Eigenvalues and eigenfunctions of the Laplacian via inverse iteration with shift

Rodney Josué BIEZUNER [a], Grey ERCOLE [a], Eder Marinho MARTINS [b*]

[a] *Departamento de Matemática - ICEx, Universidade Federal de Minas Gerais,*

Av. Antônio Carlos 6627, Caixa Postal 702, 30161-970, Belo Horizonte, MG, Brazil

[b] *Departamento de Matemática - ICEB, Universidade Federal de Ouro Preto,*

Campus Universitário Morro do Cruzeiro, 35400-000, Ouro Preto, MG, Brazil

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Abstract

In this paper we present an iterative method inspired by the inverse iteration with shift technique of finite linear algebra designed to find the eigenvalues and eigenfunctions of the Laplacian with homogeneous Dirichlet boundary condition for arbitrary bounded domains $\Omega \subset \mathbb{R}^N$. Uniform convergence away from nodal surfaces is obtained and used in order to produce a faster and more accurate algorithm for computing the eigenvalues with minimal computational requirements, instead of using the ubiquitous Rayleigh quotient in finite linear algebra. The method can also be used in order to produce the spectral decomposition of any given function $u \in L^2(\Omega)$.

Keywords: Laplacian, eigenvalues, eigenfunctions, Fourier series, inverse iteration with shift, Rayleigh quotient.

1 Introduction

In [BEM] we introduced an iterative method for computing the first eigenpair of the p-Laplacian operator $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, $p > 1$, with homogeneous Dirichlet boundary condition in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$. The technique was inspired by the inverse power method or inverse iteration of finite linear algebra. In the present paper we concentrate in the special case $p = 2$, the Laplace operator Δ, which was superficially dealt with in [BEM]. Besides clarifying some of the arguments sketched in that paper for this

E-mail addresses: rodney@mat.ufmg.br (R. J. Biezuner), grey@mat.ufmg.br (G. Ercole), eder@iceb.ufop.br (E. Martins).

case and providing some error estimates, our main purpose in this work is to show how inverse iteration with shift in the presence of uniform convergence can be used in order to obtain a fast and efficient method for computing the eigenvalues and eigenfunctions of the Laplacian operator with homogeneous Dirichlet boundary condition for any bounded domain Ω. If the eigenvalues or at least good estimates for them are a priori known, the method can produce the corresponding eigenfunctions with great speed and accuracy. The technique can alternatively also be used as a fast process to obtain the spectral decomposition of any function $u \in L^2(\Omega)$ (in other words, the Fourier series of u). We remark that the application of the method to the special case of the Laplacian operator is more natural since the Laplacian is a linear operator, $L^2(\Omega)$ is a Hilbert space and the inverse operator $-\Delta^{-1}$ is a self-adjoint and compact operator, therefore allowing the complete characterization of its spectral structure, as well as having the property that its eigenfunctions constitute a basis for $L^2(\Omega)$ (except for compactness, these properties are absent when $p \neq 2$).

Inverse iteration with shift is used in finite linear algebra in order to find the eigenvalues and eigenfunctions of a finite-dimensional linear operator. As an eigenvalue-finding procedure it is not as efficient as other methods, such as the QR algorithm. However, if the eigenvalues of the operator or at least very good estimates of them are known in advance, its rate of convergence can be very fast (see [Trefenthen-Bau], for instance). This approach can be naturally extended to self-adjoint compact linear operators in infinite-dimensional Hilbert spaces such as the Laplacian and those arising in Sturm-Liouville problems. In spite of this, we have not been able to find any reference in the literature to this approach being used in the Laplacian context. Since there is now a vast literature concerning the search for estimates for the eigenvalues of the Laplacian, as well as the gaps between eigenvalues (see, for instance, [Kutller-Sigillito], [Hile-Protter], [Yang], [Cheng-Yang]; although particularly useful in our context would be lower bounds for the difference between consecutive eigenvalues), these results can be used in connection with the inverse iteration with shif algorithm to find eigenfunctions of the Laplacian in arbitrary domains, as well as better approximations for its eigenvalues. It must be emphasized, however, that as with the finite linear method, the inverse iteration with shift method is not capable to find all the eigenfunctions associated to a non-simple eigenvalue. It can only find an eigenfunction of the associated eigenspace. In the generic sense most domains have Laplacian spectra consisting only of simple eigenvalues (see [Uhlenbeck1], [Uhlenbeck2]), although many domains of practical interest have eigenvalues with multiplicity greater than one (usually, domains which exhibit some type of symmetry, although not all of them).

Algorithm 1 below is the simplest version of the inverse iteration with shift algorithm for computing one specific eigenvalue and corresponding eigenfunction of the Laplacian.

Algorithm 1 Inverse Iteration with Shift for Laplacian Eigenvalue and Eigenfunction

1: $\phi_0 = u$
2: Set x_0 (point in Ω outside nodal surfaces, randomly chosen)
3: Set σ (shift, usually eigenvalue estimate)
4: **for** $n = 0, 1, 2, \dots$ **do**
5: Solve $-\Delta_\sigma \phi_{n+1} = \phi_n$ in Ω, $\phi_{n+1} = 0$ on $\partial\Omega$
6: **end for**
7: **return** $\phi_{n+1} / \|\phi_{n+1}\|_\infty$ (L^∞-normalized eigenfunction)
8: **return** $\phi_n(x_0)/\phi_{n+1}(x_0) + \sigma$ (eigenvalue)

In principle, the function u at the start of Algorithm 1 should be chosen so that it will have components in all eigenspaces of the Laplacian and a random choice would suffice. However, in practice, due to rounding errors any simple function can be used. In our numerical tests (see Section 6), we observed that even the unit constant function could be used in order to obtain all the eigenvalues, even in a domain where it does not have an infinite number of them in its spectral decomposition. In Line 5 of Algorithm 1 the symbol $-\Delta_\sigma$ stands for the shift operator $-\Delta - \sigma I$ and any PDE solver can be used. This allows one to choose the fastest solver available for a particular domain. In Line 8 the eigenvalue is computed according to the uniform convergence theory developed in Section 5. Since uniform convergence occurs away from nodal surfaces, a point $x_0 \in \Omega$ not in a nodal surface must be chosen; because nodal surfaces have zero N-dimensional measures, a random choice will suffice in the vast majority of cases, even taking into account that nodal surfaces change as the computed eigenfunction changes. In finite linear algebra, the approximation to the eigenvalue is usually computed using the Rayleigh quotient. In our context, the eigenvalue can also be computed via the Rayleigh quotient of the approximated eigenfunction:

$$r\left(\phi_{n+1}\right) = \frac{\langle \nabla\phi_{n+1}, \nabla\phi_{n+1}\rangle_2}{\langle \phi_{n+1}, \phi_{n+1}\rangle_2} = \frac{\|\nabla\phi_{n+1}\|_2^2}{\|\phi_{n+1}\|_2^2} = \frac{\int_\Omega |\nabla\phi_{n+1}|^2 \, dx}{\int_\Omega \phi_{n+1}^2 \, dx}. \tag{1}$$

