Intersection of stable and unstable manifolds for invariant Morse functions

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Abstract

We study the structure of the smooth manifold which is defined as the intersection of a stable manifold and an unstable manifold for an invariant Morse-Smale function.

1 Introduction

The aim of this paper is to investigate invariant Morse functions on compact smooth manifolds with action of compact Lie groups.

Let M be a compact n-dimensional Riemannian manifold, $\langle \cdot, \cdot \rangle$ its Riemannian metric, and Φ a Morse function on M. We denote by $-\nabla\Phi$ the negative gradient vector field of Φ with respect to the metric $\langle \cdot, \cdot \rangle$, and let $\gamma_p(t)$ be the corresponding negative gradient flow which passes through a point p of M at $t = 0$. For a critical point p of Φ, the **unstable manifold** and the **stable manifold** of p are defined by

$$W^u(p) = \left\{ x \in M \,\middle|\, \lim_{t \to -\infty} \gamma_x(t) = p \right\},$$
$$W^s(p) = \left\{ x \in M \,\middle|\, \lim_{t \to \infty} \gamma_x(t) = p \right\}$$

respectively. Since Φ is a Morse function, $W^u(p)$ and $W^s(p)$ are a smoothly embedded open disks of dimensions $n - \lambda(p), \lambda(p)$ respectively, where $\lambda(p)$ denotes the Morse index of p (see [BH, Theorem 4.2]). We say that a Morse function Φ is **Morse-Smale** if $W^u(p)$ and $W^s(q)$ intersect transversally for all critical points p, q. If the Morse function Φ is Morse-Smale, then $\widetilde{\mathcal{M}}(p,q) := W^u(p) \cap W^s(q)$ is also a submanifold of M which has dimension $\lambda(p) - \lambda(q)$.

$\widetilde{\mathcal{M}}(p,q)$ has a natural $\mathbb{R}$-action which is defined by $t \cdot x := \gamma_x(t)$ where $t \in \mathbb{R}, x \in \widetilde{\mathcal{M}}(p,q)$. The quotient space of $\widetilde{\mathcal{M}}(p,q)$ by the $\mathbb{R}$-action is denoted by $\mathcal{M}(p,q)$. Witten's Morse theory [W] asserts that in some cases, the homology group of M with integral coefficient is recovered from the structure of $\mathcal{M}(p,q)$'s such that $\lambda(p) - \lambda(q) = 1$. However, there is a Morse function which has no critical points p, q such that $\lambda(p) - \lambda(q) = 1$. For example, for a certain Morse function on the partial flag manifold, every unstable manifold is given by the Bruhat cell BwP/P. In particular, every Morse index is even (see [A]).

This phenomenon leads us to the study of the structure of $\widetilde{\mathcal{M}}(p,q)$ for $p,q \in \mathrm{Cr}(\Phi), \lambda(p) - \lambda(q) = 2$.

In this paper, we investigate the structure of $\mathcal{M}(p,q)$ for $p,q \in \mathrm{Cr}(\Phi)$ such that $\lambda(p) - \lambda(q) = 2$ under the assumption that M admits an action of a compact Lie group G and Φ is G-invariant.

Our main theorem is the following.

Theorem 1.1. Let Φ be a G-invariant Bott-Morse function on M. Let p,q be critical points of Φ such that $\lambda(p) - \lambda(q) = 2$. Assume the following two conditions:
(1)$M^G \subset \mathrm{Cr}(\Phi)$.
(2)$W^u(p)$ and $W^s(q)$ intersect transversally.
Then every connected component of $\widetilde{\mathcal{M}}(p,q)$ is diffeomorphic to $S^1 \times \mathbb{R}$. □

We also show that the action of G on $\widetilde{\mathcal{M}}(p,q)$ is given by the rotation of sphere (see Proposition 3.6 below). By these results geometric structure of $\widetilde{\mathcal{M}}(p,q)$ in our setting is similar to the one treated in the GKM theory [GKM].

This paper is organized as follows. In Section 2, we study the critical point set of an invariant Morse function and apply it to an invariant Morse function on a homogenious space. In Section 3, we prove Theorem 1.1.

2 Critical points

Let G be a compact Lie group and M be a compact G-manifold. Denote by M^G the fixed point set of the action of G on M. We say a smooth function $\Phi : M \longrightarrow \mathbb{R}$ is G-**invariant** if it satisfies $\Phi(g \cdot p) = \Phi(p)$ for all $g \in G, p \in M$. For a smooth function Φ on M, we denote by $\mathrm{Cr}(\Phi)$ the critical point set of Φ.

