TORIC PLURISUBHARMONIC FUNCTIONS AND ANALYTIC ADJOINT IDEAL SHEAVES

HENRI GUENANCIA

Abstract. In the first part of this paper, we study the properties of some particular plurisubharmonic functions, namely the toric ones. The main result of this part is a precise description of their multiplier ideal sheaves, which generalizes the algebraic case studied by Howald. In the second part, almost entirely independent of the first one, we generalize the notion of the adjoint ideal sheaf used in algebraic geometry to the analytic setting. This enables us to give an analogue of Howald's theorem for adjoint ideals attached to monomial ideals. Finally, using the local Ohsawa-Takegoshi-Manivel theorem, we prove the existence of the so-called generalized adjunction exact sequence, which, combined with a Nadel-like vanishing result, enables us to recover a global extension theorem of Manivel, for weakly pseudoconvex Kähler manifolds.

Contents

Introduction 1
Acknowledgments 3
1. Toric plurisubharmonic functions 3
1.1. Multiplier ideal sheaves 3
1.2. Integrability of the exponential of a concave function 4
1.3. Toric plurisubharmonic functions 6
1.4. Valuative interpretation 8
1.5. An example 10
2. The analytic adjoint ideal sheaf 10
2.1. Preliminaries 10
2.2. Adjoint ideal attached to a plurisubharmonic function 14
2.3. Adjoint ideal of a monomial ideal 16
2.4. The adjunction exact sequence 18
2.5. Back to the sheaf $\mathcal{A}dj_H^0(\varphi)$ 21
References 23

Introduction

Multiplier ideal sheaves are a fundamental tool in complex analytic geometry, for example through Nadel's vanishing theorem : attached to a plurisubharmonic (psh) function φ on a complex manifold X by $\mathcal{J}(\varphi)_x = \{f \in \mathcal{O}_{X,x}; ||f||_\varphi = |f|e^{-\varphi} \in L^2_{\text{loc}}(\text{Leb})\}$, they measure the singularity of φ.
Lazarsfeld introduced their algebraic analogue for an ideal $\mathfrak{a}$ using a log-resolution of $\mathfrak{a}$; of course both ideals coincide whenever φ is attached to $\mathfrak{a}$ (this means that $\varphi = \frac{1}{2}\log(|f_1|^2 + \cdots + |f_r|^2) + O(1)$ if $\mathfrak{a} = (f_1, \ldots, f_r)$ locally), but the conceptual gap between the analytic and algebraic

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definitions suggests that both approaches may be useful, and in some sense complementary to each others.

In the first part of this paper, we give a different analytic approach of Howald's theorem, that extends to plurisubharmonic functions.
More precisely, Howald's theorem states that the multiplier ideal $\mathcal{J}(\mathfrak{a})$ attached to a monomial ideal $\mathfrak{a} = (\mathbf{z}^{\alpha_1}, \dots, \mathbf{z}^{\alpha_r}) \subset \mathbb{C}[z_1, \dots z_r]$ is generated by monomials $\mathbf{z}^{\alpha}$ satisfying $\alpha + \mathbb{1} \in \overset{\circ}{\widehat{P(\mathfrak{a})}}$, where $P(\mathfrak{a})$ is the Newton polyhedron attached to the α_i's. Extending the notion of Newton polyhedron attached to a monomial ideal to any toric psh function φ – by a toric psh function we mean a psh function that is pointwise invariant under the compact unit torus $\mathbb{T}^n$ – and using integrability properties for concave functions, we prove the generalization of Howald's theorem concerning the description of $\mathscr{I}(\varphi)$:

Theorem A. *Let φ be a toric psh function on some polydisk of $\mathbb{C}^n$ centered at 0, and let us set $\mathbb{1} = (1, \dots, 1)$. Then $\mathscr{I}(\varphi)$ is a monomial ideal, and we have :*

$$\mathbf{z}^{\alpha} = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in \mathscr{I}(\varphi) \quad \Longleftrightarrow \quad \alpha + \mathbb{1} \in \overset{\circ}{\widehat{P(\varphi)}}.$$

This shows in passing that the (generalized) openness conjecture stated in [DK01] holds for toric psh functions. Finally, we give one example of usual psh function for which we use this result to characterize very precisely the multiplier ideal.
Let us notice that J. McNeal and Y. Zeytuncu recently gave a new proof of Howald's theorem in [MZ10] using basic analytic techniques.

In the second part of this paper, we focus on another ideal sheaf, related to the multiplier ideal sheaf, namely the adjoint ideal sheaf attached to smooth hypersurface, say H. This ideal, well-known in complex algebraic geometry, is a subsheaf of the multiplier ideal sheaf which measures how largely the restricted ideal $\mathcal{J}(\mathfrak{a})_{|H}$ contains $\mathcal{J}(\mathfrak{a}_{|H})$, as expressed in [Laz04].

Our goal is to define an analytic analogue $\mathcal{A}dj_H(\varphi)$ attached to any psh function on a smooth complex manifold X. In view of the Ohsawa-Takegoshi-Manivel theorem, the natural candidate for $\mathcal{A}dj_H(\varphi)$ would be defined by its stalks

$$\mathcal{A}dj_H^0(\varphi)_x = \left\{ f \in \mathcal{O}_{X,x}\, ; ||f||_{\varphi} \in L^2_{\mathrm{loc}}(\mathrm{Poin}_H) \right\}$$

where Poin_H is the standard Poincaré volume form attached to H; namely if H is locally given by $\{h = 0\}$, then $\mathrm{Poin}_H = \frac{1}{|h|^2 \log^2 |h|} \mathrm{Leb}$.

Unfortunately, the ideal $\mathcal{A}dj_H^0(\varphi)$ doesn't coincide in general with the algebraic adjoint : indeed, even in the algebraic case, $\mathcal{A}dj_H^0(\varphi)$ fails to satisfy the expected openness property (in general, $\mathcal{A}dj_H^0((1+\epsilon)\varphi) \neq \mathcal{A}dj_H^0(\varphi)$ for any $\epsilon > 0$). So we have to perturb a bit this ideal by setting

$$\mathcal{A}dj_H(\varphi) = \bigcup_{\epsilon > 0} \mathcal{A}dj_H^0((1+\epsilon)\varphi).$$

Once we have introduced our ideal, we need to make sure that this new ideal is coherent, and that it coincides with the algebraic adjoint ideal whenever φ is associated to an ideal $\mathfrak{a} = (f_1, \dots, f_r)$, namely $\varphi = \frac{1}{2}\log(|f_1|^2 + \cdots + |f_r|^2) + O(1)$ where the f_i's are polynomials (or even holomorphic functions).
This new point of view allows us to show an analogue of Howald's theorem for (algebraic) adjoint ideals :

Theorem B. *Let $\mathfrak{a} = (\mathbf{z}^{\alpha_1}, \dots, \mathbf{z}^{\alpha_k}) \subset \mathbb{C}[z_1, \dots, z_n]$ be a monomial ideal, $H = \{z_1 = 0\}$ such that $\mathfrak{a} \nsubseteq (z_1)$. We denote by $ri(F_1)$ the relative interior of the (infinite) face of $P(\mathfrak{a})$ contained in $\{x_1 = 0\}$ and we set $\widetilde{\mathbb{1}} = (0, 1, \dots, 1)$. Then for every $c > 0$, $\mathrm{Adj}(\mathfrak{a}^c, H)$ is a monomial ideal and*

$$\mathbf{z}^\beta \in \mathrm{Adj}(\mathfrak{a}^c, H) \quad \Longleftrightarrow \quad \beta + \widetilde{\mathbb{1}} \in c \cdot \overset{\circ}{\widehat{P(\mathfrak{a})}} \cup c \cdot \mathrm{ri}(F_1).$$

We then prove that the fundamental adjunction exact sequence given in [Laz04] extends to our setting, under the additional hypothesis that e^φ is Hölder continuous :

Theorem C. *Let X be a complex manifold, $H \subset X$ a smooth hypersurface, and φ a psh function on X, $\varphi_{|H} \not\equiv -\infty$, such that e^φ is locally Hölder continuous. Then the natural restriction map induces the following exact sequence :*

$$0 \longrightarrow \mathscr{I}_+(\varphi) \otimes \mathcal{O}_X(-H) \longrightarrow \mathcal{A}dj_H(\varphi) \longrightarrow \mathscr{I}_+(\varphi_{|H}) \longrightarrow 0$$

Finally, we give some properties of the sheaf $\mathcal{A}dj^0_H(\varphi)$, and explain what can be expected of it.

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1. Toric plurisubharmonic functions

1.1. **Multiplier ideal sheaves.** In this section, we recall the notion of multiplier ideal sheaves, introduced by Nadel, and which measures the singularity of a psh function. Their definition is rather simple :

Definition 1.1. *Let X be a complex manifold, φ a psh function on X. The multiplier ideal sheaf attached to φ, $\mathscr{I}(\varphi)$, consists in the germs of holomorphic functions $f \in \mathcal{O}_{X,x}$ such that $|f|^2 e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in any local coordinates chart near x.*

Let's recall the following fundamental result, even if we won't use it directly (for a proof, see e.g [DBIP96]) :

Theorem 1.1 (Nadel, 1989)**.** *For every psh function φ on X, the sheaf $\mathscr{I}(\varphi)$ is a coherent ideal sheaf on X.*

Now we would like to get briefly onto the openness conjecture. We need to recall the definition of a right-regularized version of the multiplier ideal sheaf $\mathscr{I}(\varphi)$, and introduced in [DEL00] :

Definition 1.2. *Let X be a complex manifold, and φ a psh function on X. We define*

$$\mathscr{I}_+(\varphi) = \bigcup_{\epsilon > 0} \mathscr{I}((1+\epsilon)\varphi)$$

Remark 1.1. *By the strong noetherian property for coherent sheaves, for all $\Omega \Subset X$, there exists $\epsilon_{\varphi,\Omega} > 0$ such that for all $0 < \epsilon \leqslant \epsilon_{\varphi,\Omega}$, we have $\mathscr{I}_+(\varphi)_{|\Omega} = \mathscr{I}((1+\epsilon)\varphi)_{|\Omega} = \mathscr{I}((1+\epsilon_{\varphi,\Omega})\varphi)_{|\Omega}$.*

The famous openness conjecture expressed in [DK01] admits a natural generalization in terms of these right-regularized multiplier ideals :

Conjecture 1.1 (Strong openness conjecture)**.** *Let φ be a plurisubharmonic function on X, the the following equality of sheaves holds :*

$$\mathscr{I}_+(\varphi) = \mathscr{I}(\varphi).$$

The only non-trivial case where this conjecture is known is the 2-dimensional one, as C. Favre and M. Jonsson proved it in their paper [FJ05], using the so-called valuation tree.
We now seize the opportunity to discuss briefly the valuative point of view concerning multiplier ideals of psh functions. This approach has been widely developed in [FJ05] in the two-variable case, and in [BFJ08] in higher dimensions. We will only evoke one important result.

