CARTESIAN APPROACH FOR CONSTRAINED MECHANICAL SYSTEMS.

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Abstract. In the history of mechanics, there have been two points of view for studying mechanical systems: Newtonian and Cartesian. According the Descartes point of view, the motion of mechanical systems is described by the first-order differential equations in the N dimensional configuration space Q. In this paper we develop the Cartesian approach for mechanical systems with constraints which are linear with respect to velocity.

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1. Introduction.

In "Philosophiae Naturalis Principia Mathematica" (1687), Newton considers that movements of celestial bodies can be described by differential equations of the second order. To determine their trajectory, it is necessary to give the initial position and velocity. To reduce the equations of motion to the investigation of a dynamics system it is necessary to double the dimension of the position space and to introduce the auxiliary phase space. Descartes in 1644 proposed that the behavior of the celestial bodies be studied from another point of view. These ideas were stated in "Principia Philosophiae" (1644) and in "Discours de la métode" (1637). According to Descarte the understanding of cosmology starts from acceptance of the initial chaos, whose moving elements are ordered according to certain fixed laws and form the Cosmo. He consider that the Universe is filled with a tenuous fluid matter (ether), which is constantly in a vortex motion. This motion moves the largest particle of matter of the vortex axis, and they subsequently form planets. Then, according to what Descartes wrote in his "Treatise on Light", "the material of the

Heaven must be rotate the planets not only about the Sun but also about their own centers...and this will hence form several small Heavens rotating in the same direction as the great Heaven."[13]. Thus the equation of motion in the Descartes theory must be of the first order equation in the configuration space Q

$$\dot{\mathbf{x}} = \mathbf{v}(x,t), \quad x \in \mathrm{Q}$$

Descartes gave no principles for constructing the field $\mathbf{v}$ for different mechanical systems. Hence, to determine the trajectory from Descartes's point of view it is necessary to give only the initial position. In the modern scientific literature the study of the Descarte ideas we can find in the monographic of V.V. Kozlov [13] in which the author affirms that "solving dynamics problem is possible inside the configuration space".

In [18, 21, 24] we developed the Cartesian approach for mechanical systems with three degrees of freedom and with constraints linear with respects to velocity. The aim of this works is to generalized the Cartesian approach for non-holonomic mechanical system with N degrees of freedom and constraints which are linear with respect to the velocity.

We shall present briefly the contents of the paper.

In section 2 we prove our main results (see Theorem 1.1, Corollary 1.3, Corollary 1.4, Corollary 1.5 below).

In section 3 Corollary 1.3 applied to determine Cartesian and lagrangian approach for non-holonomic systems with three degree of freedom. We illustrate the obtained results to study Chapliguin-Caratheodory's sleigh and to study Suslov's problem for the rigid body around a fixed point.

In section 4 we determine Cartesian and lagrangian approach in three dimensional Euclidean space.

In section 5 by applying the results of the previous section we study the integrability of the geodesic flow on the surface.

In section 6 Theorem 1.1 applied to the study Gantmacher's system and Rattleback.

In section 7 Corollary 1.4 applied to solve the inverse problem in dynamics.

For simplicity we shall assume the underlying functions to be of class $\mathcal{C}^\infty$, although most results remain valid under weaker hypotheses.

It is well known that the behavior of constrained Lagrangian system

$$\langle\ \mathrm{Q}, L = \frac{1}{2}\sum_{j,k=1}^{N} G_{kj}(x)\dot{x}^j\dot{x}^k - U(x), \quad \sum_{k=1}^{N}\alpha_{jk}(x)\dot{x}^k = 0, \quad j = 1,2,\ldots,M, \rangle$$

can be described by the differential equations deduced from the D'Alembert-Lagrange Principle[14, 19]

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}^k} - \frac{\partial T}{\partial x^k} = \frac{\partial U}{\partial x^k} + \sum_{j=1}^{M}\mu_j\alpha_{jk}(x), \quad k = 1,2,\ldots N, \tag{1}$$

where $\mu_1,\, \mu_2, \ldots, \mu_M$ are Lagrangian multipliers.

Our main results are the following

Theorem 1.1. *Let Let* Q *be a smooth manifold of the dimension* N *with local coordinates* $x = (x^1, ..., x^N)$ *and equipped by the Riemann metric* $G = (G_{kj}(x)) = (G_{kj})$ *and let*

$$\sigma = (\ \boldsymbol{v}(x), d\boldsymbol{x}) = \sum_{j,k=1}^{N} G_{kj}\dot{x}^j v^k,$$

be the 1-form associated to the vector field

$$\boldsymbol{v} = \frac{1}{\Upsilon}\begin{vmatrix} \Omega_1(\partial_1) & \Omega_1(\partial_2) & \ldots & \Omega_1(\partial_N) & 0\\ \vdots & \vdots & \ldots & \vdots & \vdots\\ \Omega_M(\partial_1) & \Omega_M(\partial_2) & \ldots & \Omega_M(\partial_N) & 0\\ \Omega_{M+1}(\partial_1) & \Omega_{M+1}(\partial_2) & \ldots & \Omega_{M+1}(\partial_N) & \lambda_{M+1}\\ \vdots & \vdots & \ldots & \vdots & \vdots\\ \Omega_N(\partial_1) & \Omega_N(\partial_2) & \ldots & \Omega_N(\partial_N) & \lambda_N\\ \partial_1 & \partial_2 & \ldots & \partial_N & 0 \end{vmatrix}, \tag{2}$$

where $\Upsilon = \Omega_1 \wedge \Omega_2 ... \wedge \Omega_N(\partial_1, \partial_2, ..., \partial_N)$, $\quad \partial_k = \dfrac{\partial}{\partial x^k}$, $\lambda_j = \lambda_j(x)$ *for* $j = M+1, \ldots N$ *are arbitrary functions and* Ω_k *for* $k = 1, \ldots N$ *are 1-forms on* Q *which we assume that satisfies the following conditions*

(i) Ω_j *for* $j = 1, \ldots, M$ *are a given 1-forms:* $\Omega_j = \sum_{k=1}^{N} \alpha_{jk} dx^k$ *where* $\alpha_{jk} = \alpha_{jk}(x)$ *are functions on* Q,

(ii) Ω_k *for* $k = M+1, \ldots, N$ *are arbitrary 1-forms which we choose in such a way that* $\Upsilon \neq 0$.

(iii) *The 2-form* $d\sigma$ *admits the development*

$$d\sigma = \frac{1}{2}\sum_{j,k=1}^{N} a_{jk}(x)\Omega_j \wedge \Omega_k$$

where $A = (a_{jk})$ *is a skew symmetric matrix such that*

$$H = M^T A M$$

where $H = (d\sigma(\partial_j,\, \partial_k))$ *and* $M = (\Omega_j(\partial_k))$ *are* $N \times N$ *matrix.*

(iv) *The contraction of the 2-form* $d\sigma$ *along the vector field* $\boldsymbol{v}$ *is such that*

$$\imath_{\boldsymbol{v}} d\sigma = \sum_{j=1}^{N} \Lambda_j(x)\Omega_j,$$

where $\Lambda = (\Lambda_1,\, \ldots, \Lambda_N)^T$ *which we can be calculated as follows*

$$\Lambda = A\tilde{\lambda} = M^{-1}\tau, \tag{3}$$

where $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_N)^T$ *and* $\tau = (\imath_{\boldsymbol{v}} d\sigma(\partial_1)\,, \ldots, \imath_{\boldsymbol{v}} d\sigma(\partial_N))^T$

Then the first order differential system on Q

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(x), \quad \textit{under the conditions}, \qquad \Lambda_j(x) = 0, \tag{4}$$

for $j = M+1, \dots N$, *is invariant relationship of the second order differential system*

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}^k} - \frac{\partial T}{\partial x^k} = \frac{\partial \frac{1}{2}||\boldsymbol{v}||^2}{\partial x^k} + \sum_{j=1}^{M} \Lambda_j \alpha_{jk}, \quad k = 1, 2, \dots N, \tag{5}$$

Comparing equations (1) and (5) we deduce that the latter can be interpreted as the equations describing the behavior of non-holonomic mechanical systems under the action of active forces with potential $U = \frac{1}{2}||\mathbf{v}||^2 + U_0, \quad U_0 = const$ and under the action of the reactive forces with the components

$$\left(\sum_{j=1}^{M} \Lambda_j \alpha_{j1}, \sum_{j=1}^{M} \Lambda_j \alpha_{j2}, ..., \sum_{j=1}^{M} \Lambda_j \alpha_{jN} \right),$$

generated by the constraints $\Omega_j(\dot{\mathbf{x}}) = \sum_{k=1}^{N} \alpha_{jk} \dot{x}^k = 0$, for $j = 1, 2, ...M$.

Of interest is that the equations $\Lambda_j = 0$ for $j = M+1, \dots, N$ or, which is the same

$$\Lambda_j = \sum_{k=1}^{N} a_{jk} \lambda_k = 0, \quad a_{jk} = -a_{kj} \tag{6}$$

for $j = M+1, \dots, N$, represent a system of partial differential equations of first order with respect to the functions λ_k for $k = 1, \dots, N$.

Definition

We call the vector field $\mathbf{v}$ which generated system (4) *Cartesian vector field.* The vector field $\check{\mathbf{v}}$ we say *Cartesian equivalent* if there exist a nonzero function κ on Q such that $\kappa \check{\mathbf{v}}$ is Cartesian vector field.

Definition

Studying the behavior of nonholonomic mechanical systems with constraints linear with respect to the velocity using the equations (1), (4) and (5) we called *Classical, Decartes* and *Lagrangian approach* respectively.

Conjecture 1.2. *There are solutions of the equations* (6) *which generate a Cartesian (or Cartesian equivalent) vector field which completely describe the behavior of the study constrained Lagrangian system.*

This conjecture supports the following facts.

First, in view of Theorem1.1 the solutions of (4) are solutions of (5), which is closely linked to the system (1). Second, the solutions of the equations (1) depend on the $2N - M$ initial conditions. The solutions of (4) depend on N initial conditions and $N - M$ functions which are solutions of the linear partial differential equations (6), therefore the solutions of the equations (6) also depend on $N-M$ arbitrary constants. Finally, also contribute to the strengthening of the conjecture the large number of applications given below.

Corollary 1.3. *Let us suppose that in Theorem 1.1 manifold* Q *is three dimensional smooth manifold with local coordinates* $\boldsymbol{x}=(x,y,z)$ *and the given 1-form is* $\Omega=a_1dx+a_2dy+a_3dz=(\boldsymbol{a},d\boldsymbol{x})$ *where* $a_j=a_j(x,y,z)$ *for* $j=1,2,3$ *are functions on* Q.

Denoting by $[\ \times\ ]$ *the vector product in* $\mathbf{R}^3$ *and by* $rot\boldsymbol{v}$, $\boldsymbol{a}$, $\boldsymbol{w}$ *the following vectors fields*

$$\begin{aligned} rot\boldsymbol{v}= &\ \frac{1}{\sqrt{detG}}\left((\partial_y p_3-\partial_z p_2),\,(\partial_z p_1-\partial_x p_3),\,(\partial_x p_2-\partial_y p_1)\right)^T, \\ \boldsymbol{w}== &\ \frac{1}{\Upsilon}\left((\lambda_2\Omega_3-\lambda_3\Omega_2)(\partial_x),(\lambda_2\Omega_3-\lambda_3\Omega_2)(\partial_y),(\lambda_2\Omega_3-\lambda_3\Omega_2)(\partial_z)\right)^T. \end{aligned}$$

where $p_k=\sum_{j=1}^{3}G_{kj}v^j$, *for* $k=1,2,3$, *then differential system* (4) *and* (5) *take the form respectively*

$$\dot{\boldsymbol{x}}=[\boldsymbol{a}\times\boldsymbol{w}]=\boldsymbol{v}(x),\quad (\boldsymbol{a},rot[\boldsymbol{a}\times\boldsymbol{w}])=0. \tag{7}$$

$$\begin{aligned} \frac{d}{dt}\frac{\partial,T}{\partial\dot{\boldsymbol{x}}}-\frac{\partial T}{\partial\boldsymbol{x}}=&\ \frac{\partial\frac{1}{2}||\boldsymbol{v}||^2}{\partial\boldsymbol{x}}-(\boldsymbol{w},\,rot[\boldsymbol{a}\times\boldsymbol{w}])\,\boldsymbol{a} \\ =&\ \frac{\partial\frac{1}{2}||\boldsymbol{v}||^2}{\partial\boldsymbol{x}}-\Omega_2\wedge\Omega_3(\boldsymbol{v},rot\boldsymbol{v})\boldsymbol{a}, \end{aligned} \tag{8}$$

where $\frac{\partial}{\partial\dot{\boldsymbol{x}}}=(\partial_{\dot{x}^1},\ldots,\partial_{\dot{x}^N})^T$, $\quad\frac{\partial}{\partial\boldsymbol{x}}=(\partial_{x^1},\ldots,\partial_{x^N})^T$.

Corollary 1.4. *Let us introduce the notation*

$$\begin{vmatrix} df_1(\partial_1) & \ldots & df_1(\partial_N) \\ \vdots & & \vdots \\ df_{N-1}(\partial_1) & \ldots & df_{N-1}(\partial_N) \\ \partial_1 & \ldots & \partial_N \end{vmatrix}=\{f_1,\,f_2,\,\ldots,f_{N-1},\,*\}$$

and suppose that in Theorem 1.1 the given independent 1-form are such that $\Omega_j=df_j(x)$ *for* $j=1,2,...,N-1$, *and the arbitrary 1-form it is also exact, i.e.* $\Omega_N=df_N$, $\quad\Omega_N(\boldsymbol{v})=\lambda_N$, *and such that* $\Upsilon=\{f_1,\,f_2,\,\ldots,f_{N-1},\,f_N\}\neq 0.$

Then the equations (4) *and* (5) *take the form respectively*

$$\begin{aligned} \dot{\boldsymbol{x}}=-\lambda_N\frac{\{f_1,\,f_2,\,\ldots,f_{N-1},\,*\}}{\{f_1,\,f_2,\,\ldots,f_{N-1},f_N\}}=\lambda\{f_1,\,f_2,\,\ldots,f_{N-1},\,*\}, \\ \frac{d}{dt}\frac{\partial T}{\partial\dot{x}^k}-\frac{\partial T}{\partial x^k}=\frac{\partial\frac{1}{2}||\boldsymbol{v}||^2}{\partial x^k}+\lambda\sum_{j=1}^{N-1}a_{Nj}df_j(\partial_k), \end{aligned} \tag{9}$$

where $\lambda_N = \lambda_N(x)$ is an arbitrary function and $a_{Nj} = a_{Nj}(x)$ for $j = 1, \dots N-1$ are element of the skew symmetric matrix A.

Corollary 1.5. *Differential equations (5) are Lagrangian with Lagrangian function $L = \frac{1}{2}||\dot{\boldsymbol{x}} - \boldsymbol{v}(x)||^2$, where $\boldsymbol{v}$ is Cartesian vector field.*

The proofs of Theorem 1.1 , Corollary 1.3, Corollary 1.4, and Corollary 1.5 are given in section 2.

2. Proof of the main results

Proof of Theorem 1.1. Firstly we shall introduce the following notation and concepts. Let $\xi \in (\mathrm{Q})$ be the Lie algebra of vector fields on Q and let ∇ be the connection

$$\begin{aligned} \nabla : \xi(\mathrm{Q}) \quad &\times \xi(\mathrm{Q}) \longmapsto \xi(\mathrm{Q}), \\ &(\mathbf{u}, \mathbf{v}) \longmapsto \nabla_{\mathbf{u}\ \mathbf{v}}, \end{aligned}$$

which is $\mathbb{R}$ lineal with respect to $\mathbf{v}$ and C^∞ lineal with respect to $\mathbf{u}$ and is compatible with metric G, i.e. $\nabla_{\mathbf{u}} G(\mathbf{v}, \mathbf{w}) = 0, \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \xi(\mathrm{Q})$.

Let $\tilde{\mathbf{v}} \in \xi(\mathrm{Q})$ be a vector field:

$$\tilde{\mathbf{v}} = \frac{1}{\Upsilon} \begin{vmatrix} \Omega_1(\partial_1) & \Omega_1(\partial_2) & \dots & \Omega_1(\partial_N) & \lambda_1 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \Omega_M(\partial_1) & \Omega_M(\partial_2) & \dots & \Omega_M(\partial_N) & \lambda_2 \\ \Omega_{M+1}(\partial_1) & \Omega_{M+1}(\partial_2) & \dots & \Omega_{M+1}(\partial_N) & \lambda_{M+1} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \Omega_N(\partial_1) & \Omega_N(\partial_N) & \lambda_N & & \\ \partial_1 & \partial_2 & \dots & \partial_N & 0 \end{vmatrix}$$

where $\Upsilon \equiv \Omega_1 \wedge \Omega_2 ... \wedge \Omega_N(\partial_1, \partial_2, ..., \partial_N) \neq 0$.

The functions λ_j for $j = 1, \dots N$, are arbitrary functions on Q such that $\Omega_j(\tilde{\mathbf{v}}) = \lambda_j$, for $j = 1, \dots N$.