However, due to the much higher oscillatory nature of the higher frequence eigenfunctions, in order to accurately compute the integral of the (squared) gradient of eigenfunctions belonging to high frequence eigenvalues a much finer grid needs to be used, which affects the efficiency of the method. Therefore, the Rayleigh quotient is not recommended for the computation of the eigenvalues of the Laplacian, unless one is prepared to incur the higher computational costs (see also further comments in Section 7). A third alternative way to compute the eigenvalues is given by the quotient

$$\frac{\|\phi_n\|_2}{\|\phi_{n+1}\|_2} + \sigma = \frac{\int_\Omega \phi_n^2 \, dx}{\int_\Omega \phi_{n+1}^2 \, dx} + \sigma, \tag{2}$$

when the shift σ is chosen below the eigenvalue. This quotient also gives accurate approximations for the eigenvalues even using relatively coarse meshes; the computation of the integrals of the approximated eigenfunctions, instead of their *gradients*, does not appear to be significantly affected by the use of coarse grids, even for higher frequence eigenfunctions, as has been observed in the numerical experiments (see Section 6).

We believe that, differently from what happens in finite dimensions, where much better and faster algorithms for finding the eigenvalues of linear operators (matrices), specially self-adjoint operators, are present, the inverse iteration with shift algorithm can be a very competitive method for finding the eigenvalues of the Laplacian. Typical algorithms for computing Laplacian eigenvalues involve the discretization of the Laplacian operator and the computation of the eigenvalues of the resulting discretization matrix. However, since only a small number of the eigenvalues of the discretization matrix are good approximations to the Laplacian eigenvalues (the smaller ones), huge matrices are necessary in order to obtain a sufficiently good number of eigenvalues. And some problems, particularly those arising in the study of quantum billiards, demand the computation of a very large number of eigenvalues. Needless to say, besides the requirements of memory, the size of the matrix makes it computationally costly to find its eigenvalues (see the classical [Hackbusch] book, the review [Kutller-Sigillito] and the more recent work [Heuveline] for details). The inverse iteration with shift method, that requires only typical relatively modest sizes for meshes in order to solve the Poisson equation with homogeneous Dirichlet boundary condition, can be very competitive in terms of memory allocation and processing time. This is true even when one considers that in order to find accurate approximations for the highest order eigenvalues and eigenfunctions sometimes one needs to refine the mesh, due to the increase of oscillations.

Even if good estimates for the eigenvalues of a particular domain are not known in advance, a few iterations of inverse iteration with shift should be able to find good approximations to them, which can work as first estimates for the shift on a second run of the algorithm. The first choices for the shift might be concentrated around the numbers given by Weil's Law (see [Weyl] or [Courant-Hilbert]).

The inverse iteration with shift method can also be used in order to find the spectral decomposition of any function $u \in L^2(\Omega)$, that is, in order to find its projections on the Laplacian eigenspaces. One has only to be careful to eliminate spurious projections, that is, projections which arise from rounding errors. This can be done through computing the Fourier coefficient associated to each eigenspace. If this coefficient becomes less than a specified very small tolerance, this projection can be safely discarded as arising from rounding errors. The two algorithms can be combined together in order to simultaneously find both the desired spectral decomposition of a given function defined on a domain and the spectrum of the Laplacian on it. The spectral decomposition algorithm is given below (Algorithm 2). Once again, Line 12 can be replaced by (1) or (2).

Although for the sake of simplicity all computations here are done for the Laplacian, the same algorithm can be used for similar elliptic operators.

This paper is organized as follows. The inverse iteration with shift sequence is defined in Section 2, where most of the notation used in this paper is also established. In Section 3 some well-known results concerning the Rayleigh quotient are recalled and proven for completeness. In Sections 4 and 5 we discuss the L^2 and uniform convergence of the inverse iterated sequence, respectively. Section 6 presents the results of a few numerical experiments.

Algorithm 2 Spectral Decomposition

$\phi_0 = u$
Set x_0 (point in Ω randomly chosen)
for $k = 0, 1, 2, \dots$ **do**
  Set σ_k (shift)
  for $n = 0, 1, 2, \dots$ **do**
    Solve $-\Delta_{\sigma_k} \phi_{n+1}^k = \phi_n^k$ in Ω, $\phi_{n+1}^k = 0$ on $\partial\Omega$
  end for
  Compute $\langle u, \phi_{n+1}^k / \|\phi_{n+1}^k\|_2 \rangle_2$ (Fourier coeficient)
  if $\left|\langle u, \phi_{n+1}^k / \|\phi_{n+1}^k\|_2 \rangle_2\right| >$ tolerance **then**
    return $\phi_{n+1}^k / \|\phi_{n+1}^k\|_2$ (L^2-normalized eigenfunction)
    return $\langle u, \phi_{n+1}^k / \|\phi_{n+1}^k\|_2 \rangle_2$ (Fourier coefficient)
    return $\phi_n(x_0)/\phi_{n+1}(x_0) + \sigma_k$ (eigenvalue)
  end if
end for

Finally, in Section 7 we discuss if the rate of convergence of the method could theoretically be improved through the use of the inverse iteration with shift given by the Rayleigh quotient, as is standard in finite linear algebra.

2 Definition of the inverse iteration with shift sequence

Let $\mathcal{H} = \{e_k\}_{k=1}^{\infty} \subset H_0^1(\Omega)$ be an orthogonal (not necessarily normalized) basis for $L^2(\Omega)$ consisting of eigenfunctions of the Laplacian operator with homogeneous Dirichlet boundary condition, that is,

$$\begin{cases} -\Delta e_k = \lambda_k e_k & \text{in } \Omega, \\ e_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\{\lambda_k\}_{k=1}^{\infty}$ is the non-decreasing sequence of eigenvalues of the Laplacian, counting multiplicities:

$$0 < \lambda_1 < \lambda_2 \leqslant \dots \tag{4}$$

Let $\sigma > 0$ and define the *shift operator* $-\Delta_\sigma = -\Delta - \sigma I$. It follows that e_k is also an eigenfunction of $-\Delta_\sigma$ corresponding to the eigenvalue

$$\lambda_k - \sigma. \tag{5}$$

Conversely, if λ is an eigenvalue of $-\Delta_\sigma$, then $\lambda = \lambda_k - \sigma$ for some k. Thus, the spectrum of the shift operator equals the spectrum of the Laplacian operator shifted to the left by σ, while the corresponding eigenspaces are the same.