We consider a psh germ with isolated singularities at $0 \in \mathbb{C}^n$, and we want to describe $\mathscr{I}(\varphi)$ or $\mathscr{I}_+(\varphi)$ in terms of valuations. Let us denote by $\mathcal{V}_{\mathrm{m}}$ the space of monomial valuations, or equivalently Kiselman numbers v_w for $w \in \mathbb{R}_+^n$, defined by :

$$v_w(\varphi) = \sup\left\{\gamma \geqslant 0,\, \varphi(z) \leqslant \gamma \max_i \left(\frac{1}{w_i}\log|z_i|\right) + \underset{z\to 0}{O}(1)\right\}.$$

For example, $v_w(\mathbf{z}^\alpha) := v_w(\log|\mathbf{z}^\alpha|) = \langle w, \alpha\rangle = \sum w_i\alpha_i$. Note that the thinness of those valuations is : $A(v_w) = |w| = \sum w_i$. The following characterization of the multiplier ideal is given in [BFJ08]:

$$f \in \mathscr{I}(\varphi) \quad \Longrightarrow \quad \forall v \in \mathcal{V}_{\mathrm{m}}\,,\, \frac{v(\varphi)}{v(f)+A(v)} < 1.$$

If we consider now the quasi-monomial valuations $v \in \mathcal{V}_{\mathrm{qm}}$ (this means monomial valuations on some birational model of $(\mathbb{C}^n, 0)$), one can also define their thinness, and get a full description of $\mathscr{I}_+(\varphi)$:

Theorem 1.2 ([BFJ08])**.** *Let φ be a psh germ at $0 \in \mathbb{C}^n$. Then*

$$f \in \mathscr{I}_+(\varphi) \quad \Longleftrightarrow \quad \sup_{v\in\mathcal{V}_{\mathrm{qm}}} \frac{v(\varphi)}{v(f)+A(v)} < 1.$$

1.2. **Integrability of the exponential of a concave function.** In the next section, we are going to focus on a very particular type of functions, the toric psh functions. The results we will state about them involve convergence properties for integrals of the form $\int_D e^g$ where $D = \mathbb{R}_+^n$ is the first octant, and g is any concave function on D. So this part is devoted to the study of such integrals.
The key-object appears in the following definition :

Definition 1.3. *Let g be a concave function on $D = \{(x_1, \ldots, x_n) \in \mathbb{R}^n, \forall i, x_i \geqslant 0\}$. The Newton convex body $P(g)$ is :*

$$P(g) = \{\lambda \in \mathbb{R}^n;\, g - \langle \lambda, \cdot\rangle \leqslant O(1)\}.$$

Remark 1.2. *The set $P(g)$ is the domain of the Legendre transform $g^*(y) = \sup_x(g(x) - \langle y, x\rangle)$.*

It is clear that for any real number $c > 0$, $P(cg) = c \cdot P(g)$. Moreover, it is important to notice that $P(g)$ is a convex set, which is in general neither open nor closed (take $g(x) = \frac{-1}{x+1}$ and $g(x) = \log(x+1)$ respectively).

Before going into the important results of this section, let us fix some convenient notations :

(1) We define a partial ordering on $\mathbb{R}^n$ by

$$(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n) \Longleftrightarrow \forall i \in \{1, \ldots, n\},\ x_i \leqslant y_i.$$

In the same way we define

$$(x_1, \ldots, x_n) \prec (y_1, \ldots, y_n) \Longleftrightarrow \forall i \in \{1, \ldots, n\},\ x_i < y_i.$$

(2) We set $D = \mathbb{R}_+^n$ and $\mathbb{1} := (1, \ldots, 1) \in \mathbb{R}^n$.

(3) We know that the set E of points $v \in D$ such that g is differentiable at v has full measure in D (see [RV73] e.g.).

(4) We set $Gr(g) := \mathrm{Conv}\left(\left\{\nabla g(v) + \mu;\ (v, \mu) \in E \times (\mathbb{R}_+^*)^n\right\}\right)$.

Now take some $\lambda = \nabla g(v) + \mu \in Gr(g)$. Then for every $x \in D$, we have $g(x) - g(v) \leqslant \langle \nabla g(v), x - v\rangle$ so that $\lambda \in \overset{\circ}{\widehat{P(g)}}$. By convexity of $\overset{\circ}{\widehat{P(g)}}$, we thus have $Gr(g) \subset \overset{\circ}{\widehat{P(g)}}$. The crucial result of this section is given in the next proposition :

Proposition 1.1. *Let g be a concave function on D. Then :*

$$\int_D e^g < +\infty \quad \Longleftrightarrow \quad 0 \in \overset{\circ}{\widehat{P(g)}}.$$

Proof. The direction $\Leftarrow$ is easy : there exists $\epsilon > 0$ and some constant $C > 0$ such that for all x, $g(x) + \langle \epsilon \mathbb{1}, x\rangle \leqslant C$. Therefore we have :

$$\begin{aligned} \int_D e^g &\leqslant C' \int_D e^{-\epsilon \sum_i x_i} dx_1 \cdots dx_n \\ &= C' \prod_{i=1}^n \int_D e^{-\epsilon x_i} dx_i \\ &< +\infty. \end{aligned}$$

As for the other direction, we suppose that the integral $\int_D e^g$ converges. If $0 \notin \overset{\circ}{\widehat{P(g)}}$, by Hahn-Banach's theorem we can find some vector $w \in \mathbb{R}^n$ such that for all $u \in \overset{\circ}{\widehat{P(g)}}$, we have $\langle u, w\rangle \geqslant 0$ (this implies that w has positive coordinates since $P(g)$ contains a translated of $\mathbb{R}_+^n$). By Fubini's theorem we may find $a \in D$ such that g is differentiable at almost every point of the ray $R = a + \mathbb{R}_+ w$ and $\int_R e^g < +\infty$. As g is a 1-variable concave function on R, it is easy to see that $Dg_x(w)$ (Gâteaux-derivative along w at x) decreases to some $\ell \in \mathbb{R} \cup \{-\infty\}$ when $x \in R$ tends to infinity. Then the integrability of e^g along R shows that $\ell < 0$, so that there exists some $x \in R$, at which g is differentiable, and which satisfies $\langle \nabla g(x), w\rangle = Dg_x(w) < 0$. Thus we can find $\epsilon > 0$ with $\langle \nabla g(x) + \epsilon \mathbb{1}, w\rangle < 0$; this is absurd because $\nabla g(x) + \epsilon \mathbb{1} \in Gr(g) \subset \overset{\circ}{\widehat{P(g)}}$ and the linear form $\langle \cdot\,, w\rangle$ is non-negative on $P(g)$. □

Rewriting the proof using the open convex set $Gr(g)$ instead of $\overset{\circ}{\widehat{P(g)}}$, we see that the convergence of $\int_D e^g$ implies that $0 \in Gr(g)$. As $P(g + \langle \lambda, \cdot\rangle) = \lambda + P(g)$ and $Gr(g + \langle \lambda, \cdot\rangle) = \lambda + Gr(g)$ for all $\lambda \in \mathbb{R}^n$, we see that $\overset{\circ}{\widehat{P(g)}} \subset Gr(g)$. So we have proved :

Proposition 1.2. *For any concave function g on D, we have :*

$$Gr(g) = \overset{\circ}{\widehat{P(g)}}.$$

To finish this section, let us stress that what we proved is an openness property; namely if $e^g \in L^1(D)$, then $e^{(1+\epsilon)g} \in L^1(D)$ for ϵ small enough. As any locally uniformly upper bounded sequence of psh functions converging pointwise to a psh function converges in fact in *the* topology of psh functions, the argument given in section 5.4 of [DK01] applies here to show that any small perturbation $g+h$, where h is any sufficiently small concave function, satisfies the integrability condition $e^{g+h} \in L^1(D)$.

1.3. **Toric plurisubharmonic functions.** Now we get back to toric plurisubharmonic functions on a polydisk $D(0,r) = \{(z_1,\dots,z_n) \in \mathbb{C}^n \,|\, \forall i \in \{1,\dots,n\}, |z_i| < r\}$. This is a special kind of polydisk (we fix the same radius r for every coordinate), but since all the upcoming results are purely local, there will not be any loss of generality (we could even fix $r=1$). The dimension, n, is fixed for the rest of the paper.
Let us recall that a toric function φ on $D(0,r)$ is a function which is invariant under the torus action on $\mathbb{C}^n$: $(e^{i\theta_1},\dots,e^{i\theta_n})\cdot z := (e^{i\theta_1}z_1,\dots,e^{i\theta_n}z_n)$. In more elementary terms, $\varphi(z)$ depends only on $|z_1|,\dots,|z_n|$. In the psh case, we can say more (e.g. [Dem]) :

Proposition 1.3. *Let φ be a toric psh function on $D(0,r)$. Then there exists a convex function f, non-decreasing in each variable, defined on $]-\infty,\log r[^n$ such that for all $z=(z_1,\dots,z_n)\in D(0,r)$, we have $\varphi(z) = f(\log|z_1|,\dots,\log|z_n|)$.*

For the convenience of the reader, we now state and give an elementary proof of the following well-known result, that will be useful in the following.

Lemma 1.1. *If $\mathscr{I}$ is an ideal of the ring $\mathcal{O}_{\mathbb{C}^n,0}$ of the germs of holomorphic functions at $0\in\mathbb{C}^n$ such that for every $f\in\mathscr{I}$, all monomials appearing in f are also in $\mathscr{I}$, then $\mathscr{I}$ is generated by monomials (ie it is a monomial ideal).*

Proof. The first step is to see that, given a (countable) set I of monomials in n variables, we can always extract some finite subset J such that each element of I can be divided by an element of J.
To see this, we use a *reductio ad absurdum*. So, if this property fails, there exists a sequence $(u_k)_{k\geqslant 1}$ with values in $\mathbb{N}^n$ such that $z^{u_{k+1}}$ cannot be divided by any z^{u_p} with $p\leqslant k$. Stated with quantifiers, the property becomes :

$$\exists\,\sigma : \mathbb{N}^2 \to \{1,\dots,n\}; \quad \forall k\geqslant 2, \forall j<k,\ (u_k)_{\sigma(j,k)} < (u_j)_{\sigma(j,k)},$$

where $(u_k)_i$ denotes the i-th component of the vector u_k.
As the sequence $\sigma(k-1,k)$ has values in a finite set, we can extract some subsequence, given by $\psi : \mathbb{N}^* \to \mathbb{N}^*$ increasing, such that $\sigma(\psi(k)-1,\psi(k))$ is a constant, say 1. But then, for every $k\geqslant 2$, we have : $(u_{\psi(k)})_1 < (u_{\psi(k)-1})_1$, which is impossible because $(u_k)_1$ is always a non-negative integer.
The second step is the result of the lemma itself.
As $\mathcal{O}_{\mathbb{C}^n,0}$ is noetherian, $\mathscr{I}$ is finitely generated, so we can consider a finite generating family $(f_1,\dots,f_p)$. For each index k, we consider the monomial ideal $\mathcal{I}_k$ of $\mathbb{C}[z_1,\dots,z_n]$ generated by the monomials appearing in f_k. From the first point, there exists a finite number of minimal monomials appearing in f_k, such that the others ones can be divided by the minimal ones. Therefore we have shown that each f_k lies in the ideal of $\mathcal{O}_{\mathbb{C}^n,0}$ generated by a finite number of monomials appearing in the expansion of f_k. If we put all those minimal monomials for $f_1,\dots,f_k$ together, we see that $\mathscr{I}$ is generated by (a finite number of) monomials.

□

For the following, if φ is a toric psh function on $D(0,r)$ attached to the convex function f, we denote by g the concave function defined on $[\log(r),+\infty[^n$ by $g(x) = -f(-x)$. Moreover, if g is attached to φ, we define $P(\varphi)$ to be the Newton convex body $P(g)$ of g.

Now we can state the precise description of the multiplier ideal sheaf attached to any toric psh function, which can be seen as the analogue or generalization in the analytic setting of Howald's theorem (see [Laz04]) :

Theorem 1.3. *Let φ be a toric psh function on $D(0,r) \subset \mathbb{C}^n$. Then the multiplier ideal $\mathscr{I}(\varphi)$ is a monomial ideal, and we have :*

$$\mathbf{z}^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in \mathscr{I}(\varphi) \quad \Longleftrightarrow \quad \alpha + \mathbb{1} \in \overset{\circ}{\widehat{P(\varphi)}}.$$

We want to apply this theorem to the psh function attached to $g = \min_i \langle \alpha_i, \cdot \rangle$ for some $\alpha_i \in \mathbb{R}^n$. But thanks to proposition 1.2, $P(\mathfrak{a})$ and $P(g)$ have same interiors, so we obtain :

Corollary 1.1 (Howald's theorem, [How01])**.** *Let $\mathfrak{a} = (\mathbf{z}^{\alpha_1}, \ldots, \mathbf{z}^{\alpha_k})$ be a monomial ideal of $\mathbb{C}[z_1, \ldots, z_n]$, and let $P(\mathfrak{a})$ be the Newton polyhedron attached to the set $\{\alpha_1, \ldots, \alpha_k\}$. Then for every $c > 0$:*

$$\mathbf{z}^\beta \in \mathcal{J}(\mathfrak{a}^c) \quad \Longleftrightarrow \quad \beta + \mathbb{1} \in c \, \overset{\circ}{\widehat{P(\mathfrak{a})}}.$$

Remark 1.3. *J. McNeal and Y. Zeytuncu gave recently another analytic proof of this last result in* [MZ10].