Let $\tilde{\sigma}$ be the 1-form associated with the vector field $\tilde{\mathbf{v}}$, i.e.

$$\tilde{\sigma} = (\tilde{\mathbf{v}}(x), dx) = \sum_{j,k=1}^{N} G_{jk} \tilde{v}^j(x) dx^k \equiv \sum_{k=1}^{N} \tilde{p}_k dx^k,$$

then, in view of the condition $\Upsilon \neq 0$ the 2-form $d\tilde{\sigma}$ admits the development $d\tilde{\sigma} = \frac{1}{2} \sum_{j,k=1}^{N} \tilde{a}_{jk}(x) \Omega_j \wedge \Omega_k$, where $A = (\tilde{a}_{jk})$ is a $N \times N$ skew-symmetric matrix such that

$$\tilde{a}_{jk} = (-1)^{j+k-1} \frac{1}{\Upsilon} d\sigma \wedge \Omega_1 \wedge ... \wedge \widehat{\Omega}_k .... \wedge \widehat{\Omega}_j .... \wedge \Omega_N(\partial_1, \partial_2, ..., \partial_N),$$

$\widehat{\Omega}_j$, $\widehat{\Omega}_k$, means that these elements are omitted.

In view of the relations

$$\iota_{\tilde{\mathbf{v}}} d\tilde{\sigma} = \frac{1}{2}\sum_{k,j=1}^{N}\left(\tilde{a}_{jk}\Omega_j(\tilde{\mathbf{v}}) - \tilde{a}_{kj}\Omega_j(\tilde{\mathbf{v}})\right)\Omega_k = \sum_{k,j=1}^{N}\tilde{a}_{jk}\Omega_j(\tilde{\mathbf{v}})\Omega_k, = \sum_{k,j=1}^{N}\tilde{a}_{jk}\lambda_j\Omega_k$$

we obtain that the contraction of $d\tilde{\sigma}$ along $\tilde{\mathbf{v}}$ is

$$\iota_{\tilde{\mathbf{v}}} d\tilde{\sigma} = \sum_{j=1}^{N}\tilde{\Lambda}_j\Omega_j, \tag{10}$$

where $\tilde{\Lambda} = \left(\tilde{\Lambda}_1, \tilde{\Lambda}_2, ..., \tilde{\Lambda}_N\right)^T$. It is easy to check that this vector can be calculated as follows

$$\tilde{\Lambda} = A\tilde{\lambda} = M^{-1}\tilde{\tau}, \tag{11}$$

where $M = (\Omega_j(\partial_k))$ and $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_N)^T$, $\tilde{\tau} = (\iota_{\tilde{\mathbf{v}}} d\tilde{\sigma}(\partial_1), \ldots, \iota_{\tilde{\mathbf{v}}} d\tilde{\sigma}(\partial_N))^T$.

Now we prove that the differential system

$$\dot{\mathbf{x}} = \tilde{\mathbf{v}}(x), \quad x \in \mathrm{Q}, \tag{12}$$

is invariant relationship of the Lagrangian equations with Lagrangian function

$$\tilde{L} = \frac{1}{2}||\dot{\mathbf{x}} - \tilde{\mathbf{v}}||^2 = \frac{1}{2}\sum_{j,k=1}^{N} G_{kj}(\dot{x}^j - \tilde{v}^j)(\dot{x}^k - \tilde{v}^k).$$

Indeed after covariant differentiation we obtain $\nabla_{\dot{\mathbf{x}}}(\dot{\mathbf{x}} - \tilde{\mathbf{v}}) = 0$, or, what is the same

$$\nabla_{\dot{\mathbf{x}}} G\left(\dot{\mathbf{x}} - \tilde{\mathbf{v}}\right) = \nabla_{\dot{\mathbf{x}}}\left(\frac{\partial \tilde{L}}{\partial \dot{\mathbf{x}}}\right) = 0,$$

hence , by considering that

$$\begin{aligned}
\nabla_{\dot{\mathbf{x}}}\left(G\dot{\mathbf{x}}\right) = &\quad \frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{x}}} - \frac{\partial T}{\partial \mathbf{x}}, \quad T = \frac{1}{2}||\dot{\mathbf{x}}||^2 \\
\nabla_{\dot{\mathbf{x}}}\tilde{p}_j = &\quad \sum_{j=1}^{N}\dot{x}^j\left(\nabla_{\partial_j}\tilde{p}_k - \nabla_{\partial_k}\tilde{p}_j\right) + \sum_{j=1}^{N}\dot{x}^j\nabla_{\partial_k}\tilde{p}_j \\
= &\quad \sum_{j=1}^{N}\dot{x}^j\left(\partial_j\tilde{p}_k - \partial_k\tilde{p}_j\right) + \sum_{j=1}^{N}v^j\nabla_{\partial_k}\tilde{p}_j + \sum_{j=1}^{N}\left(\dot{x}^j - \tilde{v}^j\right)\nabla_{\partial_k}\tilde{p}_j \\
= &\quad \frac{d}{dt}\frac{\partial V}{\partial \dot{x}^k} - \frac{\partial V}{\partial x^k} + \sum_{j=1}^{N}\left(\dot{x}^j - \tilde{v}^j\right)\nabla_{\partial_k}\tilde{p}_j
\end{aligned}$$

where $\tilde{p}_k = \sum_{j=1}^{N} G_{kj}\tilde{v}^j$, $V = (\dot{\mathbf{x}}, \tilde{\mathbf{v}}) - \frac{1}{2}||\tilde{\mathbf{v}}||^2$, and (,) is the scalar product.

Then along the solutions of (12) we give $\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{x}^k} - \frac{\partial \tilde{L}}{\partial x^k} = 0$, for $k = 1, 2, \ldots N$.

It is easy to show that these Lagrangian equations admits the representation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}^k}-\frac{\partial T}{\partial x^k}=\tilde{\omega}(\partial_k)+\nabla_{\dot{\mathbf{x}}-\tilde{\mathbf{v}}}\tilde{p}_k,$$

where $\tilde{\omega}=\frac{1}{2}d||\tilde{\mathbf{v}}||^2+\imath_{\tilde{\mathbf{v}}}d\tilde{\sigma}$. In view of (10) and (12) we finally deduce the differential equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}^k}-\frac{\partial T}{\partial x^k}=\frac{\partial\frac{1}{2}||\tilde{\mathbf{v}}||^2}{\partial x^k}+\sum_{j=1}^{N}\tilde{\Lambda}_j\Omega_j(\partial_k). \tag{13}$$

If we determine the vector field $\mathbf{v}$ and functions $\Lambda_1,\,\Lambda_2,\ldots\Lambda_N$ as follows

$$\mathbf{v}=\tilde{\mathbf{v}}|_{\lambda_1=\lambda_2=\ldots=\lambda_M=0},\quad \Lambda_j=\tilde{\Lambda}_j|_{\lambda_1=\lambda_2=\ldots=\lambda_M=0},\quad j=1,2,\ldots N \tag{14}$$

and require that $\Lambda_j=0,\quad j=M+1,...,N$, then we obtain the Cartesian vector field and differential system (13) coincide with system (5). In short Theorem 1.1 is proved. $\square$

Proof of Corollary 1.3. The vector field (2) in this case takes the form

$$\begin{aligned}
\Upsilon\mathbf{v} &= \begin{vmatrix}\Omega_1(\partial_x) & \Omega_1(\partial_y) & \Omega_1(\partial_z) & 0\\ \Omega_2(\partial_x) & \Omega_2(\partial_y) & \Omega_2(\partial_z) & \lambda_2\\ \Omega_3(\partial_x) & \Omega_3(\partial_y) & \Omega_3(\partial_z) & \lambda_3\\ \partial_x & \partial_y & \partial_z & 0\end{vmatrix}\\
&=\lambda_2\begin{vmatrix}\Omega_1(\partial_x) & \Omega_1(\partial_y) & \Omega_1(\partial_z)\\ \Omega_3(\partial_x) & \Omega_3(\partial_y) & \Omega_3(\partial_z)\\ \partial_x & \partial_y & \partial_z\end{vmatrix}\\
&-\lambda_3\begin{vmatrix}\Omega_1(\partial_x) & \Omega_1(\partial_y) & \Omega_1(\partial_z)\\ \Omega_2(\partial_x) & \Omega_2(\partial_y) & \Omega_2(\partial_z)\\ \partial_x & \partial_y & \partial_z\end{vmatrix}\\
&=\begin{vmatrix}a_1 & a_2 & a_3\\ w_1 & w_2 & w_3\\ \partial_x & \partial_y & \partial_z\end{vmatrix},
\end{aligned}$$

thus $\mathbf{v}(x)=[\mathbf{a}\times\mathbf{w}]$.

On the other hand considering that

$$d\sigma=(\mathrm{rot}\mathbf{v})_x dz\wedge dy+(\mathrm{rot}\mathbf{v})_y dx\wedge dz+(\mathrm{rot}\mathbf{v})_z dy\wedge dx$$

where $(\mathrm{rot}\mathbf{v})_x=dx(\mathrm{rot}\mathbf{v}),\ (\mathrm{rot}\mathbf{v})_y=dy(\mathrm{rot}\mathbf{v}),\ (\mathrm{rot}\mathbf{v})_z=dz(\mathrm{rot}\mathbf{v})$, we get

$$\imath_{\mathbf{v}}d\sigma=([\mathbf{v}\times\mathrm{rot}\mathbf{v}],d\mathbf{x})=\Lambda_1\Omega_1+\Lambda_2\Omega_2+\Lambda_3\Omega_3$$

which lead to

$$\begin{aligned}\Upsilon\Lambda_1 &= \Omega_2\wedge\Omega_3(\mathbf{v},\,\mathrm{rot}\mathbf{v}) = \lambda_2\Omega_3(\mathrm{rot}\mathbf{v}) - \lambda_3\Omega_2(\mathrm{rot}\mathbf{v}) = \\ &= (\mathbf{w},\,\mathrm{rot}\mathbf{v}),\\ \Upsilon\Lambda_2 &= \Omega_3\wedge\Omega_1(\mathbf{v},\,\mathrm{rot}\mathbf{v}) = \lambda_3\Omega_1(\mathrm{rot}\mathbf{v}) - \lambda_1\Omega_3(\mathrm{rot}\mathbf{v}),\\ &= \lambda_3\Omega_1(\mathrm{rot}\mathbf{v}) = \lambda_3\,(\mathbf{a},\mathrm{rot}\mathbf{v}),\\ \Upsilon\Lambda_3 &= -\Omega_2\wedge\Omega_1(\mathbf{v},\,\mathrm{rot}\mathbf{v}) = -\lambda_2\Omega_1(\mathrm{rot}\mathbf{v}) + \lambda_1\Omega_2(\mathrm{rot}\mathbf{v}),\\ &= -\lambda_2\Omega_1(\mathrm{rot}\mathbf{v}) = -\lambda_2\,(\mathbf{a},\mathrm{rot}\mathbf{v}),\end{aligned}$$

here we put $\Omega_1(\mathbf{v}) = \lambda_1 = 0$.

From the conditions $\Lambda_2 = \Lambda_3 = 0$ we obtain

$$(\mathbf{a},\mathrm{rot}\mathbf{v}) = (\mathbf{a},\mathrm{rot}[\mathbf{a}\times\mathbf{w}]) = 0,$$

hence we easily deduce differential system (7) and (8).

In short Corollary 1.3 is proved. □

Proof of Corollary 1.4. In this case we obtain that the vector field $\mathbf{v}$ takes the form

$$\mathbf{v} = \frac{1}{\Upsilon}\begin{vmatrix} df_1(\partial_1) & \dots & df_1(\partial_N) & 0\\ \vdots & \dots & \vdots & \vdots\\ df_{N-1}(\partial_1) & \dots & df_{N-1}(\partial_N) & 0\\ df_N(\partial_1) & \dots & df_N(\partial_N) & \lambda_N\\ \partial_1 & \dots & \partial_N & 0\end{vmatrix} = -\frac{\lambda_N}{\Upsilon}\begin{vmatrix} df_1(\partial_1) & \dots & df_1(\partial_N)\\ \vdots & & \vdots\\ df_{N-1}(\partial_1) & \dots & df_{N-1}(\partial_N)\\ \partial_1 & \dots & \partial_N\end{vmatrix}$$

$$= -\lambda_N\frac{\{f_1,\,f_2\dots,f_{N-1},\,*\}}{\{f_1,\,f_2\dots,f_{N-1},\,f_N\}} = \lambda\{f_1,\,f_2\dots,f_{N-1},\,*\}.$$

On the other hand by considering that $\lambda_j = 0$ for $j = 1,2,\dots N-1$, from (3) follows that $\Lambda_j = a_{Nj}\lambda_N,\quad j = 1,\dots,N-1,\quad \Lambda_N = a_{NN}\lambda_N = 0.$

Clearly the last equation is satisfied identically in view that (a_{jk}) are elements of the skew-symmetric matrix A.

Therefore we easily deduced the differential equations generated from Cartesian and Lagrangian approach.

Thus Corollary 1.4 has been proven. □

Proof of Corollary 1.5. Follows from the proof of Theorem 1.1 by considering (14) □

3. Decartes approach for non-holonomic system with three degree of freedom and one constraints .

In this section we apply the corollary 1.3 to study the Chapliguin-Catatheodory sleight and Suslov's problem for the rigid body around a fixed point.

Chapliguin-Carathodory's sleigh

We shall now analyze one of the classical nonholonomic systems Chapliguin-Carathodory's sleigh (which we call a sleigh). [16]

The idealized sleigh is a body that has three points of contact with the plane. Two of them slide freely but the third, A, behaves like a knife edge subjected to a constraining force $\mathbf{R}$ which does not allow transversal velocity. More precisely, let yoz be an inertial frame and $\xi\, A\eta$ a frame moving with the sleigh. Take as generalized coordinates the Decartes coordinates of the center of mass C of the sleigh and the angle x between the y and the ξ axis. The reaction force $\mathbf{R}$ against the runners is exerted laterally at the point of application A in such a way that the η component of the velocity is zero. Hence, one has the constrained system with the configuration space $\mathrm{Q} = S^1 \times \mathbb{R}^2$, with the kinetic energy $T = \frac{m}{2}(\dot{y}^2 + \dot{z}^2) + \frac{I_c}{2}\dot{x}^2$, and with the constraint $\epsilon\dot{x} + \sin x\dot{y} - \cos x\dot{z} = 0$, where m is the mass of the system and I_c is the moment of inertia about a vertical axis through C and $\epsilon = |AC|$.

Observe that the "javelin" (or arrow or Chapliguin's skate) is a particular case of a sleigh and can be obtained when $\epsilon = 0$.

To apply the Decartes approach for this system, first we introduce the 1-form Ω_j, for $j = 1, 2, 3$ in such a way that the determinant $\Upsilon \neq 0$. This condition holds in particular if

$$\Omega_1 = \epsilon dx + \sin x dy - \cos x dz, \quad \Omega_2 = \sin x dz + \cos x dy, \quad \Omega_3 = dx,$$

so $\Upsilon = \Omega_1 \wedge \Omega_2 \wedge \Omega_3(\partial_x,\, \partial_y,\, \partial_z) = 1$.

The Descartes approach produce the differential equations (see formula (7)) [24]

$$\dot{x} = \lambda_3, \quad \dot{y} = \lambda_2 \cos x - \epsilon\lambda_3 \sin x, \quad \dot{z} = \lambda_2 \sin x + \epsilon\lambda_3 \cos x, \tag{15}$$

here $\lambda_j = \lambda_3(x, y, z, \epsilon)$ for $j = 2, 3$ are solutions of the equation

$$\sin x(J\partial_z\lambda_3 + \epsilon m\partial_y\lambda_2) + \cos x(J\partial_y\lambda_3 - \epsilon m\partial_z\lambda_2) - m(\partial_x\lambda_2 - \epsilon\lambda_3) = 0, \tag{16}$$

where $J = J_C + \epsilon^2 m, \quad ||\mathbf{v}||^2 = J\lambda_3^2 + m\lambda_2^2$.

Now we show that there are solutions of (15) and (16) fully describes the inertial movements of the sleigh.