Proposition 2.1. Let G be a compact connected Lie group, M be a compact smooth G-manifold, and $\Phi : M \longrightarrow \mathbb{R}$ be a G-invariant Morse function on M. Assume that there exist only finitely many G-fixed points on M. Then we have $\mathrm{Cr}(\Phi) = M^G$. □

Since G and M are both compact, there exists a G-invariant metric $\langle \cdot, \cdot \rangle$ on M. Consider the negative gradient flow equation

$$\gamma(0) = p, \quad \frac{d}{dt}\gamma(t) = -(\nabla \Phi)_{\gamma(t)}.$$

Here, we denote by $\nabla \Phi$ the gradient vector field for Φ with respect to the G-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ on M. Let $\gamma_p(t)$ be the unique solution of this equation. By the uniqueness of the solution we see easily the following.

Lemma 2.2. We have $\gamma_{g \cdot p}(t) = g \cdot \gamma_p(t)$ for all $g \in G, p \in M$. □

Proof of Proposition 2.1. Take $p \in \mathrm{Cr}(\Phi)$. By Lemma 2.2, we have

$$\lim_{t\to-\infty}\gamma_{g\cdot p}(t) = \lim_{t\to-\infty} g\cdot\gamma_p(t) = g\cdot p.$$

This means $g\cdot p$ is also a critical point for Φ, so we have $G\cdot p \subset \mathrm{Cr}(\Phi)$. However, since M is compact, $\mathrm{Cr}(\Phi)$ is a finite set. Thus by the connectedness of G, we have $G\cdot p = \{p\}$. This shows $p \in M^G$.

Take $p \in M^G$. By Lemma 2.2 we have

$$g\cdot\gamma_p(t) = \gamma_{g\cdot p}(t) = \gamma_p(t)$$

for all $g \in G$. This means $\{\gamma_p(t) | t \in \mathbb{R}\} \subset M^G$. Since M^G is a finite set, this implies $\{\gamma_p(t) | t \in \mathbb{R}\} = \{p\}$. Thus we have $p \in \mathrm{Cr}(\Phi)$. □

Corollary 2.3. Let p_0 be a point of M and H be its stabilizer. Assume the following three conditions:
(1) H is connected.
(2) $W_H := N_G(H)/H$ is a finite group.
(3) The Fixed point set of the H-action on M is contained in the G-orbit of p_0.
Then, we have

$$\mathrm{Cr}(\Phi) = W_H \cdot p_0$$

for any H-invariant Morse function $\Phi : M \longrightarrow \mathbb{R}$.

Proof. First, we prove $M^H = W_H \cdot p_0$. The inclusion $M^H \supset W_H \cdot p_0$ is clear. Take $p \in M^H$. Then by the condition (3), it is contained in the G-orbit of p_0. So we can write $p = g \cdot p_0$ where g is an element of G. Since $p \in M^H$, we have $h \cdot (g \cdot p_0) = g \cdot p_0$ for all $h \in H$. So we have $g^{-1}Hg \subset H$. Since $g^{-1}Hg$ and H are connected Lie subgroups with the same Lie algebra, the inclusion implies $g^{-1}Hg = H$. Thus we have $p = g \cdot p_0 \in W_H \cdot p_0$, as desired.

In particular, by the condition (2), $M^H = W_H \cdot p_0$ is a finite set. Thus by Proposition 2.1, we have $\mathrm{Cr}(\Phi) = W_H \cdot p_0$. □

As an application to homogeneous spaces, we have the following corollaries:

Corollary 2.4. Let G be a compact Lie group and H be its connected closed subgroup. If $N_G(H)/H$ is a finite group, we have

$$\mathrm{Cr}(\Phi) = N_G(H)/H$$

for any H-invariant Morse function $\Phi : G/H \longrightarrow \mathbb{R}$. □

Corollary 2.5. Let G be a compact Lie group and T be a maximal torus. Then, the critical point set of any T-invariant Morse function on the flag manifold G/T is given by its Weyl group. □

3 Intersections

Let G be a compact connected Lie group and M be a compact smooth G-manifold. The following is our main result in this paper.