Proof of theorem 1.3. We first have to check that $\mathscr{I}(\varphi)$ is monomial, so we consider $f = \sum a_I z^I \in \mathscr{I}(\varphi)$. This means that for some $r > 0$,

$$\int_{D(0,r)} |f|^2 e^{-2\varphi(|z_1|,\ldots,|z_n|)} dV(z)$$

is finite. Thanks to Parseval's theorem, this is equivalent to

$$\sum_I |a_I|^2 \int_{D(0,r)} |z^I|^2 e^{-2\varphi(|z_1|,\ldots,|z_n|)} dV(z) < +\infty$$

so that each monomial of f already belongs to $\mathscr{I}(\varphi)$. Then we are done applying lemma 1.1. We are now interested in the convergence of the integral

$$\int_{D(0,r)} |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n} e^{-2\varphi(z_1,\ldots,z_n)} dV(z).$$

So we first perform the change of variables variables $z_j = r_j e^{i\theta_j}$, and up to a multiplicative factor, the integral equals :

$$\int_{[0,r]^n} r_1^{2\alpha_1+1} \cdots r_n^{2\alpha_n+1} e^{-2f(\log r_1,\ldots,\log r_n)} dr_1 \cdots dr_n$$

We set then $t_i = -\log(r_i)$ so that the previous integral becomes

$$\int_{[\log(r),+\infty[^n} e^{-(2\alpha_1+2)u_1} \cdots e^{-(2\alpha_n+2)u_n} e^{2g(u_1,\ldots,u_n)} du_1 \cdots du_n$$

or also

$$\int_{[\log(r),+\infty[^n} e^{2g(x)-2\langle A,x\rangle} dx.$$

Now we just have to apply proposition 1.1 to the concave function $2(g - \langle A, \cdot \rangle)$, and we are done. □

As $P((1+\epsilon)\varphi) = (1+\epsilon) \cdot P(\varphi)$, the characterization of the multiplier ideal given in theorem 1.3 implies a (very) particular case of the generalized openness conjecture, recalled in this paper as the conjecture 1.1 :

Corollary 1.2. *The generalized openness conjecture $\mathscr{I}_+(\varphi) = \mathscr{I}(\varphi)$ holds for any toric psh function φ.*

Remark 1.4. *If $X = (\mathbb{C}^n, 0)$ and $\mathbf{z} = z_1 \cdots z_n$, then for any toric psh germ φ and any holomorphic germ f on X, theorem 1.3 implies the following property :*

$$f\, e^{-\varphi} \in L^2(X) \quad \Longrightarrow \quad (\mathbf{z} \cdot f)\, e^{-\varphi} \in L^\infty(X).$$

1.4. **Valuative interpretation.** We now want give the valuative interpretation of theorem 1.3, keeping in mind the end of section 1.1.
For this, let us briefly recall some classical facts about Kiselman numbers, that can be found in [Dem]. We fix φ a psh germ at $0 \in \mathbb{C}^n$, $w \in \mathbb{R}^n_+$, and we define $\psi_w(z) = \max_i \frac{1}{w_i} \log |z_i|$ and also $\chi_w(t) = \sup_{\{\psi_w < t\}} \varphi$, which is a convex function. Then we have :

$$\begin{aligned} v_w(\varphi) &= \sup \{\mu \geqslant 0,\, \varphi \leqslant \mu \psi_w + O(1)\} \\ &= \max \{\mu \geqslant 0,\, \varphi \leqslant \mu \psi_w + O(1)\} \\ &= \chi'_w(-\infty) \\ &= \lim_{-\infty} \frac{\chi_w(t)}{t}. \end{aligned}$$

Definition 1.4. *Let g be a concave function on $D = \mathbb{R}^n_+$. Then we define $\hat{g}$ the homogenization of g on $D \setminus \{0\}$ by*

$$\hat{g}(w) = \lim_{t \to +\infty} \frac{g(tw)}{t}.$$

One reason for which we introduced the homogenization function lies in the following lemma :

Lemma 1.2. *Let φ be a toric psh function on $D(0,r) \subset \mathbb{C}^n$, and g its attached concave function. Then*

$$v_w(\varphi) = \hat{g}(w).$$

Proof. We write

$$\begin{aligned} \chi_w(t) &= \sup\{\varphi(z);\, \forall i, \log|z_i| < tw_i\} \\ &= \sup\{-g(x);\, \forall i, x_i > -tw_i)\} \quad [x_i := -\log|z_i|] \\ &= -\inf\{g(x);\, \forall i, x_i > -tw_i\} \\ &= -g(-tw) \end{aligned}$$

because g is non-decreasing in each variable. Therefore $\frac{\chi_w(t)}{t} = \frac{g(-tw)}{-t}$ and passing to the limit when $t \to -\infty$, we obtain the desired result. □

The next result gives a precise description of the closure $\overline{P(g)}$ of the Newton convex body attached to g in terms of $P(\hat{g})$.

Lemma 1.3. *Let g_a be a non-decreasing in each variable concave function on $D_a = a + \mathbb{R}^n_+$ for some $a \prec 0$. Setting $g = g_{a|D}$, we have the following equalities :*

$$\overline{P(g)} = P(\hat{g}) = \{\lambda \in \mathbb{R}^n;\, \hat{g} \leqslant \langle \lambda, \cdot \rangle\}.$$

Proof. As $\hat{g}$ is homogeneous, if $\lambda \in P(\hat{g})$, then there exists $C > 0$ such that for all $x \in D \setminus \{0\}$ and all $t > 0$, we have $\hat{g}(x) = \frac{1}{t}\hat{g}(tx) \leqslant \langle \lambda, x \rangle + \frac{C}{t}$ so that when t tends to $+\infty$, we obtain the second identity of the lemma, which shows that $P(\hat{g})$ is closed.
Now, we choose $\lambda \in P(g)$, and write for all $x \in D$, and $t > 0$: $\frac{1}{t} g(tx) \leqslant \langle \lambda, x \rangle + \frac{C}{t}$, and then $\lambda \in P(\hat{g})$. So we have proved $\overline{P(g)} \subset P(\hat{g})$.
As any convex set with non-empty interior has the same closure than its interior, it is enough to

show that $P(g)$ and $P(\hat{g})$ have the same interior. So we choose λ in the interior of $P(\hat{g})$. This means that there exists $\epsilon > 0$ such that for all $x \in D$, $\hat{g}(x) \leqslant \langle \lambda - \epsilon \mathbb{1}, x \rangle$. We write $x = tw$ where $t > 0$ and $w \in \Delta_n = \{(w_1, \ldots, w_n) \in \mathbb{R}_+^n; \sum w_i = 1\}$ is the standard n-simplex, which is obviously compact. An important remark is that g and $\hat{g}$ are restrictions of concave functions g_a and $\widehat{g_a}$ to $D \subset \widehat{\overset{\circ}{D_a}}$, so they are both continuous on D.
We know that g is non-decreasing in each variable, so $t \mapsto g(tw)$ is a non-decreasing concave function. Clearly, g is bounded below on D, thus there exists C such that $h = g + C$ is non-negative on D. Then $h_t(w) := \frac{1}{t} h(tw)$ for $t > 0, w \in \Delta_n$, defines a non-increasing (in t) family of continuous functions on Δ_n converging to $\hat{h} = \hat{g}$. Indeed, if $t_2 > t_1$, then $h_{t_1}(w) - h_{t_2}(w) \geqslant h(0)(\frac{1}{t_1} - \frac{1}{t_2})$.
By Dini's theorem, the convergence is uniform, as $|h_t(w) - g_t(w)| \leqslant C/t$, then for $t \geqslant t_0(\epsilon)$, $||\hat{g} - g_t||_{\Delta_n, \infty} \leqslant \epsilon$. Thus, for such a t, we have :

$$\begin{aligned} g_t(w) &\leqslant \hat{g}(w) + \epsilon \\ &\leqslant \langle \lambda - \epsilon \mathbb{1}, w \rangle + \epsilon \\ &= \langle \lambda, w \rangle. \end{aligned}$$

If $C = \sup\{g(x) - \langle \lambda, x \rangle;\, x \in D \,\text{and} \sum x_i \leqslant t_0\}$, then we have for all $x \in D$, $g(x) \leqslant \langle \lambda, x \rangle + C$ and therefore $\lambda \in P(g)$. So the interior of $P(\hat{g})$ is contained in $P(g)$ thus in $\widehat{\overset{\circ}{P(g)}}$, and as $P(g) \subset P(\hat{g})$, this concludes the proof of the lemma. □

These two lemmas give now almost immediately the valuative version of theorem 1.3 :

Theorem 1.4. *Let φ be a toric psh germ at $0 \in \mathbb{C}^n$. Then $\mathscr{I}(\varphi)$ is monomial, and :*

$$\mathbf{z}^\alpha \in \mathscr{I}(\varphi) \quad \Longleftrightarrow \quad \sup_{w \in \mathbb{R}_+^n} \frac{v_w(\varphi)}{v_w(\mathbf{z}^\alpha) + A(w)} < 1.$$

Proof. First of all, we attach to φ its concave function g, and as the singularity is isolated at 0, we may suppose (by shrinking the domain of φ) that g is the restriction to $D = \mathbb{R}_+^n$ of a concave function on some $D_a = a + \mathbb{R}_+^n$ with $a \prec 0$, so that the preceding lemma applies here, and in particular, $P(g)$ and $P(\hat{g})$ have same interiors.
Thus, using theorem 1.3 and both preceding lemmas, we have :

$$\begin{aligned} \mathbf{z}^\alpha \in \mathscr{I}(\varphi) &\Longleftrightarrow \alpha + \mathbb{1} \in \widehat{\overset{\circ}{P(g)}} \\ &\Longleftrightarrow \exists \delta \in]0,1[;\, \forall w \in D,\, \hat{g}(w) \leqslant (1-\delta)\langle \alpha + \mathbb{1}, w \rangle \\ &\Longleftrightarrow \exists \delta \in]0,1[;\, \forall w \in D,\, v_w(\varphi) \leqslant (1-\delta)\langle \alpha + \mathbb{1}, w \rangle \\ &\Longleftrightarrow \exists \delta \in]0,1[;\, \sup_{w \in D} \frac{v_w(\varphi)}{v_w(\mathbf{z}^\alpha) + A(w)} \leqslant 1 - \delta \end{aligned}$$

which concludes the proof of the theorem. □

Remark 1.5. *Compared to theorem 1.2, this result tells us that for toric psh functions, multiplier ideals satisfy the openness property, and that they are totally determined by the datum of all monomial valuations; namely we don't need to look at divisors lying in some birational model of $(\mathbb{C}^n, 0)$ to understand the singularities of toric psh functions.*

1.5. **An example.** To finish this first part, we illustrate theorem 1.3 with a particular example, for which some computations lead to a rather simple result.
Let us define $g(x_1, \dots, x_n) = k\, x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, with $k > 0$ and $\alpha_i \geqslant 0$ for all i. First, we must know whether this function is concave or not. But we can see rather easily that g is concave if and only if $\alpha_1 + \cdots + \alpha_n \leqslant 1$.
Then, following the method suggested by theorem 1.3 and proposition 1.2, some computations give rise to the following description of the multiplier ideal :

Proposition 1.4. *Let $\varphi(z) = -k\,|\log|z_1||^{\alpha_1} \cdots |\log|z_n||^{\alpha_n}$ where the α_i are non-negative real numbers, of sum less or equal than 1, and $k > 0$ be a real number. Then φ is psh on $D(0,1)$, and :*

(i) Either $\sum \alpha_i < 1$, and then $\mathscr{I}(\varphi) = \mathcal{O}_{D(0,1)}$;
(ii) Or $\sum \alpha_i = 1$ and then $\mathscr{I}(\varphi)_0$ is generated by the $\mathbf{z}^\beta$ such that :

$$\prod_{\alpha_i > 0} \left(\frac{\beta_i + 1}{k\alpha_i} \right)^{\alpha_i} > 1.$$

2. The analytic adjoint ideal sheaf

2.1. **Preliminaries.** The adjoint ideal attached to an ideal was introduced in the algebraic setting to deal with extension problems for functions belonging to some multiplier ideals. A general and detailed approach can be found in [Tak07] or [Eis10], so we are just going to recall some elementary facts about adjoint ideals.