Corollary 3.1. *All the inertial trajectories of Chapliguin -Carathodory's sleigh can be obtained from Cartesian approach.*

Proof. Let us suppose that $\lambda_j = \lambda_j(x, \epsilon)$ for $j = 2, 3$. Clearly that in this case (16) takes the form $\partial_x\lambda_2 - \epsilon\lambda_3 = 0$ and all paths of the equation (15) can be obtained from the formula

$$\begin{aligned} y &= y_0 + \int \frac{(\lambda_2(x,\epsilon)\cos x - \epsilon\lambda_3 \sin x)dx}{\lambda_3(x,\epsilon)}, \\ z &= z_0 - \int \frac{(\lambda_2(x,y,z,\epsilon)\sin x - \epsilon\lambda_3 \cos x)dx}{\lambda_3(x,\epsilon)}, \\ t &= t_0 + \int \frac{dx}{\lambda_3(x,\epsilon)}. \end{aligned}$$

On the other hand, the inertial motions of the sleigh can be obtained from the equations deduce from the classical approach

$$J_C\ddot{x} = \epsilon\mu, \quad m\ddot{y} = \mu\sin x, \quad m\ddot{z} = -\mu\cos x, \quad \epsilon\dot{x} + \sin x\dot{y} - \cos x\dot{z} = 0.$$

Hence, after straightforward calculations we get the first order ordinary differential equations

$$\dot{x} = qC\cos\theta,\ \dot{y} = C(\sin\theta\cos x - q\epsilon\cos\theta\sin x),\ \dot{z} = C(\sin\theta\sin x + q\epsilon\cos\theta\cos x),$$

where $\theta = q\epsilon x + C, \quad q = \sqrt{\frac{m}{J}}$

This system can be obtained from (15) if we choose

$$\lambda_2 = C\sin\theta, \quad \lambda_3 = Cq\cos\theta.$$

Is clear that in this case

$$2||\mathbf{v}||^2 = J\lambda_3^2(x,\epsilon) + m\lambda_2^2(x,\epsilon) = mC^2,$$

therefore the sleigh moves by inertia. So the corollary is proved. □

Now we study Chapliguin's skate. Cartesian approach in this case produce the differential equations, which can be obtained from (15) and (16) by putting $\epsilon = 0$.

$$(17) \qquad \begin{aligned} \dot{x} = &\ \lambda_3(x,y,z,0), \quad \dot{y} = \lambda_2(x,y,z,0)\cos x, \quad \dot{z} = \lambda_2(x,y,z,0)\sin x, \\ &J(\sin x\partial_z\lambda_3 + \cos x\partial_y\lambda_3) - m\partial_x\lambda_2 = 0 \end{aligned}$$

Corollary 3.2. *All the trajectories of Chapliguin's skate with the initial condition $\dot{x}(t_0) = C_0 \neq 0$ and under the action of the potential field of force with potential function $U = mgy$ can be obtained from the Cartesian approach.*

Proof. In fact, for the case when $\epsilon = 0$ the classical approach for Chapliguin-Carathodory's sleigh gives (Chapliguin's skate) the following equations of motion

$$\ddot{x} = 0, \quad \ddot{y} = g + \mu\sin x, \quad \ddot{z} = -\mu\cos x, \quad \sin x\dot{y} - \cos x\dot{z} = 0$$

Hence, integrating we deduce the following differential system of first order (see for instance [24])

$$\dot{x} = C_0 \neq 0, \quad \dot{y} = (\frac{g\sin x}{C_0} + C_1)\cos x, \quad \dot{z} = (\frac{g\sin x}{C_0} + C_1)\sin x.$$

Let $\mathbf{b}$ be the vector field associated with this differential system, i.e.

$$\mathbf{b} = \left(C_0,\ (\frac{g\sin x}{C_0} + C_1)\cos x,\ (\frac{g\sin x}{C_0} + C_1)\sin x\right).$$

Whereas $(\mathbf{a}, \mathrm{rot}\mathbf{b}) = \dfrac{g\cos x}{C_0} \neq 0.$

Denoting by $\kappa = \dfrac{C_0}{g\sin xC_0 + C_1)}$, we easily obtain that $(\mathbf{a}, \mathrm{rot}(\kappa\ \mathbf{b})) = 0$, so $\mathbf{b}$ is Cartesian equivalent vector field.

The vector field $\kappa\mathbf{b}$ can be obtained from (17) by choosing $\lambda_3 = \kappa$ and $\lambda_2 = C_0$.

Summarizing the corollary is proved. $\square$

The rigid body around a fixed point in the Suslov case

in this section we study one classical problem of non-holonomic dynamics formulated by Suslov [25, 14]. We consider the rotational motion of a rigid body around a fixed point and subject to the non-holonomic constraints $(\tilde{\mathbf{a}}, \omega) = 0$ where $\omega = (\omega_1, \omega_2, \omega_3)$ is a body angular velocity and $\tilde{\mathbf{a}}$ is a constant vector. Suppose the body rotates in an force field with potential $U(\gamma_1, \gamma_2, \gamma_3)$. Applying the method of Lagrange multipliers we write the equations of motion in the form

$$
\begin{aligned}
I\dot{\omega} = \quad & [I\omega \times \omega] + [\gamma \times \tfrac{\partial U}{\partial \gamma}] + \mu\tilde{\mathbf{a}}, \quad \dot{\gamma} = [\gamma \times \omega] \\
& (\tilde{\mathbf{a}}, \omega) = 0
\end{aligned}
\tag{18}
$$

Where $\gamma = \gamma_1, \gamma_2, \gamma_3) = (\sin z \sin x, \, \sin z \cos x, \, \gamma_3 = \cos z),$, I is the inertial tensor of the body, μ is the Lagrange multiplier which can be expressed as a function of ω and γ as follows

$$
\mu = -\frac{\Big(\mathbf{a}, [I\omega, \, \omega] + [\gamma, \, \frac{\partial U}{\partial \gamma}]\Big)}{(\mathbf{a}, I^{-1}\mathbf{a})}.
$$

It is well-known the following result [14]

Proposition 3.3. *If $\boldsymbol{a}$ is an eigenvector of operator I, i.e.*

$$
I\boldsymbol{a} = \kappa\boldsymbol{a}, \tag{19}
$$

then the phase flow of system (18) *preserves the "standard"measure in* $\mathbb{R}^6 = \mathbb{R}^3\{\omega\} \times \mathbb{R}^3\{\gamma\}$.

G.K.Suslov has considered a particular case when the body is not under action of exterior forces: $U \equiv 0$. If (19) holds then the equations (18) have the additional first integral $K_4 = (I\omega, I\omega)$. E.I.Kharlamova in [11] study the case when the body rotates in the homogenous force field with the potential $U = (\mathbf{b}, \gamma)$ where $\mathbf{b}$ is an orthogonal to $\mathbf{a}$ vector. Under these conditions the equation of motion have the first integral $K_4 = (I\omega, \mathbf{b})$. V.V. Kozlov in [13] consider an opposite case, when $\mathbf{b} = \epsilon\mathbf{a}$, $\epsilon \neq 0$. The integrability problem in this case was study in particular in [14, 17]. The case when $U = \epsilon \det I(I^{-1}\gamma, \, \gamma)$ the system (18) have the Clebsch-Tisseran first integral $K_4 = \frac{1}{2}(I\omega, I\omega) - \dfrac{1}{2}\epsilon \det I(I^{-1}\gamma, \, \gamma)$, [14].

From now on, we suppose that equality (19) is fulfilled. We assume that vector $\mathbf{a}$ coincides with one of the principal axes and without loss of generality we can choose it as the third axis, i.e., $\mathbf{a} = (0, 0, 1)$ (see for more details [14]) Equations of motion have the following form

$$
\left\{
\begin{array}{l}
I_1\dot{\omega}_1 = \gamma_3\partial_{\gamma_2}U - \gamma_2\partial_{\gamma_3}U, \quad I_2\dot{\omega}_2 = \gamma_1\partial_{\gamma_3}U - \gamma_3\partial_{\gamma_1}U, \\
(I_1 - I_2)\omega_1\omega_2 + \gamma_2\partial_{\gamma_1}U - \gamma_1\partial_{\gamma_2}U + \mu = 0, \\
\dot{\gamma}_1 = -\gamma_3\omega_2, \quad \dot{\gamma}_2 = \gamma_3\omega_1, \quad \dot{\gamma}_3 = \gamma_1\omega_2 - \gamma_2\omega_1,
\end{array}
\right. \tag{20}
$$

where I_1, I_2 are the principal moments of inertia of the body with respect to the first and the second axis. We observe that the above mentioned choice of $\mathbf{a}$ guarantees that the phase flow of system (20) preserves the standard measure in $\mathbb{R}^5\{\omega_1, \omega_2, \gamma\}$.

In [20] we prove the following result

Theorem 3.4. *Let us suppose that the body rotates within the force field defined by the potential*

$$
U = \frac{1}{2I_1I_2}(I_1\mu_1^2 + I_2\mu_2^2) - h, \tag{21}
$$

where h is a constant and μ_1, μ_2 are the solutions of the following first order partial differential equation

$$
\gamma_3\left(\frac{\partial\mu_1}{\partial\gamma_2} - \frac{\partial\mu_2}{\partial\gamma_1}\right) - \gamma_2\frac{\partial\mu_1}{\partial\gamma_3} + \gamma_1\frac{\partial\mu_2}{\partial\gamma_3} = 0, \tag{22}
$$

then the following statements hold:

(I) *The equations* (20) *have the first integrals*

$$
I_1\omega_1 - \mu_2 = 0, \quad I_2\omega_2 + \mu_1 = 0, \tag{23}
$$

consequently, they are integrable by quadratures. In particular

$$
\begin{aligned}
I_1\omega_1 = \; & \frac{\partial S(\gamma_1, \gamma_2, K_2)}{\partial\gamma_2} + \Psi_2(\gamma_1^2 + \gamma_3^2, K_2, \gamma_2) \\
& + \gamma_1\Omega(\gamma_1^2 + \gamma_2^2, K_2, \gamma_3), \\
I_2\omega_2 = \; & -\frac{\partial S(\gamma_1, \gamma_2, K_2)}{\partial\gamma_1} - \Psi_1(\gamma_1^2 + \gamma_3^2, K_2, \gamma_1) \\
& - \gamma_2\Omega(\gamma_1^2 + \gamma_2^2, K_2, \gamma_3),
\end{aligned} \tag{24}
$$

are constants on the solutions of (20)*, where $K_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2$ and $S, \Psi_1, \Psi_2, \Omega$ are arbitrary smooth functions .*

(II) *The Suslov's, Kharlamova-Zabelina's, Kozlov's , Clebsch-Tisseran's and Tisseran-Okunova's first integrals can be obtained from* (24)*.*

(III) *The dependence $\gamma = \gamma(t)$ we determine by quadratures of the Poisson equations which in this case take the form:*

$$
\dot{\gamma}_1 = \frac{\gamma_3\mu_1}{I_2}, \quad \dot{\gamma}_2 = \frac{\gamma_3\mu_2}{I_1}, \quad \dot{\gamma}_3 = -\frac{\gamma_1\mu_1}{I_2} - \frac{\gamma_2\mu_2}{I_1}. \tag{25}
$$

It is interesting to note that the proof of the Theorem 3.4 first was obtained using Cartesian approach (for more details see [21]) which we proposed below

.

Let us suppose that manifold Q is the special orthogonal group of rotations of $\mathbb{E}^3$, i.e. $\mathrm{Q} = SO(3)$, with the Riemann metric

$$G = \begin{pmatrix} I_3 & I_3\cos z & 0 \\ I_3\cos z & (I_1\sin^2 x + I_2\cos^2 x)\sin^2 z + I_3\cos^2 z & (I_1 - I_2)\sin x\cos x\sin z \\ 0 & (I_1 - I_2)\sin x\cos x\sin z & I_1\cos^2 x + I_2\sin^2 x \end{pmatrix}$$

$\det G = I_1 I_2 I_3 \sin^2 z$.

In this case we have that the constraint is $\omega_3 = \dot{x} + \cos z\dot{y} = 0$.

Hence $\mathbf{a} = (1,\, \cos z,\, 0)$. By choosing the 1-form Ω_j for $j = 1, 2, 3$ as follow

$$\Omega_1 = dx + \cos z dy, \quad \Omega_2 = dy, \quad \Omega_3 = dz.$$

Consequently $\Upsilon = d\Omega_1 \wedge d\Omega_2 \wedge d\Omega_3(\partial_x,\, \partial_y,\, \partial_z) = 1$, and

$$\mathbf{v} = \lambda_2(\cos\, z\,\partial_x - \partial_y) - \lambda_3\partial_z$$

Thus we obtain that

$$\begin{aligned} p_1 &= 0, \\ p_2 &= (I_3 - I_1 + (I_1 - I_2)\cos^2 x)\cos z\sin^2 z\lambda_2 + (I_1 - I_2)\cos x\sin x\sin z\lambda_3, \\ p_3 &= (I_2\sin^2 x + I_1\cos^2 x)\lambda_3 + (I_2 - I_1)\sin x\cos x\sin z\cos z\lambda_2 \end{aligned}$$

The differential equations (7) in this cases take the form respectively

$$\dot{x} = \cos z\,\lambda_2, \quad \dot{y} = -\lambda_2, \quad \dot{z} = -\lambda_3 \tag{26}$$

and

$$(\mathbf{a}, \mathrm{rot}\mathbf{v}) = \partial_z p_2 - \partial_y p_3 + \cos z\partial_x p_3 = 0 \tag{27}$$

After the change $\gamma_1 = \sin z\sin x, \quad \gamma_2 = \sin z\cos x, \quad \gamma_3 = \cos z$ the system (26) and condition (27) can be written as follow

$$\dot{\gamma_1} = \frac{1}{I_2}\mu_1\gamma_3, \quad \dot{\gamma_2} = \frac{1}{I_1}\mu_2\gamma_3, \quad \dot{\gamma_3} = -\frac{1}{I_1 I_2}\left(I_1\mu_1\gamma_1 + I_2\mu_2\gamma_2\right) \tag{28}$$

$$\sin z(\gamma_3(\frac{\partial\mu_1}{\partial\gamma_2} - \frac{\partial\mu_2}{\partial\gamma_1}) - \gamma_2\frac{\partial\mu_1}{\partial\gamma_3} + \gamma_1\frac{\partial\mu_2}{\partial\gamma_3}) - \cos x\partial_y\mu_2 - \sin x\partial_y\mu_1 = 0 \tag{29}$$

where $\mu_2 = -I_1(\cos x\lambda_3 + \sin x\lambda_2), \quad \mu_1 = I_2(-\sin x\lambda_3 + \cos x\lambda_2)$.

Clearly if $\mu_j = \mu_j(x, z)$, for $j = 1, 2$, then the equation (29) coincide with equation (22) and (28) coincide with (25).

4. Cartesian vector field in three dimensional Euclidean space

Let $\mathbb{E}^3$ be the three dimensional Euclidian space with Cartesian coordinates $\mathbf{x} = (x_1, x_2, x_3)$.

We consider a particle with Lagrangian function

$$L = \frac{1}{2}||\dot{\mathbf{x}}||^2 - U(x),$$

and constraint $(\mathbf{a}, \dot{\mathbf{x}}) = 0$, where $(\ ,\ )$ denotes the scalar product in $\mathbb{E}^3$, $\dot{\mathbf{x}} = (\dot{x}_1,\, \dot{x}_2,\, \dot{x}_3)$ and $\mathbf{a} = \mathbf{a}(x) = (a_1(x)\,, a_2(x),\, a_3(x))$ is a smooth vector field in $\mathbb{E}^3$. Below we shall use the following notation $\partial_{\mathbf{x}} f = (\partial_{x_1} f, \partial_{x_2} f, \partial_{x_3} f)^T$.

The equations of motion in particular for constrained particle in $\mathbb{R}^3$, can be deduced from the d$^{'}$Alembert-Lagrange Principle and are such that

$$\ddot{\mathbf{x}} = \partial_{\mathbf{x}} U + \mu \mathbf{a}, \quad (\mathbf{a}, \dot{\mathbf{x}}) = 0,$$

where μ is the Lagrangian multiplier.

Cartesian and Lagrangian approach produces the following differential equations respectively (see formula (7) and (8))

$$\begin{aligned} \dot{\mathbf{x}} = &\ [\mathbf{a} \times \mathbf{w}], \qquad (\mathbf{a}, \mathrm{rot}[\mathbf{a} \times \mathbf{w}]) = 0, \\ \ddot{\mathbf{x}} = &\ \partial_{\mathbf{x}} \left(\frac{1}{2}||[\mathbf{a} \times \mathbf{w}]||^2\right) + (\mathrm{rot}[\mathbf{a} \times \mathbf{w}], \mathbf{w})\mathbf{a}, \end{aligned} \tag{30}$$

Example

Suppose that $\mathbf{a} = f_{\mathbf{x}}, \quad \mathbf{w} = \dfrac{\mathbf{c}}{c^2},$

where $f = r + (\mathbf{x}, \mathbf{b}) = c^2$ and $\mathbf{b}$ and $\mathbf{c}$ are constants vector field such that

$$(\mathbf{x}, \mathbf{c}) = 0, \quad (\mathbf{b}, \mathbf{c}) = 0, \quad ||\mathbf{c}||^2 = c^2,\ r = ||\mathbf{x}||.$$

Then Cartesian and Lagrangian approach generate the following differential equations respectively

$$\dot{\mathbf{x}} = \frac{[f_{\mathbf{x}} \times \mathbf{c}]}{c^2}, \qquad \ddot{\mathbf{x}} = -\frac{\mathbf{x}}{r^3}.$$

Indeed in view of the relation $\mathrm{rot}[f_{\mathbf{x}} \times \mathbf{c}] = \dfrac{\mathbf{c}}{r}$, we get

$$(\mathbf{a}, \mathrm{rot}[\mathbf{a} \times \mathbf{w}]) = \left(f_{\mathbf{x}}, \frac{\mathbf{c}}{c^2}\right) = 0$$

thus the given vector field is Cartesian.

From the relations $(\mathrm{rot}[\mathbf{a} \times \mathbf{w}], \mathbf{w}) = -\dfrac{1}{c^4}(\mathbf{c}, \mathrm{rot}[f_{\mathbf{x}} \times \mathbf{c}]) = -\dfrac{1}{rc^2}$, we obtain that the second order differential system (30) in this case takes the form

$$\ddot{\mathbf{x}} = \frac{1}{c^2}\left(\partial_{\mathbf{x}}(\frac{f}{r}) - \frac{\partial_{\mathbf{x}} f}{r}\right) = -\frac{f}{c^2}\frac{\mathbf{x}}{r^3} = -\frac{\mathbf{x}}{r^3}$$

Below we study the particular case when the vector field $\mathbf{a}$ satisfies the equation

$$\mathrm{rot}(\mathbf{a}) = \nu(x)\mathbf{a}, \tag{31}$$

where ν is certain function.