Given $u \in L^2(\Omega)$, let

$$u = \sum_{k=1}^{\infty} \alpha_k e_k \tag{6}$$

be the Fourier expansion of u, so that the Fourier coefficients α_k are given by

$$\alpha_k = \frac{\langle u, e_k \rangle_2}{\|e_k\|_2^2} = \frac{\int_\Omega u e_k \, dx}{\int_\Omega e_k^2 \, dx}.$$

Denote by λ_u^1 the least eigenvalue whose associated eigenspace is not orthogonal to u. That is,

$$\lambda_u^1 := \lambda_{k_1} \quad \text{where } k_1 = \min \{k : \alpha_k \neq 0\}.$$

In other words, λ_u^1 is the first eigenvalue such that u has a non-zero component in the corresponding eigenspace. Note that if r_1 is the multiplicity of λ_{k_1} then

$$\lambda_u^1 = \lambda_{k_1} = \lambda_{k_1+1} = \cdots = \lambda_{k_1+r_1-1}.$$

We will denote by e_u^1 the orthogonal projection of u on the λ_u^1-eigenspace, that is

$$e_u^1 := \alpha_{k_1} e_{k_1} + \alpha_{k_1+1} e_{k_1+1} + \cdots + \alpha_{k_1+r_1-1} e_{k_1+r_1-1} = \sum_{\lambda_k = \lambda_u^1} \alpha_k e_k.$$

Thus, the Fourier expansion of u can be rewritten as

$$u = \sum_{\lambda_k \geqslant \lambda_u^1} \alpha_k e_k = e_u^1 + \sum_{\lambda_k > \lambda_u^1} \alpha_k e_k = e_u^1 + \sum_{k \geqslant k_1 + r_1} \alpha_k e_k. \tag{7}$$

Proceeding in this way, denoting by e_u^j the orthogonal projection of u on the λ_u^j-eigenspace which is the j^{th}-eigenspace not orthogonal to u, the eigenfunction expansion of u can be written in terms of its non-zero components in the eigenspaces of the Laplacian as

$$u = \sum_{j=1}^{M} e_u^j \tag{8}$$

where either M is a positive integer or, as in most cases, $M = \infty$, and the corresponding sequence of eigenvalues $\{\lambda_u^j\}_{j=1}^\infty$ is (strictly) increasing

$$0 < \lambda_u^1 < \lambda_u^2 < \ldots$$

As is well known from the theory of compact linear operators, if σ does not belong to the spectrum of $-\Delta$ we have that $(-\Delta_\sigma)^{-1} : L^2(\Omega) \longrightarrow L^2(\Omega)$ is a continuous, compact and invertible operator. Therefore, whenever σ is not an eigenvalue of the Laplacian, we can define a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ by inverse iteration setting $\phi_0 = u$ and

$$\begin{cases} -\Delta_\sigma \phi_{n+1} = \phi_n & \text{in } \Omega, \\ \phi_{n+1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

3 Properties of the Rayleigh quotient

We present here some results concerning the Rayleigh quotient. For ease of consultation, short proofs of some well-known results are given.

Proposition 1. (Rayleigh's Principle) *Let* $r : H_0^1(\Omega) \setminus \{0\} \longrightarrow \mathbb{R}$ *defined by*

$$r(v) = \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_\Omega v^2 \, dx}$$

denote the Rayleigh quotient. Then u *is a critical point of* r *if and only if* u *is an eigenfunction of the Laplacian with homogeneous Dirichlet boundary condition and* $r(u)$ *is the corresponding eigenvalue.*

Proof: Given $v \in H_0^1(\Omega)$, we have

$$r'(u)\, v = \frac{2}{\|u\|_2^2} \left[\langle \nabla u, \nabla v \rangle_2 - r(u) \langle u, v \rangle_2 \right].$$

Therefore, $r'(u) = 0$ if and only if

$$\int_\Omega \nabla u \cdot \nabla v = r(u) \int_\Omega uv$$

for all $v \in W_0^{1,2}(\Omega)$, that is, u is a weak solution of

$$\begin{cases} -\Delta u = r(u)\, u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

■

Corollary 1. *The Rayleigh quotient gives a quadratically accurate estimate for the Dirichlet Laplacian eigenvalues, that is, if* u *is an eigenfunction of the Laplacian with homogeneous Dirichlet boundary condition with* $r(u)$ *as the corresponding eigenvalue, then*

$$r(v) - r(u) = O\left(\|v - u\|^2 \right)$$

as $v \to u$ *in* $L^2(\Omega)$.

Proof: If follows immediately from Taylor's formula, since $r'(u) = 0$. ■

4 L^2-convergence of inverse iteration with shift

For each $u \in L^2(\Omega)$ and $\sigma > 0$ not in the Laplacian spectrum consider the sequence $\{\phi_n\}$ defined by inverse iteration in (9). Since $\phi_{n+1} = (-\Delta_\sigma)^{-1} \phi_n$, it follows from (8) that the eigenfunction expansion of ϕ_n is given by

$$\phi_n = \sum_{j=1}^{M} \frac{1}{\left(\lambda_u^j - \sigma\right)^n} e_u^j. \tag{10}$$

Let λ_u^σ be the Laplacian eigenvalue appearing in the spectral decomposition of u which is closest to σ, i.e.,

$$|\lambda_u^\sigma - \sigma| = \min_{j \in \mathbb{N}} \left|\lambda_u^j - \sigma\right|. \tag{11}$$

Denoting by e_u^σ the projection of u on the λ_u^σ-eigenspace, we can write

$$\begin{aligned}
\phi_n &= \frac{1}{(\lambda_u^\sigma - \sigma)^n} e_u^\sigma + \sum_{\left|\lambda_u^j - \sigma\right| > |\lambda_u^\sigma - \sigma|}^{M} \frac{1}{(\lambda_u^j - \sigma)^n} e_u^j \\
&= \frac{1}{(\lambda_u^\sigma - \sigma)^n} \left(e_u^\sigma + \sum_{\left|\lambda_u^j - \sigma\right| > |\lambda_u^\sigma - \sigma|}^{M} \left(\frac{\lambda_u^\sigma - \sigma}{\lambda_j - \sigma}\right)^n e_u^j \right),
\end{aligned}$$

or

$$\phi_n = \frac{1}{(\lambda_u^\sigma - \sigma)^n} \left(e_u^\sigma + \psi_n\right), \tag{12}$$

where

$$\psi_n = \sum_{\left|\lambda_u^j - \sigma\right| > |\lambda_u^\sigma - \sigma|}^{M} \left(\frac{\lambda_u^\sigma - \sigma}{\lambda_u^j - \sigma}\right)^n e_u^j. \tag{13}$$

In particular,

$$\frac{\phi_n}{\|\phi_n\|_2} = \pm \frac{e_u^\sigma + \psi_n}{\|e_u^\sigma + \psi_n\|_2}, \tag{14}$$

where the sign of the right-hand side will depend on whether the shift is taken above or below the eigenvalue and on n: if the shift is taken below the eigenvalue, the sign will always be positive, while if the shift is chosen above the eigenvalue the sign will be $(-1)^n$. Throughout this paper, we will denote by λ_u^τ the Laplacian eigenvalue appearing in the spectral decomposition of u which is second closest to σ, i.e.,

$$|\lambda_u^\tau - \sigma| = \min_{\substack{j \in \mathbb{N} \\ \lambda_u^j \neq \lambda_u^\sigma}} \left|\lambda_u^j - \sigma\right|. \tag{15}$$

Theorem 1. *Let $u \in L^2(\Omega)$. Then*

(i)

$$\|\psi_n\|_2 \leqslant \left|\frac{\lambda_u^\sigma - \sigma}{\lambda_u^\tau - \sigma}\right|^n \|u\|_2 .$$

In particular, $\psi_n \to 0$ *in* $L^2(\Omega)$ *with an exponential rate.*

(ii) *There exists* $n_0 \in \mathbb{N}$ *such that*

$$\left\| \frac{e_u^\sigma + \psi_n}{\|e_u^\sigma + \psi_n\|_2} - \frac{e_u^\sigma}{\|e_u^\sigma\|_2} \right\|_2 \leqslant \frac{4}{\|e_u^\sigma\|_2} \|\psi_n\|_2 ,$$

for all $n \geqslant n_0$*. In particular*

$$\frac{e_u^\sigma + \psi_n}{\|e_u^\sigma + \psi_n\|_2} \to \frac{e_u^\sigma}{\|e_u^\sigma\|_2} \quad in\ L^2(\Omega)$$

with an exponential rate.

