# Time reparametrization symmetry in a short-range p-spin model

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**Abstract.** We explore the existence of time reparametrization symmetry in the p-spin model. Using the Martin-Siggia-Rose generating functional, we analytically probe the long-time dynamics. We perform a Renormalization Group analysis where we systematically integrate over short-time scale fluctuations. We find that the RG flow converges to a fixed point that is invariant under reparametrizations of the time variable. This continuous symmetry is broken in the glass state and we argue that this gives rise to the presence of Goldstone modes. We expect the Goldstone modes to determine the properties of fluctuations in the glass state, in particular predicting the presence of dynamical heterogeneity.

PACS numbers: 64.70.Q-, 61.20.Lc, 61.43.Fs, 05.40.-a

## 1. Introduction

Very slow dynamics is an essential feature of glasses [1]. In both structural glasses and spin glasses slow dynamics is manifested through a dramatic increase in the relaxation times. The slowdown has been captured in mean field theories, such as the mode coupling theory for supercooled liquids [2] and the dynamical theory for mean field spin glass models [3, 4, 5, 6]. Even though mean field theories are useful in describing some aspects of glassy dynamics they do not completely capture phenomena associated with fluctuations. Fluctuations have been shown, with the discovery of dynamical heterogeneities, to be central to an understanding of glassy dynamics [7].

Dynamical heterogeneities - mesoscopic regions that evolve differently from each other as well as from the bulk - have been shown to be present in experimental studies of materials close to the glass transition [8, 9, 10] and in simulations of both spin glasses and structural glasses [11, 12, 13]. Their presence has been directly observed at the microscopic level in experiments on colloidal glasses [10] and granular systems [14]. Understanding the onset of heterogeneities without an apparent structural trigger is believed to be key to an understanding of the glass transition [7]. Theoretical attempts to explain the emergence of heterogeneous dynamics as the glass transition is approached include among others a geometrical explanation. In the geometrical picture dynamical heterogeneities result from non-trivial structure in the space of trajectories due to dynamical constraints [15]. Another explanation is provided by the Random First Order transition (RFOT) approach, in which a liquid freezes into a mosaic of aperiodic crystals [16]. Here we will explore a different theoretical avenue to explain dynamical heterogeneities, which is based on time reparametrization symmetry [17, 18, 19].

Time reparametrization symmetry (TRS), the invariance under transformations of the time variable $t \to h(t)$, was discovered some years ago in the non-equilibrium mean field dynamics of spin glasses (Sherrington-Kirkpatrick and p-spin glass models) [5, 6]. The symmetry was shown to be present in the long-time limit of the mean field evolution equations and implies that the asymptotic equations do not have a unique solution [5, 6, 20]. In more recent studies, TRS has been proved to be present in the long time dynamics of a short range spin glass model, the Edwards-Anderson model [17, 18, 19]. In these studies, the proof of the symmetry is at the level of the generating functional, including all fluctuations. Using the renormalization group (RG), it was shown that the stable fixed point of the generating functional is invariant under reparametrizations of the time variable.

The explanation for dynamical heterogeneities from TRS is derived from the fact that TRS is spontaneously broken by the correlations and responses in the glass state. In the absence of long range interactions or gauge fields, a spontaneously broken continuous symmetry is expected to give rise to Goldstone modes [21], and these modes are associated with spatially correlated fluctuations of the time variable, which give rise to heterogeneous dynamics. In this work we go beyond mean field theory and give a proof that the p-spin model is invariant under time reparametrizations regardless of the spatial extent of the interactions. We discuss how this may carry over to models of structural glasses which have been shown to be connected to the p-spin model [22, 23]. Furthermore, we discuss the connection between TRS and heterogeneities. The proposal that dynamical heterogeneities originate in fluctuations of the time variable is supported by positive evidence from numerical studies in spin glasses [18] and in structural glasses [24, 25].

In this work we consider a system of soft spins on a lattice, with p-spin interactions. The spin couplings are assumed to be uncorrelated Gaussian random variables with zero mean. We assume Langevin dynamics for the spins with a white noise term that represents the coupling of the spins to a heat reservoir. We set up the calculation by writing the generating functional of the spin correlations and responses using the Martin-Siggia-Rose (MSR) approach and introduce two-time fields that are associated with the spin correlations and responses. To study the long time dynamics we start the renormalization group procedure by introducing a short time cutoff $\tau_0$. We systematically increase the short time cutoff by integrating over the two-time fields associated with the shortest time differences, thus following a procedure analogous to Wilson's approach to the RG. In our case, however, we integrate over fluctuations that are fast in *time*, not in space. We find that the RG flow converges to a fixed point generating functional which is invariant under reparametrizations of the time variable.

The rest of the paper is organized as follows: We start by giving a description of the model in Sec. 2. Then, we illustrate how we obtain the generating functional through the MSR formalism in Sec. 3. In Sec. 4 we show how we use Wilson's approach to the renormalization group to get the stable fixed point. We then show, in Sec. 5, that the stable fixed point generating functional is invariant under reparametrizations of the time variable. We end with a discussion of our results and conclusions in Sec. 6.