Optical-mechanical analogy

From the standpoint of geometric optics, propagation of light in $\mathbb{E}^3$ can be represented as a flow of particles. Trajectories of particle are called *rays.* It is known [13] that the vector field $\mathbf{K}$ of an arbitrary system of rays an a homogeneous optical medium satisfies the relation $\mathbf{K} \times \mathrm{rot}\mathbf{K} = \mathbf{0}$. System of rays such that $\mathrm{rot}\mathbf{K} \neq \mathbf{0}$ are called *Kummer systems*

Proposition 4.1. *Let $\boldsymbol{a}$ be the Kummer vector field i.e., satisfies the partial differential equation* (31) *with $\nu \neq 0$. Then Cartesian and Lagrangian approach generate the following differential equations respectively*

$$\dot{\boldsymbol{x}} = \boldsymbol{V} - \gamma \boldsymbol{a} = \frac{1}{||\boldsymbol{a}||^2}[\boldsymbol{a} \times rot\boldsymbol{W}] = \boldsymbol{v}(x) \tag{32}$$

and

$$\ddot{\boldsymbol{x}} = \partial_{\boldsymbol{x}}\left(\frac{1}{2}||[\boldsymbol{a} \times \boldsymbol{w}]||^2\right) + (\boldsymbol{R}, [\ \boldsymbol{a} \times \boldsymbol{w}])\, \boldsymbol{a}, \tag{33}$$

where $\gamma = \dfrac{(\boldsymbol{a}, \boldsymbol{V})}{||\boldsymbol{a}||^2}$, $\boldsymbol{V}$ *and* $\boldsymbol{R}$ *are the vector fields:*

$$rot\boldsymbol{V} = \nu\boldsymbol{V} + [\boldsymbol{a} \times \tilde{\boldsymbol{V}}], \qquad \boldsymbol{R} = \tilde{\boldsymbol{V}} - \partial_{\boldsymbol{x}}\gamma, \tag{34}$$

where $\tilde{\boldsymbol{V}}$ is an arbitrary smooth vector field.

Proof. Indeed, taking the well-known relations

$$div[\mathbf{A} \times \mathbf{B}] = (\mathbf{A}, \mathrm{rot}\mathbf{B}) - (\mathrm{rot}\mathbf{A}, \mathbf{B}).$$

into account, we obtain

$$\mathrm{div}[\mathbf{a} \times\ [\mathbf{a} \times \mathbf{w}]] = \big(\mathbf{a}, \mathrm{rot}[\mathbf{a} \times \mathbf{w}]\big) - \big(\mathrm{rot}\mathbf{a}, [\mathbf{a} \times \mathbf{w}]\big)$$

Hence in view of this identity, and considering that $(\mathbf{a}, \mathrm{rot}[\mathbf{a} \times \mathbf{w}]) = 0$ and (31) we obtain that $[\mathbf{a} \times [\mathbf{a} \times \mathbf{w}]]$ is a solenoidal vector field, so

$$[\mathbf{a} \times [\mathbf{a} \times \mathbf{w}]] = (\mathbf{a}, \mathbf{a})\mathbf{w} - (\mathbf{a}, \mathbf{w})\mathbf{a} = \mathrm{rot}\mathbf{W}, \tag{35}$$

where $\mathbf{W}$ is a smooth vector field such that $(\mathbf{a}, \mathrm{rot}\mathbf{W}) = 0$, consequently $\mathrm{rot}\mathbf{W} = [\mathbf{a} \times \mathbf{V}]$, for a smooth vector field $\mathbf{V}$ which must satisfy the partial differential equation

$$\begin{aligned}&\mathrm{div}(\mathrm{rot}\mathbf{W}) = \mathrm{div}([\mathbf{a} \times \mathbf{V}])\\ &= (\mathbf{a}, \mathrm{rot}\mathbf{V}) - (\mathbf{V}, \mathrm{rot}\mathbf{a}) = (\mathbf{a}, \mathrm{rot}\mathbf{V} - \nu\mathbf{V}) = 0,\end{aligned}$$

hence we obtain the representation (34)

In view of relation (35) we get the following representation for $\mathbf{w}$

$$\mathbf{w} == \frac{(\mathbf{a},\mathbf{w})}{||\mathbf{a}||^2}\mathbf{a} + \frac{[\mathbf{a}\times\mathbf{V}]}{||\mathbf{a}||^2} = \frac{(\mathbf{a},\mathbf{w})}{||\mathbf{a}||^2}\mathbf{a} + \frac{\mathrm{rot}\mathbf{W}}{||\mathbf{a}||^2}$$

From the relation $\mathbf{v} = [\mathbf{a}\times\mathbf{w}]$ we get formula (32).

On the other hand, after some calculations we obtain

$$\begin{aligned} \mathrm{rot}\mathbf{v} = &\quad \mathrm{rot}\mathbf{V} - [\partial_{\mathbf{x}}\gamma \times \mathbf{a}] - \gamma\mathrm{rot}\mathbf{a} \\ = &\quad \nu\,(\mathbf{V} - \gamma\mathbf{a}) + [\tilde{\mathbf{V}} - \partial_{\mathbf{x}}\gamma \times \mathbf{a}] \end{aligned}$$

or, what is the same

$$\mathrm{rot}\mathbf{v} = \nu\mathbf{v} + [\mathrm{R}\times\mathbf{a}]. \tag{36}$$

Inserting (36) in the relation $(\mathbf{w},\mathrm{rot}\mathbf{v})$ we obtain

$$\begin{aligned} (\mathbf{w},\mathrm{rot}\mathbf{v}) &= (\mathbf{w},\nu\mathbf{v} + [\mathbf{R}\times\mathbf{a}]) \\ &= (\mathbf{w},\ [\mathbf{R}\times\mathbf{a}]) = (-[\ \mathbf{w}\times\mathbf{a}],\mathbf{R}) \\ &= ([\ \mathbf{a}\times\mathbf{w}],\mathbf{R})\,. \end{aligned}$$

Hence from (30) we obtain the second order differential equations (33). □

Corollary 4.2. *Let us suppose that Cartesian vector field is such that*

$$rot\boldsymbol{v} = \nu\boldsymbol{v}, \quad \nu \neq 0, \tag{37}$$

then Lagrangian approach generated the differential system

$$\ddot{\boldsymbol{x}} = \partial_{\boldsymbol{x}}\left(\frac{1}{2}||\boldsymbol{v}||^2\right)$$

which describe the motion of a material point of unit mass in the potential field with the force function $\frac{1}{2}||\boldsymbol{v}||^2$.

Proof. From (37) and (36) follows that $[\mathrm{R}\times\mathbf{a}] = \mathbf{0}$, based on this relation and considering that

$$(\mathbf{R},\ [\mathbf{a}\times\mathbf{w}])) = (\mathbf{w},\ [\mathbf{R}\times\mathbf{a}]))$$

we deduce that $(\mathbf{R},\ [\mathbf{a}\times\mathbf{w}])) = 0$, thus, in view of (33), we obtain the proof of the proposition.

This results coincide with theorem J.Bernoulli (1696.) " Light rays in an isotropic optical medium with the refraction index $n(x)$ coincide with the trajectories of a material point in a potential field with the force function $U = \frac{1}{2}n^2(x)$." □

Example

Consider a particle with Lagrangian function $L = \frac{1}{2}\big(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\big)$ and constraint $\omega_2 = \sin x\,\dot{y} - \cos x\,\dot{z} = 0$,

The vector field $\mathbf{a} = (0,\ \sin x,\ -\cos x)$, satisfies the equation $\mathrm{rot}\mathbf{a} = \mathbf{a}$.

Classical approach generates the differential equations

$$\ddot{x} = 0, \quad \ddot{y} = \mu \sin x, \quad \ddot{z} = -\mu \cos x.$$

Integrating these equations we obtain

$$\dot{x} = C_1, \quad \dot{y} = C_2 \cos x, \quad \dot{z} = C_2 \sin x, \tag{38}$$

where C_1 and C_2 are arbitrary constants. The vector field associated to this system is Cartesian type.

Cartesian approach generate the differential system

$$\dot{x} = \lambda, \quad \dot{y} = \varrho \cos x, \quad \dot{z} = \varrho \sin x$$

where $\lambda = w_3 \sin x + w_2 \cos x$, $\varrho = -w_1$, and w_1, w_2, w_3 are components of the arbitrary vector field $\mathbf{w}$.

From the condition $(\mathbf{a}, \mathrm{rot}[\mathbf{a} \times \mathbf{w}]) = 0$ follows that the functions λ and ϱ are solutions of the linear partial differential equation

$$\partial_z \lambda \sin x + \partial_y \lambda \cos x - \partial_x \varrho = 0. \tag{39}$$

In particular the vector field

$$\mathbf{v} = \left(\lambda(x),\ \varrho(y,z) \cos x,\ \varrho(y,z) \sin x\right), \tag{40}$$

satisfies (39), where $\lambda = \lambda(x)$ and $\varrho = \varrho(y,z)$ are arbitrary real functions, .

Solving equations $\lambda(x) = w_3 \sin x + w_2 \cos x$, $\varrho(y,z) = -w_1$, with respect to the components of the vector $\mathbf{w}$ we obtain

$$\mathbf{w} = (-\varrho(y,z),\ \lambda(x) \cos x - \psi(x,y,z) \sin x,\ \lambda(x) \sin x + \psi(x,y,z) \cos x)$$

where $\psi = \psi(x,y,z)$ is an arbitrary function on Q.

It is easy to check that

$$\mathrm{rot}\mathbf{v} = \mathbf{v} + \mathbf{u},$$

where $\mathbf{u} = (\partial_z \varrho \cos x - \partial_y \varrho \sin x - \lambda, 0, 0)^T$, is a vector orthogonal to vector $\mathbf{a}$. Hence we obtain the relations

$$[\mathbf{R} \times \mathbf{a}] = \mathbf{u}, \quad (\mathbf{w}, [\mathbf{R} \times \mathbf{a}]) = \varrho \left(\partial_y \varrho \sin x + \lambda - \partial_z \varrho \cos x\right), \quad ||[\mathbf{w} \times \mathbf{a}]||^2 = \varrho^2 + \lambda^2.$$

Lagrangian approach generates the equations

$$\ddot{\mathbf{x}} = \partial_{\mathbf{x}} \left(\frac{1}{2}(\varrho^2 + \lambda^2)\right) - \varrho \left(\partial_z \varrho \cos x - \partial_y \varrho \sin x - \lambda\right) \mathbf{a}$$

The vector field (38) can be obtained from (40) if we choose $\lambda = C_1$, $\varrho = C_2$. For these parameter values Lagrangian approach generates the differential equations $\ddot{\mathbf{x}} = C_1 C_2 \mathbf{a}$.

If we choose ϱ and λ: $(\partial_z \varrho \cos x - \partial_y \varrho \sin x - \lambda = 0$ Then $\mathrm{rot}\mathbf{v} = \mathbf{v}$ and as a consequence Lagrangian approach generate the equation $\ddot{\mathbf{x}} = \partial_{\mathbf{x}} \left(\frac{1}{2}(\varrho^2 + \lambda^2)\right)$.

Below we study the case when the constraints are integrable.

Corollary 4.3. *Let* $\boldsymbol{V}$ *and* $\boldsymbol{a}$ *are the vector field such that*

$$\text{(41)} \qquad rot\boldsymbol{a} = \boldsymbol{0}, \quad \boldsymbol{V} = \partial_f G \partial_{\boldsymbol{x}} \Phi$$

where $G = G(f, \Phi)$ *and* Φ *are an arbitrary smooth functions, then Cartesian and Lagrangian approach for a particle in* $\mathbb{E}^3$ *which is constrained to move on the surface* $f = f(x) = c$ *generate the following differential equations*

$$\text{(42)} \qquad \dot{\boldsymbol{x}} = \partial_f G \left(\partial_{\boldsymbol{x}} \Phi - \gamma \partial_{\boldsymbol{x}} f\right) = \boldsymbol{v}(x),$$

and

$$\text{(43)} \qquad \begin{aligned} \ddot{\boldsymbol{x}} &= \partial_{\boldsymbol{x}} \left(\frac{\partial_f G^2}{2} ||\partial_{\boldsymbol{x}} f \times \partial_{\boldsymbol{x}} \Phi||^2 \right) \\ &-\partial_f G \left(\partial^2_{ff} G \partial_{\boldsymbol{x}} \Phi - \partial_{\boldsymbol{x}} \gamma \, , \partial_{\boldsymbol{x}} \Phi - \gamma \partial_{\boldsymbol{x}} f \right) \partial_{\boldsymbol{x}} f, \end{aligned}$$

where $g = ||\partial_{\boldsymbol{x}} f||^2$, *and* $\gamma = \dfrac{(\partial_{\boldsymbol{x}} f, \, \partial_{\boldsymbol{x}} \Phi)}{g}$,

If $\gamma = 0$ *then the equations* (42) *and* (43) *take the form respectively*

$$\text{(44)} \qquad \dot{\boldsymbol{x}} = \partial_f G \partial_{\boldsymbol{x}} \Phi, \quad \ddot{\boldsymbol{x}} = \partial_{\boldsymbol{x}} \left(\frac{G_f^2}{2} ||\partial_{\boldsymbol{x}} \Phi||^2 \right) - \left(G_{ff} G ||\partial_{\boldsymbol{x}} \Phi||^2 \right) \partial_{\boldsymbol{x}} f$$

Proof. Suppose that (41) hold, then

$$\mathbf{a} = \partial_{\mathbf{x}} f, \quad \text{rot} \mathbf{V} = [\partial_{\mathbf{x}} f \times \left(\partial^2_{ff} G \partial_{\mathbf{x}} \Phi \right) = [\partial_{\mathbf{x}} f \times \tilde{V}],$$

, hence the vector fields $\mathbf{W}$, $\tilde{\mathbf{V}}$ and $\mathbf{R}$ admit the representation

$$\mathbf{W} = G \partial_{\mathbf{x}} \Phi, \quad \tilde{\mathbf{V}} = \partial^2_{ff} G \partial_{\mathbf{x}} \Phi + \upsilon \partial_{\mathbf{x}} f, \quad \mathbf{R} = \partial^2_{ff} G \partial_{\mathbf{x}} \Phi + \upsilon \partial_{\mathbf{x}} f - \partial_{\mathbf{x}} \gamma,$$

where υ is an arbitrary function, hence we easily obtain (43).

If $\gamma = 0$ then $(\mathbf{R}, \partial_f G \left(\partial_{\mathbf{x}} \Phi - \gamma \partial_{\mathbf{x}} f \right)) = \partial_f G \partial^2_{ff} G ||\partial_{\mathbf{x}} \Phi||^2$, thus formula (44) follow . ☐

5. Integrability of the geodesic flow on the surface.

It is well known that the differential equations $\ddot{\mathbf{x}} = \mu \, \partial_{\mathbf{x}} f$, where μ is Lagrangian multiplier, determine the geodesic flows on the surface $f = c$, and admits the energy integral

$$||\dot{\mathbf{x}}||^2 = 2h(f).$$

If there is an additional first integral, functionally independent with the energy integral , then the *geodesic flow is integrable.*

Proposition 5.1. *Let us suppose that* (41) *holds. Then Lagrangian geodesic flow of the constrained particle on the surface* $f = c$, $g = ||\partial_{\boldsymbol{x}} f||^2 > 0$ *is integrable if there exist a solution of the following non-lineal partial differential equation respectively*

$$\text{(45)} \qquad G_f^2(f, \Phi) ||[\partial_{\boldsymbol{x}} f \times \partial_{\boldsymbol{x}} \Phi]||^2 = 2h(f) g, \quad \textit{if} \quad (\partial_{\boldsymbol{x}} f, \partial_{\boldsymbol{x}} \Phi) \neq 0,$$

and

$$\text{(46)} \qquad G_f^2(f, \Phi) ||\partial_{\boldsymbol{x}} \Phi||^2 = 2h(f), \quad \textit{if} \quad (\partial_{\boldsymbol{x}} f, \, \partial_{\boldsymbol{x}} \Phi) = 0, \quad \partial_{\boldsymbol{x}} \Phi \neq \kappa \boldsymbol{x}.$$

Proof. Indeed, if (45) holds then the system (43)takes the form

$$\ddot{\mathbf{x}} = \left(\partial_f h(f) - \partial_f G\left(\partial^2_{ff} G \partial_{\mathbf{x}}\Phi - \partial_{\mathbf{x}}\gamma\,, \partial_{\mathbf{x}}\Phi - \gamma\partial_{\mathbf{x}} f\right)\right)\partial_{\mathbf{x}} f, \tag{47}$$

which determine the geodesic Lagrangian flow and admits the following complementary first integral

$$F_1 = \frac{g||[\partial_{\mathbf{x}}\Phi \times \dot{\mathbf{x}}]||^2}{(\partial_{\mathbf{x}} f,\, \partial_{\mathbf{x}}\Phi)^2} = C_1,$$

which it is easy to obtain from (42). Clearly this first integral is independent of energy integral.

If $(\partial_{\mathbf{x}} f,\, \partial_{\mathbf{x}}\Phi) = 0$ then Cartesian and Lagrangian approach generated the differential equations respectively

$$\dot{x} = G_f(f, \Phi\,)\partial_{\mathbf{x}}\Phi, \quad \ddot{x} = \left(h_f - G_f(f, \Phi\,)G_{ff}(f, \Phi\,)||\partial_{\mathbf{x}}\Phi||^2\right)\partial_{\mathbf{x}} f,$$

The condition (45) under the given condition of orthogonality takes the form (46).