Definition 2.1. *Let $\mathfrak{a} \subset \mathcal{O}_X$ be a non-zero ideal sheaf on a complex variety X, $c > 0$ a real number, and D a reduced divisor on X such that $\mathfrak{a}$ is not included in any ideal $\mathscr{I}_{D_i}$ of D_i an irreducible component of D. We fix $\mu : \widetilde{X} \to X$ a log resolution of $\mathfrak{a}$ such as $\mathfrak{a} \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-F)$ is such that $F + \mu^* D + K_{\widetilde{X}/X} + \mathrm{Exc}(\mu)$ is a simple normal crossing divisor. Then the adjoint ideal $\mathrm{Adj}(\mathfrak{a}^c, D)$ attached to c and $\mathfrak{a}$ is defined by :*

$$\mathrm{Adj}(\mathfrak{a}^c, D) = \mu_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/X} - [c \cdot F] - \mu^* D + D')$$

where $K_{\widetilde{X}/X} = K_{\widetilde{X}} - \mu^* K_X$, $[\ ]$ denotes the integral part of a divisor, and D' is the strict transform of D, defined by linearity.

Remark 2.1. *To obtain such a resolution, we compose a log resolution $(\mu', X', \mathcal{O}_{X'}(-F'))$ of $\mathfrak{a}$ with a log resolution of $F' + \mu'^* D$.*
Furthermore, one can show that the previously defined sheaf does not depend on such a log resolution.

We then have the so-called adjunction exact sequence, given in [Laz04], theorem 9.5.1 :

Theorem 2.1. *With the previous notations, and in the case where $D = H$ is a non-singular hypersurface, the following short sequence is exact :*

$$0 \longrightarrow \mathscr{I}(\mathfrak{a}^c) \otimes \mathcal{O}_X(-H) \longrightarrow \mathrm{Adj}(\mathfrak{a}^c, H) \longrightarrow \mathscr{I}((\mathfrak{a}^c)_{|H}) \longrightarrow 0$$

So what we are willing to construct is an analogue of the adjoint ideal which would be attached to any psh function φ. Just as multiplier ideals can be defined using the space of holomorphic germs in $L^2(e^{-\varphi}, \mathrm{Leb})$ (it is even their original definition), we would like to find some volume form Ω such that the space of holomorphic germs in $L^2(e^{-\varphi}, \Omega)$ defines adjoint ideals. To find Ω, the intuition is given by the famous Ohsawa-Takegoshi-Manivel theorem ([Dem01]) :

Theorem 2.2 (Ohsawa-Takegoshi-Manivel). *Let $X \subset \mathbb{C}^n$ be a bounded pseudoconvex open set, and let $Y \subset X$ be a complex submanifold of codimension r, defined by a section s of a holomorphic hermitian line bundle with bounded curvature tensor. We suppose that s is everywhere transverse to the zero section, and that the inequality $|s| \leqslant e^{-1}$ holds on X. Then there exists a constant $C > 0$ (depending only on E) such that : for all psh function φ on X, for all holomorphic function f on Y such that $\int_Y |f|^2 |\Lambda^r(ds)|^{-2} e^{-2\varphi} dV_Y < +\infty$, there exists a holomorphic function F on X extending f such that*

$$\int_X \frac{|F|^2}{|s|^2 \log^2 |s|} e^{-2\varphi} dV_X \leqslant C \int_Y \frac{|f|^2}{|\Lambda^r(ds)|^2} e^{-2\varphi} dV_Y.$$

So it seems very natural that choosing Ω to be the standard Poincaré volume form attached to H (this means that if H is locally given by $\{h = 0\}$, then $\mathrm{Poin}_H = \frac{1}{|h|^2 \log^2 |h|}\mathrm{Leb}$) will be the right way to define the analytic adjoint ideal. In this section, we are going to check if things happen as well as predicted.

Let us now give more general and precise setting. We take a complex manifold X and a simple normal crossing (SNC) divisor $D = \sum D_i$; in the following, we will identify the divisor with its support. Then, for all $x \in X$, there exists a neighborhood U_x of x, an integer $0 \leqslant p \leqslant n$ and coordinates $z_1, \dots z_n$ such that $D \cap U_x = \{(z_1, \dots, z_n) \in U_x; z_1 \cdots z_p = 0\}$. In these coordinates, we have obviously :

$$U_x \smallsetminus D \simeq (\Delta^*)^p \times \Delta^{n-p},$$

where Δ is the open unit disk in $\mathbb{C}$, and Δ^* the punctured disk. If $x \notin D$, then $p = 0$.

The fundamental object, which is a growth's class of volume forms, is described in the following definition :

Definition 2.2. *Let X be a complex manifold of dimension n, $D = \sum D_i$ a simple normal crossing divisor on X, and $X_0 = X \smallsetminus D$. We say that a positive $(1,1)$-form ω_P on X_0 is D-Poincaré if for all sufficiently small open set $U \subset X$ there exists some coordinates $z_1, \dots, z_n$, $U \cap D = \{(z_1, \dots, z_n) \in U; z_1 \cdots z_p = 0\}$, and some positive constant C such that :*

$$C^{-1}\omega_P \leqslant \frac{i}{2}\left(\sum_{i=1}^{p} \frac{dz_i \wedge d\bar{z}_i}{|z_i|^2 \log^2 |z_i|} + \sum_{i=p+1}^{n} dz_i \wedge d\bar{z}_i\right) \leqslant C\omega_P.$$

The associated volume form $\frac{\omega_P^n}{n!}$, which we will denote by Ω_P, is then said to be D-Poincaré. So locally, we have up to equivalence :

$$\Omega_P = \prod_{i=1}^{p} \frac{1}{|z_i|^2 \log^2 |z_i|} \ \mathrm{Leb},$$

and the density of Ω_P is integrable with respect to the Lebesgue measure on $\mathbb{R}^{2n}$.

Remark 2.2. *According to the definition, it is clear that there is a unique D-Poincaré volume form on X, up to equivalence.*

Let us remark that for a sufficiently small open set $U \subset X$, if we set $U_0 = U \cap X_0$, then the manifold (U_0, ω_P) is complete and Kähler.

Definition 2.3. *Let φ be a psh function on a complex manifold X, D an SNC divisor. We define the ideal sheaf $\mathcal{A}dj_D^0(\varphi)$ to be made up of the germs $f \in \mathcal{O}_{X,x}$ such that $|f|^2 e^{-2\varphi}$ is integrable with respect to some (hence any) D-Poincaré volume form near x.*

Remark 2.3. *We always have* $\mathcal{A}dj^0_D(\varphi) \subset \mathscr{I}(\varphi)$*, and if* $x \notin D$*, then* $\mathcal{A}dj^0_D(\varphi)_x = \mathscr{I}(\varphi)_x$*.*

The next result, which tells us that our new tool is a reasonable object for analytic geometry, has a similar proof to the one of Nadel's theorem on the coherence of the multiplier ideal sheaves given in [DBIP96].

Proposition 2.1. *For any function* φ *on* X *and for all SNC divisor* D *on* X*, the sheaf* $\mathcal{A}dj^0_D(\varphi)$ *is a coherent ideal sheaf.*

Proof. To simplify the notations, we set $\mathscr{A} := \mathcal{A}dj^0_H(\varphi)$.
As the result is purely local, we may suppose that X is the ball $B(0,\frac{1}{2})$ in $\mathbb{C}^n$ and that $D = \{(z_1,\dots,z_n) \in X \,|\, z_1\cdots z_p = 0\}$, we fix then $\Omega_p = \prod_{i=1}^p \frac{1}{|z_i|^2\log^2|z_i|}$ Leb the canonical D-Poincaré volume form on X. For the following, $\mathcal{H}(X,\varphi)$ will refer to the set of holomorphic function f on X such that $\int_X |f(z)|^2 e^{-2\varphi(z)}\Omega_P(z) < +\infty$. By the strong noetherian property for coherent sheaves, the set $\mathcal{H}(X,\varphi)$ generates a coherent ideal sheaf $\mathscr{J} \subset \mathcal{O}_X$. Clearly, $\mathscr{J} \subset \mathscr{A}$. To show the converse, we are going to check that the equality $\mathscr{J}_x + \mathscr{A}_x \cap \mathfrak{m}^{s+1}_{X,x} = \mathscr{A}_x$ holds for all $x \in X$ and for all integer s, . The Krull lemma will then show that the intersection on all integer s of the left hand side is equal to $\mathscr{J}_x$, which will conclude.
To do this, we pick up a germ $f_x \in \mathscr{A}_x$ defined on a neighborhood V of x, and then we choose χ a smooth function with support included in V identically equal to 1 near x. As $\bar\partial(\chi f) = (\bar\partial\chi)f$ is L^2 with respect to $e^{-2\varphi}\Omega_P$, we may use Hörmander estimates (cf [DBIP96]) on the complete Kähler manifold $(X \smallsetminus D, \omega_P)$ equipped with the trivial line bundle $(X \smallsetminus D)\times\mathbb{C}$ and the strictly psh weight (thanks to $|z|^2$)

$$\tilde\varphi(z) = \varphi(z) + (n+s)\log|z-x| + |z|^2$$

thus we find u on $X \smallsetminus D$ satisfying $\bar\partial u = \bar\partial(\chi f)$ such that

$$\int_{X\smallsetminus D} \frac{|u|^2 e^{-2\varphi}}{|z-x|^{2(n+s)}}\Omega_P(z) < +\infty.$$

Furthermore, $F = \chi f - u$ is holomorphic on $X \smallsetminus D$. It extends to the whole Ω : in fact, we only need to see that F is holomorphic in each variable $z_1,\dots,z_p$ near D. We do it for z_1, using the classical approach : setting $z' = (z_2,\dots,z_n)$, we write the Laurent series expansion $F(z_1,z') = \sum_{-\infty}^{+\infty} a_n(z')z_1^n$. As the integral $\int_{X\smallsetminus D}|F|^2e^{-2\varphi}\Omega_P$ converges, the integral $\int_{X\smallsetminus D}|F|^2\Omega_P$ converges too, and using Fubini's theorem, the integral $\int_{B_{\mathbb{C}}(0,\epsilon)\smallsetminus\{0\}} \frac{|F|^2}{|z_1|^2\log^2|z_1|}dV(z_1)$ is convergent. But then, thanks to Parseval's theorem :

$$\int_{B_{\mathbb{C}}(0,\epsilon)\smallsetminus\{0\}} \frac{|F|^2}{|z_1|^2\log^2|z_1|}dV(z_1) = C\sum_{-\infty}^{+\infty}|a_n(z')|^2\int_{B(0,1)}\frac{|z_1|^{2(n-1)}}{\log^2|z_1|}dV(z_1)$$

and necessarily, we have : $\forall n \leqslant -1, a_n(z') = 0$, which shows that $F(\cdot\,,z')$ admits an holomorphic continuation at 0.
Therefore, $F \in \mathcal{H}(X,\varphi)$, and as φ is has an upper bound near x, $f_x - F_x = u_x \in \mathscr{A}_x \cap \mathfrak{m}^{s+1}_{X,x}$, which concludes. □

Unfortunately, our sheaf $\mathcal{A}dj^0_D(\varphi)$ fails to coincide in general with the algebraic adjoint, as the next example shows :

Counterexample. *Let* $X = (\mathbb{C}^2, 0)$, $\mathfrak{a} = \mathfrak{m}^6$, $H = \{z_1 = 0\}$*, and* $f(z_1,z_2) = z_1^3z_2^3$*. If* $\varphi_{\mathfrak{a}} = 3\log(|z_1|^2+|z_2|^2)$ *is a psh function attached to* $\mathfrak{a}$*, then we have :*

$$f \in \mathcal{A}dj_H(\varphi_{\mathfrak{a}}) \smallsetminus \mathrm{Adj}(\mathfrak{a}, H).$$

Indeed, we are in case (ii) *of the next proposition 2.2, with equality in the first large inequality. Therefore, setting* $D = D(0,1)$, $\int_D \frac{|f|^2 e^{-2\varphi}}{|z_1|^2 \log^2 |z_1|} dV < +\infty$ *but for all* $\epsilon > 0$,

$$\int_D \frac{|f|^2}{|z_1|^2 \log^2 |z_1|} e^{-2(1+\epsilon)\varphi} dV = +\infty.$$

As the algebraic adjoint satisfies the openness property, f *cannot belong to* $\mathrm{Adj}(\mathfrak{a}, H)$.