## 2. Model

The p-spin Hamiltonian is given by

$$H = -\frac{1}{p!}\sum_{i_1 \dots i_p} J_{i_1 \dots i_p}\phi_{i_1}\dots\phi_{i_p}, \tag{1}$$

where the $\{\phi\}_{i=1,\cdots,N}$ are soft spins with the spherical constraint $\sum_{i=1}^{N}[\phi_i(t)]^2 = N$ and the couplings are assumed to be uncorrelated, Gaussian distributed, zero mean random variables, $P\{J\} = \prod_{i_1<\dots<i_p} \frac{1}{\sqrt{4\pi K_{i_1\dots i_p}}} \exp[-J^2_{i_1\dots i_p}/4K_{i_1\dots i_p}]$. The dynamics is given by the Langevin equation

$$\Gamma_0^{-1}\partial_t\phi_i(t) = -\frac{\delta H}{\delta\phi_i(t)} + \eta_i(t), \tag{2}$$

where the $\{\eta_i(t)\}_{i=1,\dots,N}$ are assumed to be zero mean gaussian random variables with the correlation $\langle\eta_i(t)\eta_j(t')\rangle = 2T\delta_{ij}\delta(t-t')$ that couple the spins to a thermal bath at temperature $T$. Then the Langevin equation can be written as

$$\Gamma_0^{-1}\partial_t\phi_i(t) = \frac{p}{p!}\sum_{i_1\dots i_{p-1}} J_{i,i_1\dots i_{p-1}}\phi_{i_1}\dots\phi_{i_{p-1}} + \eta_i(t). \tag{3}$$

## 3. MSR generating functional

From the Langevin equation we use the Martin-Siggia-Rose formalism [26] and write down the noise averaged generating functional

$$\langle Z[\{l_i\},\{h_i\}]\rangle = \int D\phi D\hat{\phi} D\hat{\varphi} D\hat{N} \exp\Bigg\{ L[\phi,\hat{\phi}] + \sum_{i=1}^{N} \int_{t_0}^{t_f} dt [l_i(t)\phi_i(t)$$
$$+ih_i(t)\hat{\phi}_i(t)] + i\sum_{i=1}^{N} \hat{\varphi}_i[\phi_i(t_0)-\varphi_i] + i\int_{t_0}^{t_f} dt\hat{N}(t)\left[\sum_{i=1}^{N}\phi_i^2(t) - N\right]\Bigg\}, \quad (4)$$

where the $l$ and $h$ are sources and the last two terms in the exponent are due to the initial condition and spherical constraint, respectively. The action $L[\phi,\hat{\phi}]$ is given by

$$L[\phi,\hat{\phi}] = -i\sum_{i=1}^{N}\int_{t_0}^{t_f} dt\; \hat{\phi}_i(t)\Big[\Gamma_0^{-1}\partial_t\phi_i(t) - iT\hat{\phi}_i(t)$$
$$-\frac{p}{p!}\sum_{i_1\ldots i_{p-1}} J_{i,i_1\ldots i_{p-1}}\phi_{i_1}\ldots\phi_{i_{p-1}}\Big]. \quad (5)$$

We now average the generating functional over the disorder in the system. The action contains only one term with an explicit dependence on the disorder, which we call $L_J[\phi,\hat{\phi}]$,

$$L_J[\phi,\hat{\phi}] = i\frac{p}{p!}\sum_{i_1\ldots i_p}\int_{t_0}^{t_f} dt J_{i_1\ldots i_p}\hat{\phi}_{i_1}(t)\phi_{i_2}(t)\cdots\phi_{i_p}(t). \quad (6)$$

We now compute the part of the generating functional affected by disorder averaging,

$$\overline{\exp(L_J[\phi,\hat{\phi}])} = \int DJ P\{J\}\exp[L_J], \quad (7)$$

with $DJ \equiv \prod_{i_1<\cdots<i_p} dJ_{i_1\ldots i_p}$. Therefore, after integrating over the disorder we have $\overline{\exp(L_J[\phi,\hat{\phi}])}$ given by

$$\overline{\exp(L_J[\phi,\hat{\phi}])} = \exp\Bigg[-\frac{p^2}{p!}\sum_{i_1\ldots i_p} K_{i_1\ldots i_p}\int_{t_0}^{t_f}\int_{t_0}^{t_f} dtdt'$$
$$\times \sum_{\alpha_{i_r},\alpha'_{i_r}\in\{0,1\}}^{C=1,C'=1}\;\sum_{i_1\ldots i_p}\prod_{r=1}^{p}\phi_{i_r}^{\alpha_{i_r}}(t)\phi_{i_r}^{\alpha'_{i_r}}(t')\Bigg], \quad (8)$$

where we have re-labeled the fields using the definitions $\phi_i^0(t)\equiv\hat{\phi}_i(t)$ and $\phi_i^1(t)\equiv\phi_i(t)$. The constrained variables $C$ and $C'$ are given by $C\equiv\sum_{r=1}^{p}(1-\alpha_{i_r})$, $C'\equiv\sum_{r=1}^{p}(1-\alpha'_{i_r})$, and the constraints $C=C'=1$ enforce the condition that for each of the two times $t$ and $t'$, there is a product of fields, of which only one is a $\hat{\phi}$ and all others are $\phi$ fields. We are also interested in introducing two-time fields $Q_i^{\alpha,\alpha'}(t_1,t_2)$, physically associated with two-time correlations and responses. In order to do this we write the number one in terms of an integral of a product of delta functions that enforce the condition $Q_i^{\alpha,\alpha'}(t_1,t_2)=\phi_i^{\alpha}(t_1)\phi_i^{\alpha'}(t_2)$,

$$1 = \int DQ\prod_{i,t_1,t_2}\;\prod_{\alpha_i,\alpha'_i\in\{0,1\}}\delta\left(Q_i^{\alpha_i,\alpha'_i}(t_1,t_2)-\phi_i^{\alpha_i}(t_1)\phi_i^{\alpha'_i}(t_2)\right). \quad (9)$$