The complementary first integral is

$$F_2 = \frac{||\Phi_{\mathbf{x}}||^2||[\mathbf{x} \times \dot{\mathbf{x}}]||^2}{||[\mathbf{x} \times \partial_{\mathbf{x}}\Phi]||^2} = C_2,$$

which we can obtain from (42), by considering that $\gamma = 0$. This complete the proof of the proposition. □

Now we apply the above results.

Now we consider the surface

$$f(x) = c, \quad (\mathbf{x},\, \partial_{\mathbf{x}} f) = mf, \quad c \neq 0 \tag{48}$$

which we call *homogeneous surface of degree m*.

We are interested in studying the integrability of Lagrangian geodesic flow on the homogenous surface.

Euler's formula shows that $c = 0$ is the unique critical value of f, hence for $c \neq 0$ the function $g = ||\partial_{\mathbf{x}} f||^2 > 0$, on the surface $f(x) = c$.

Taking into account formula (48) it follows that

$$(\mathbf{x},\, \partial_{\mathbf{x}} g) = 2(m-1)g.$$

Below we use the notation

$$\{F, G, H\} = \begin{vmatrix} \partial_x F & \partial_y F & \partial_z F \\ \partial_x G & \partial_y G & \partial_z G \\ \partial_x H & \partial_y H & \partial_z H \end{vmatrix}.$$

Clearly, if F, G, H are independent functions then $\{F, G, H\} \neq 0$.

The integrability of the geodesic flow on the homogeneous surface we study in the following two cases

$$\{f, g, r^2\} = 0, \quad \{f, g, r^2\} \neq 0, \tag{49}$$

where $r^2 = x^2 + y^2 + z^2$.

We analyze the first case. We study only the particular subcase when the homogeneous surface satisfies the condition

$$g = g(f, r). \tag{50}$$

Hence, in view of (48) we obtain $mf\partial_f g + r\partial_r g = 2(m-1)g$.

We assume that function Φ is such that $\Phi_{\mathbf{x}} = \mathbf{x}$, thus the differential equation generated by the Cartesian and Lagrangian approach are respectively

$$\dot{\mathbf{x}} = \frac{G_f}{g}\big(g\mathbf{x} - m\, f\partial_{\mathbf{x}} f\big), \quad \ddot{\mathbf{x}} = \frac{m\partial_r g f h(f)}{r^2 g^2}\partial_{\mathbf{x}} f, \tag{51}$$

Proposition 5.2. *Lagrangian geodesic flow on the homogeneous surface under the assumption* (50) *is integrable*

Proof. Let us suppose that $\partial_{\mathbf{x}}\Phi = \mathbf{x}$, then $(\partial_{\mathbf{x}}\Phi,\, \partial_{\mathbf{x}} f) = mf \neq 0$. On the other hand if we choose $G(f, r)$ as

$$G_f^2(f, r) = \frac{2h(f)g(f, r)}{g(f, r)r^2 - m^2 f^2},$$

then we obtain that (45) holds. Thus there exist an additional first integral

$$g(f, r)||[\mathbf{x} \times \dot{\mathbf{x}}]||^2 = m^2 f^2 h(f)$$

□

Example

Lagrangian geodesic flow on the homogeneous surface of degree one

$$f(x) = r + (\mathbf{b}, \mathbf{x}) = c, \quad c \neq 0$$

is integrable, where $\mathbf{b}$ is a constant vector field.

Proof. In this case we have $g = \dfrac{2f}{r} + ||\mathbf{b}||^2 - 1 = g(f, r)$.

The complementary first integrals is

$$\big(\frac{2f}{r} + ||\mathbf{b}||^2 - 1\big)||[\mathbf{x} \times \dot{\mathbf{x}}]||^2 = 2f^2 h(f).$$

□

We have studied the case in which $\{f, g, r^2\} = 0$. Now we study the case in which the functions f, g, r^2 are independent, i.e., $\{f, g, r^2\} \neq 0$. Hence, we obtain that

$$x = x(r, f, g), \quad y = y(r, f, g), \quad z = z(r, f, g). \tag{52}$$

To establish the integrability or non-integrability of the Lagrangian geodesic flow on the surface in this case it is necessary to determine de existence or non-existence of the solution of the equation (45) or (46). We illustrate this case for the third-order surface

$$f(x) = x\, y\, z = c, \quad c \neq 0. \tag{53}$$

First we determine the dependence $x=x(r,f,g)$, $y=y(r,f,g)$, $z=z(r,f,g)$. By considering that in this case

$$g=(x\,y)^2+(x\,z)^2+(y\,z)^2$$

thus the functions f,g and r^2 are independent. Indeed if we introduce the cubic polynomial in Z :

$$P(z)=Z^3-r^2Z^2+gZ-f^2=(Z-x^2)(Z-y^2)(Z-z^2),$$

then by using Cardano's formula we obtain the dependence (52).

This case was examined already by Riemann in his study of motion of a homogeneous liquid ellipsoid. More exactly, Riemann examined the integrability of the geodesic flow on (53).

In [15] the author raised the problem.

"Is it true that the geodesic flow on a generic third-order algebraic surface is not integrable?. In particular I do not know a rigorous proof of non-integrability for the surface (53)"

To prove the integrability of Lagrangian geodesic flow it is necessary to solve the non-lineal partial differential equation

$$G_f(f,\Phi)\big(g||\Phi_{\mathbf{x}}||^2-\big(y\,z\Phi_x+z\,x\Phi_y+x\,y\Phi_z\big)^2\big)=2h(f)g$$

Now we are not able to provide an answer to this question.

6. Decartes approach for non-holonomic system with four and five degree of freedom and two constraints

In this section we apply Theorem 1.1 to study the non-holonomic system study in [8] and the well known non-holonomic system-the rattleback .

Gantmacher's system

Two material points M_1, M_2 with equal mass are linked by a metal rod with fixed long and small mass. The system can move only in the vertical plane and so that the speed of the midpoint of the rod is directed along the rod. It is necessary to determine the trajectories of material points M_1, M_2.

Let $(x_1,\,y_1)$ and $(x_2,\,y_2)$ are the coordinates of the points M_1, M_2.

Introducing the following change of coordinates

$$2u_1=x_2-x_1,\quad 2u_2=y_1-y_2,\quad 2u_3=y_2+y_1,\quad 2u_4=x_1+x_2$$

we obtain the mechanical system with configuration space $\mathrm{Q}=\mathbb{R}^4$, and Lagrangian function $L=\dfrac{1}{2}\displaystyle\sum_{j=1}^{4}\dot{u}_j^2-gu_3$.

The equations of the constraints can be rewritten as

$$u_1\dot{u}_1+u_2\dot{u}_2=0,\quad u_1\dot{u}_3-u_2\dot{u}_4=0.$$

To construct Cartesian approach in this case we firstly determine the 1-forms Ω_j for $j = 1, 2, 3, 4$ as follow

$$\begin{aligned}\Omega_1 &= u_1 du_1 + u_2 du_2, \quad \Omega_2 = u_1 du_3 - u_2 du_4,\\ \Omega_3 &= u_1 du_2 - u_2 du_1, \quad \Omega_4 = u_2 du_3 + u_1 du_4\end{aligned}$$

hence we obtain that $\Upsilon = u_1^2 + u_2^2$.

After some calculations we obtain that the vector field (2) takes the form

$$\mathbf{v} = \nu_3(u_1\partial_2 - u_2\partial_1) + \nu_4(u_2\partial_3 + u_1\partial_4), \quad \Upsilon = (u_1^2 + u_2^2)^2,$$

where $\nu_j = \lambda_j(u_1^2 + u_2^2)$ for $j = 3, 4$.

The 1-form associated to vector $\mathbf{v}$ is the following

$$\sigma = \nu_3(-u_2 du_1 + u_1 du_2) + \nu_4(u_2 du_3 + u_1 du_4).$$

Thus the 1-form $\iota_{\mathbf{v}} d\sigma$ admits the representation

$$\begin{aligned}&\iota_{\mathbf{v}} d\sigma = \Lambda_1\Omega_1 + \Lambda_2\Omega_2 + \Lambda_3\Omega_3 + \Lambda_4\Omega_4\\ &= (u_1^2 + u_2^2)\Big(- u_2\partial_{u_2}(\frac{\nu_3^2 + \nu_4^2}{2}) - u_1\partial_{u_1}(\frac{\nu_3^2 + \nu_4^2}{2}) - 2\nu_3^2 - \nu_4^2\Big)\Omega_1\\ &+(u_1^2 + u_2^2)\Big(u_2\partial_{u_4}(\frac{\nu_3^2 + \nu_4^2}{2}) - u_1\partial_{u_3}(\frac{\nu_3^2 + \nu_4^2}{2}) + \nu_3\nu_4\Big)\Omega_2\\ &+(u_1^2 + u_2^2)\nu_4\Big(u_2\partial_{u_3}\nu_3 + u_1\partial_{u_4}\nu_3 + u_2\partial_{u_1}\nu_4 - u_1\partial_{u_2}\nu_4\Big)\Omega_3\\ &+(u_1^2 + u_2^2)\nu_3\Big(u_2\partial_{u_3}\nu_3 + u_1\partial_{u_4}\nu_3 + u_2\partial_{u_1}\nu_4 - u_1\partial_{u_2}\nu_4\Big)\Omega_4.\end{aligned}$$

If $\Lambda_3 = \Lambda_4 = 0$ then $u_2\partial_{u_3}\nu_3 + u_1\partial_{u_4}\nu_3 + u_2\partial_{u_1}\nu_4 - u_1\partial_{u_2}\nu_4 = 0$.

Cartesian approach generate the following differential equations respectively

$$\dot{u}_1 = -\nu_3 u_2, \quad \dot{u}_2 = \nu_3 u_1, \quad \dot{u}_3 = \nu_4 u_2, \quad \dot{u}_4 = \nu_4 u_1 \tag{54}$$

and

$$u_2\partial_{u_3}\nu_3 + u_1\partial_{u_4}\nu_3 + u_2\partial_{u_1}\nu_4 - u_1\partial_{u_2}\nu_4 = 0 \tag{55}$$

It is easy to show that the functions ν_3, ν_4 :

$$\nu_3 = g_3(u_1^2 + u_2^2), \qquad \nu_4 = \sqrt{\frac{2(-gu_3 + h)}{(u_1^2 + u_2^2)} - g_3^2(u_1^2 + u_2^2)}, \tag{56}$$

where g, h are constants, are solutions of (55), (54)as a consequence

$$2||\mathbf{v}||^2 = (u_1^2 + u_2^2)(\nu_3^2 + \nu_4^2) = 2(-gu_3 + h).$$

Under these restrictions Lagrangian approach generate the differential system

$$\ddot{u}_1 = \Lambda_1 u_1, \quad \ddot{u}_2 = \Lambda_1 u_2, \quad \ddot{u}_3 = -g + \Lambda_2 u_1, \quad \ddot{u}_4 = \Lambda_2 u_2. \tag{57}$$

The solutions of (55) are

$$
\begin{aligned}
&u_1 = r\cos\alpha, \quad u_2 = r\sin\alpha, \quad \alpha = \alpha_0 + g_3(r)t,\\
&u_3 = u_3^0 + \frac{g}{2g_3(r)}t - \frac{g}{4g_3^2(r)}\sin 2\alpha - -\frac{\sqrt{2g}C}{g_3(r)}\cos\alpha,\\
&u_4 = -h + \frac{r^2g_3^2(r)}{2g} + \Big(\frac{\sqrt{g}}{\sqrt{2}g_3(r)}\sin\alpha + C)^2,
\end{aligned}
$$

where $C,\ r,\ \alpha_0,\ u_3^0,\ h,$ are arbitrary constants, g_3 is an arbitrary on r function.

To compare these solutions with the solutions obtained from the classical approach, we determine the equations of motion obtained from the d'Alembert-lagrange principle

$$
\ddot{u}_1 = \mu_1 u_1, \quad \ddot{u}_2 = \mu_1 u_2, \quad \ddot{u}_3 = -g + \mu_2 u_1, \quad \ddot{u}_4 = -\mu_2 u_2 \tag{58}
$$

where $\mu_1,\ \mu_2$ are the Lagrangian multipliers.

After the integration of the system (58) we obtain [8]

$$
\dot{u}_1 = -\dot{\varphi}u_2, \quad \dot{u}_2 = \dot{\varphi}u_1, \quad \dot{u}_3 = \frac{f}{r}u_2, \quad \dot{u}_4 = \frac{f}{r}u_1 \tag{59}
$$

where $(\varphi,\ r)$ are the polar coordinates: $u_1 = r\cos\varphi, \quad u_2 = r\sin\varphi$ and f is a solution of the equation

$$
\dot{f} = -\frac{2g}{r}u_2 \tag{60}
$$

The solution of (60) is $f = \dfrac{2g\ \cos\varphi}{\dot{\varphi}} + 2\gamma$ where γ is an arbitrary constants.

Clearly if we choose $\nu_3 = \dot{\varphi}, \quad \nu_4 = \dfrac{f}{r}$ then we the vector field associated to system (59) can be obtained from Cartesian vector field.

The rattleback.

The rattleback's amazing mechanical behaviour is a convex asymmetric rigid body rolling without sliding on a horizontal plane. It is known for its ability to spin in one direction and to resist spinning in the opposite direction for some parameters values, and for others values to exhibit multiple reversals. Basic references on the rattleback are [28, 9, 12, 2].

Introduce the Euler angles ψ, ϕ, θ using the principal axis body frame relative to an inertial reference frame. These angles together with two horizontal coordinates x, y of the center of mass are coordinates in the configuration space $\mathrm{Q} = SO(3)\times\mathbb{R}^2$ of the rattleback.

The Lagrangian of the rattleback is computed to be

$$
\begin{aligned}
L = \; & \tfrac{1}{2}(I_1\cos^2\psi + I_2\sin^2\psi + m(\Gamma_1\cos\theta - \zeta\sin\theta)^2)\dot{\theta}^2 \\
& \tfrac{1}{2}(I_1\sin^2\psi + I_2\cos^2\psi)\sin^2\theta) + I_3\cos^2\theta)\dot{\phi}^2 \\
& +\tfrac{1}{2}(I_3 + m\Gamma_2^2\sin^2\theta)\dot{\psi}^2 + \tfrac{m}{2}(\dot{x}^2 + \dot{y}^2) \\
& +m(\Gamma_1\cos\theta - \zeta\sin\theta)\Gamma_2\sin\theta\dot{\theta}\dot{\psi} + (I_1 - I_2)\sin\theta\sin\psi\cos\psi\dot{\theta}\dot{\phi} \\
& C\cos\theta\dot{\phi}\dot{\psi} + mg(\Gamma_1\sin\theta + \zeta\cos\theta)
\end{aligned}
$$

where I_1, I_2, I_3 are the principal moments of inertia of the body, m is the total mass of the body,

$$\Gamma_1 = \xi\sin\psi + \eta\cos\psi, \quad \Gamma_2 = \xi\cos\psi - \eta\sin\psi$$

and $(\xi = \xi(\theta, \psi),\, \eta = \eta(\theta, \psi),\, \zeta = \zeta(\theta, \psi))$ are the coordinates of the point of contact relative to the body frame.The shape of the body is encoded by the functions ξ, η and ζ.

The constraints are

$$\dot{x} - \alpha_1\dot{\theta} - \alpha_2\dot{\psi} - \alpha_3\dot{\phi} = 0, \quad \dot{y} + \beta_1\dot{\theta} + \beta_2\dot{\psi} + \beta_3\dot{\phi} = 0,$$

where

$$\alpha_1 = (-\Gamma_1\sin\theta - \zeta\cos\theta)\sin\phi, \quad \alpha_2 = \Gamma_2\cos\theta\sin\phi + \Gamma_1\cos\phi,$$
$$\alpha_3 = \Gamma_2\sin\phi + (\Gamma_1\cos\theta - \zeta\sin\theta)\cos\phi, \quad \beta_1 = \frac{\partial\alpha_1}{\partial\phi},\, \beta_2 = \frac{\partial\alpha_2}{\partial\phi},\, \beta_3 = \frac{\partial\alpha_3}{\partial\phi}.$$

Clearly that the rattleback equations of motion in this particular case formally contain the equations of the heavy rigid body in the singular case $m \to 0, \quad mg \to l, \quad l \neq 0$

To determine Cartesian approach for the rattleback we first determine the 1-forms Ω_j, for $j = 1, \ldots, 5$. In this case we determine as follows

$$\Omega_1 = dx - \alpha_1 d\theta - \alpha_2 d\psi - \alpha_3 d\phi, \quad \Omega_2 = dy + \beta_1 d\theta + \beta_2 d\psi + \beta_3 d\phi,$$
$$\Omega_3 = d\theta, \quad \Omega_4 = d\psi, \quad \Omega_5 = d\phi$$

Hence $\Upsilon = 1$ and the vector field $\mathbf{v}$:

$$\mathbf{v} = \lambda_3 X_3 + \lambda_4 X_4 + \lambda_5 X_5, \tag{61}$$

where

$$X_3 = \alpha_1\partial_x - \beta_1\partial_y + \partial_\theta, \quad X_4 = \alpha_2\partial_x - \beta_2\partial_y + \partial_\psi, \quad X_5 = \alpha_3\partial_x - \beta_3\partial_y + \partial_\phi.$$

We now proceed to the consideration of the particular case for which ξ, η and ζ admits the development

$$\xi = \xi_0 + \epsilon\xi_1(\theta,\, \psi), \quad \eta = \eta_0 + \epsilon\eta_1(\theta,\, \psi), \quad \zeta = \zeta_0 + \epsilon\zeta_1(\theta,\, \psi)$$

where ξ_0, η_0, ζ_0, are constants and ϵ is a small parameter. Under this consideration we obtain that the Lagrangian function can be represented as follow

$$L = L_0 + \epsilon L_1 + \epsilon^2 L_2.$$

Below we study the case when $\epsilon = 0$.