Theorem 3.1. Let Φ be a G-invariant Bott-Morse function on M. Let p, q be critical points of Φ such that $\lambda(p) - \lambda(q) = 2$. Assume the following two conditions:
(1) $M^G \subset \mathrm{Cr}(\Phi)$.
(2) $W^u(p)$ and $W^s(q)$ intersect transversally.
Then every connected component of $\widetilde{\mathcal{M}}(p,q)$ is diffeomorphic to $S^1 \times \mathbb{R}$.

Proof. Let C be a connected component of $\widetilde{\mathcal{M}}(p,q)$. By Lemma 2.2 and the connectedness of G, C is a G-invariant subset of $\widetilde{\mathcal{M}}(p,q)$. We note that C is non-compact. To see this, assume that C is compact. Take $c' \in C$. Since the negative gradient flow $\gamma_{c'}(\mathbb{R})$ is connected, it must be contained in C. Therefore the assumption implies that $p = \lim_{t \to -\infty} \gamma_c(t) \in C$. This is a contradiction, because $p \notin \widetilde{\mathcal{M}}(p,q)$. So C is non-compact. Since $\mathrm{Cr}(\Phi) \cap C = \emptyset$, the assumption (1) implies that $M^G \cap C = \emptyset$. Let us show the following.

(3.1) $\dim G \cdot c = 1$.

Assume that $\dim G \cdot c = 2$. Then $G \cdot c$ is a codimension 0 submanifold of C. Therefore $G \cdot c$ is an open subset of C. On the other hand, by the compactness of G, $G \cdot c$ is a closed subset of C. So we have $C = G \cdot c$ since C is connected. This is a contradiction, because C is non-compact. Assume that $\dim G \cdot c = 0$. Then by the connectedness of G, we have $G \cdot c = \{c\}$. This is also a contradiction, because $c \notin M^G$. Hence we have $\dim G \cdot c = 1$. The proof of (3.1) is complete.

Define an action of $G \times \mathbb{R}$ on C by $(g,t) \cdot c = g \cdot \gamma_c(t)$. In fact, this gives an action on C, because

$$\begin{aligned}(gg', t+t') \cdot c &= gg' \cdot \gamma_c(t+t') \\ &= g \cdot \gamma_{g' \cdot c}(t+t') \\ &= g \cdot \gamma_{\gamma_{g' \cdot c}(t')}(t) \\ &= (g,t) \cdot \gamma_{g' \cdot c}(t') \\ &= (g,t) \cdot ((g',t') \cdot c)\end{aligned}$$

for all $(g,t), (g',t') \in G \times \mathbb{R}$. We next show the following.

(3.2) $(G \times \mathbb{R})_c = G_c \times \{0\}$.

Here, $(G \times \mathbb{R})_c$ (resp. G_c) is the stabilizer of c for the action of $G \times \mathbb{R}$ (resp. G) on C. It is enough to show that $(G \times \mathbb{R})_c \subset G_c \times \{0\}$. Let (g,t) be an element of $(G \times \mathbb{R})_c$. It is sufficient to show $t = 0$. Assume that $t > 0$. Since

$(g^n, nt) \in (G \times \mathbb{R})_c$ for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty} g^n \cdot c = \lim_{n\to\infty} \gamma_c(-nt) = p$. This implies that $p \in C$ since $G \cdot c$ is a closed subset of C. This is a contradiction. If we assume that $t < 0$, a similar argument implies the same contradiction. The proof of (3.2) is complete.

Let us consider the natural embedding $G \times \mathbb{R}/(G \times \mathbb{R})_c \longrightarrow \widetilde{\mathcal{M}}(p, q)$. By (3.1) and (3.2), we have $\dim G \times \mathbb{R}/(G \times \mathbb{R})_c = \dim \widetilde{\mathcal{M}}(p, q) = 2$. Thus $G \cdot \gamma_c(\mathbb{R})$ is open in $\widetilde{\mathcal{M}}(p, q)$. In particular, every orbit of the action of $G \times \mathbb{R}$ on C is open. Since C is connected, this implies that $C = G \cdot \gamma_c(\mathbb{R})$. Therefore we obtain the following isomorphisms:

$$C \cong G \times \mathbb{R}/G_c \times \{0\} \cong G/G_c \times \mathbb{R} \cong G \cdot c \times \mathbb{R}.$$

By (3.1), $G \cdot c$ is a compact connected 1-dimensional manifold. Thus $G \cdot c$ is diffeomorphic to S^1. Hence C is diffeomorphic to $S^1 \times \mathbb{R}$.