By writing the delta function in exponential form we get

$$1 = \int DQD\hat{Q} \exp \left\{ i \sum_i \sum_{\alpha_i, \alpha'_i \in \{0,1\}} \int dt_1 dt_2 \hat{Q}_i^{\overline{\alpha_i}, \overline{\alpha'_i}}(t_1, t_2) \right. \\ \left. \times \left( Q_i^{\alpha_i, \alpha'_i}(t_1, t_2) - \phi_i^{\alpha_i}(t_1) \phi_i^{\alpha'_i}(t_2) \right) \right\}, \tag{10}$$

where we have introduced the auxiliary two-time fields $\hat{Q}_i^{\alpha_i, \alpha'_i}(t_1, t_2)$ and the notation $\overline{0} = 1$, $\overline{1} = 0$. We now obtain the noise and disorder averaged generating functional

$$\overline{\langle Z[\{l_i\}, \{h_i\}] \rangle} = \int DQD\hat{Q}D\phi^0 D\phi^1 D\hat{\varphi} D\hat{N} \exp(\mathcal{S}), \\ \mathcal{S} = S_1 + S_J + S_{spin} + S_{ext} + S_{BC} + S_{SC}. \tag{11}$$

Here we have written the different terms of the action separately:

$$S_1[Q, \hat{Q}, \phi^0, \phi^1] = i \sum_i \sum_{\alpha_i, \alpha'_i \in \{0,1\}} \int_{t_0}^{t_f} dt_1 dt_2 \hat{Q}_i^{\overline{\alpha_i}, \overline{\alpha'_i}}(t_1, t_2) \\ \times \left( Q_i^{\alpha_i, \alpha'_i}(t_1, t_2) - \phi_i^{\alpha_i}(t_1) \phi_i^{\alpha'_i}(t_2) \right), \tag{12}$$

$$S_J[Q] = -\frac{p^2}{p!} \sum_{i_1 \ldots i_p} K_{i_1 \ldots i_p} \int_{t_0}^{t_f} dt_1 dt_2 \sum_{\alpha_{i_r}, \alpha'_{i_r} \in \{0,1\}}^{C=1, C'=1} \prod_{r=1}^{p} Q_{i_r}^{\alpha_{i_r}, \alpha'_{i_r}}(t_1, t_2), \tag{13}$$

$$S_{spin}[\phi^0, \phi^1] = -i \sum_{i=1}^{N} \int_{t_0}^{t_f} dt \phi_i^0(t) \left[ \Gamma^{-1} \partial_t \phi_i^1(t) + \gamma_{00} \phi_i^0(t) \right] - \frac{p^2}{p!} \sum_{i_1 \ldots i_p} K_{i_1 \ldots i_p} \\ \times \int_{t_0}^{t_f} dt_1 dt_2 \, g(t_1 - t_2) \sum_{\alpha_{i_r}, \alpha'_{i_r} \in \{0,1\}}^{C=1, C'=1} \prod_{r=1}^{p} \phi_{i_r}^{\alpha_{i_r}}(t_1) \phi_{i_r}^{\alpha'_{i_r}}(t_2), \tag{14}$$

$$S_{ext}[\phi^0, \phi^1; l, h] = \int_{t_0}^{t_f} dt \left[ l_i(t) \phi_i^1(t) + i h_i(t) \phi_i^0(t) \right], \tag{15}$$

$$S_{BC}[\phi^1, \hat{\varphi}; \varphi] = i \sum_{i=1}^{N} \hat{\varphi}_i \left[ \phi_i^1(t_0) - \varphi_i \right], \tag{16}$$

$$S_{SC}[\phi^1, \hat{N}; N] = i \int_{t_0}^{t_f} dt \hat{N}(t) \left[ \sum_{i=1}^{N} (\phi_i^1(t))^2 - N \right], \tag{17}$$

and we have $\Gamma = \Gamma_0$, $\gamma_{00} = -iT$ and $g(t - t') = 0$ at the start of the RG flow.