(iii)

$$\frac{\|\phi_n\|_2}{\|\phi_{n+1}\|_2} \to |\lambda_u^\sigma - \sigma| .$$

(iv)

$$\left(\int_\Omega u \frac{\phi_n}{\|\phi_n\|_2} dx\right) \frac{\phi_n}{\|\phi_n\|_2} \to e_u^\sigma \quad in\ L^2(\Omega) .$$

(v)

$$r(\phi_n) \to \lambda_u^\sigma$$

with

$$r(\phi_n) - \lambda_u^\sigma = O\left(\left|\frac{\lambda_u^\sigma - \sigma}{\lambda_u^\tau - \sigma}\right|^{2n}\right).$$

Proof. Write

$$\begin{aligned}
\|\psi_n\|_2^2 &= \sum_{\left|\lambda_u^j - \sigma\right| > |\lambda_u^\sigma - \sigma|}^{M} \left|\frac{\lambda_u^\sigma - \sigma}{\lambda_u^j - \sigma}\right|^{2n} \left\|e_u^j\right\|_2^2 \\
&\leqslant \left|\frac{\lambda_u^\sigma - \sigma}{\lambda_u^\tau - \sigma}\right|^{2n} \sum_{\left|\lambda_u^j - \sigma\right| > |\lambda_u^\sigma - \sigma|}^{M} \left\|e_u^j\right\|_2^2 \\
&\leqslant \left|\frac{\lambda_u^\sigma - \sigma}{\lambda_u^\tau - \sigma}\right|^{2n} \|u\|_2^2 .
\end{aligned}$$

Since

$$\left|\frac{\lambda_u^\sigma - \sigma}{\lambda_u^\tau - \sigma}\right| < 1,$$

it follows that $\|\psi_n\|_2 \to 0$ as $n \to \infty$, which proves (i).

Let $n_0 \in \mathbb{N}$ be such that

$$\|\psi_n\|_2 \leqslant \frac{1}{2} \|e_u^\sigma\|$$

for all $n \geqslant n_0$. If $n \geqslant n_0$ it follows that

$$\frac{1}{2} \|e_u^\sigma\|_2 = \|e_u^\sigma\|_2 - \frac{1}{2} \|e_u^\sigma\|_2 \leqslant \|e_u^\sigma + \psi_n\|_2 + \|\psi_n\|_2 - \|\psi_n\|_2 = \|e_u^\sigma + \psi_n\|_2$$

and

$$\begin{aligned}\left\| \frac{e_u^\sigma + \psi_n}{\|e_u^\sigma + \psi_n\|_2} - \frac{e_u^\sigma}{\|e_u^\sigma\|_2} \right\|_2 &= \left\| \frac{\|e_u^\sigma\|_2 (e_u^\sigma + \psi_n) - e_u^\sigma \|e_u^\sigma + \psi_n\|_2}{\|e_u^\sigma + \psi_n\|_2 \|e_u^\sigma\|_2} \right\|_2 \\ &\leqslant \left\| \frac{e_u^\sigma (\|e_u^\sigma\|_2 - \|e_u^\sigma + \psi_n\|_2) + \|e_u^\sigma\|_2 \psi_n}{(1/2) \|e_u^\sigma\|_2^2} \right\|_2 \\ &\leqslant 2 \frac{\|e_u^\sigma\|_2 \|\psi_n\|_2 + \|e_u^\sigma\|_2 \|\psi_n\|_2}{\|e_u^\sigma\|_2^2} \\ &= \frac{4}{\|e_u^\sigma\|_2} \|\psi_n\|_2 ,\end{aligned}$$

which proves (ii).

Since

$$\frac{\|\phi_n\|_2}{\|\phi_{n+1}\|_2} = |\lambda_u^\sigma - \sigma| \frac{\|e_u^\sigma + \psi_n\|_2}{\|e_u^\sigma + \psi_{n+1}\|_2},$$

(iii) follows from (i).

As

$$\left(\int_\Omega u \frac{\phi_n}{\|\phi_n\|_2} dx \right) \frac{\phi_n}{\|\phi_n\|_2} = \left\langle u, \frac{\phi_n}{\|\phi_n\|_2} \right\rangle_2 \frac{\phi_n}{\|\phi_n\|_2} = \left\langle u, \frac{e_u^\sigma + \psi_n}{\|e_u^\sigma + \psi_n\|_2} \right\rangle_2 \frac{e_u^\sigma + \psi_n}{\|e_u^\sigma + \psi_n\|_2},$$

and, from (ii),

$$\left\langle u, \frac{\phi_n}{\|\phi_n\|_2} \right\rangle_2 \frac{\phi_n}{\|\phi_n\|_2} \to \left\langle u, \frac{e_u^\sigma}{\|e_u^\sigma\|_2} \right\rangle_2 \frac{e_u^\sigma}{\|e_u^\sigma\|_2} = e_u^\sigma,$$

there follows (iv).

In order to prove (v), we notice that if $\lambda_u^\sigma \geqslant \sigma$ then

$$\begin{aligned}\lim \left\langle \frac{\phi_{n-1}}{\|\phi_{n-1}\|_2}, \frac{\phi_n}{\|\phi_n\|_2} \right\rangle_2 \frac{\|\phi_{n-1}\|_2}{\|\phi_n\|_2} &= \lim \left\langle \frac{e_u^\sigma + \psi_{n-1}}{\|e_u^\sigma + \psi_{n-1}\|_2}, \frac{e_u^\sigma + \psi_n}{\|e_u^\sigma + \psi_n\|_2} \right\rangle_2 \frac{\|\phi_{n-1}\|_2}{\|\phi_n\|_2} \\ &= \left\langle \frac{e_u^\sigma}{\|e_u^\sigma\|_2}, \frac{e_u^\sigma}{\|e_u^\sigma\|_2} \right\rangle_2 |\lambda_u^\sigma - \sigma| \\ &= \lambda_u^\sigma - \sigma,\end{aligned}$$

while if $\lambda_u^\sigma < \sigma$ then

$$\lim \left\langle \frac{\phi_{n-1}}{\|\phi_{n-1}\|_2}, \frac{\phi_n}{\|\phi_n\|_2} \right\rangle_2 \frac{\|\phi_{n-1}\|_2}{\|\phi_n\|_2} = \lim \left\langle (-1)^{n-1} \frac{e_u^\sigma + \psi_{n-1}}{\|e_u^\sigma + \psi_{n-1}\|_2}, (-1)^n \frac{e_u^\sigma + \psi_n}{\|e_u^\sigma + \psi_n\|_2} \right\rangle_2 \frac{\|\phi_{n-1}\|_2}{\|\phi_n\|_2}$$
$$= -\left\langle \frac{e_u^\sigma}{\|e_u^\sigma\|_2}, \frac{e_u^\sigma}{\|e_u^\sigma\|_2} \right\rangle_2 |\lambda_u^\sigma - \sigma|$$
$$= \lambda_u^\sigma - \sigma.$$