Let us now give the following result we used in our counterexample, and which gives a precise description of the "zero" adjoint ideal attached to some coordinates monomials :

Proposition 2.2. *Let* $\varphi = \frac{k}{2}\log(\sum_{i=1}^n |z_i|^{2\alpha_i})$, *with* α_i *some positive real numbers, just as* k, *and let* H *be the hyperplane* $\{z_1 = 0\}$. *Then the stalk at* 0 *of* $\mathcal{A}dj^0_H(\varphi)$ *is a monomial ideal, generated by the* z^β *satisfying one of the following conditions :*

(i) $\sum \frac{\beta_i+1}{\alpha_i} > k + \frac{1}{\alpha_1}$

(ii) $\sum \frac{\beta_i+1}{\alpha_i} \geqslant k + \frac{1}{\alpha_1}$ *and* $\beta_1 > 0$.

Proof. The fact that the ideal is monomial can be easily deduced from the same reasoning as the one made to show that multiplier ideals attached to toric psh functions are monomial.
We set $N = \sum \frac{\beta_i+1}{\alpha_i}$.
As for the computation of the ideal, after a first polar, then toric change of variables, it boils down to the convergence, for $U \subset D(0,\delta)$, $\delta < 1$ (resp. V) nieghborhood of 0 in $\mathbb{C}^n$ (resp. $\mathbb{R}^n_+$) of the integral :

$$\begin{aligned} \int_U \frac{\prod_{i=1}^n |z_i|^{2\beta_i}}{|z_1|^2 \log^2 |z_1| \left(\sum_{i=1}^n |z_i|^{2\alpha_i}\right)^k} dV_{\mathbb{C}^n} &= C \int_V \frac{\prod_{i=1}^n r_i^{2\beta_i+1}}{r_1^2 \log^2 r_1 \left(\sum_{i=1}^n r_i^{2\alpha_i}\right)^k} dV_{\mathbb{R}^n} \\ &= C' \int_{t=0}^{\delta} \int_{u \in \mathbb{S}^{n-1}_+} \frac{t^{2(N-k-1/\alpha_1)-1} \prod_{i=1}^n u_i^{2(\beta_i+1)/\alpha_i - 1}}{u_1^{2/\alpha_1} \log^2(tu_1)} du\, dt \end{aligned}$$

où $\mathbb{S}^{n-1}_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n_+ ; x_1^2 + \dots + x_n^2 = 1\}$.
To simplify the computations, we introduce the following notations : $r = 2(N - k - 1/\alpha_1) - 1$, $\lambda_1 = 2\beta_1/\alpha_1 - 1$, and for $i \geqslant 2, \lambda_i = 2(\beta_i + 1)/\alpha_i - 1$. So we always have $\lambda_1 \geqslant -1$, and for $i \geqslant 2, \lambda_i > -1$. We now have to estimate the following integral :

$$I(r, \underline{\lambda}) = \int_{t=0}^{\delta} \int_{u \in \mathbb{S}^{n-1}_+} \frac{t^r \prod_{i=1}^n u_i^{\lambda_i}}{\log^2(tu_1)} du\, dt$$

An obvious necessary condition of convergence is $r \geqslant -1$, which is equivalent to $N \geqslant k + \frac{1}{\alpha_1}$.

- Let us suppose that we have $r > -1$. Then the integral is bounded above by :

$$\int_{t=0}^{\delta} \int_{u \in [0,1]^n} \frac{t^r \prod_{i=2}^n u_i^{\lambda_i}}{u_1 \log^2(tu_1)} du\, dt$$

and integrating with respect to u_1, the last integral becomes :

$$\int_{t=0}^{\delta} \int_{u \in [0,1]^{n-1}} \frac{t^r \prod_{i=2}^n u_i^{\lambda_i}}{-\log t} du\, dt < +\infty$$

- We now suppose that $r \geqslant -1$ and $\lambda_1 > 0$. Then the integral $I(r, \underline{\lambda})$ is less than :

$$\int_{t=0}^{\delta} \int_{u \in [0,1]^n} \frac{\prod_{i=1}^n u_i^{\lambda_i}}{t \log^2(tu_1)} du\, dt$$

which in its turn equals to

$$\int_{u\in[0,1]^n} \frac{\prod_{i=1}^n u_i^{\lambda_i}}{-\log(\delta u_1)} du\, dt < +\infty$$

• Reciprocally, let us assume that $I(r, \underline{\lambda})$ is finite. Thus $r \geqslant -1$, and it remains to show that if $r = -1$, then we necessarily have $\lambda_1 > -1$. We use the following equality :

$$I(-1, \underline{\lambda}) = \int_{u\in\mathbb{S}_+^{n-1}} \frac{\prod_{i=1}^n u_i^{\lambda_i}}{-\log(\delta u_1)} du$$

Then, fixing $\epsilon = \sqrt{3}/2\sqrt{n-1}$, if $u = (u_1, \ldots, u_n) \in \mathbb{S}_+^{n-1}$ satisfies $u_1 \in [0, \epsilon]$ and $u_2, \ldots, u_{n-1} \in [\epsilon/2, \epsilon]$, then $x_n \geqslant 1/2$. In fact, $x_n^2 \geqslant 1 - (n-1)\epsilon^2 = 1/4$.
So we have the minoration :

$$\begin{aligned} I(-1, \underline{\lambda}) &\geqslant \int_{u_1=0}^{\epsilon} \int_{u_2=\epsilon/2}^{\epsilon} \cdots \int_{u_{n-1}=\epsilon/2}^{\epsilon} 2^{-\lambda_n} \frac{\prod_{i=1}^{n-1} u_i^{\lambda_i}}{-\log(\delta u_1)} du_1 \cdots du_{n-1} \\ &\geqslant C \int_{u_1=0}^{\delta\epsilon} \frac{u_1^{\lambda_1}}{-\log(u_1)} du_1 \end{aligned}$$

where C is a positive constant. But the right hand side is finite if and only if $\lambda_1 > -1$, which concludes the proof of the proposition. □

The last counterexample shows us that we have to modify the definition of the analytic adjoint ideal if we want it to extend the usual algebraic adjoint. The goal of the next section is thus to find the correct way to define analytically the adjoint ideal, and to check if this new ideal fits to the generalized adjunction exact sequence.

2.2. **Adjoint ideal attached to a plurisubharmonic function.** As we saw in the preceding counterexample, our "zero" adjoint ideal doesn't satisfy the expected openness property even in the algebraic case. So the idea is to regularize our ideal : more precisely, we know that on our complex manifold X, every non-decreasing sequence of coherent sheaves is stationary on every compact set because of the coherence of $\mathcal{O}_X$. Therefore, for $\Omega \Subset X$, there exists $\epsilon_{\varphi,\Omega} > 0$ such that for all $0 < \epsilon \leqslant \epsilon_{\varphi,\Omega}$, we have $\mathcal{A}dj_D^0((1+\epsilon)\varphi)_{|\Omega} = \mathcal{A}dj_D^0((1+\epsilon_\varphi)\varphi)_{|\Omega}$.

Definition 2.4. *With the preceding notations, and those from definition 2.3, we define the analytic adjoint sheaf $\mathcal{A}dj_D(\varphi)$ to be :*

$$\mathcal{A}dj_D(\varphi) = \bigcup_{\epsilon>0} \mathcal{A}dj_D^0((1+\epsilon)\varphi).$$

In more analytic terms, we can rephrase the definition by saying that $\mathcal{A}dj_D(\varphi)$ is made up of the germs $f \in \mathcal{O}_{X,x}$ such that for $\epsilon > 0$ small enough, $|f|^2 e^{-2(1+\epsilon)\varphi}$ is integrable with respect to any D-Poincaré volume form near x.

Since coherence is checked locally, the next proposition is a straightforward consequence of the previous proposition 2.1 :

Proposition 2.3. *For all psh function φ on X a complex manifold, and for all SNC divisor D on X, the sheaf $\mathcal{A}dj_D(\varphi)$ is a coherent ideal sheaf.*

We are now going to show that the sheaf $\mathcal{A}dj_D(\varphi)$ generalizes the usual adjoint ideal sheaf, in the sense that $\mathcal{A}dj_H(\varphi)$ coincides with the algebraic adjoint ideal whenever φ has analytic singularities, and that it fits the adjunction exact sequence.

Proposition 2.4. *Let D be an SNC divisor on a smooth complex manifold X, and φ be a psh function attached to an analytic ideal sheaf $\mathfrak{a}$, non identically $-\infty$ on any irreducible component of D. Then the following equality of sheaves holds :*

$$\mathcal{A}dj_D(\varphi) = \mathrm{Adj}(\mathfrak{a}, D).$$

Proof. We write

$$\varphi = \frac{1}{2}\log(|f_1|^2 + \cdots + |f_N|^2) + O(1)$$

in the neighborhood of the poles, for f_i local generators of $\mathfrak{a}$, and we write $D = \sum_{i=1}^p D_i$. There exists a modification $\mu : X' \to X$, with exceptional divisors $E_1, \ldots, E_m$ such as $\mu^*\mathfrak{a} = \mathcal{O}_{X'}(F)$ where $F = \sum_{j=p+1}^m a_j E_j$ is such that $F + \mu^* D + K_{X'/X} + \mathrm{Exc}(\mu)$ has simple normal crossings, and satisfies for all $j \geqslant p+1, a_j > 0$ (for $j \in \{1, \ldots, p\}$, we set $a_j = 0$). Moreover, for $i \in \{1, \ldots, p\}$, E_i denotes the strict transform of D_i. To sum up, we use the following notations :

$$\begin{aligned} \mu^*\mathfrak{a} &= \sum_{j=p+1}^m a_j E_j \\ \mu^* D_i &= E_i + \sum_{j=p+1}^m b_{i,j} E_j \\ K_{X'} &= \mu^* K_X + \sum_{j=1}^m c_j E_j \end{aligned}$$

We choose $x \in X$, which will be 0 in our chart. To simplify the notations, we suppose that p is chosen such that $x \in D_1 \cap \cdots \cap D_p$. We take the local generators $x_1, \ldots, x_p$ of $\mathcal{O}_X(-D_1), \ldots, \mathcal{O}_X(-D_p)$ respectively. Similarly, z_k will be a local generator of $\mathcal{O}_{X'}(-E_k)$. If f is a germ of holomorphic function near x, defined on a sufficiently small neighborhood U of 0, we have to compute the following expression :

$$\int_U \frac{|f|^2 e^{-2(1+\epsilon)\varphi}}{\prod_{k=1}^p |x_k|^2 \log^2 |x_k|} dV = \int_{U'=\mu^{-1}(U)} \frac{|f\circ\mu|^2 e^{-2(1+\epsilon)\varphi\circ\mu}}{\prod_{k=1}^p |x_k\circ\mu|^2 \log^2 |x_k\circ\mu|} \, |J_\mu|^2 \, dV'$$