## 4. Renormalization group analysis

We perform a Renormalization Group analysis on the time variables. For simplicity we take $t_0 = 0$ and $t_f = \infty$ from now on. We focus on the two-time fields. First, we introduce a cutoff in the integration of two-time fields, $\tau_0 \leq |t_1 - t_2|$. We then write the terms of the action affected by the cutoff:

$$S_1[Q,\hat{Q}] = i\sum_i \int_{\substack{0\leq t_1,t_2<\infty \\ \tau_0\leq|t_1-t_2|}} dt_1 dt_2 \sum_{\alpha_i,\alpha_i'} \hat{Q}_i^{\overline{\alpha_i},\overline{\alpha_i'}}(t_1,t_2) \times \left(Q_i^{\alpha_i,\alpha_i'}(t_1,t_2) - \phi_i^{\alpha_i}(t_1)\phi_i^{\alpha_i'}(t_2)\right), \tag{18}$$

$$S_J[Q] = -\frac{p^2}{p!}\sum_{i_1\ldots i_p} K_{i_1\ldots i_p} \int_{\substack{0\leq t_1,t_2<\infty \\ \tau_0\leq|t_1-t_2|}} dt_1 dt_2 \sum_{\alpha_{i_r},\alpha_{i_r}'\in\{0,1\}}^{C=1,C'=1} \prod_{r=1}^{p} Q_{i_r}^{\alpha_{i_r},\alpha_{i_r}'}(t_1,t_2). \tag{19}$$

We define fast and slow fields respectively by $Q_{>i}^{\alpha_i,\alpha_i'}(t_1,t_2) = Q_i^{\alpha_i,\alpha_i'}(t_1,t_2)$, for $\tau_0 \leq |t_1 - t_2| < b\tau_0$ and $Q_{<i}^{\alpha_i,\alpha_i'}(t_1,t_2) = Q_i^{\alpha_i,\alpha_i'}(t_1,t_2)$, for $b\tau_0 \leq |t_1 - t_2|$, with $b > 1$. This separation of fast and slow parts of the fields results in a separation in the terms:

$$S_1[Q,\hat{Q},\phi^0,\phi^1] = S_1[Q_>,\hat{Q}_>,\phi^0,\phi^1] + S_1[Q_<,\hat{Q}_<,\phi^0,\phi^1], \tag{20}$$

$$S_J[Q] = S_J[Q_>] + S_J[Q_<]. \tag{21}$$

Next we calculate the integral $I_>$ over fast fields. To do this we use the fact that there are no cross-terms between fast and slow fields in the integral:

$$I_> = \int DQD\hat{Q}\exp\left\{ i\sum_i \int_{\substack{\tau_0\leq|t_1-t_2|<b\tau_0 \\ 0\leq t_1,t_2<\infty}} dt_1 dt_2 \sum_{\alpha_i,\alpha_i'} \hat{Q}_{>i}^{\overline{\alpha_i},\overline{\alpha_i'}}(t_1,t_2) \times \left(Q_{>i}^{\alpha_i,\alpha_i'}(t_1,t_2) - \phi_i^{\alpha_i}(t_1)\phi_i^{\alpha_i'}(t_2)\right) - \frac{p^2}{p!}\sum_{i_1\ldots i_p} K_{i_1\ldots i_p} \int_{\substack{\tau_0\leq|t_1-t_2|<b\tau_0 \\ 0\leq t_1,t_2<\infty}} dt_1 dt_2 \sum_{\alpha_{i_r},\alpha_{i_r}'\in\{0,1\}}^{C=1,C'=1} \prod_{r=1}^{p} Q_{>i_r}^{\alpha_{i_r},\alpha_{i_r}'}(t_1,t_2) \right\}. \tag{22}$$

Calculating the delta function integral constitutes undoing the delta function integral transformation we used to introduce the two-time fields for the fast modes. Hence,

$$I_> = \exp\left\{ -\frac{p^2}{p!}\sum_{i_1\ldots i_p} K_{i_1\ldots i_p} \int_{\substack{\tau_0\leq|t_1-t_2|<b\tau_0 \\ 0\leq t_1,t_2<\infty}} dt_1 dt_2 \sum_{\alpha_{i_r},\alpha_{i_r}'\in\{0,1\}}^{C=1,C'=1} \prod_{r=1}^{p} \phi_{i_r}^{\alpha_{i_r}}(t_1)\phi_{i_r}^{\alpha_{i_r}'}(t_2) \right\}. \tag{23}$$

Next we re-scale all the one-time and two-time fields

$$Q_{<i}^{\alpha_i,\alpha_i'}(bt_1',bt_2') = b^{\lambda_{\alpha_i\alpha_i'}} Q'^{\alpha_i,\alpha_i'}_i(t_1',t_2'), \tag{24}$$

$$\hat{Q}_{<i}^{\alpha_i,\alpha_i'}(bt_1',bt_2') = b^{\hat{\lambda}_{\alpha_i\alpha_i'}} \hat{Q}'^{\alpha_i,\alpha_i'}_i(t_1',t_2'), \tag{25}$$

$$bt' = t, \tag{26}$$

$$\phi_i^{\alpha_i}(bt') = b^{\lambda_{\alpha_i}} \phi'^{\alpha_i}_i(t'), \tag{27}$$

$$l_i(bt') = b^{\lambda_l} l_i'(t'), \tag{28}$$

$$h_i(bt') = b^{\lambda_h} h_i'(t'), \tag{29}$$

$$\hat{\varphi}_i = b^{\lambda_\varphi} \hat{\varphi}_i'. \tag{30}$$

By rescaling the fields in the part of the action arising from the delta function ($S_1$) we get

$$S'_1[Q,\hat{Q},\phi^0,\phi^1] = i \int\limits_{\tau_0 \le |b(t_1-t_2)|} dt'_1 dt'_2 \hat{Q}'^{\overline{\alpha_i},\overline{\alpha'_i}}_i(t'_1,t'_2)$$
$$\times \left( b^{\hat{\lambda}_{\overline{\alpha_i},\overline{\alpha'_i}}+\lambda_{\alpha_i,\alpha'_i}+2} Q'^{\alpha_i,\alpha'_i}_i(t'_1,t'_2) - b^{\hat{\lambda}_{\overline{\alpha_i},\overline{\alpha'_i}}+\lambda_{\alpha_i}+\lambda_{\alpha_i}+2} \phi'^{\alpha_{ir}}_i(t'_1)\phi'^{\alpha'_i}_i(t'_2) \right). \quad (31)$$