From the weak formulation of (9) we have that

$$\frac{\int_\Omega |\nabla \phi_n|^2 dx}{\int_\Omega |\phi_n|^2 dx} = \sigma + \frac{\int_\Omega \phi_n \phi_{n-1} dx}{\int_\Omega |\phi_n|^2 dx}$$
$$= \sigma + \frac{\|\phi_{n-1}\|_2}{\|\phi_n\|_2} \left\langle \frac{\phi_{n-1}}{\|\phi_{n-1}\|_2}, \frac{\phi_n}{\|\phi_n\|_2} \right\rangle_2 ,$$

whence

$$\lim r(\phi_n) = \sigma + \lambda_u^\sigma - \sigma = \lambda_u^\sigma.$$

Set

$$u_n = \left\langle u, \frac{\phi_n}{\|\phi_n\|_2} \right\rangle_2 \frac{\phi_n}{\|\phi_n\|_2}.$$

From (iv) and Corollary 1 it follows that

$$r(\phi_n) - r(e_u^\sigma) = r(u_n) - r(e_u^\sigma) = O\left(\|u_n - e_u^\sigma\|_2^2\right) = O\left(\left|\frac{\lambda_u^\sigma - \sigma}{\lambda_u^\tau - \sigma}\right|^{2n}\right).$$

■

5 Uniform convergence of inverse iteration with shift

We begin this section by stating a L^∞-estimate for an eigenfunction of the Laplacian in terms of its L^2-norm. In the following, we denote by $|\Omega|$ the Lebesgue measure of Ω.

Lemma 1. *Let $e \in H_0^1(\Omega)$ be an eigenfunction of $-\Delta$ corresponding to the eigenvalue λ. Then*

$$\|e\|_\infty \leqslant 4^N |\Omega|^{1/2} \lambda^{N/2} \|e\|_2 . \tag{16}$$

Proof. It is shown in [Lindqvist], without any smoothness assumption on $\partial\Omega$, that if e is an eigenfunction corresponding to a variational eigenvalue λ of the homogeneous Dirichlet problem for the p-Laplacian then

$$\|e\|_\infty \leqslant 4^N \lambda^{N/p} \|e\|_{L^1(\Omega)} .$$

Choosing $p=2$, (16) follows from Hölder's inequality. ■
Estimates for eigenfunctions of the Laplacian with the exponent $N/2$ in the eigenvalue replaced by $N/4$ can be found in [Egorov-Kondrat'ev], [Yakubov1], [Yakubov2] and [Yakubov3]. See also [Burenkov-Lamberti, Remark 5.21] for more references.

The following result refers to the nondecreasing sequence (4) of eigenvalues of the Laplacian.

Lemma 2. *If* $k > N/2$, *then*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^k} \leqslant \frac{NC^k}{2k-N} < \infty, \tag{17}$$

where C *is a positive constant which depends only on* N *and* $|\Omega|$.

Proof. It is well known (see [Li-Yau], [Lieb]) that

$$\lambda_j \geqslant \frac{1}{C} j^{2/N},$$

where

$$C = \frac{N+2}{N} \frac{(\omega_N |\Omega|)^{2/N}}{4\pi^2} \tag{18}$$

and ω_N is the volume of the N-dimensional unit ball. Hence, if $j > N/2$ we have

$$\sum_{j=1}^{\infty} \lambda_j^{-k} \leqslant C^k \sum_{j=1}^{\infty} j^{-2k/N} < C^k \int_1^{\infty} s^{-2k/N} ds = \frac{NC^k}{2k-N}.$$

■

In the next lemma we show that the convergence of a series formed by the eigenvalues λ_u^j which appear in the spectral decomposition of a function $u \in L^2(\Omega)$ follows from the convergence of a series formed by all eigenvalues of the Laplacian.

Lemma 3. *Let* k *be chosen so that the series*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^k}$$

is convergent. Then the series

$$\sum_{j=1}^{M} \frac{(\lambda_u^j)^{N/2}}{\left|\lambda_u^j - \sigma\right|^{N/2+k+1}}$$

is also convergent.

Proof. Assume that the expansion of u is not finite, i.e., $M=\infty$ (otherwise the result is trivial). Since

$$\sum_{j=1}^{\infty}\frac{1}{\left(\lambda_u^j\right)^k}\leqslant\sum_{j=1}^{\infty}\frac{1}{\lambda_j^k},$$

it suffices to show that

$$\frac{\left(\lambda_u^j\right)^{N/2}}{\left|\lambda_u^j-\sigma\right|^{N/2+k+1}}\leqslant\frac{1}{\left(\lambda_u^j\right)^k} \tag{19}$$

for all sufficiently large j. Define $f:(\sigma,+\infty)\longrightarrow\mathbb{R}$ by

$$f(t)=\left(\frac{t}{t-\sigma}\right)^{N/2+k}=\left(1+\frac{\sigma}{t-\sigma}\right)^{N/2+k}.$$

Since f is decreasing, there exists t_0 such that

$$f(t)<t-\sigma$$

for all $t>t_0$. As $\lambda_u^j\to\infty$, there also exists j_0 such that $|\lambda_u^j-\sigma|=\lambda_u^j-\sigma$ for all $j\geqslant j_0$. Thus, if j is sufficiently large, we can write

$$\frac{\left(\lambda_u^j\right)^{N/2+k}}{\left|\lambda_u^j-\sigma\right|^{N/2+k}}=\frac{\left(\lambda_u^j\right)^{N/2+k}}{\left(\lambda_u^j-\sigma\right)^{N/2+k}}=f\left(\lambda_u^j\right)<\lambda_u^j-\sigma=\left|\lambda_u^j-\sigma\right|,$$

whence (19) follows. ■

In order to prove the uniform convergence of the inverse iteration sequence $\{\phi_n\}_{n\in\mathbb{N}}$, we return to (12) and write

$$\frac{\phi_n}{\|\phi_n\|_\infty}=\pm\frac{e_u^\sigma+\psi_n}{\|e_u^\sigma+\psi_n\|_\infty}. \tag{20}$$

As in (14), the sign of the right-hand side will depend on whether the shift is taken above or below the eigenvalue and on n.