The proof is complete. □

Corollary 3.2. Let Φ be a G-invariant Morse-Smale function on M. Let p, q be critical points of Φ such that $\lambda(p) - \lambda(q) = 2$. If M^G is a finite set, every connected component of $\widetilde{\mathcal{M}}(p, q)$ is diffeomorphic to $S^1 \times \mathbb{R}$.

Proof. By Proposition 2.1, we have $M^G = \mathrm{Cr}(\Phi)$. So this corollary follows from Theorem 3.1. □

In the rest of this section, we study the stabilizer G_c. Let G be a compact connected Lie group which acts smoothly on S^1. We denote by $\mathfrak{g}$ the Lie algebra of G. Consider the following commutative diagram:

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Diff}(S^1) \\ \uparrow & & \uparrow \\ \mathfrak{g} & \longrightarrow & \Gamma(TS^1). \end{array}$$

Here, vertical arrows are exponential maps and horizontal arrows are induced by the action of G on S^1. Since G is a compact connected Lie group, the exponential map $\mathfrak{g} \longrightarrow G$ is surjective. Thus the image of $G \longrightarrow \mathrm{Diff}(S^1)$ is completely determined by the image of $\mathfrak{g} \longrightarrow \Gamma(TS^1)$. We need the following result of Plante [P, Theorem 1.2].

Lemma 3.3. Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Assume that G acts smoothly and transitively on S^1. Then the image of $\mathfrak{g} \longrightarrow \Gamma(TS^1)$ is conjugate via a diffeomorphism to one of the following subalgebras of $\Gamma(TS^1)$

(1) $\left\langle \frac{\partial}{\partial x} \right\rangle$,

(2) $\left\langle (1 + \cos x)\frac{\partial}{\partial x}, (\sin x)\frac{\partial}{\partial x}, (1 - \cos x)\frac{\partial}{\partial x} \right\rangle$. □

Note that we have the isomorphism

$$\left\langle (1+\cos x)\frac{\partial}{\partial x}, (\sin x)\frac{\partial}{\partial x}, (1-\cos x)\frac{\partial}{\partial x} \right\rangle \cong \mathfrak{sl}_2(\mathbb{R})$$

of Lie algebras.

Proposition 3.4. In the setting of Theorem 3.1, let C be a connected component of $\widetilde{\mathcal{M}}(p,q)$. Then there is a surjective group homomorphism $\alpha : G \longrightarrow S^1$ and a diffeomorphism $C \cong S^1 \times \mathbb{R}$ such that the action of $G \times \mathbb{R}$ on $C \cong S^1 \times \mathbb{R}$ is given by

$$(g,t)\cdot(x,s) = (\alpha(g)x, t+s)$$

for all $(g,t) \in G \times \mathbb{R}, (x,s) \in S^1 \times \mathbb{R}$.

Proof. Take $c \in C$. We consider the action of G on $G \cdot c$ and identify $G \cdot c$ with S^1. Let $\alpha_0 : G \longrightarrow \mathrm{Diff}(S^1)$ be the representation of the action of G on S^1, $\alpha_0' : \mathfrak{g} \longrightarrow \Gamma(TS^1)$ the corresponding Lie algebra homomorphism.

Since $\mathfrak{g}$ is the Lie algebra of the compact Lie group G, it does not admit $\mathfrak{sl}_2$ as a quotient Lie algebra . Hence by Lemma 3.3 we can take $\varphi \in \mathrm{Diff}(S^1)$ such that

$$\varphi_*(\alpha_0'(\mathfrak{g})) = \left\langle \frac{\partial}{\partial x} \right\rangle.$$

This shows that $\varphi(\alpha_0(G))\varphi^{-1}$ consists of rotations of S^1. Now we define a group homomorphism $\alpha : G \longrightarrow S^1$ by $\alpha(g) := \varphi \circ \alpha_0(g) \circ \varphi^{-1}$. This map satisfies the required properties. □

Corollary 3.5. In the setting of Theorem 3.1, let C be a connected component of $\widetilde{\mathcal{M}}(p,q)$. Then the stabilizer of $c \in C$ is independent of choice of c and is a codimension 1 closed normal Lie subgroup of G. □

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