Since this term represents the constant one, it must be marginal in the RG transformation. So then we obtain the two equations

$$\hat{\lambda}_{\overline{\alpha_i},\overline{\alpha'_i}} + \lambda_{\alpha_i,\alpha'_i} + 2 = 0 \quad (32)$$

$$\hat{\lambda}_{\overline{\alpha_i},\overline{\alpha'_i}} + \lambda_{\alpha_i} + \lambda_{\alpha'_i} + 2 = 0 \quad (33)$$

By rescaling the fields in the part of the action arising from the disorder average ($S_J$) we get

$$S_J[Q] = -\frac{p^2}{p!} b^{\lambda_J} \sum_{i_1\ldots i_p} K_{i_1\ldots i_p} \int\limits_{\tau_0 \le |b(t'_1-t'_2)|} dt'_1 dt'_2$$
$$\times \sum_{\alpha_{i_r},\alpha'_{i_r}\in\{0,1\}}^{C=1,C'=1} \sum_{i_1\ldots i_p} \prod_{r=1}^{p} Q'^{\alpha_{i_r},\alpha'_{i_r}}_{i_r}(t'_1,t'_2). \quad (34)$$

Using the relation between $\lambda_{\alpha,\alpha'}$, $\lambda_\alpha$ and $\lambda_{\alpha'}$ implied by Eq. (32) and Eq. (33) together with the constraints $C = C' = 1$ we get $\lambda_J \equiv 2\lambda_0 + 2(p-1)\lambda_1 + 2$. We argue that since this is the term that produces the glassy freezing, it must be present in the long time dynamics. Therefore, it must be marginal, i.e.

$$\lambda_J \equiv 2\lambda_0 + 2(p-1)\lambda_1 + 2 = 0 \quad (35)$$

There are an infinite number of possible solutions to this equation. The most simple of these solutions is given by

$$\lambda_1 = -\overline{1} = 0 \quad (36)$$
$$\lambda_0 = -\overline{0} = -1. \quad (37)$$

We note that this solution is the only one consistent with the two-time spin correlations being frozen, which is what we expect in the glass state. Rescaling the fields in the source terms yields

$$S'_{ext}[\phi^0,\phi^1;l,h] = \sum_{i=1}^{N} \int_0^\infty dt' \left[ b^{1+\lambda_l+\lambda_1} l'_i(t')\phi'^1_i(t') + b^{1+\lambda_h+\lambda_0} i h'_i(t')\phi'^0_i(t') \right]. \quad (38)$$

The source terms are marginal by virtue of being derived from external fields, hence

$$\lambda_l = -1, \quad (39)$$
$$\lambda_h = 0. \quad (40)$$

Rescaling the fields in the boundary term we get;

$$S'_{BC}[\phi^1,\hat{\varphi};\varphi] = i \sum_{i=1}^{N} b^{\lambda_\varphi} \hat{\varphi}'_i \left[ b^{\lambda_1}\phi'^1_i(t_0) - \varphi'_i \right] \quad (41)$$

Since $\lambda_1 = 0$, the boundary term is marginal if $\lambda_\varphi = 0$. Next we consider the constraint term and obtain

$$S'_{CS}[\phi^1, \hat{N}; N] = ib^{1+\lambda_N} \int_0^\infty dt' \hat{N}'(t) \left[ \sum_{i=1}^{N} (b^0 \phi'^1_i(t))^2 - N' \right]. \tag{42}$$

There is no condition on this term. But requiring this term to be marginal so that the constraint is still enforced at long times gives $\lambda_N = -1$. Finally we re-scale the fields in the spin term

$$S_{spin}[\phi^0, \phi^1] = -i \sum_{i=1}^{N} \int_0^\infty dt' \phi'^0_i(t') \left[ b^{-1}\Gamma^{-1}\partial_{t'}\phi'^1_i(t') + b^{-1}\gamma_{00}\phi'^0_i(t') \right]$$
$$-\frac{p^2}{p!} \sum_{i_1 \dots i_p} K_{i_1 \dots i_p} \int_0^\infty dt'_1 dt'_2 b^0 \, g(t'_1 - t'_2) \sum_{\alpha_{i_r}, \alpha'_{i_r} \in \{0,1\}}^{C=1, C'=1} \prod_{r=1}^{p} \phi'^{\alpha_{i_r}}_{i_r}(t'_1) \phi'^{\alpha'_{i_r}}_{i_r}(t'_2) \tag{43}$$

where we have used the fact that $\lambda_0 = -1$, $\lambda_1 = 0$, $\lambda_J = 0$ and the constraints $C = C' = 1$ to determine the powers of $b$ in the terms. The result is the following set of flow equations