Lemma 4. *The inequality*

$$\|\psi_n\|_\infty\leqslant K\left|\frac{\lambda_u^\sigma-\sigma}{\lambda_u^\tau-\sigma}\right|^{n-\theta} \tag{21}$$

holds for all sufficiently large n, for some $\theta>0$ and a positive constant $K=K\left(u,\Omega,|\lambda_u^\sigma-\sigma|\right)$. In particular, $\psi_n\to 0$ uniformly in $\overline{\Omega}$ with an exponential rate.

Proof. From (13) and Lemma 1 we obtain

$$|\psi_n|\leqslant\sum_{\left|\lambda_u^j-\sigma\right|>\left|\lambda_u^\sigma-\sigma\right|}^{M}\left|\frac{\lambda_u^\sigma-\sigma}{\lambda_u^j-\sigma}\right|^n\left|e_u^j\right|\leqslant 4^N\left|\Omega\right|^{1/2}\|u\|_2\sum_{\left|\lambda_u^j-\sigma\right|>\left|\lambda_u^\sigma-\sigma\right|}^{M}\left|\frac{\lambda_u^\sigma-\sigma}{\lambda_u^j-\sigma}\right|^n\left(\lambda_u^j\right)^{N/2}.$$

But, taking $\theta = N/2 + k + 1$, we have

$$\begin{aligned}\sum_{\left|\lambda_u^j-\sigma\right|>\left|\lambda_u^\sigma-\sigma\right|}^{M} \left|\frac{\lambda_u^\sigma - \sigma}{\lambda_u^j - \sigma}\right|^n \left(\lambda_u^j\right)^{N/2} &= \left|\lambda_u^\sigma - \sigma\right|^\theta \sum_{\left|\lambda_u^j-\sigma\right|>\left|\lambda_u^\sigma-\sigma\right|}^{M} \left|\frac{\lambda_u^\sigma - \sigma}{\lambda_u^j - \sigma}\right|^{n-\theta} \frac{\left(\lambda_u^j\right)^{N/2}}{\left|\lambda_u^j - \sigma\right|^\theta} \\ &\leqslant \left|\lambda_u^\sigma - \sigma\right|^\theta \left|\frac{\lambda_u^\sigma - \sigma}{\lambda_u^\tau - \sigma}\right|^{n-\theta} \sum_{\left|\lambda_u^j-\sigma\right|>\left|\lambda_u^\sigma-\sigma\right|}^{M} \frac{\left(\lambda_u^j\right)^{N/2}}{\left|\lambda_u^j - \sigma\right|^\theta} \\ &\leqslant \left|\frac{\lambda_u^\sigma - \sigma}{\lambda_u^\tau - \sigma}\right|^{n-\theta} \left|\lambda_u^\sigma - \sigma\right|^\theta \sum_{j=1}^{M} \frac{\left(\lambda_u^j\right)^{N/2}}{\left|\lambda_u^j - \sigma\right|^\theta},\end{aligned}$$

and by Lemma 3 the last series converges. Thus, (21) follows if we take

$$K = 4^N \left|\Omega\right|^{1/2} \left\|u\right\|_2 \left|\lambda_u^\sigma - \sigma\right|^\theta \sum_{j=1}^{M} \frac{\left(\lambda_u^j\right)^{N/2}}{\left|\lambda_u^j - \sigma\right|^\theta}.$$

■

Theorem 2. *Let $u \in L^2(\Omega)$. Then*

(i) *There exists $n_0 \in \mathbb{N}$ such that*

$$\left\| \frac{e_u^\sigma + \psi_n}{\left\|e_u^\sigma + \psi_n\right\|_\infty} - \frac{e_u^\sigma}{\left\|e_u^\sigma\right\|_\infty} \right\|_\infty \leqslant \frac{4}{\left\|e_u^\sigma\right\|_\infty} \left\|\psi_n\right\|_\infty$$

for all $n \geqslant n_0$. In particular,

$$\frac{e_u^\sigma + \psi_n}{\left\|e_u^\sigma + \psi_n\right\|_\infty} \to \frac{e_u^\sigma}{\left\|e_u^\sigma\right\|_\infty} \quad \textit{uniformly in } \Omega$$

with an exponential rate.

(ii)

$$\frac{\left\|\phi_n\right\|_\infty}{\left\|\phi_{n+1}\right\|_\infty} \to \left|\lambda_u^\sigma - \sigma\right|.$$

(iii) $\dfrac{\phi_n}{\phi_{n+1}} \to \lambda_u^\sigma - \sigma$ *uniformly on any compact $\mathcal{K} \subset\subset \operatorname{supp} e_u^\sigma$ with an exponential rate.*

(iv)

$$\left(\int_\Omega u \frac{\phi_n}{\left\|\phi_n\right\|_2} dx\right) \frac{\phi_n}{\left\|\phi_n\right\|_2} \to e_u^\sigma \quad \textit{uniformly in } \Omega.$$

Proof. Let n_0 be such that $\|\psi_n\|_\infty \leqslant \frac{1}{2}\|e_u^\sigma\|_\infty$ for all $n \geqslant n_0$. Then, as in the proof of Theorem 1 (ii), we have for all $n \geqslant n_0$ that

$$\frac{1}{2}\|e_u^\sigma\|_\infty \leqslant \|e_u^\sigma + \psi_n\|_\infty$$

and

$$\left\| \frac{e_u^\sigma + \psi_n}{\|e_u^\sigma + \psi_n\|_\infty} - \frac{e_u^\sigma}{\|e_u^\sigma\|_\infty} \right\|_\infty \leqslant \frac{4}{\|e_u^\sigma\|_\infty}\|\psi_n\|_\infty .$$

The remaining of (i) follows from Lemma 4.

Since from (12) we have

$$\frac{\|\phi_n\|_\infty}{\|\phi_{n+1}\|_\infty} = |\lambda_u^\sigma - \sigma| \frac{\|e_u^\sigma + \psi_n\|_\infty}{\|e_u^\sigma + \psi_{n+1}\|_\infty},$$

(ii) also follows from Lemma 4.

Now, let $\mathcal{K} \subset\subset \operatorname{supp} e_u^\sigma$ be compact so that

$$m := \min_{\mathcal{K}} |e_u^\sigma| > 0$$

and fix $n_0 \in \mathbb{N}$ such that

$$\|\psi_n\|_\infty < \frac{m}{2} \quad \text{for all } n \geqslant n_0.$$

Thus if $n \geqslant n_0$ we have on $\mathcal{K}$

$$|e_u^\sigma + \psi_n| \geqslant |e_u^\sigma| - |\psi_n| \geqslant m - \|\psi_n\|_\infty > \frac{m}{2}.$$

Therefore, the quotient

$$\frac{\phi_n}{\phi_{n+1}} = (\lambda_u^\sigma - \sigma)\frac{e_u^\sigma + \psi_n}{e_u^\sigma + \psi_{n+1}}$$

makes sense on $\mathcal{K}$ for all sufficiently large n and again (iii) follows from Lemma 4 since

$$\begin{aligned} \left| \frac{\phi_n}{\phi_{n+1}} - (\lambda_u^\sigma - \sigma) \right| &= |\lambda_u^\sigma - \sigma| \left| \frac{e_u^\sigma + \psi_n}{e_u^\sigma + \psi_{n+1}} - 1 \right| \\ &= |\lambda_u^\sigma - \sigma| \left| \frac{\psi_n - \psi_{n+1}}{e_u^\sigma + \psi_{n+1}} \right| \\ &\leqslant \frac{2|\lambda_u^\sigma - \sigma|}{m} |\psi_n - \psi_{n+1}| . \end{aligned}$$