Let $(x^1,\, x^2,\, x^3,\, x^4,\, x^5)$ be a new set of variables derived from $x,,\, y,\, \theta,\, \psi,\, \phi$ by the transformation

$$\begin{gathered}\psi = x^1, \quad \phi = x^2, \quad \theta = x^3,\\ y + \zeta_0 \sin\theta\cos\phi + \Gamma_1^0\cos\theta\sin\phi - \Gamma_2^0\sin\phi = x^4,\\ x + \zeta_0\sin\theta\sin\phi - \Gamma_1^0\cos\theta\cos\phi + \Gamma_2^0\cos\phi = x^5,\end{gathered}$$

where $\Gamma_1^0 = \xi_0\sin\psi + \eta_0\cos\psi, \quad \Gamma_2^0 = \xi_0\cos\psi - \eta_0\sin\psi$.

The vector field $\mathbf{v}$ and the constraints on account of this change, take respectively the form respectively

$$\mathbf{v} = (a,\, b,\, c,\, 0,\, 0) \quad \dot{x}^4 = 0, \quad \dot{x}^5 = 0$$

where $a = a(x^1, .., x^5)),\ b = b(x^1, .., x^5)),\ c = c(x^1, .., x^5))$ are the $\mathcal{C}^1$ functions.

In the coordinates $x = (x^1,\, x^2,\, x^3,\, x^4,\, x^5)$ the Lagrangian function L_0 becomes to the function

$$\tilde{L} = \frac{1}{2}\sum_{j,k=1}^{5} G_{jk}\dot{x}^j\dot{x}^k + mg(\Gamma_1^0\sin x^3 + \zeta_0\cos x^3),$$

where $G = (G_{jk}(x)) = (G_{jk})$ is the Riemann metric.

We shall now determine Cartesian approach under the given conditions.

Proposition 6.1. *The vector field $\tilde{\mathbf{v}}(x) = (a,\, b,\, c)$ is a Kummer vector field.*

Proof. Indeed the 1-form associated to the vector field $\mathbf{v}$ is

$$\sigma = p_1dx^1 + p_2dx^2 + p_3dx^3, \quad p_k = G_{k1}a + G_{k2}b + G_{k3}c, \quad k = 1, 2, .., 5$$

then

$$\imath_{\mathbf{v}}d\sigma = \sum_{j=1}^{5}\Lambda_j dx^j$$

where

$$\begin{aligned}&\Lambda_1 = (\frac{\partial p_1}{\partial x^2} - \frac{\partial p_2}{\partial x^1})b + (\frac{\partial p_1}{\partial x^3} - \frac{\partial p_3}{\partial x^1})c, \quad \Lambda_2 = (\frac{\partial p_2}{\partial x^3} - \frac{\partial p_3}{\partial x^2})c + (\frac{\partial p_2}{\partial x^1} - \frac{\partial p_1}{\partial x^2})a\\ &\Lambda_3 = (\tfrac{\partial p_3}{\partial x^2} - \tfrac{\partial p_2}{\partial x^3})b + (\frac{\partial p_3}{\partial x^1} - \frac{\partial p_1}{\partial x^3})a, \quad \Lambda_4 = -\frac{\partial p_1}{\partial x^4}a - \frac{\partial p_2}{\partial x^4}b - \frac{\partial p_3}{\partial x^4}c\\ &\Lambda_5 = -\frac{\partial p_1}{\partial x^5}a - \frac{\partial p_2}{\partial x^5}b - \frac{\partial p_3}{\partial x^5}c\end{aligned}$$

We have therefore that the differential equations generated by Cartesian approach are respectively

$$
\begin{aligned}
&\dot{x}^1 = a, \quad \dot{x}^2 = b, \quad \dot{x}^3 = c,\\
&\Lambda_1 = (\frac{\partial p_1}{\partial x^2} - \frac{\partial p_2}{\partial x^1})b + (\frac{\partial p_1}{\partial x^3} - \frac{\partial p_3}{\partial x^1})c = 0,\\
&\Lambda_2 = (\frac{\partial p_2}{\partial x^3} - \frac{\partial p_3}{\partial x^2})c + (\frac{\partial p_2}{\partial x^1} - \frac{\partial p_1}{\partial x^2})a = 0,\\
&\Lambda_3 = (\frac{\partial p_3}{\partial x^2} - \frac{\partial p_2}{\partial x^3})b + (\frac{\partial p_3}{\partial x^1} - \frac{\partial p_1}{\partial x^3})a = 0,
\end{aligned}
\tag{62}
$$

where $a = a(x^1, x^2, x^3, C_4, C_5),\ b = b(x^1, x^2, x^3, C_4, C_5),\ c = c(x^1, x^2, x^3, C_4, C_5)$
Let $\mathrm{rot}\tilde{\mathbf{v}}(x)$ be the vector field

$$rot\tilde{\mathbf{v}} = \frac{1}{\sqrt{detG}}(\frac{\partial p_3}{\partial x^2} - \frac{\partial p_2}{\partial x^3}, \frac{\partial p_1}{\partial x^3} - \frac{\partial p_3}{\partial x^1}, \frac{\partial p_2}{\partial x^1} - \frac{\partial p_1}{\partial x^2})^T,$$

then the last three equations in (62) can be rewritten as

$$[\tilde{\mathbf{v}}(x) \times rot\tilde{\mathbf{v}}(x)] = \mathbf{0}.$$

Thus the vector field $\tilde{\mathbf{v}}(x) = (a, b, c)$ is a Kummer vector field. $\square$

For the general case, i.e., when the ξ, η and ζ are functions on the variables θ and ψ Cartesian approach produce the following equations respectively

$$\dot{\mathbf{x}} = \mathbf{v}(x)$$

$$
\begin{aligned}
&\sum_{j=1}^{5}\Big(\frac{\partial p_1}{\partial x^j} - \frac{\partial p_j}{\partial x^1} + \alpha_2(\frac{\partial p_4}{\partial x^j} - \frac{\partial p_j}{\partial x^4}) - \beta_2(\frac{\partial p_5}{\partial x^j} - \frac{\partial p_j}{\partial x^5})\Big)v^j = 0,\\
&\sum_{j=1}^{5}\Big(\frac{\partial p_2}{\partial x^j} - \frac{\partial p_j}{\partial x^2} + \alpha_3(\frac{\partial p_4}{\partial x^j} - \frac{\partial p_j}{\partial x^4}) - \beta_3(\frac{\partial p_5}{\partial x^j} - \frac{\partial p_j}{\partial x^5})\Big)v^j = 0,\\
&\sum_{j=1}^{5}\Big(\frac{\partial p_3}{\partial x^j} - \frac{\partial p_j}{\partial x^1} + \alpha_1(\frac{\partial p_4}{\partial x^j} - \frac{\partial p_j}{\partial x^4}) - \beta_1(\frac{\partial p_5}{\partial x^j} - \frac{\partial p_j}{\partial x^5})\Big)v^j = 0
\end{aligned}
$$

where $\psi = x^1, \quad \phi = x^2, \quad \theta = x^3, \quad y = x^4, \quad x = x^5$ and $\mathbf{v}$ is given by the formula (61).

7. Inverse problem of dynamics

Introduction

This section is devoted to apply Corollary 1.4 to study the problem of finding the field of force that generates a given $(N-1)$-parametric family of orbits for a mechanical system with N degrees of freedom. This problem is usually referred to as the inverse problem of dynamics. We study this problem in relation to the problems of Celestial Mechanics.

One of the fundamental classical problems in celestial mechanics is to determine the potential-energy function U such that every curve from a given family of curves will be a possible trajectory of a particle moving under the action of potential forces $\mathbf{F}$, admitting U; i. e. $\mathbf{F} = \dfrac{\partial U}{\partial \mathbf{x}}$.

In the modern scientific literature the importance of this problem was already acknowledged by Szebehely [3], [27]

The first inverse problem in Celestial Mechanics was stated and solved by Newton (1687) and concerns the determination of the potential field of force that ensures the planetary motion in accordance to the observed properties, namely to Kepler's laws.

Bertrand (1877) [1] proved that the expression for Newton's force of attraction can be obtained directly from the Kepler first law to within a constant multiplier.

Bertrand stated also a more general problem of determining a positional force, under which a particle describes a conic section under any initial conditions. Bertarnd's ideas were developed by [5] [26], [10], [6], and [7].

Dainelli in [5] essentially states a more general problem of how to determine the most general field of force (the force being supposed to depend only on the position of the particle on which it acts) under which a given family of planar curves is a family of orbits of a particle.

The solution proposed by Dainelli is the following .

Theorem 7.1. *The most general field of force* $\boldsymbol{F} = (F_x,\, F_y)$ *capable of generating the family of planar orbits* $f(x,y) = const$ *can be determine as follows* [5], [29]

$$F_x = -\lambda^2\{f, \partial_y f\} - \lambda\{f,\, \lambda\}\partial_y f, \quad F_y = \lambda^2\{f, \partial_x f\} + \lambda\{f,\, \lambda\}\partial_x f, \tag{63}$$

where $\{f, \quad\} = \partial_x f \partial_y - \partial_y f \partial_x$ *and* λ *is an arbitrary function which depends on the velocity with which the given orbits are described.*

By considering that the components F_x *and* F_y *are to be functions of the position of the particle, we can take* λ *to be an arbitrary function on* x *and* y.

The above expressions for the field of force under which the curves of the given family are orbits were first given by Dainelli [5].

After some calculations we can prove that (63) can be rewritten as follows

$$\mathbf{F} = \frac{\partial \frac{1}{2}\|\mathbf{v}\|^2}{\partial \mathbf{x}} - \lambda\left(\partial_x(\lambda\partial_x f) + \partial_y(\lambda\partial_y f)\right)\frac{\partial f}{\partial \mathbf{x}} \tag{64}$$

where $\mathbf{v} = (-\lambda\partial_y f,\, \lambda\partial_x f)$.

Suslov in [26] stated and solved a problem which was a further development of Bertrand's problem. He shows that, given a $(N-1)$-parametric family of orbits $f_j = f_j(x) = c_j$ for $j = 1, 2, \dots N-1$ in the configuration space of a holonomic system with N degrees of freedom and a kinetic energy

$T=\frac{1}{2}\sum_{j,k=1}^{N}G_{jk}(x)\dot{x}^j\dot{x}^k=\frac{1}{2}||\dot{\mathbf{x}}||^2$, it is necessary to determine the potential field of force under which any trajectory of the family can be traced by the representative point of the system. Suslov deduced the following system of linear partial differential equations with respect to the require potential function:

$$\frac{\partial\theta}{\partial\triangle_k}\frac{\partial U}{\partial x^N}-\frac{\partial\theta}{\partial\triangle_N}\frac{\partial U}{\partial x^k}=\frac{U+h}{\theta}\Big(\frac{\partial\theta}{\partial\triangle_N}\frac{\partial\theta}{\partial x^k}-\frac{\partial\theta}{\partial\triangle_k}\frac{\partial\theta}{\partial x^N}+\sum_{m=1}^{N}\triangle^m(\frac{\partial\theta}{\partial\triangle_k}\frac{\partial^2\theta}{\partial\triangle_N\partial x^m}-\frac{\partial\theta}{\partial\triangle_N}\frac{\partial^2\theta}{\partial\triangle_k\partial x^m})\Big)$$

for $k=1,2,..,N-1$. where θ, $\triangle^1$, $\triangle^2,...,\triangle^N$ are functions:

$$\sum_{k=1}^{N}\frac{\partial f(x)_\alpha}{\partial x^k}\triangle^k=0,\quad \theta=\frac{1}{2}\sum_{k,j=1}^{N}G_{kj}(x)\triangle^k\triangle^j$$

$$\triangle_k=\sum_{j=1}^{N}G_{jk}(x)\triangle^j,\quad k=1,2,..,N,\ \alpha=1,2,..,N-1,$$

and proved that theses equations represented the necessary and sufficient conditions under which the equations of motion of the study mechanical system admits the given $N-1$ partial integrals.

Assuming that given trajectories admit a family of the orthogonal surfaces, Joukovski in [10] constructed the potential-energy functions in explicit forms for systems with two and three degrees of freedom.

The following theorem was enunciated by Joukovsky in 1890

Theorem 7.2. *If $q=const$ is the equation of the family of curves on a surface, and $p=const$ denotes the family of curves orthogonal to these, then the curves $q=const$ can be freely described by a particle under the influence of forces derived from the potential-energy function*

$$V=\Delta_1(p)\Big(g(p)+\int h(q)\frac{\partial}{\partial q}(\frac{1}{\Delta_1(p)})dq\Big)$$

where h and g are arbitrary functions, and Δ_1 denotes the first differential parameter.

A new approach to the problem of constructing the potential field of force was proposed by Ermakov in [6], who integrated the equations for the potential-energy function for several particular cases.

In the most general form the inverse problem in dynamics was studied in [24, 22]. By applying the results presented in that work we propose the following new results:

(i) Statement and solution of inverse Dainelli's problem for a mechanical system with N degree of freedom.

(ii) New approach to solve the Suslov problem.
(iii) Statement and solution of inverse Joukovski's problem for mechanical system with $N \geq 3$ degree of freedom.
(iv) Generalization of Theorem 7.2 for mechanical system with $N \geq 3$ degree of freedom
(v) Statement and solution of inverse Stäckel's problem.
(vi) General solution of Bertrand's inverse problem.

The results listed above are obtained by applying Corollary 1.4.

Statement of the generalized inverse Dainelli problem.

Given a $N-1$ -parametric family of orbits $f_j = f_j(x) = c_j$ for $j = 1, 2, \ldots, N-1$ in the configuration space Q of a holonomic system with N degrees of freedom and kinetic energy $T = \frac{1}{2}||\dot{\mathbf{x}}||^2$. *Generalized Dainelli's problem* is the problem of determining the most general field of force that depends only on the position of the system under which any trajectory of the family can be traced by a representative point of the system.

Solution of the generalized inverse Dainelli problem

The following proposition provides a solution to the problem above

Proposition 7.3. *Given a mechanical system with a configuration space* Q *and a kinetic energy* $T = \frac{1}{2}||\dot{\boldsymbol{x}}||^2$, *then the most general field of force* $\boldsymbol{F}$ *that depends only on the position of the system and is capable of generating the given orbits* $f_j(x) = c_j$, *for* $j = 1, \ldots, N-1$ *where* $f_1, \ldots, f_{N-1}$ *are independent functions can be determine from the formula*

$$\boldsymbol{F} = \frac{\partial \left(\frac{1}{2}||\boldsymbol{v}||^2\right)}{\partial \boldsymbol{x}} + \lambda \sum_{j=1}^{N-1} a_{Nj} \frac{\partial f_j}{\partial \boldsymbol{x}}, \tag{65}$$

where

$$\boldsymbol{v} = -\lambda_N \frac{\{f_1, \ldots, f_{N-1}, *\}}{\{f_1, \ldots, f_{N-1}, f_N\}} = \lambda \{f_1, \ldots, f_{N-1}, *\}, \tag{66}$$

$a_{Nj} = a_{Nj}(x)$ *for* $j = 1, 2, \ldots N$ *are functions:*

$a_{Nj} = (-1)^{N+j-1} d\sigma \wedge df_1 \wedge df_2 \wedge \ldots \wedge df_{j-1} \wedge df_{j+1} \wedge \ldots \wedge df_{N-1}(\partial_1, \ldots, \partial_N)$,

and λ_N *and* f_N *are arbitrary functions such that* $\{f_1, \ldots, f_{N-1}, f_N\} \neq 0$.

Proof. Let us suppose that are given the $N-1$ parametric family of trajectories, hence $(\partial_{\mathbf{x}} f_j, \dot{\mathbf{x}}) = 0$ for $j = 1, \ldots N-1$. In view of independence of functions $f_1, \ldots, f_{N-1}$ we can solve these equations respect to velocity, thus we obtain the system $\dot{\mathbf{x}} = \mathbf{v}(x)$, where $\mathbf{v}$ is determine by the formula (66).

After covariant derivation we obtain the equations of motion of the mechanical system (see proof of Theorem 1.1)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{x}}} - \frac{\partial T}{\partial \mathbf{x}} = \frac{\partial\left(\frac{1}{2}||\mathbf{v}||^2\right)}{\partial \mathbf{x}} + \lambda\sum_{j=1}^{N-1} a_{Nj}\frac{\partial f_j}{\partial \mathbf{x}} = \mathbf{F},$$

therefore the proposition has been proved. □

The following proposition shows that Theorem 7.1 is a particular case of Proposition 7.3.