$$\Gamma^{-1} \to \Gamma'^{-1} = (b\Gamma)^{-1}, \tag{44}$$
$$\gamma_{00} \to \gamma'_{00} = \gamma_{00} b^{-1}, \tag{45}$$
$$g(t_1 - t_2) \to g'(t'_1 - t'_2) = g(b(t'_1 - t'_2)) + \mathcal{C}_{\tau_0 \le |bt'_1 - bt'_2| < b\tau_0}, \tag{46}$$

where $\mathcal{C}_\mathcal{P}$ is defined by $\mathcal{C}_\mathcal{P} = 1$ if $\mathcal{P}$ is true and $\mathcal{C}_\mathcal{P} = 0$ if $\mathcal{P}$ is not true. If we let $b = e^{dl} \cong 1 + dl$ then the flow equations can be written as follows

$$\frac{d\Gamma}{dl} = \Gamma, \tag{47}$$
$$\frac{d\gamma_{00}}{dl} = -\gamma_{00}. \tag{48}$$

From the flow equations we obtain the following fixed points

$$\Gamma = 0, \infty, \tag{49}$$
$$\gamma_{00} = 0, \infty, \tag{50}$$
$$g(t_1 - t_2) = \mathcal{C}_{|t_1 - t_2| < \tau_0}. \tag{51}$$

The stable fixed points are $\Gamma = \infty$, $\gamma_{00} = 0$ and $g(t_1 - t_2) = \mathcal{C}_{|t_1 - t_2| < \tau_0}$. The stable fixed point values indicate that the derivative term and the noise term are not present in the fixed point generating functional. In the next section we will show that these are the only terms that break time reparametrization invariance and this will lead to the fixed point generating functional being invariant under reparametrizations of the time variable. The stable fixed point generating functional we have obtained is

$$
\begin{aligned}
&\mathcal{Z}_{fp}[l,h] = \overline{\langle Z[\{l_i\},\{h_i\}]\rangle}_{fp} = \int DQD\hat{Q}D\phi^0 D\phi^1 D\hat{\varphi} D\hat{N} \\
&\times \exp\Bigg\{ i\sum_i \int_0^\infty dt_1 dt_2 \sum_{\alpha_i,\alpha_i'} \hat{Q}_i^{\overline{\alpha_i},\overline{\alpha_i'}}(t_1,t_2)\left(Q_i^{\alpha_i,\alpha_i'}(t_1,t_2) - \phi_i^{\alpha_i}(t_1)\phi_i^{\alpha_i'}(t_2)\right) \\
&-\frac{p^2}{p!}\sum_{i_1\ldots i_p} K_{i_1\ldots i_p}\int_0^\infty dt_1 dt_2 \sum_{\alpha_{i_r},\alpha_{i_r}'\in\{0,1\}}^{C=1,C'=1} \prod_{r=1}^p Q_{i_r}^{\alpha_{i_r},\alpha_{i_r}'}(t_1,t_2) \\
&+\int_0^\infty dt[l_i(t)\phi_i^1(t) + ih_i(t)\phi_i^0(t)] \\
&+i\sum_{i=1}^N \hat{\varphi}_i[\phi_i^1(t_0) - \varphi_i] + i\int_0^\infty dt\hat{N}(t)\left[\sum_{i=1}^N(\phi_i^1(t))^2 - N\right]\Bigg\}.
\end{aligned}
\tag{52}
$$

## 5. Time reparametrization symmetry

We now evaluate the effect of a reparametrization $t \to s(t)$ of the time variable on the fixed point generating functional. For this purpose we consider a monotonously increasing function with the boundary conditions $s(0) = 0$ and $s(\infty) = \infty$, which induces the following transformations on the sources,

$$
\tilde{l}_i(t) = \frac{\partial s}{\partial t} l_i(s(t)), \tag{53}
$$

$$
\tilde{h}_i(t) = h_i(s(t)). \tag{54}
$$

We now evaluate the fixed point generating functional for the new sources

$$
\begin{aligned}
&\mathcal{Z}_{fp}[\tilde{l},\tilde{h}] = \int D\tilde{Q}D\tilde{\hat{Q}}D\psi^0 D\psi^1 D\tilde{\varphi} D\tilde{N} \\
&\times \exp\Bigg\{ i\sum_i \int_0^\infty dt_1 dt_2 \sum_{\alpha_i,\alpha_i'} \tilde{\hat{Q}}_i^{\overline{\alpha_i},\overline{\alpha_i'}}(t_1,t_2)\left(\tilde{Q}_i^{\alpha_i,\alpha_i'}(t_1,t_2) - \psi_i^{\alpha_i}(t_1)\psi_i^{\alpha_i'}(t_2)\right) \\
&-\frac{p^2}{p!}\sum_{i_1\ldots i_p} K_{i_1\ldots i_p}\int_0^\infty dt_1 dt_2 \sum_{\alpha_{i_r},\alpha_{i_r}'\in\{0,1\}}^{C=1,C'=1} \prod_{r=1}^p \tilde{Q}_{i_r}^{\alpha_{i_r},\alpha_{i_r}'}(t_1,t_2) \\
&+\int_{t_0}^{t_f} dt[\tilde{l}_i(t)\psi_i^0(t) + i\tilde{h}_i(t)\psi_i^1(t)] \\
&+i\sum_{i=1}^N \tilde{\hat{\varphi}}_i[\psi_i^1(t_0) - \tilde{\varphi}_i] + i\int_0^\infty dt\tilde{N}(t)\left[\sum_{i=1}^N(\psi_i^1(t))^2 - N\right]\Bigg\}.
\end{aligned}
\tag{55}
$$