Thanks to Parseval's theorem, if a function f is such that the right hand side is finite, then all monomials in the Taylor expansion of f satisfy the same property. So there is no loss of generality in supposing that $f \circ \mu = \prod z_j^{d_j}$. Thus, up to a non-zero multiplicative constant, the right hand side is (we may suppose that U' is a included in a polydisk $D(0,R)$ with $R < 1$) :

$$\int_{U'} \frac{\prod_{k=1}^m |z_k|^{2(c_k + d_k - (1+\epsilon)ca_k)}}{\prod_{k=1}^p \left[|z_k|^2 \log^2(|z_k| \prod_{j>p} |z_j|^{b_{k,j}})\right] \cdot \prod_{k>p} |z_k|^{2e_k}} dV'$$

where we set, for $k > p$, $e_k = \sum_{i=1}^p b_{i,k}$. Setting then $k \in \{1, \ldots, p\}, e_k = 1$, the previous integral can be written :

$$\int_{U'} \frac{\prod_{k=1}^m |z_k|^{2(c_k + d_k - e_k - (1+\epsilon)ca_k)}}{\prod_{k=1}^p \log^2(|z_k| \prod_{j=1}^m |z_j|^{b_{k,j}})} dV'$$

We set $\lambda_k(\epsilon) = 2(c_k + d_k - e_k - (1+\epsilon)ca_k) + 1$ for all $1 \leqslant k \leqslant m$, and changing to polar coordinates leads us to estimate the following integral, on V a neighborhood of 0 in $\mathbb{R}_+^m$:

$$I(\epsilon) = \int_V \frac{\prod_{k=1}^m x_k^{\lambda_k(\epsilon)}}{\prod_{k=1}^p \log^2(x_k \prod_{b_{k,j}>0} x_j)} dx_1 \ldots dx_m$$

and $V \subset B(0,r)$ for some $r<1$. The question of the convergence is answered by the lemma 2.1 given at the end of the proof.

Furthermore, we already know that for $k \in \{1,\dots,p\}$, we have $\lambda_k(\epsilon) = 2(c_k+d_k-1)+1 = 2(c_k+d_k)-1 \geqslant -1$. About the condition concerning $k>p$, it is equivalent to :

$$c_k + d_k \geqslant e_k + [(1+\epsilon)ca_k].$$

But for all real number $x \geqslant 0$, we have $[(1+\epsilon)x] = [x]$ for $\epsilon>0$ small enough, and more precisely for $\epsilon < ([x]+1)/x - 1$.
Putting all these results together, we have shown that $f \in \mathcal{A}dj_D(\varphi)$ if and only if for all k, we have $d_k \geqslant -(c_k - [ca_k] - e_k)$. Now, let us remind that $\mu^*D - D' = \sum_{k>p} e_k E_k$, so that the previous condition is equivalent to : $f \in \mu_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/X} - [c\cdot F] - \mu^* D + D')$, which shows the proposition.

□

Lemma 2.1. *The integral $I(\epsilon')$ converges for all $0<\epsilon' \leqslant \epsilon$ if and only if for all $k \in \{1,\dots,m\}$, we have $\lambda_k(\epsilon) \geqslant -1$.*

Proof. The condition is obviously necessary by the Bertrand criterion .
Reciprocally, we suppose that for all k, we have $\lambda_k(\epsilon) \geqslant -1$. Then, as for all $k>p$, we have $a_k>0$, the following inequality holds for all $0<\epsilon'<\epsilon$: $\lambda_k(\epsilon') > -1$. To conclude, we are going to use the identity

$$\int_{]0,\delta[^2} \frac{x^a y^{-1}}{\log^2(xy)} dy dx = \int_0^{\delta} \frac{x^a}{-\log(\delta x)} dx = -\delta^{1-a} \int_0^{\delta^2} \frac{x^a}{\log x} dx$$

in the following computation :

$$\begin{aligned}
I(\epsilon') &= \int_V \frac{\prod_{k=1}^m x_k^{\lambda_k(\epsilon')}}{\prod_{k=1}^p \log^2(x_k \prod_{b_{k,j}>0} x_j)} dx_1 \dots dx_m \\
&\leqslant \int_V \frac{\prod_{k=1}^p x_k^{-1} \prod_{k>p} x_k^{\lambda_k(\epsilon')}}{\prod_{k=1}^p \log^2(x_k \prod_{b_{k,j}>0} x_j)} dx_1 \dots dx_m \\
&\leqslant C \int_{V'} \frac{\prod_{k>p} x_k^{\lambda_k(\epsilon')}}{|\prod_{k=1}^p \log(\prod_{b_{k,j}>0} x_j)|} dx_{p+1} \dots dx_m \\
&< +\infty
\end{aligned}$$

where V' is a neighborhood of 0 in $\mathbb{R}_+^{m-p}$. □

2.3. **Adjoint ideal of a monomial ideal.** We would like to give a precise description of the adjoint ideal attached to a monomial ideal, just as Howald's theorem does for multiplier ideals. Unfortunately, the statement corresponding to the adjoint ideal is a little more complicated.
So we work locally and we are given an ideal $\mathfrak{a} = (\mathbf{z}^{\alpha_1},\dots,\mathbf{z}^{\alpha_k}) \subset \mathbb{C}[z_1,\dots,z_n]$, together with the hypersurface $H \subset \mathbb{C}^n$ defined by $\{z_1=0\}$. We know that the Newton polyhedron $P(\mathfrak{a})$ attached to $\mathfrak{a}$ has exactly n infinite faces $F_1,\dots,F_n$ which are orthogonal to $e_1=(1,0,\dots,0),\dots,e_n=(0,\dots,0,1)$ respectively, and all other faces of $P(\mathfrak{a})$ are finite and not included in any affine hyperplane $\{x_p = \text{const}\}$. We recall that the relative interior $\text{ri}(F_p)$ of a face F_p is the interior of F_p as embedded in some affine hyperplane $\{x_p=\text{const}\}$. Finally, we define $\widetilde{\mathbb{1}} := (0,1,\dots,1)$.
Now we can state the desired result :

Theorem 2.3. *Let $\mathfrak{a} = (\mathbf{z}^{\alpha_1}, \ldots, \mathbf{z}^{\alpha_k}) \subset \mathbb{C}[z_1, \ldots, z_n]$ be a monomial ideal, $H = \{z_1 = 0\}$ such that $\mathfrak{a} \nsubseteq (z_1)$. Then, for every $c > 0$, $\mathrm{Adj}(\mathfrak{a}^c, H)$ is a monomial ideal, and*

$$\mathbf{z}^\beta \in \mathrm{Adj}(\mathfrak{a}^c, H) \quad \Longleftrightarrow \quad \beta + \widetilde{\mathbb{1}} \in c \cdot \overset{\circ}{\widehat{P(\mathfrak{a})}} \cup c \cdot \mathrm{ri}(F_1).$$

Proof. We are going to use the analytic definition of $\mathrm{Adj}(\mathfrak{a}, H) = \mathcal{A}dj_H(\varphi)$ where φ is attached to the concave function $g = c\ \min_i \langle \alpha_i, \cdot \rangle$. Then $\mathbf{z}^\beta \in \mathcal{A}dj_H(\varphi)$ if and only if there exists $\epsilon > 0$ such that on a neighborhood U of 0 in $\mathbb{C}^n$, the integral

$$\int_V \frac{|\mathbf{z}|^{2\beta} e^{-2(1+\epsilon)g(-\log|z_1|, \ldots, -\log|z_n|)}}{|z_1|^2 \log^2|z_1|} dV$$

converges. But after performing the usual changes of variables, the convergence of this integral is equivalent to the one of

$$\int_{[1,+\infty[^n} \frac{e^{2((1+\epsilon)g(t) - \langle A, t\rangle)}}{t_1^2} dt_1 \cdots dt_n,$$

where $A = \beta + \widetilde{\mathbb{1}}$. There is no loss of generality in replacing α_i by $c\,\alpha_i$, so we shall suppose that $c = 1$ in the following.

- First, we suppose that this integral converges for some $\epsilon > 0$. This implies that for all $\eta > 0$, the integral

$$\int_{[1,+\infty[^n} e^{2((1+\epsilon)g - \langle A + (\eta, 0, \ldots, 0), \cdot\rangle)} dt$$

converges, so that, thanks to proposition 1.1, we have :

$$\forall \eta > 0, \quad A + (\eta, 0, \ldots, 0) \in (1+\epsilon)\overset{\circ}{\widehat{P(g)}} \tag{1}$$

We claim that (1) is equivalent to

$$A \in \overset{\circ}{\widehat{P(\mathfrak{a})}} \cup \mathrm{ri}(F_1) \tag{2}$$

The implication (2) $\Rightarrow$ (1) is clear because as $\mathfrak{a} \nsubseteq (z_1)$, $F_1 \subset \{z_1 = 0\}$ contains thus the infinite face orthogonal to e_1 attached to $(1+\epsilon)P(\mathfrak{a})$, so that (1) holds.
As for the other direction, we first show that if A belongs to some finite (closed) face of $P(\mathfrak{a})$, then for all $\epsilon > 0$, (1) fails to be true. Indeed, as each α_i has non-negative components, any finite face of $P(\mathfrak{a})$ is included in some affine hyperplane $a + w^\perp$ where a, w have *positive* components. Thus the corresponding face for $(1+\epsilon)P(\mathfrak{a})$ is included in $(1+\epsilon)a + w^\perp$, and we have of course $(1+\epsilon)\overset{\circ}{\widehat{P(\mathfrak{a})}} \subset \{x; \langle w, x\rangle > (1+\epsilon)\langle w, a\rangle\}$.
Therefore we should have for all $\eta > 0$: $\langle A + (\eta, 0, \ldots, 0), w\rangle > (1+\epsilon)\langle w, a\rangle$, or equivalently $\eta w_1 > \epsilon\langle w, a\rangle$, which is absurd because η can be arbitrarily small.
The case where A belongs to one of the faces $F_2, \ldots, F_n$ is immediate, so we have proved that if $\mathbf{z}^\beta \in \mathcal{A}dj_H(\varphi)$, then (2) holds.

- Conversely, if $A \in \overset{\circ}{\widehat{P(\mathfrak{a})}}$, then $A \in (1+\epsilon)\overset{\circ}{\widehat{P(\mathfrak{a})}}$ for $\epsilon > 0$ sufficiently small, so that $e^{2[(1+\epsilon)g - \langle A, \cdot\rangle]}$ is integrable. In the case where $A \in \mathrm{ri}(F_1)$, then there exists some $\lambda = (0, \lambda_2, \ldots, \lambda_n) \in \mathbb{R} \times (\mathbb{R}_+^*)^{n-1}$ and some barycentric coefficients t_i such that $A = \sum t_i\alpha_i + \lambda$. As $g \leqslant \sum t_i\alpha_i$, we have : $(1+\epsilon)g - \langle A, \cdot \rangle \leqslant \langle \epsilon A - \lambda, \cdot \rangle$. As $F_1 \subset \{z_1 = 0\}$, the first component A_1 of A is zero, so that if we choose $0 < \epsilon < \min\{\frac{\lambda_i}{2A_i}; i \geqslant 2\}$, then

$$e^{(1+\epsilon)g(t)-\langle A,t\rangle} \leqslant e^{-\lambda_2 t_2/2} \cdots e^{-\lambda_n t_n/2}$$

and thus the integral

$$\int_{[1,+\infty[^n} \frac{e^{2((1+\epsilon)g(t)-\langle A,t\rangle)}}{t_1^2} dt_1 \cdots dt_n$$

is convergent. Therefore $\mathbf{z}^\beta \in \mathcal{A}dj_H(\varphi)$, which concludes the proof of the theorem. □

2.4. **The adjunction exact sequence.** We turn now to the generalized adjunction exact sequence. To prove the validity of the adjunction exact sequence in the analytic setting, we are going to use in a essential manner the proof of the so-called inversion of adjunction, that we may find in [DK01]. We will face two difficulties : the first one is to show that the restriction map is well-defined, and the second one is to show that this restriction induces a surjection, which is a consequence of the Ohsawa-Takegoshi-Manivel theorem.
Before going into the proof, we give an easy but useful result :

Lemma 2.2. *Let $\Omega \subset \mathbb{C}^n$ an open set that is relatively compact in the unit polydisk, let φ a psh function on Ω such that for all $z \in \Omega, \varphi(z) \leqslant -1$, let f be an holomorphic function on Ω, and $\alpha > 0$ a real number.*
If there exists $\epsilon > 0$ such that $\int_\Omega \frac{|f|^2 e^{-2(1+\epsilon)\varphi}}{(-\varphi)^\alpha} dV_\Omega$ converges, then there exists $\epsilon' > 0$ such that the integral $\int_\Omega |f|^2 e^{-2(1+\epsilon')\varphi} dV_\Omega$ converges.
In particular, if $\int_\Omega \frac{|f|^2 e^{-2(1+\epsilon)\varphi}}{\log^2 |z_n|} dV_\Omega$ converges, then $\int_{\Omega'} |f|^2 e^{-2(1+\epsilon')\varphi} dV_\Omega$ converges too, for some $\epsilon' > 0$ and all $\Omega' \Subset \Omega$.