Corollary 7.4. *For $N = 2$ and $\mathrm{Q} = \mathbb{R}^2$ the force field (65) coincides with the solution proposed by Dainelli .*

Proof. Indeed for $N = 2$ the field of force $\mathbf{F}$ takes the form

$$\mathbf{F} = \frac{\partial\left(\frac{1}{2}||\mathbf{v}||^2\right)}{\partial \mathbf{x}} + \lambda\, a_{21}\frac{\partial f}{\partial \mathbf{x}}.$$

On the other hand by considering that $\mathbf{v} = (-\lambda\partial_y f,\, \lambda\partial_x f)$ thus

$$|\mathbf{v}||^2 = (\lambda\partial_x f)^2 + (\lambda\partial_y f)^2, \quad a_{21} = d\sigma(\partial_x,\, \partial_y) = \partial_x(\lambda\partial_x f) + \partial_y(\lambda\partial_y f)$$

hence we obtain the formula (64). □

Corollary 7.5. *For $N = 3$ the force field (65) takes the form*

$$\boldsymbol{F} = \frac{\partial\left(\frac{1}{2}||\boldsymbol{v}||^2\right)}{\partial \boldsymbol{x}} + \lambda\left(df_1(rot\boldsymbol{v})df_2 - df_2(rot\boldsymbol{v})df_1\right) \tag{67}$$

Example

Given a particle with $\mathrm{Q} = \mathbb{R}^3$ and kinetic energy $T = \frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)$. Construct the field of force capable of generating the two-parametric family of trajectories defined as intersections of the two families of surfaces

$$f_1 = \zeta = c_1, \qquad f_2 = H(\xi, \eta, \zeta) = c_2. \tag{68}$$

The solution of this problem can easily be derived from Corollary 7.5. The vector field $\mathbf{v}$, rot$\mathbf{v}$ are the following

$$\begin{aligned}
\mathbf{v} &= \lambda\left(\frac{\partial H}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial H}{\partial \xi}\frac{\partial}{\partial \eta}\right)\\
rot\,\mathbf{v} &= \frac{\partial}{\partial \zeta}\left(\lambda\frac{\partial H}{\partial \xi}\right)\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \zeta}\left(\lambda\frac{\partial H}{\partial \eta}\right)\frac{\partial}{\partial \eta} - \mu\frac{\partial}{\partial \zeta}
\end{aligned}$$

Hence the require field of force is such that

$$
\begin{aligned}
\ddot{\xi} &= \frac{\partial}{\partial \xi}\left(\frac{\lambda^2}{2}\left((\frac{\partial H}{\partial \xi})^2 + (\frac{\partial H}{\partial \eta})^2\right)\right) + \lambda\mu\frac{\partial H}{\partial \xi} \\
\ddot{\eta} &= \frac{\partial}{\partial \eta}\left(\frac{\lambda^2}{2}\left((\frac{\partial H}{\partial \xi})^2 + (\frac{\partial H}{\partial \eta})^2\right)\right) + \lambda\mu\frac{\partial H}{\partial \eta} \\
\ddot{\zeta} &= 0,
\end{aligned} \tag{69}
$$

where $\mu = \frac{\partial}{\partial \xi}\Big(\lambda\frac{\partial H}{\partial \xi}\Big) + \frac{\partial}{\partial \eta}\Big(\lambda\frac{\partial H}{\partial \eta}\Big)$

In the next section we make use of the solution of the generalized Dainelli inverse problem for solving the Suslov and generalized Joukovski problems.

Statement of Suslov's problem [26]

Given a $N-1$ -parametric family of orbits $f_j = f_j(x) = c_j$ for $j = 1, 2, \dots, N-1$ in the configuration space Q of a holonomic system with N degrees of freedom and kinetic energy $T = \frac{1}{2}||\dot{\mathbf{x}}||^2$. *Suslov's problem* is the problem of determining the potential field of force that under which any trajectory of the family can be traced by a representative point of the system.

Solution of Suslov's problem

We now propose a new solution to the Suslov problem. This solution we have obtained as a special case of the previous solution to the generalized inverse Dainelli problem.

Proposition 7.6. *Given a mechanical system with a configuration space* Q *and a kinetic energy* $T = \frac{1}{2}||\dot{\boldsymbol{x}}||^2$, *then the potential field of force* $\boldsymbol{F} = \frac{\partial U}{\partial \boldsymbol{x}}$, *capable of generating the given orbits* $f_j(x) = c_j$, *for* $j = 1, \dots, N-1$, *can be determine from the formula*

$$
\frac{\partial U}{\partial \boldsymbol{x}} = \frac{\partial\left(\frac{1}{2}||\boldsymbol{v}||^2\right)}{\partial \boldsymbol{x}} + \lambda\sum_{j=1}^{N-1} a_{Nj}\frac{\partial f_j}{\partial \boldsymbol{x}}, \tag{70}
$$

if and only if

$$
\lambda\sum_{j=1}^{N-1} a_{Nj}(x)df_j = dh(f_1, f_2, \dots, f_{N-1}) \tag{71}
$$

Clearly if (71) *holds then the potential function* U *is such that*

$$
U(x) = \frac{1}{2}\|\boldsymbol{v}\|^2 + h(f_1, f_2, \dots, f_{N-1})\,,
$$

where $\boldsymbol{v}$ *is determined by the formula* (66), *and* λ *is an arbitrary function.*

Proof. From (65) follows that $\mathbf{F} = \dfrac{\partial U}{\partial \mathbf{x}}$, if and only if $\lambda \sum_{j=1}^{N-1} a_{Nj}(x) df_j$ is exact 1-form dh. In view of the relations $df_j(\mathbf{v}) = 0$ we deduce that $dh(\mathbf{v}) = 0$, hence in view of independence of functions f_j for $j = 1, 2, \dots, N-1$ we get $h = (f_1, f_2, \dots, f_{N-1})$.

□

Example(Ermakov's problem).

Given a mechanical system with configuration space $\mathrm{Q} = \mathbb{R}^4$ and kinetic energy

$$T = \left(\frac{1}{2}(m_1(\dot{x}_1^2 + \dot{y}_1^2) + m_2(\dot{x}_2^2 + \dot{y}_2^2) \right).$$

Construct the potential field of force capable of generating the three-parametric family of trajectories defined as intersections of the families of hyper-surfaces

$$f_1 = x_1^2 + y_1^2 = c_1, \qquad f_2 = x_2^2 + y_2^2 = c_2, \qquad f_3 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = c_3$$

Corollary 7.7. *Under the assumptions of Proposition 7.6 for $N = 2$ we obtain that*

$$\frac{\partial U}{\partial \boldsymbol{x}} = \frac{\partial \left(\frac{1}{2} ||\boldsymbol{v}||^2 \right)}{\partial \boldsymbol{x}} + \lambda \left(\partial_x(\lambda \partial_x f) + \partial_y(\lambda \partial_y f) \right) \frac{\partial f}{\partial \boldsymbol{x}}. \tag{72}$$

if and only if

$$\lambda \left(\partial_x(\lambda \partial_x f) + \partial_y(\lambda \partial_y f) \right) df = dh(f) \tag{73}$$

where $\boldsymbol{v} = (-\lambda \partial_y f,\, \lambda \partial_x f)$

Another interesting application of the solution to the generalized Dainelli problem is the determination of the solution of the generalized Joukovski problem.

Statement of generalized Joukovski problem

Given a $N-1$ -parametric family of orbits $f_j = f_j(x) = c_j$ for $j = 1, 2, \dots, N-1$ in the configuration space Q of a holonomic system with N degrees of freedom and kinetic energy $T = \frac{1}{2}||\dot{\mathbf{x}}||^2$. Assuming complementary that the given trajectories admit a family of orthogonal hyper-surface $S = S(x) = c_N$ then *Generalized Joukovski problem* is the problem of determining the potential field of force that under which any trajectory of the family can be traced by a representative point of the system.

Solution of Generalized Joukovski problem

Proposition 7.8. *Given a mechanical system with a configuration space* Q *and kinetic energy* $T = \frac{1}{2}\|\dot{\boldsymbol{x}}\|^2$, *then the potential field of force* $\boldsymbol{F} = \frac{\partial U}{\partial \boldsymbol{x}}$, *capable of generating the given orbits* $f_j(x) = c_j$, *for* $j = 1, \ldots, N-1$, *which admit a family of orthogonal hyper-surface* $S = S(x) = c_N$ *can be determine from the formula*

$$\frac{\partial U}{\partial \boldsymbol{x}} = \frac{\partial \left(\frac{\nu}{\sqrt{2}}\|\frac{\partial S}{\partial \boldsymbol{x}}\|\right)^2}{\partial \boldsymbol{x}} + (\frac{\partial \frac{\nu^2}{2}}{\partial \boldsymbol{x}}, \frac{\partial S}{\partial \boldsymbol{x}})\frac{\partial S}{\partial \boldsymbol{x}} - \|\frac{\partial S}{\partial \boldsymbol{x}}\|^2\frac{\partial \frac{\nu^2}{2}}{\partial \boldsymbol{x}}$$

if and only if

$$(\frac{\partial \nu^2}{\partial \boldsymbol{x}}, \frac{\partial S}{\partial \boldsymbol{x}})dS - \|\frac{\partial S}{\partial \boldsymbol{x}}\|^2 d\nu^2 = -2dh(f_1,\, f_2, \ldots,\, f_{N-1}) \tag{74}$$

where ν *is an arbitrary function. Clearly if* (74) *holds then the potential function* U *can be determined as follow*

$$U = \left(\frac{\nu}{\sqrt{2}}\|\frac{\partial S}{\partial \boldsymbol{x}}\|\right)^2 + h(f_1,\, f_2, \ldots,\, f_{N-1}).$$

Proof. In view of the condition that there is an orthogonal hyper-surface to the given trajectories then the following relations hold $\left(\frac{\partial S}{\partial \mathbf{x}}, \frac{\partial f_j}{\partial \mathbf{x}}\right) = 0$ for $j = 1, 2, \ldots, N-1$, thus

$$\rho G^{-1}\frac{\partial S}{\partial \mathbf{x}} = \{f_1,\, f_2, \ldots,\, f_{N-1},\, *\} \tag{75}$$

where ρ is an arbitrary nonzero function and $G^{-1} = (G^{jk})$ is the inverse matrix of the Riemann metric G. Hence we obtain that the vector field (66) takes the form

$$\mathbf{v} = \nu G^{-1}\frac{\partial S}{\partial \mathbf{x}},$$

where $\nu = \lambda\rho$, therefore the 1-form associated to this vector field is such that $\sigma = \nu\left(\frac{\partial S}{\partial \mathbf{x}}, d\mathbf{x}\right)$ consequently

$$\imath_{\mathbf{v}} d\sigma = d\nu(\mathbf{v})dS - dS(\mathbf{v})d\nu = \nu(\frac{\partial \nu}{\partial \mathbf{x}}, \frac{\partial S}{\partial \mathbf{x}})dS - \nu|\frac{\partial S}{\partial \mathbf{x}}\|^2 d\nu$$

hence we easily obtain the proof of the proposition (see for more details proof of Theorem (1.1)). □

As a first application of the above proposition we have the following results

Corollary 7.9. *If in Proposition 7.8 we suppose that* $\nu = \frac{d\Phi(S)}{dS}$ *then the potential function* U *can be determine as follows*

$$U = \left(\frac{1}{\sqrt{2}}\|\frac{\partial \Phi(S)}{\partial \boldsymbol{x}}\|\right)^2 - h_0,$$

where h_0 *is an arbitrary constant.*

Proof. Indeed if $\nu = \dfrac{d\Phi(S)}{dS}$ then then the 1-form σ associated to vector field $\mathbf{v}$ is exact, thus $\imath_{\mathbf{v}}d\sigma == 0$, consequently (see formula (74)) $dh = 0$. Hence

$$U = \left(\frac{\nu}{\sqrt{2}}\|\frac{\partial S}{\partial \mathbf{x}}\|\right)^2 - h = \left(\frac{1}{\sqrt{2}}\|\frac{\partial \Phi(S)}{\partial \mathbf{x}}\|\right)^2 - h_0.$$

□

Example(Bertrand's Problema [1])

Given a particle with configuration space $\mathrm{Q} = \mathbb{R}^3$ and kinetic energy

$$T = \frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2).$$

Construct the potential field of force capable of generating the two-parametric family of trajectories defined as intersections of the two families of surfaces

$$f_1 = \zeta = c_1, \qquad f_2 = \sqrt{\xi^2 + \eta^2} + b\xi = c_2.$$

From (75) we obtain that the function S :

$$\frac{\partial S}{\partial \xi} = \frac{1}{\rho}\frac{\eta}{\sqrt{\xi^2 + \eta^2}}, \qquad \frac{\partial S}{\partial \eta} = \frac{1}{\rho}\left(-\frac{\xi}{\sqrt{\xi^2 + \eta^2}} - b\right).$$

Hence, by choosing $\rho = \eta$ we obtain that $S = \ln(\xi + \sqrt{\xi^2 + \eta^2}) - (b+1)\ln\eta$ and $\nu = \eta\lambda$. Clearly the field of force is potential in particular if $\nu = \Phi(S)$. The general solution of this problem we give below.

Example

Given a particle with configuration space $\mathrm{Q} = \mathbb{R}^3$ and kinetic energy

$$T = \frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2).$$

Construct the potential field of force capable of generating the two-parametric family of trajectories defined as intersections of the two families of surfaces

$$f_1 = xz = c_1, \qquad f_2 = yz = c_2$$

From (75) we obtain that the function $S = \dfrac{1}{2}(x^2 + y^2 - z^2)$, thus the condition (74) takes the form

$$(x\partial_x\nu^2 + y\partial_y\nu^2 - z\partial_z\nu^2)(xdx + ydy - zdz) - (x^2 + y^2 + z^2)d\nu^2$$
$$= -2dh(f_1, f_2).$$

There are two obvious solutions $\nu = \nu(x^2 + y^2 - z^2)$ and $\nu = z$. The first solution produces the potential function $U = U(x^2 + y^2 + z^2) - h_0, \quad h = h_0$ which coincide with the solution obtained by Joukovski and the second gives the potential $U = \dfrac{1}{2}z^4 - h_0, \quad h = \dfrac{1}{2}(f_1^2 + f_2^2) - h_0$.

The following result is a generalization of Theorem 7.2.

Corollary 7.10. *If $x^j = C_j = const$, for $j = 1, 2, .., N-1$ are the equations of the $N-1$ parametric family of curves on Q, and $x^N = const$ denotes the family of curves orthogonal to these, then the curves $x^j = C_j = const$ can be freely described by a particle under the influence of forces derived from the potential-energy function*

$$U = \frac{1}{G_{NN}(x^1, x^2, .., x^N)}\Big(g(x^N) + \sum_{j=1}^{N-1}\int h(x^1, x^2, .., x^{N-1})\frac{\partial G_{NN}(x^1, x^2, ..x^N)}{\partial x^j}dx^j\Big)$$

where $h = h(x^1,\, x^2,\, \ldots, x^{N-1})$ and $g = g(x^N)$ are arbitrary functions.

Clearly, for $N = 2$ we obtain the Joukovski theorem given in the introduction.

Proof. Whereas in the study case $f_j = x^j$, $j = 1, 2, \ldots N-1$ and $S = x^N$ then from Proposition 7.8 we obtain that the relations hold

$$\begin{aligned}
\frac{\partial 2U}{\partial x^j} &= \frac{\partial\left(\nu^2 G^{NN}\right)}{\partial x^j} - G^{NN}\frac{\partial \nu^2}{\partial x^j}, \quad j = 1, 2, \ldots N-1,\\
\frac{\partial 2U}{\partial x^N} &= \frac{\partial\left(\nu^2 G^{NN}\right)}{\partial x^N} - \left(\sum_{j=1}^{N-1} G^{Nj}\frac{\partial \nu^2}{\partial x^j}\right)
\end{aligned}$$

where $G_{kj} = G_{kj}(x^1,\, x^2, \ldots x^N)$ for $k, j = 1, 2, \ldots N$, if and only if

$$(\sum_{j=1}^{N} G^{jN}\frac{\partial \nu^2}{\partial x^j})dx^N - G^{NN}d\nu^2 = 2dh(f_1,\, f_2, \ldots,\, f_{N-1}), \tag{76}$$

assuming that the Riemann metric is orthogonal then

$d\nu^2 = -2G_{NN}dh$, where $G^{NN} = \dfrac{1}{G_{NN}}$.

$$\begin{aligned}
\nu^2 &= 2g(x^N) - 2\int G_{NN}dh\\
&= 2g(x^N) - 2G_{NN}h + \sum_{j=1}^{N-1} 2\int h\frac{\partial G_{NN}}{\partial x^j}dx^j,
\end{aligned}$$

where $g = g(x^N)$ is an arbitrary function.

Consequently in view of the formula $U = \dfrac{1}{2}\nu^2 G^{NN} + h = \dfrac{\nu^2}{2G_{NN}} + h$ we obtain the proof of the proposition. □

Now we apply Theorem 7.8 to solve the inverse problem which we will call the inverse *Stäckel problem.*

Statement of inverse Stäckel problem.

Given a $N-1$ -parametric family of orbits

$$f_\mu = f_\mu(x) \equiv \sum_{k=1}^{n}\int \frac{\varphi_{k\mu}(x^k)}{\sqrt{K_k(x^k)}}dx^k = c_\mu, \quad \mu = 1, 2, ..., N-1, \tag{77}$$

where $K_k(x^k) = 2\Psi_k(x^k) + 2\sum_{j=1}^{N} \alpha_j \varphi_{kj}(x^k)$, α_k, for $k = 1, 2, ..N$ are constants, in the configuration space Q of a holonomic system with N degrees of freedom and kinetic energy

$$T = \frac{1}{2}||\dot{\mathbf{x}}||^2 = \frac{1}{2}\sum_{j=1}^{N} \frac{(\dot{x}^j)^2}{A^j}, \tag{78}$$

where $A^j = A^j(x)$ for $j = 1, 2, \ldots, N$ are functions such that

$$\frac{\{\varphi_1, \varphi_2, \ldots, \varphi_{N-1}, *\}}{\{\varphi_1, \varphi_2, \ldots, \varphi_{N-1}, \varphi_N\}} = \sum_{j=1}^{N} A^j \partial_j$$

$d\varphi_\alpha = \sum_{k=1}^{N} \varphi_{k\alpha}(x^k) dx^k$, $\varphi_{k\alpha} = \varphi_{k\alpha}(x^k)$, for $k = 1, ..., N$, $\alpha = 1, \ldots, N$ are arbitrary functions.

