexes as the other flexes are not possible in generic frameworks.

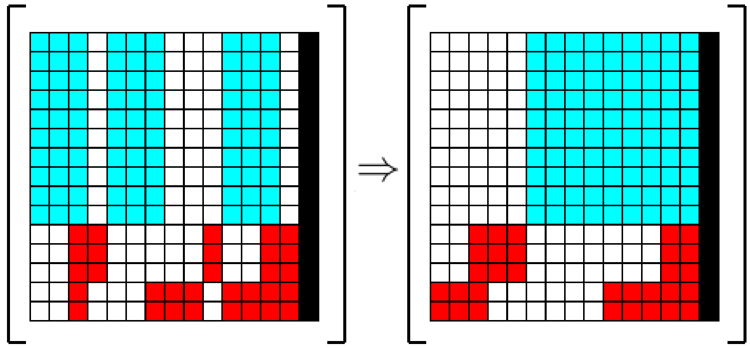

FIGURE 10. The induction step on H_3 to H_4, where non-empty entries are marked. The matrix columns are rearranged to illustrate the block triangular form.

We may choose any coordinates that makes the submatrix invertible and has $c_d \neq 0$. Since the $(d+1)\times(d+1)$ submatrix is of maximal rank, the entire quadric rigidity matrix is of maximal rank, as well. □

By applying Proposition 5.4, we obtain generic local rigidity for the graph attachments in consideration. The remainder of the proof is almost identical to the specific case of $\mathcal{C}(3,5)$ in $\mathbb{R}^5$.[3]

Sketch of Proof. We must show that the attached graph is $(d+1)$-connected, generically redundantly rigid, and not generically globally rigid.

Since G is $(d+1)$-connected, the only possibility for disconnecting the graph is deleting the vertices from $\mathcal{C}(i,d)$. However, we would have to delete all the vertices of X_1 and X_4, X_1 and X_3, or X_2 and X_4, and each of those pairs has at least $d+1$ vertices. Thus, the attached graph is $(d+1)$-connected.

When $G = K_j$, where $j \geq d+1$, we have generic local rigidity by repeated application of Proposition 4.4. For the general case, $|V_G| \geq d+1$, so we can compare the flexes of $\mathcal{C}(i,d) \lhd G$ to $\mathcal{C}(i,d) \lhd K_{|V_G|}$. Suppose a non-trivial flex of $\mathcal{C}(i,d) \lhd G$ exists. Since G is assumed to be generically locally rigid, that flex must be a Euclidean motion on G. However, this same flex could be applied to $\mathcal{C}(i,d) \lhd K_{|V_G|}$ and still be non-trivial, so no such flex exists.

By Proposition 5.4, $\mathcal{C}(i,d) \lhd K_{d+1}$ is GLR in $\mathbb{R}^d$ by moving vertices with no extraneous edges to a different bipartite class. The space of stresses for $\mathcal{C}(i,d) \lhd K_{d+1}$ has dimension

$$\begin{aligned}\Omega(\mathcal{C}(i,d) \lhd K_{d+1}, p) &= e - vd + \binom{d+1}{2} \\ &= (2d-i)(d+i-1) + \binom{d-1}{2} + 1 - (3d-1)d + \binom{d+1}{2} \\ &= (i-2)(d-i-1).\end{aligned}$$

However, from Theorem 4.6, the space of stresses for the complete bipartite graph $K_{2d-i,d-i-1}$ is also $(i-2)(d-i-1)$. That implies that any non-zero stress on $\mathcal{C}(i,d) \lhd K_{d+1}$ is zero on extraneous edges and non-zero on bipartite edges. By Proposition 4.5, the extraneous edges are not redundant, while the bipartite edges are. The edge set of $\mathcal{C}(i,d) \lhd G$ is the disjoint union of the edge sets of $\mathcal{C}(i,d)$ and G. Removing an edge from G leaves the graph locally rigid since G is redundantly rigid. Every edge e of $\mathcal{C}(i,d)$ is redundant in the graph $\mathcal{C}(i,d) \lhd K_j$, so $\mathcal{C}(i,d) - \{e\} \lhd K_j$ is generically

[3] See Frank and Jiang [6] for a complete proof.

locally rigid. By the same argument, any flex of $\mathcal{C}(i,d)-\{e\}\lhd G$ is a flex of $\mathcal{C}(i,d)-\{e\}\lhd K_{|V_G|}$, so $\mathcal{C}(i,d)-\{e\}\lhd G$ is generically locally rigid. Thus, we have generic redundant rigidity for $\mathcal{C}(i,d)\lhd G$.

Since not all edges of $\mathcal{C}(i,d)\lhd K_{d+1}$ are redundant, the graph is not generically globally rigid by Theorem 3.4. In general, for any two equivalent frameworks of G, the two subframeworks of a globally rigid subgraph are congruent. Adding a vertex to a graph and attaching it to vertices of a globally rigid subgraph then preserves global non-rigidity, so $\mathcal{C}(i,d)\lhd K_j$ where $j\geq d+1$ is not generically globally rigid either. Since G is a factor of $K_{|V_G|}$, $\mathcal{C}(i,d)\lhd G$ is not generically globally rigid either. □

We can also exhibit more graph attachments based on different families of 4-chains. In general, we found that the sum of the middle two arguments of the 4-chain must be $x(d-x+1)$, which we denote $v(d,x)$. The following generalization can be proven using the same techniques.

Theorem 7.4. *Let G be a generically redundantly rigid, $(d+1)$-connected graph. Then*

$$C_{x,i,v(d,x)-i,(d+1)-x}\lhd G,$$

where $x<i<v(d,x)+x-d-1$, is generically almost-globally rigid in $\mathbb{R}^d$.

However, there are still 4-chain graph attachments that escape this characterization, namely those that attach to $d+2$ or more vertices (there cannot exist any that attach to only d vertices since this violates $(d+1)$-connectivity). The smallest such outlier we found was $C_{3,3,5,5}$ in $\mathbb{R}^6$.

Problem 7.5. *Characterize all 4-chain graph attachments.*

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