Here we have used new dummy variables $\psi^\alpha$, $\tilde{\hat{\varphi}}$, $\tilde{N}$, $\tilde{\hat{Q}}$ and $\tilde{Q}$ instead of $\phi^\alpha$, $\hat{\varphi}$, $\hat{N}$, $\hat{Q}$ and $Q$, respectively, in the functional integral. We now perform the following change of variables

$$
\psi_i^\alpha(t) = \left(\frac{\partial s}{\partial t}\right)^{\overline{\alpha}} \phi_i^\alpha(s(t)), \tag{56}
$$

$$
\tilde{Q}_i^{\alpha,\alpha'}(t,t') = \left(\frac{\partial s}{\partial t}\right)^{\overline{\alpha}} \left(\frac{\partial s}{\partial t'}\right)^{\overline{\alpha'}} Q_i^{\alpha,\alpha'}(s(t),s(t')), \tag{57}
$$

$$\tilde{\hat{Q}}_i^{\alpha,\alpha'}(t,t') = \left(\frac{\partial s}{\partial t}\right)^{\alpha}\left(\frac{\partial s}{\partial t'}\right)^{\alpha'}\hat{Q}_i^{\alpha,\alpha'}(s(t),s(t')), \tag{58}$$

$$\tilde{N}(t) = \frac{\partial s}{\partial t}\hat{N}(s(t)), \tag{59}$$

$$\tilde{\hat{\varphi}} = \hat{\varphi}. \tag{60}$$

The change of variables results in Jacobians in the differentials,

$$D\tilde{Q}D\tilde{\hat{Q}} = DQD\hat{Q}\mathcal{J}_1\left[\frac{D\tilde{Q}}{DQ}\frac{D\tilde{\hat{Q}}}{D\hat{Q}}\right], \tag{61}$$

$$D\psi^0 D\psi^1 D\tilde{N} = D\phi^0 D\phi^1 D\hat{N}\mathcal{J}_2\left[\frac{D\psi^0}{D\phi^0}\frac{D\psi^1}{D\phi^1}\frac{D\tilde{N}}{D\hat{N}}\right], \tag{62}$$

$$D\tilde{\hat{\varphi}} = D\hat{\varphi}. \tag{63}$$

Since the field transformations are linear, the Jacobians only depend on the reparametrization $s(t)$ and are independent of the fields and sources, and can be taken outside the integral as common factors.

By inserting the values of transformed sources and dummy variables back into the fixed point generating functional we obtain,

$$\begin{aligned}
&\mathcal{Z}_{fp}[\tilde{l},\tilde{h}] = \mathcal{J}_1\mathcal{J}_2\int DQD\hat{Q}D\phi^0 D\phi^1 D\hat{\varphi}D\hat{N} \\
&\exp\left\{i\sum_i\int_0^\infty dtdt'\sum_{\alpha_i,\alpha_i'}\left(\frac{\partial s}{\partial t}\right)^{\overline{\alpha_i}+\alpha_i}\left(\frac{\partial s}{\partial t'}\right)^{\overline{\alpha_i'}+\alpha_i'}\hat{Q}_i^{\overline{\alpha_i},\overline{\alpha_i'}}(s(t),s(t'))\right. \\
&\times\left(Q_i^{\alpha_i,\alpha_i'}(s(t),s(t')) - \phi_i^{\alpha_i}(s(t))\phi_i^{\alpha_i'}(s(t'))\right) \\
&-\frac{p^2}{p!}\sum_{i_1\ldots i_p}K_{i_1\ldots i_p}\int_0^\infty dtdt'\sum_{\alpha_{i_r},\alpha_{i_r}'\in\{0,1\}}^{C=1,C'=1}\prod_{r=1}^{p}\left(\frac{\partial s}{\partial t}\right)^{\overline{\alpha_{i_r}}}\left(\frac{\partial s}{\partial t}\right)^{\overline{\alpha_{i_r}'}}Q_{i_r}^{\alpha_{i_r},\alpha_{i_r}'}(s(t),s(t')) \\
&+\int_0^\infty dt\left(\frac{\partial s}{\partial t}l_i(s(t))\phi_i^1(s(t)) + ih_i(s(t))\frac{\partial s}{\partial t}\phi_i^0(s(t))\right) \\
&\left.+i\sum_{i=1}^{N}\hat{\varphi}_i[\phi_i^1(s(0)) - \varphi_i] + i\int_0^\infty dt\frac{\partial s}{\partial t}\hat{N}(s(t))\left[\sum_{i=1}^{N}(\phi_i^1(s(t)))^2 - N\right]\right\}.
\end{aligned} \tag{64}$$

We now use the fact that $\alpha + \overline{\alpha} = 1$ and that the constraints $C$ and $C'$ ensure that