Finally, in order to prove (iv), write

$$\frac{\phi_n}{\|\phi_n\|_2} \int_\Omega u \frac{\phi_n}{\|\phi_n\|_2} dx = \left(\frac{\|\phi_n\|_\infty}{\|\phi_n\|_2} \right)^2 \frac{\phi_n}{\|\phi_n\|_\infty} \int_\Omega u \frac{\phi_n}{\|\phi_n\|_\infty} dx.$$

Since

$$\lim \frac{\|\phi_n\|_\infty}{\|\phi_n\|_2} = \lim \frac{\|e_u^\sigma + \psi_n\|_\infty}{\|e_u^\sigma + \psi_n\|_2} = \frac{\|e_u^\sigma\|_\infty}{\|e_u^\sigma\|_2}$$

and, from (i),

$$\frac{\phi_n}{\|\phi_n\|_\infty} \int_\Omega u \frac{\phi_n}{\|\phi_n\|_\infty} dx \to \frac{e_u^\sigma}{\|e_u^\sigma\|_\infty} \int_\Omega u \frac{e_u^\sigma}{\|e_u^\sigma\|_\infty} dx$$

uniformly in Ω, it follows that

$$\begin{aligned} \frac{\phi_n}{\|\phi_n\|_2} \int_\Omega u \frac{\phi_n}{\|\phi_n\|_2} dx &\to \left(\frac{\|e_u^\sigma\|_\infty}{\|e_u^\sigma\|_2} \right)^2 \frac{e_u^\sigma}{\|e_u^\sigma\|_\infty} \int_\Omega u \frac{e_u^\sigma}{\|e_u^\sigma\|_\infty} dx \\ &= \frac{e_u^\sigma}{\|e_u^\sigma\|_2^2} \int_\Omega u e_u^\sigma \, dx \\ &= e_u^\sigma. \end{aligned}$$

■

6 Numerical Tests

In this section we present some numerical tests in the unit interval and the unit disk. More specifically, we present the graphs of the first nine eigenfunctions of the unit constant function $u \equiv 1$ on these domains obtained by inverse iteration with shift, as well as eigenvalue approximations for the eigenvalues λ_k of the Laplacian for several values of k. For the computation of each eigenvalue and eigenfunction, the shift was set in both cases to the corresponding exact eigenvalue minus 0.1. Eigenvalue approximations were computed via the three sequences considered in this paper, which we denote:

$$\mu_n := \frac{\phi_n}{\phi_{n+1}}(x_0) + \sigma,$$

$$\nu_n := \frac{\|\phi_n\|_2}{\|\phi_{n+1}\|_2} + \sigma$$

and

$$r(\phi_n) = \frac{\int_\Omega |\nabla \phi_n(x)|^2 dx}{\int_\Omega |\phi_n(x)|^2 dx}.$$

We chose $x_0 = 0.01$ for both domains. On each domain we used neither the most efficient available method for solving the underlying partial differential equation nor a fine grid, but one of the most basic methods and a relatively coarse grid, in order to show the efficiency of inverse iteration with shift. Namely, the method of finite differences was applied with a grid containing only 101 nodes. Integrals were computed via the Simpson composite method.

6.1 Eigenvalues and eigenfunctions for the unit interval $[0, 1]$

In this case, (9) becomes the Sturm-Liouville problem

$$\begin{cases} -\phi''_{n+1} - \sigma\phi_{n+1} = \phi_n \\ \phi_{n+1}(0) = \phi_{n+1}(1) = 0 \end{cases} \tag{22}$$

The unit constant function does not have components corresponding to λ_k for k even. Nonetheless, as mentioned in the Introduction, due to rounding errors we were able to obtain these values as well. The graphs of the first nine (L^∞-normalized) approximated eigenfunctions of the Laplacian on $[0, 1]$ obtained by the algorithm are shown in Figure 1. In Table 1 we present the corresponding exact eigenvalue and its approximations given by the three sequences for several values of k, up to one billion, after 10 iterations of inverse iteration with shift.

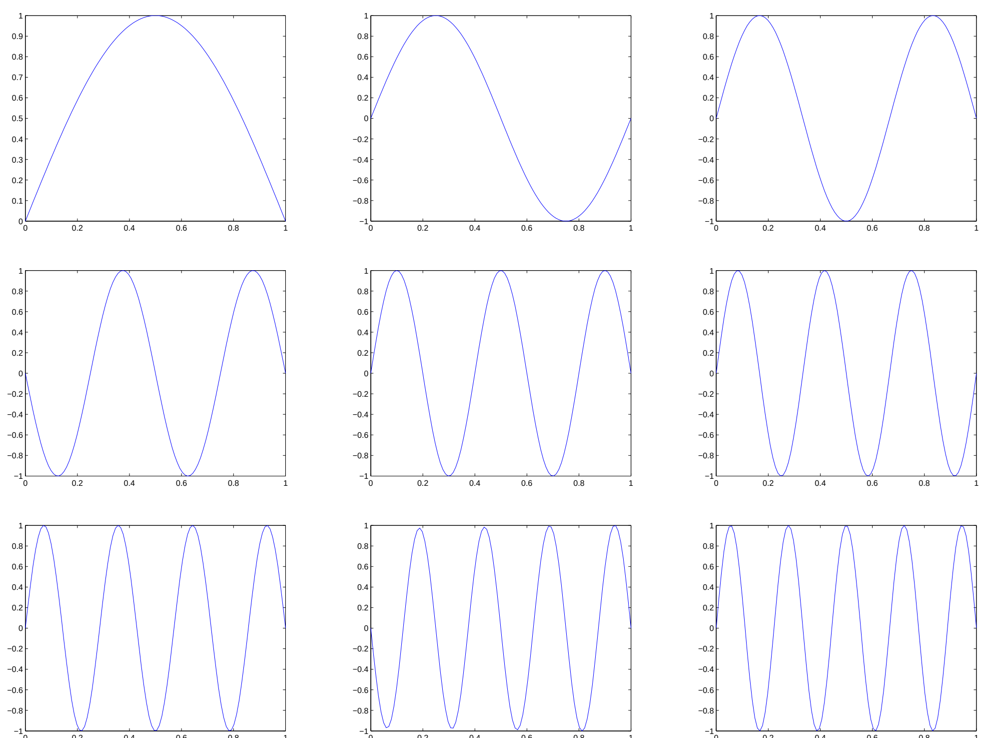

Figure 1. First nine approximated eigenfunctions of the Laplacian on $[0, 1]$.