The *inverse Stäckel problem* is the problem of determining the potential field of force that under which any trajectory of the family can be traced by a representative point of the system.

Solution of inverse Stäckel problem

Proposition 7.11. *Given a mechanical system with a configuration space* Q *and kinetic energy* (78)*, then the potential field of force* $\boldsymbol{F} = \dfrac{\partial U}{\partial \boldsymbol{x}}$*, capable of generating the given orbits* (77) *is the field of force with potential function*

$$U = \nu^2(S)\left(\frac{\{\varphi_1, \varphi_2, \ldots, \varphi_{N-1}, \Psi\}}{\{\varphi_1, \varphi_2, \ldots, \varphi_{N-1}, \varphi_N\}} + \alpha_1\right) - h_0, \tag{79}$$

where $S = 2\int \sum_{j=1}^{N} \sqrt{\Psi_k(x^k) + \sum_{j=1}^{N} \alpha_j \varphi_{kj}(x^k)}\; dx^k.$

Proof. In view of the equality

$$\frac{\{f_1, f_2, \ldots, f_{N-1}, *\}}{\{f_1, f_2, \ldots, f_{N-1}, f_N\}} = \frac{\begin{vmatrix} q_1(x^1)d\varphi_1(\partial_1) & \ldots & q_N(x^N)d\varphi_1(\partial_N) \\ \vdots & & \vdots \\ q_1(x^1)d\varphi_{N-1}(\partial_1) & \ldots & q_N(x^N)d\varphi_{N-1}(\partial_N) \\ \partial_1 & \ldots & \partial_N \end{vmatrix}}{\prod_{j=1}^{N} q_j(x^j)\{\varphi_1, \varphi_2, \ldots, \varphi_{N-1}, \varphi_N\}}$$

$$= \sum_{j=1}^{N}\left(\frac{A^j}{q_j(x^j)}\partial_j\right) = \sum_{j=1}^{N}\left(A^j \frac{\partial S}{\partial x^j}\partial_j\right)$$

where $q_j(x^j) = \dfrac{1}{\sqrt{K_k(x^k)}}$, consequently from (75) and (77) follows that

$$\mathbf{v} = -\lambda_N G^{-1}\frac{\partial S}{\partial \mathbf{x}} = \nu G^{-1}\frac{\partial S}{\partial \mathbf{x}},$$

hence

$$\begin{aligned}
\|\mathbf{v}\|^2 &= \nu^2 \sum_{j=1}^{N} A^j (K_j(x^j))^2 = \nu^2 \sum_{k=1}^{N} A^k \left(2\Psi_k(x^k) + 2\sum_{j=1}^{N} \alpha_j \varphi_{kj}(x^k)\right) \\
&= 2\nu^2 \sum_{k=1}^{N} A^k \Psi_k(x^k) + 2\nu^2 \sum_{j=1}^{N} \alpha_j \sum_{k=1}^{N} A^k \varphi_{kj}(x^k) \\
&= 2\nu^2 \left(\frac{\{\varphi_1,\, \varphi_2,\, \ldots,\, \varphi_{N-1},\, \Psi\}}{\{\varphi_1,\, \varphi_2,\, \ldots,\, \varphi_{N-1},\, \varphi_N\}} + \sum_{j=1}^{N} \alpha_j \frac{\{\varphi_1,\, \varphi_2,\, \ldots,\, \varphi_{N-1},\, \varphi_j\}}{\{\varphi_1,\, \varphi_2,\, \ldots,\, \varphi_{N-1},\, \varphi_N\}}\right) \\
&= .2\nu^2 \left(\frac{\{\varphi_1,\, \varphi_2,\, \ldots,\, \varphi_{N-1},\, \Psi\}}{\{\varphi_1,\, \varphi_2,\, \ldots,\, \varphi_{N-1},\, \varphi_N\}} + \alpha_1\right).
\end{aligned}$$

where $d\Psi = \sum_{j=1}^{N} \Psi_k(x^k) dx^k$.

On the other hand if we choose $\nu = \nu(S)$ then from Corollary 7.9 follows the proof of the proposition. □

Corollary 7.12. *If in the previous proposition we suppose that $\nu(S) = 1$ and $\alpha_1 = h_0$ then the potential function* (79) *coincide with the Stäckel potential* [4].

Proof. Indeed, if $\nu(S) = 1$ and $\alpha_1 = h_0$ then [4]

$$U = \frac{\{\varphi_1,\, \varphi_2,\, \ldots,\, \varphi_{N-1},\, \Psi\}}{\{\varphi_1,\, \varphi_2,\, \ldots,\, \varphi_{N-1},\, \varphi_N\}} = \sum_{k=1}^{N} A^k \Psi_k(x^k),$$

□

Statement of Bertrand's problem

Let a particle with configuration space $\mathrm{Q} = \mathbb{R}^2$ and kinetic energy $T = \dfrac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right)$. *Bertrand's inverse problem* is the problem of determining the most general potential field of force capable of generating the one-parametric family of conics $f = \sqrt{x^2 + y^2} + bx = c,$ where b is the eccentricity.

Solution of Bertrand's inverse problem

Now we study the problem of constructing the potential field of force, which is capable to generate given conics. We prove the following proposition.

Proposition 7.13. *The potential-energy function U capable of generating a one-parameter family of conics with eccentricity b is the function*

$$U = a_{-1}H_{-1}(\cos\theta) + K_{-1}\log r(1+b\cos\theta) + \sum_{j\in\mathbb{Z}\setminus\{-1\}} a_j r^{j+1}\left(H_j(\cos\theta) + K_j\frac{(1+b\cos\theta)^{j+1}}{j+1}\right)$$

if $b \neq 0$ where $a_j \quad j\in\mathbb{Z}$, K_j are real constants and H_j, $j\in\mathbb{Z}$ are solutions of the Heun equations with singularities at the points

$$0,\ 1\,,\frac{1+b}{2b},\infty$$

and with the exponents

$$(0,\frac{j+3+b(j+1)}{2b});\ (0,j-\frac{j+3+b(j+1)}{2b});\ (0,j+1);(-1-j,1-j)$$

and

$$U = \frac{\Psi(\cos\theta)}{r^2} - \frac{2}{r^2}\int h(r)dr$$

if $b=0$, where $\Psi=\Psi(\cos\theta)$ and $h=h(r)$ are arbitrary functions.

Proof. From Proposition 7.7 follows that the require potential field of force exist if and only if

$$\left(\frac{x}{\sqrt{x^2+y^2}}+b\right)\frac{\partial\lambda^2}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial\lambda^2}{\partial y} + \frac{2\lambda^2}{r} = 2\frac{\partial h}{\partial f}$$

By introducing the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, we find that previous condition condition takes the form

$$(1+b\cos\theta)\frac{\partial\lambda^2}{\partial r} - \frac{b\sin\theta}{r}\frac{\partial\lambda^2}{\partial\theta} + \frac{2\lambda^2}{r} = 2\frac{\partial h}{\partial f}$$

or, equivalently,

$$(1+b\tau)\frac{\partial\lambda^2}{\partial r} + \frac{b(1-\tau^2)}{r}\frac{\partial\lambda^2}{\partial\tau} + \frac{2\lambda^2}{r} = 2\frac{\partial h}{\partial f} \tag{80}$$

where $f = r(1+b\tau)$, $\tau=\cos\theta$. We embark now upon the study of the case when $b\neq 0$ and h is such that

$$h(f) = \nu_{-1}\ln|f| + \sum_{\substack{j\in\mathbb{Z}\\ j\neq -1}}\nu_j\frac{f^{j+1}}{j+1}, \tag{81}$$

where ν_j, $j\in\mathbb{Z}$, are real constants, and λ is determined in such a way that

$$\lambda^2 = \sum_{j\in\mathbb{Z}}\psi_j(r)H_j(\tau) \tag{82}$$

It is clear that the series (81) and (82) are formal series.

By inserting (81) and (82) into (80) we obtain

$$\sum_{j\in\mathbb{Z}}\left((1+b\tau)\frac{d\psi_j(r)}{dr}H_j(\tau)+\frac{\psi_j(r)}{r}\left(b(1-\tau^2)\frac{dH_j(\tau)}{d\tau}+2\right)-2\nu_j r^j(1+b\tau)^j\right)=0$$

This equation holds if

$$\psi_j(r)=a_j r^{j+1},\quad \nu_j=-a_jK_j$$

for $j\in\mathbb{Z}$ and we determine H_j as a solution to the equation

$$b(1-\tau^2)H_j'(\tau)+\big((j+1)b\tau+j+3\big)H_j(\tau)+2K_j(1+b\tau)^j=0 \tag{83}$$

for $j\in\mathbb{Z}$.

The general solution of this equation is

$$H_j(\tau)=\xi_j(\tau)\left(C_j-\frac{2K_j}{b}\int\frac{(1+b\tau)^j}{(1-\tau^2)\xi_j(\tau)}d\tau\right)$$

$$\xi_j(\tau)=(1-\tau)^{\frac{j+1}{2}+\frac{j+3}{2b}}(\tau+1)^{\frac{j+1}{2}-\frac{j+3}{2b}}$$

where C_j, $j\in\mathbb{Z}$ are arbitrary constants.

Under these conditions, the required potential-energy function U results in the form

$$U(r,\tau)=\frac{1}{2}\lambda^2(1+b^2+2b\tau)-h(f)=\sum_{j\in\mathbb{Z}}a_jU_j(r,\tau)$$

where

$$\begin{aligned}U_j(r,\tau)&=\frac{1}{2}r^{j+1}H_j(\tau)(1+b^2+2b\tau)+\frac{K_j}{j+1}f^{j+1}\quad \text{if } j\neq-1\\ U_{-1}(r,\tau)&=\frac{1}{2}H_{-1}(\tau)(1+b^2+2b\tau)+K_{-1}\ln|f|.\end{aligned}$$

We will study the subcase when $b=1$ separately from the subcase when $b\neq1$.

If $b=1$,

$$U(r,\tau)=\lambda^2(1+\tau)-h(f)=\sum_{j\in\mathbb{Z}}a_jU_j(r,\tau).$$

where

$$\begin{aligned}U_j(r,\tau)&=r^{j+1}(1-\tau)^{j+2}\left(C_j-2K_j\int\frac{(1+\tau)^j}{(1-\tau)^{j+3}}d\tau\right)+\frac{K_j}{j+1}f^{j+1},\quad if\ j\neq-1\\ U_{-1}(r,\tau)&=(1-\tau)\left(C_{-1}-2K_{-1}\int\frac{d\tau}{(1-\tau)^2(1+\tau)}\right)+K_{-1}\ln|f|.\end{aligned}$$

Easily verifies that

$$U_{-2}=\frac{C_{-2}}{r}-2\frac{K_{-2}}{r}\Big(\int\frac{d\tau}{(1+\tau)^2(1-\tau)}+\frac{1}{1+\tau}\Big)=\frac{C_{-2}}{r}+\frac{K_{-2}}{r}g(\tau)$$

where $g(\tau) = \ln\sqrt{\dfrac{1-\tau}{1+\tau}}$. Therefore, if $b = 1$,

$$U(r,\tau) = \frac{a_{-2}C_{-2}}{r} + \frac{a_{-2}K_{-2}g(\tau)}{r} + \sum_{\substack{j\in\mathbb{Z}\\ j\neq -2}} a_j U_j(r,\tau)$$

If $b \neq 1$, $b \neq 0$, it is easy to prove that

$$H_{-2}(\tau) = \frac{(1-\tau)^{\frac{1-b}{2b}}}{(1+\tau)^{\frac{1+b}{2b}}} C_{-2} - \frac{2K_{-2}}{(b\tau+1)(1-b^2)}$$

$$U_{-2}(r,\tau) = \frac{H_{-2}}{2r}(1+b^2+2b\tau) - \frac{K_{-2}}{r(b\tau+1)} = \frac{2K_{-2}}{r(b^2-1)} + \frac{C_{-2}}{r}G(\tau)$$

where

$$G(\tau) = \frac{1}{2}\sqrt{\left(\frac{1-\tau}{1+\tau}\right)^{\frac{1}{b}}\frac{1}{1-\tau^2}(1+b^2+2b\tau)}$$

Under these conditions, the potential function U takes the form

$$U(r,\tau) = \frac{a_{-2}C_{-2}}{r}G(\tau) + \frac{2a_{-2}K_{-2}}{r(b^2-1)} + \sum_{\substack{j\in\mathbb{Z}\\ j\neq -2}} a_j U_j(r,\tau)$$

Summarizing the above computations, we deduce that if $b \neq 0$ the function U is represented as follows:

$$U(r,\tau) = \frac{\alpha}{r} + \frac{\beta(\tau)}{r} + \sum_{\substack{j\in\mathbb{Z}\\ j\neq -2}} a_j U_j(r,\tau)$$

where α is a constant and $\beta = \beta(\tau)$ is a certain function.

If $b = 0$, then $f = r$ and condition (80) takes the form

$$\partial_r\lambda^2 + 2\frac{\lambda^2}{r} = 2\partial_f h(f)\,.$$

Therefore,

$$r^2\lambda^2 = 2\int r^2\partial_r h(r)dr + 2\Psi(\tau),$$

which rearranged results in the expression:

$$\lambda^2 = \frac{2}{r^2}\int r^2\partial_r h(r)dr + \frac{2\Psi(\tau)}{r^2} = 2h(r) - \frac{4}{r^2}\int h(r)rdr + \frac{2\Psi(\tau)}{r^2}$$

where Ψ is an arbitrary function.

Hence,

$$U(r,\tau) = \frac{\Psi(\tau)}{r^2} - \frac{2}{r^2}\int h(r)dr.$$

To end the proof of the claim will establish the relationship between the functions H_j for $j \in \mathbb{Z}$ and the solutions of Heuns equations.

The canonical form of Heun's general equation will be taken as [23]

$$\frac{d^2x}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right)\frac{dx}{dz} + \frac{\alpha\beta z - B}{z(z-1)(z-a)}x = 0, \tag{84}$$

here x and z are regarded as complex variables and $\alpha, \beta, \gamma, \delta, \epsilon, a, b$ are parameters, generally complex and arbitrary, with the only condition that $a \neq 0, 1$. The first five parameters are linked by the relation $\alpha + \beta + 1 = \gamma + \delta + \epsilon$.

The equation is, therefore, of the Fuchsian type , with regular singularities at the points $z = 0, 1, a, \infty$. The exponents at these singularities are, respectively, $(0, 1-\gamma), (0, 1-\epsilon), (0, 1-\delta), (\alpha, \beta)$

Now we establish the relation between equation (83) and Heun's equation.

By fulfilling the replacement

$$z = \frac{1}{2}(\tau + 1)$$

we can easily obtain the following representation for (83):

$$z(z-1)\frac{dH_j}{dz} + \frac{1}{2b}\big((1+b-2bz)(j+1)+2\big)H_j(z) - \frac{K_j}{b}(1+b-2bz)^j = 0$$

By differentiating and fulfilling some straightforward calculations, we deduce that the functions

$$F_j(z) = \left(z(z-1)\frac{dH_j}{dz} + \frac{1}{2b}\big((1+b-2bz)(j+1)+2\big)H_j(z)\right)(1+b-2bz)^{-j} \quad (j \in \mathbb{Z})$$

are first integral of the Heun equation:

$$\begin{aligned}&\frac{d^2H_j}{dz^2} + \left(\frac{1-a(1+j)-\dfrac{1}{b}}{z} + \frac{a(1+j)+\dfrac{1}{b}-j}{z-1} + \frac{-j}{z-a}\right)\frac{dH_j}{dz}\\ &+\frac{(j^2-1)z - \dfrac{j}{b} - a(j^2-1)}{z(z-1)(z-a)}H_j(z) = 0\end{aligned} \tag{85}$$

where $a = \dfrac{1+b}{2b}$.

By comparison with classical Heuns equation, we obtain:

$$\begin{aligned}&\gamma_j = 1 - \frac{1+b}{2b}(1+j) - \frac{1}{b}, \quad \delta_j = \frac{1+b}{2b}(1+j) + \frac{1}{b} - j,\\ &\alpha_j\beta_j = j^2 - 1 = -(1+\epsilon_j)(2-\gamma_j-\delta_j), \qquad \epsilon_j = -j\\ &B_j = \frac{j}{b} + \frac{1+b}{2b}(j^2-1) = -a(2-\gamma_j-\delta_j) - (1-\gamma_j)\epsilon_j\end{aligned}$$

Evidently, when the given conics are parabolas then in (85) we have the confluence of singularities. In fact, in this case $b = 1$ so $a = 1$, and as a consequence the Heun equation is transformed into hypergeometric differential equation

$$\frac{d^2H_j}{dz^2}+\left(\frac{-1-j}{z}+\frac{2-j}{z-1}\right)\frac{dH_j}{dz}+\frac{(j^2-1)(z-1)-j}{z(z-1)^2}H_j(z)=0,$$

for $j\in\mathbb{Z}$. Which completed the proof of Proposition. $\square$

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