$\prod_{r=1}^{p}\left(\frac{\partial s}{\partial t}\right)^{\overline{\alpha}_{i_r}}\left(\frac{\partial s}{\partial t'}\right)^{\overline{\alpha'}_{i_r}}=\frac{\partial s}{\partial t}\frac{\partial s}{\partial t'}$, to write the transformed fixed point generating functional

$$\begin{aligned}
&\mathcal{Z}_{fp}[\tilde{l},\tilde{h}] = \mathcal{J}_1\mathcal{J}_2\int DQD\hat{Q}D\phi^0 D\phi^1 D\hat{\varphi}D\hat{N} \\
&\exp\left\{ i\sum_i \int_0^\infty dsds' \sum_{\alpha_i,\alpha'_i} \hat{Q}_i^{\overline{\alpha_i},\overline{\alpha'_i}}(s,s')\left(Q_i^{\alpha_i,\alpha'_i}(s,s') - \phi_i^{\alpha_i}(s)\phi_i^{\alpha'_i}(s')\right)\right. \\
&-\frac{p^2}{p!}\sum_{i_1\ldots i_p} K_{i_1\ldots i_p}\int_0^\infty dsds' \sum_{\alpha_{i_r},\alpha'_{i_r}\in\{0,1\}}^{C=1,C'=1}\prod_{r=1}^{p} Q_{i_r}^{\alpha_{i_r},\alpha'_{i_r}}(s,s') \\
&+\int_0^\infty ds[l_i(s)\phi_i^1(s) + ih_i(s)\phi_i^0(s)] \\
&\left.+i\sum_{i=1}^{N}\hat{\varphi}_i[\phi_i^1(0)-\varphi_i] + i\int_0^\infty ds\hat{N}(s)\left[\sum_{i=1}^{N}(\phi_i^1(s))^2 - N\right]\right\}.
\end{aligned} \tag{65}$$

In other words, we have shown that

$$\mathcal{Z}_{fp}[\tilde{l},\tilde{h}] = \mathcal{J}_1\mathcal{J}_2\mathcal{Z}_{fp}[l,h]. \tag{66}$$

We know that in the absence of sources, the transformation leaves the generating functional unchanged. This implies that $\mathcal{J}_1\mathcal{J}_2 = 1$, but since $\mathcal{J}_1$ and $\mathcal{J}_2$ are independent of the values of the sources, then for *any* value of the sources the fixed point generating functional is unchanged by the transformation, i.e.,

$$\mathcal{Z}_{fp}[\tilde{l},\tilde{h}] = \mathcal{Z}_{fp}[l,h]. \tag{67}$$

Therefore, the long-time fixed point dynamics of the p-spin model is invariant under RpG transformations.

## 6. Discussion and conclusion

We have shown that the MSR generating functional of the p-spin model is invariant under global time reparametrizations in the long time limit. The proof of invariance only assumes that the couplings $J_{i_1\ldots i_p}$ are uncorrelated Gaussian random variables with zero mean, but no condition is imposed on the variance $K_{i_1\ldots i_p}$ of the couplings, thus allowing them to have an arbitrary space dependence. In particular, the proof applies to both short-range and long-range models. Since the long-range version of the p-spin model shares some of the main features of structural glass phenomenology [22, 23], we expect that analytical tools similar to the ones used here can uncover the presence of time reparametrization symmetry in models of structural glass systems.

As discussed in Refs. [17, 18, 19], time reparametrization symmetry is a spontaneously broken symmetry in a glass. The symmetry is broken by correlations and responses. To illustrate the spontaneous breaking of the symmetry, we consider the correlation function $C(t_1,t_2)$. If correlations were invariant under the transformation we would have $C(t,t') = C(h(t),h(t'))$ for all $t$ and $t'$ and all reparametrizations and the only way this is possible is when the correlation function is independent of time. This is not the case in glasses because the correlation decays with time. The presence of a broken continuous symmetry in the absence of long range interactions or gauge potentials is expected to give rise to Goldstone modes [21]. In the case of the glass problem, the Goldstone modes should be associated with smoothly varying local fluctuations $t \to h_r(t)$ in the time reparametrization [17, 18, 19]. These fluctuations can be interpreted as representing local fluctuations of the age of the sample [17, 18]. Support for this point of view comes from simulation results both in the Edwards-Anderson model of spin glasses [18] and in models of structural glasses [24, 25]. It is expected that an analysis similar to the one done on simulation results can be done on data from confocal microscopy experiments of colloidal systems [24].

We conclude by noting that the present work proves the presence of time reparametrization symmetry in the p-spin model with arbitrary range interactions, and that we expect that similar methods to the ones used here can be applied in the future to attempt a proof of time reparametrization symmetry for structural glass models. By investigating the Goldstone modes predicted as a consequence of the symmetry we expect to be able to compute detailed predictions for probability distributions and correlation functions that describe the behavior of dynamical heterogeneity in glassy systems.

## Acknowledgments

We thank C. Chamon, L. Cugliandolo and M. Kennett for discussions. This work was supported in part by DOE under grant DE-FG02-06ER46300, by NSF under grants PHY99-07949 and PHY05-51164, and by Ohio University. H. E. C. acknowledges the hospitality of the Aspen Center for Physics and the Kavli Institute for Theoretical Physics, where parts of this work were performed.

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