k	λ_k	μ_{10}	ν_{10}	$r(\phi_{10})$
1	9.87	9.869	9.869	9.866
2	39.478	39.465	39.465	39.427
3	88.826	88.761	88.761	88.564
4	157.914	157.706	157.921	157.084
5	246.74	246.233	247.047	244.717
6	355.306	354.255	356.157	351.118
7	483.611	481.665	485.356	475.865
8	631.655	628.335	634.773	618.466
9	799.438	800.338	800.338	778.369
10	986.96	987.86	987.86	954.908
20	3947.84	3948.74	3948.74	3459.76
30	8882.64	8883.54	8883.54	6840.84
40	15791.4	15792.3	15792.3	9524.33
50	24674	24674.9	24674.9	9447.28
60	35530.6	35531.5	35531.5	4061.1
70	48361.1	48362	48362	3256.69
80	63165.5	63166.4	63166.4	3711.9
90	79943.8	79944.7	79944.7	3670.04
100	98696	98696.9	98696.9	4049.97
10^3	9.8696×10^6	9.8696×10^6	9.8696×10^6	7856.16
10^4	9.8696×10^8	9.8696×10^8	9.8696×10^8	3742.66
10^5	9.8696×10^{10}	9.8696×10^{10}	9.8696×10^{10}	3742.66
10^6	9.8696×10^{12}	9.8696×10^{12}	9.8696×10^{12}	3742.66
10^7	9.8696×10^{14}	9.8696×10^{14}	9.8696×10^{14}	3742.66
10^8	9.8696×10^{16}	9.8696×10^{16}	9.8696×10^{16}	3742.23
10^9	9.8696×10^{18}	9.8696×10^{18}	9.8696×10^{18}	3329.09

Table 1: Exact and approximated eigenvalues on the unit interval $[0, 1]$ for several values of k with 10 iterations of inverse iteration with shift.

6.2 Eigenvalues and eigenfunctions for the unit disk

In this case we computed only the radial components and (9) becomes the Sturm-Liouville problem

$$\begin{cases} -\phi''_{n+1} - \frac{1}{r}\phi'_{n+1} - \sigma\phi_{n+1} = \phi_n \\ \phi'_{n+1}(0) = \phi_{n+1}(1) = 0. \end{cases} \tag{23}$$

The graphs of the first nine approximated radial eigenfunctions of the Laplacian obtained by the algorithm are displayed in Figure 2. In Table 2, the corresponding exact eigenvalue and its approximations given by the three sequences for the first ten radial eigenvalues, after 10 iterations of inverse iteration with shift, are presented.

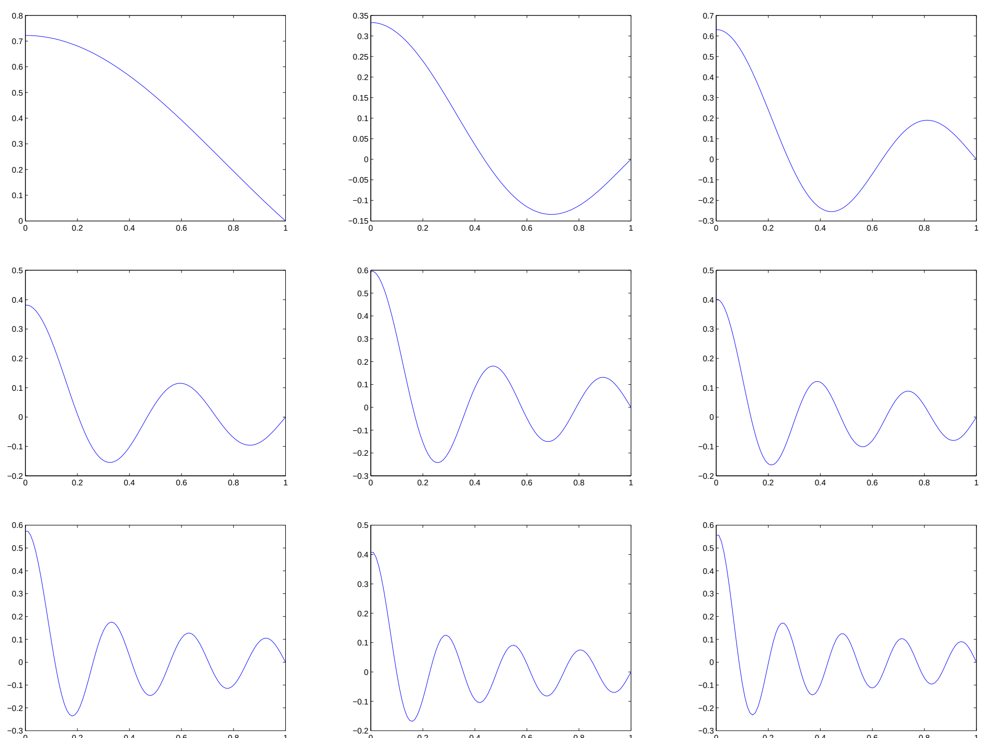

Figure 2. First nine approximated radial eigenfunctions of the Laplacian on the unit disk.

7 Final comments

In finite linear algebra, the iterative process itself is often used in order to generate increasingly better estimates for the eigenvalue at each iteration, meaning that the approximation obtained at any given iteration is used as the shift in the next iteration. It turns out that instead of using the estimates for the eigenvalue obtained in the process, the Rayleigh quotient of the estimates for the *eigenvector* obtained at each iteration give much better estimates

k	λ_k	μ_{10}	ν_{10}	$r(\phi_{10})$
1	5.78319	5.7834	5.7834	5.78279
2	30.4713	30.4698	30.4698	30.4491
3	74.887	74.865	74.865	74.7338
4	139.04	138.942	138.942	138.481
5	222.932	222.646	223.018	221.45
6	326.563	325.901	327.025	323.32
7	449.934	448.611	451.057	443.697
8	593.043	590.663	595.223	582.113
9	755.891	751.922	759.660	738.026
10	938.479	932.234	944.524	910.824

Table 2: Exact and approximated first ten radial eigenvalues on the unit disk with 10 iterations of inverse iteration with shift.

for the eigenvalue. Indeed, if the eigenvalues of the operator or at least very good estimates of them are known in advance, inverse iteration with shift given by the Rayleigh quotient is the standard method for computing eigenvectors due to its cubic rate of convergence (see [Trefenthen-Bau]). It would be only natural to extend such ideas to the Laplacian, but we were not able to do it. Instead, our (admitedly preliminary) numerical tests did not indicate convergence to the correct eigenvalues. As previously discussed, the Rayleigh quotient may not be a good way to approximate the eigenvalue of higher frequency eigenfunctions unless the grid is much further refined, due to higher oscillations, and the computational cost of using too fine grids can seriously limit the efficiency of the method. Further investigation is warranted. So it remains an open problem to us if inverse iteration with shift given by the Rayleigh quotient is a method that can be successfully applied to the Laplacian. The only reference we could find where the Rayleigh quotient was used in computing the eigenvalues of the Laplacian, and only for polygonal domains, was the work [Descloux-Tolley]; however the Rayleigh quotient was only indirectly used there, as one component of another algorithm and in a very different way from the direct approach we follow here.

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