Proof. We set $C = \inf\{e^{\epsilon x}/x^\alpha; x \geqslant 1\}$, it is a positive number. Then the inequality

$$\int_\Omega \frac{|f|^2 e^{-2(1+\epsilon)\varphi}}{(-\varphi)^\alpha} dV_\Omega \geqslant C \int_\Omega |f|^2 e^{-2(1+\epsilon/2)\varphi} dV_\Omega$$

shows the first assertion.
As for the second, we define $A = \{z \in \Omega; \varphi(z) \leqslant \frac{1}{4} \log |z_n|\}$ and $B = \{z \in \Omega; \varphi(z) \geqslant \frac{1}{4} \log |z_n|\}$. Then

$$\int_A \frac{|f|^2 e^{-2(1+\epsilon)\varphi}}{\log^2 |z_n|} dV_\Omega \geqslant \int_A \frac{|f|^2 e^{-2(1+\epsilon)\varphi}}{16\, \varphi^2} dV_\Omega$$

and using the first part, this implies that $\int_A |f|^2 e^{-2(1+\epsilon')\varphi} dV_\Omega$ is finite for some $\epsilon' > 0$.
Furthermore, setting $\delta = \min(\epsilon', 1)$, the following inequality holds on B : $-2(1+\delta)\varphi \leqslant -(1+\delta)/2 \log |z_n|$, thus :

$$\int_{B \cap \Omega'} |f|^2 e^{-2(1+\delta)\varphi} dV_\Omega \leqslant ||f||_{L^\infty(\Omega')} \int_\Omega |z_n|^{-\frac{1+\delta}{2}} dV_\Omega < +\infty$$

which concludes the proof of the lemma. □

Now we can prove the main result of this section :

Theorem 2.4. *Let X be a complex manifold, $H \subset X$ a smooth hypersurface, and φ a psh function on X, $\varphi_{|H} \neq -\infty$, such that e^φ is locally Hölder continuous, and let $i : H \hookrightarrow X$ be the inclusion. The the natural restriction map induces the following exact sequence :*

$$0 \longrightarrow \mathscr{I}_+(\varphi) \otimes \mathcal{O}_X(-H) \longrightarrow \mathcal{A}dj_H(\varphi) \longrightarrow i_* \mathscr{I}_+(\varphi_{|H}) \longrightarrow 0$$

Proof. What we have to check is that the restriction map is well-defined, that it is surjective, and that this sequence is exact. We proceed in the order we just described. As everything is purely local, we may assume that H is the hyperplane $z_n = 0$ in the polydisk $U = D(0, r), r < 1$ in $\mathbb{C}^n$. Moreover, since changing φ into $\varphi - C$ does not affect the questions of integrability, and since φ is locally upper bounded, we may assume that $\varphi \leqslant -1$ on U, so that we can apply the preceding lemma 2.2.

So, we choose a holomorphic non-zero function F, defined on a neighborhood U of 0, and satisfying $F \in \mathcal{A}dj_H(\varphi)(U)$. We write then $F(z) = F(z', z_n) = (F(z', z_n) - F(z', 0)) + F(z', 0)$, and as F is holomorphic, there exists a constant $C_1 > 0$ such that $|F(z', 0)|^2 \leqslant C_1|z_n|^2 + |F(z)|^2$ and therefore $|F(z)|^2 \geqslant |F(z', 0)|^2 - C_1|z_n|^2$.

Furthermore, as e^{φ} is Hölder, there exists $\alpha \in]0, 1]$ and $C_2 > 0$ such that

$$e^{2\varphi(z)} \leqslant \left(e^{\varphi(z',0)} + C_2|z_n|^{\alpha}\right)^2 \leqslant C_3(e^{2\varphi(z',0)} + |z_n|^{2\alpha})$$

with $C_3 = 4\max(1, C_2)$. Setting $f(z') = F(z', 0)$, we obtain the following inequalities :

$$\begin{aligned}
\frac{|F(z)|^2 e^{-2(1+\epsilon)\varphi(z)}}{|z_n|^2 \log^2 |z_n|} &\geqslant C_3^{-1} \frac{|F(z)|^2}{\log^2 |z_n|} \cdot \frac{1}{|z_n|^2 (e^{2\varphi(z',0)} + |z_n|^{2\alpha})^{1+\epsilon}} \\
&\geqslant \frac{C_3^{-1} |f(z')|^2}{|z_n|^2 \log^2 |z_n| (e^{2\varphi(z',0)} + |z_n|^{2\alpha})^{1+\epsilon}} \\
&\quad - \frac{C_3^{-1} C_1}{\log^2 |z_n| (e^{2\varphi(z',0)} + |z_n|^{2\alpha})^{1+\epsilon}}
\end{aligned}$$

Now we suppose that $U = U' \times D(0, r_n)$ (if it's not the case, we just have to restrict U a bit), and we partially integrate with respect to the last variable, in the family of disks $|z_n| < \rho(z')$ with $\rho(z') = \delta e^{(1+\epsilon)\alpha^{-1}\varphi(z',0)}$ where $\delta > 0$ is small enough so that $\rho(z') < r_n$ for all $z' \in U'$.

The right term in the right hand side is easily estimated when integrated, because $\log^2 |z_n| \geqslant \log^2 r > 0$, z_n being of module $\leqslant r < 1$, we have :

$$\int_{|z_n| < \rho(z')} \frac{C_1}{\log^2 |z_n| (e^{2\varphi(z',0)} + |z_n|^{2\alpha})^{1+\epsilon}} dV(z_n) \leqslant C_4 \delta^2 e^{(\frac{2}{\alpha} - 2)(1+\epsilon)\varphi(z',0)}$$

which is bounded because $\alpha \leqslant 1$.

As for the remaining term, we write :

$$\begin{aligned}
\int_{|z_n| < \rho(z')} \frac{dV(z_n)}{|z_n|^2 \log^2 |z_n| (e^{2\varphi(z',0)} + |z_n|^{2\alpha})^{1+\epsilon}} &\geqslant C_5 \int_{|z_n| < \rho(z')} \frac{e^{-2(1+\epsilon)\varphi(z',0)} dV(z_n)}{|z_n|^2 \log^2 |z_n|} \\
&\geqslant C_6 e^{-2(1+\epsilon)\varphi(z',0)} \int_0^{\rho(z')} \frac{dt}{t \log^2 t} \\
&= -C_6 \frac{e^{-2(1+\epsilon)\varphi(z',0)}}{\log \rho(z')}
\end{aligned}$$

Then, as $\log \rho(z') = \log \delta + (1+\epsilon)\alpha^{-1}\varphi(z', 0)$, the lemma 2.2 gives the expected result (instead of integrating, we could have written directly $(|z_n|^2 \log^2 |z_n|)^{-1} \geqslant (\rho(z')^2 \log^2 \rho(z'))^{-1}$).

To show the surjectivity of the last map, we use the local version of Ohsawa-Takegoshi-Manivel, with the weight $(1+\epsilon)\varphi$.

Finally, to show that the sequence is exact, if $f \in \mathcal{A}dj_H(\varphi)$ vanishes on $H \cap U$, then we write locally $f = g \cdot z_n$ where g is holomorphic, and satisfies on an open set $W \subset U$:

$$\int_W \frac{|g|^2 e^{-2(1+\epsilon)\varphi}}{\log^2 |z_n|} dV < +\infty$$

and using again lemma 2.2, we can conclude that $g \in \mathscr{I}_+(\varphi)(W)$, which had to be proved. □

Remark 2.4. *In the case where e^{φ} is not Hölder continuous, the restriction map may not be well-defined anymore : on the polydisk of radius $\frac{1}{2}$ in $\mathbb{C}^2$, we choose $f = 1$ and*

$$\varphi(z_1, z_2) = \max(-\lambda \log(-\log|z_1|), \log|z_2|)$$

with $0 < \lambda < \frac{1}{2}$. We then have $\varphi(z) \geqslant -\lambda \log(-\log|z_1|)$ thus $\frac{e^{-2\varphi(z)}}{|z_1|^2 \log^2|z_1|} \leqslant \frac{1}{|z_1|^2 |\log|z_1||^{2(1-\lambda)}}$ which is integrable on the polydisk. But on the hyperplane $\{z_1 = 0\}$, $e^{-2\varphi(z)} = |z_2|^{-2}$ is not integrable.

Remark 2.5. *If φ has analytic singularities, we know that $\mathscr{I}_+(\varphi) = \mathscr{I}(\varphi)$, and we have proved previously that $\mathcal{A}dj_H(\varphi)$ coincide with the algebraic ideal. Moreover, e^{φ} is clearly Hölder continuous near the poles, so theorem 2.4 is a generalization of the algebraic adjunction exact sequence given in* [Laz04].

Definition 2.5. *Let X be a complex manifold, and H an hypersurface of X. We pick an almost-psh function φ (ie it is locally the sum of a psh function and of a smooth function) non identically equal to $-\infty$ on H, T a positive closed current of bidegree $(1,1)$ on X well-defined on H, and h a singular hermitian metric of some holomorphic line bundle, satisfying $h_{|H} \not\equiv +\infty$. The we define :*

- *If locally, $\varphi = \psi + f$ with ψ psh and f smooth, then we set $\mathcal{A}dj_H(\varphi) := \mathcal{A}\mathrm{dj}_H(\psi)$, which makes sense globally;*
- *If locally $T = S + dd^c\varphi$ where S is smooth, and thus φ is almost-psh, we set $\mathcal{A}\mathrm{dj}_H(T) := \mathcal{A}dj_H(\varphi)$, which makes sense globally;*
- *If the metric h has an almost positive curvature current Θ_h, we set $\mathcal{A}\mathrm{dj}_H(h) := \mathcal{A}\mathrm{dj}_H(\Theta_h)$.*

Of course, we can make the same definition with the multiplier ideals instead of the adjoint ideals.

Combining the adjunction exact sequence and a variant of Nadel vanishing theorem, we can give a global result for extending holomorphic functions with some finite L^2 norms. So we have to prove the following result, which seems very natural in view of the openness conjecture :

Proposition 2.5. *Let (X, ω) be a weakly pseudoconvex Kähler manifold, and let (E, h) be a line bundle on X, where h is a singular hermitian metric, whose curvature tensor T satisfies $T \geqslant \eta\, \omega$ for some number $\eta > 0$. Then*

$$\forall q > 0, \quad H^q(X, \mathcal{O}_X(K_X + E) \otimes \mathscr{I}_+(h)) = 0.$$

Proof. We are going to show that we can solve the $\bar{\partial}$-operator with $L^2_+(\varphi)$-estimates; then it will suffice to follow the proof of Nadel's theorem in [DBIP96] to conclude. So we want to prove that provided an E-valued $\bar{\partial}$-closed (n, q)-form g with coefficients in $\mathscr{I}_+(T)$ –this property does not depend of the trivialisation of course–, there exists some E-valued $(n, q-1)$-form f with coefficients in $\mathscr{I}_+(T)$ such that $g = \bar{\partial} f$.
We write X as an increasing reunion $X = \bigcup_i U_i$ of relatively compact open subsets U_i. By the strong noetherian property, we know that there exists $\epsilon_i > 0$ such that $g_i := g_{|U_i}$ has coefficients in $\mathscr{I}((1+\epsilon_i)T)_{|U_i}$, so that the classical Hörmander's estimates applied to $(E, h^{1+\epsilon_i} \otimes h_0^{-\epsilon_i})$ (for some smooth metric h_0 on E, with local weight φ_0, and choosing ϵ_i sufficiently small so that $\epsilon_i\, i\partial\bar{\partial}\varphi_0 \geqslant -\frac{\eta}{2}$) give a $(n, q-1)$-form f_i on U_i (which is Kähler pseudoconvex) satisfying $g_i = \bar{\partial} f_i$ and

$$\int_{U_i} |f_i|^2 e^{-2(1+\epsilon_i)\varphi + 2\epsilon_i\varphi_0} dV_\omega \leqslant \frac{2}{q\eta} \int_{U_i} |g|^2 e^{-2(1+\epsilon_i)\varphi + 2\epsilon_i\varphi_0} dV_\omega$$

if φ is a local weight for h.
What we want to do is to glue all f_i's together, but a priori, they do not coincide on common definition sets. So we start by considering $v_{1,i} := (f_i - f_1)_{|U_1}$; as φ_0 is smooth, the $v_{1,i}$'s form a $L^2(U_1, e^{-2(1+\epsilon_1)\varphi})$-bounded family of holomorphic forms, so that we can find some subsequence $(v_{1,\sigma(i)})_i$ uniformly converging on U_1 to some holomorphic form v_1 which has finite $L^2(U_1, e^{-2(1+\epsilon_1)\varphi})$ norm too.
Now we iterate : we consider $v_{2,i} := (f_{\sigma(i)} - f_2)_{|U_2}$ for $i \geqslant 2$. We can extract a subsequence $(v_{2,\psi(i)})_i$ converging uniformly on U_2 to an holomorphic form v_2 with finite $L^2(U_2, e^{-2(1+\epsilon_2)\varphi}d)$ norm. Moreover, passing to the limit in the equation

$$(v_{2,\psi(i)} + f_2)_{|U_1} = f_{\sigma(\psi(i))|U_1} = v_{1,\sigma(\psi(i))} + f_1$$

we obtain $(v_2 + f_2)_{|U_1} = v_1 + f_1$.
By repeating the construction, we can construct (v_n) a sequence of holomorphic forms on U_n with finite $L^2(U_n, e^{-2(1+\epsilon_n)\varphi})$ norm such that

$$\begin{array}{rcl} \forall i \leqslant n,\ (v_n + f_n)_{|U_i} & = & v_i + f_i \\ \bar{\partial}(v_n + f_n) & = & g_{|U_n}. \end{array}$$

Therefore there exists some E-valued $(n, q-1)$-form f with coefficients in $\mathscr{I}_+(T)$ such that $g = \bar{\partial} f$, which concludes. □

Corollary 2.1. *Let (X, ω) be a weakly pseudoconvex Kähler manifold, $H \subset X$ a smooth hypersurface, (E, h) a holomorphic line bundle equipped with a singular hermitian metric h, $h_{|H} \not\equiv +\infty$, whose curvature current has local potentials φ such that e^φ is Hölder continuous, and such that there exists $\eta > 0$ satisfying $i\partial\bar{\partial}\varphi \geqslant \eta\,\omega$.*
Then every holomorphic section $s \in H^0(H, \mathcal{O}_H(K_H + E_H) \otimes \mathscr{I}_+(h_{|H}))$ extends to a section $\tilde{s} \in H^0(X, \mathcal{O}_X(K_X + H + E) \otimes \mathcal{A}\mathrm{dj}_H(h))$.

Proof. Tensorizing the adjunction exact sequence by $K_X + E + H$, we obtain :

$$0 \longrightarrow \mathscr{I}_+(h) \otimes \mathcal{O}_X(K_X{+}E) \longrightarrow \mathcal{A}\mathrm{dj}_H(h) \otimes \mathcal{O}_X(K_X{+}E{+}H) \longrightarrow i_*\mathscr{I}_+(h_{|H}) \otimes \mathcal{O}_H(K_H{+}E_H) \longrightarrow 0$$

If T is the Chern curvature of (E, h), then the last proposition show that :

$$H^1(X, \mathcal{O}_X(K_X + E) \otimes \mathscr{I}_+(h)) = H^1(X, \mathcal{O}_X(K_X + E) \otimes \mathscr{I}_+(T)) = 0$$

Therefore the restriction maps induces a surjection

$$H^0(X, \mathcal{O}_X(K_X + E + H) \otimes \mathcal{A}\mathrm{dj}_H(h)) \twoheadrightarrow H^0(H, \mathcal{O}_H(K_H + E_H) \otimes \mathscr{I}_+(h_{|H}))$$

which had to be proved. □

The approached we used to show this result, which relies in a essential manner on the local version of Manivel's theorem, is a natural way to obtain the global version of Manivel's theorem [Dem01]. Nevertheless, the result we obtain is a quite weaker version of the original Manivel's theorem, in the sense that it is qualitative (we don't have any control on the L^2 norm anymore), and is only given for "regular" currents (more precisely with Hölder psh local potentials).

2.5. **Back to the sheaf $\mathcal{A}dj^0_H(\varphi)$.** To finish, we would like to give one positive result concerning the ideal $\mathcal{A}dj^0_H(\varphi)$. For this, the crucial fact is given in the following lemma, which assumptions are unfortunately really restrictive :

Lemma 2.3. *Let φ be a psh function which has only analytic or toric singularities on a bounded open set $B \subset \mathbb{C}^n$, satisfying $\varphi < C < 0$ on B, and let f be an holomorphic function on B. If the integral*

$$\int_B |f|^2 \frac{e^{-2\varphi}}{\varphi} dV$$

converges, then so does the integral

$$\int_B |f|^2 e^{-2\varphi} dV.$$

Proof. We start with the case where φ has analytic singularities. As B is bounded, the question is actually local, and using a log resolution, and using the notations of section 2.2, the integrability assumption becomes :

$$\int_{U'} \frac{\prod |z_j|^{2(c_j+d_j-a_j)}}{|\log(\prod |z_j|^{a_j})|} dV' < +\infty,$$

and performing a toric change of variable, the Bertrand criterion shows that

$$\int_{U'} \prod |z_j|^{2(c_j+d_j-a_j)} dV' < +\infty,$$

which we had to show.

Let us now get to the toric case. Again, the question is local, so we borrow the techniques of the first part. The calculus appearing in the proof of theorem 1.3 shows that we are boiled down to show the integrability of e^h for some h concave (more precisely, f can be chosen to be a monomial z^α, and $h = g - \langle \alpha, \cdot \rangle$ for g the concave function attached to φ). As for non-zero concave function h of 1-variable (say on $\mathbb{R}^+$) the integrability of e^h/g for g a non-zero concave function implies the one of e^h (indeed, $h(x) = O_{x \to +\infty}(x)$ or more precisely, either $h(x)$ tends to 0 when x goes to $+\infty$, or it is equivalent to ℓx where $\ell = \lim \frac{d^+}{dx} h(x, \cdot)$ is non-zero; idem for g), we can follow the proof of proposition 1.1 to show that this extends to all dimensions.

□

Denoting $\widetilde{\mathscr{I}(\varphi)}$ the analogue of the multiplier ideal sheaf where we replace the integrability condition by the local integrability of $\frac{|f|^2 e^{-2\varphi}}{\log^2 |s|}$, where s is a (local) section defining H, with $ds_{|H}$ never zero. Then we have the following result :

Theorem 2.5. *Let φ be a Hölder psh function whose singularities are only analytic or toric, and let $i : H \hookrightarrow X$ the inclusion. The the natural restriction map induces the following exact sequence :*

$$0 \longrightarrow \widetilde{\mathscr{I}(\varphi)} \otimes \mathcal{O}_X(-H) \longrightarrow \mathcal{A}dj^0_H(\varphi) \longrightarrow i_* \mathscr{I}(\varphi_{|H}) \longrightarrow 0$$

Remark 2.6. *In particular, under these assumptions, $\widetilde{\mathscr{I}(\varphi)}$ is a coherent ideal sheaf.*

Proof. The proof is very similar to the one of the adjunction exact sequence. The only difference appearing here concerns the restriction map, which has *a priori* no reason to be well-defined. So we take $F \in \mathcal{A}dj^0_H(\varphi)$, and as previously, we have :

$$\begin{aligned} \frac{|F(z)|^2 e^{-2\varphi(z)}}{|z_n|^2 \log^2 |z_n|} &\geqslant C_3^{-1} \frac{|F(z)|^2}{\log^2 |z_n|} \cdot \frac{1}{|z_n|^2 (e^{2\varphi(z',0)} + |z_n|^{2\alpha})} \\ &\geqslant \frac{C_3^{-1} |f(z')|^2}{|z_n|^2 \log^2 |z_n| (e^{2\varphi(z',0)} + |z_n|^{2\alpha})} - \frac{C_3^{-1} C_1}{\log^2 |z_n| (e^{2\varphi(z',0)} + |z_n|^{2\alpha})} \end{aligned}$$

We also suppose that $U = U' \times D(0, r_n)$, and we partially integrate with respect to the last variable, in the family of disks $|z_n| < \rho(z')$ avec $\rho(z') = \epsilon e^{\alpha^{-1}\varphi(z',0)}$ with $\epsilon > 0$ small enough so that $\rho(z') < r_n$ for all $z' \in U'$.

The right term in the right hand side is easily estimated when integrated, because $\log^2 |z_n| \geqslant \log^2 r > 0$, z_n being of module $\leqslant r < 1$, we have :

$$\int_{|z_n| < \rho(z')} \frac{C_1}{\log^2 |z_n| (e^{2\varphi(z',0)} + |z_n|^{2\alpha})} dV(z_n) \leqslant C_4 \epsilon^2 e^{(\frac{2}{\alpha} - 2)\varphi(z',0)}$$

which is bounded because $\alpha \leqslant 1$ and φ is upper bounded. For the remaining term :

$$\begin{aligned}\int_{|z_n|<\rho(z')} \frac{dV(z_n)}{|z_n|^2 \log^2 |z_n|(e^{2\varphi(z',0)} + |z_n|^{2\alpha})} &\geqslant C_5 \int_{|z_n|<\rho(z')} \frac{e^{-2\varphi(z',0)} dV(z_n)}{|z_n|^2 \log^2 |z_n|} \\ &\geqslant C_6 e^{-2\varphi(z',0)} \int_0^{\rho(z')} \frac{dt}{t \log^2 t} \\ &= -C_6 \frac{e^{-2\varphi(z',0)}}{\log \rho(z')}\end{aligned}$$

Then we write $\log \rho(z') = \log \epsilon + \alpha^{-1}\varphi(z', 0)$, and using the lemma 2.3, the proof is finished. $\square$

Remark 2.7. *So if we knew that the lemma 2.3 still holds under the general assumption that e^{φ} is Hölder continuous, we would have a general twisted adjunction exact sequence for $\mathcal{A}dj^0_H(\varphi)$.*

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Henri Guenancia, Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie, Paris

E-mail address: `guenancia@math.jussieu.fr`

URL: `www.math.jussieu.fr/~guenancia`