# A conjecture on Iterated Integrals and application to higher order invariants

Anton Deitmar & Ivan Horozov

**Abstract:** We formulate the conjecture that the restriction morphism from free closed iterated integrals to closed iterated integrals on loops is onto. The conjecture has the consequence that the module of higher order invariants of smooth functions is generated by free closed iterated integrals.

## Contents

# Introduction

Iterated integrals as a tool of differential topology have been introduced in the nineteen-fifties [1]. They have been applied to many problems, including the construction of Hodge-structures on fundamental groups [5] and the description of knot invariants [6], just to name examples.

In most applications, iterated integrals are restricted to loops, but recently interest rose in iterated integrals of free paths. These give rise to functions on the universal covering. In [8] it was found that for Riemann surfaces these functions give higher order invariants in the sense of [4]. In the present paper, we generalize this to arbitrary manifolds.

An iterated integral is called *closed*, if its values only depend on the homotopy classes of paths, where homotopy means homotopy with fixed endpoints. The central conjecture, which is formulated in section 2 of this paper, is that the restriction map from closed free iterated integrals to closed iterated integrals on loops should be surjective. We prove the conjecture in low degrees and for general degrees under the condition that the cup product

$$H^1(X) \times H^1(X) \ \to \ H^2(X)$$

is the zero map, which is the case for instance if $H^2(X) = 0$ already. In section 3 finally we prove the central result of this paper, which says that, given the conjecture holds, then free closed iterated integrals generate the module of higher order invariants of smooth functions.

# 1 Generalities on iterated integrals

In this section we fix notations. Let $X$ be a smooth connected manifold and $x_0, x \in X$ points. We write $PX$ for the *path space*, i.e., the set of all smooth maps $p : [0,1] \to X$. We also write $PX_{x_0}$ for the subset of all paths that start at $x_0$ and $PX_{x_0,x}$ for the subset of all smooth paths from $x_0$ to $x$. The space $LX_{x_0} = PX_{x_0,x_0}$ is also called the *loop space* at $x_0$.

For a path $p$ and 1-forms $\omega_1, \dots, \omega_r$ we define the *iterated integral:*

$$\int_p \omega_1 \cdots \omega_r = \int_0^1 \int_0^{t_r} \dots \int_0^{t_2} p^*\omega_1(t_1)\, p^*\omega_2(t_2) \dots p^*\omega_r(t_r).$$

For an integer $s$, let $B_s(X)$ denote the space of all maps $\omega : PX \to \mathbb{C}$ which are linear combinations of iterated integrals of length $\leq s$. Here we include constants as they may be considered as iterated integrals of length zero. We also write $B(X)$ for the union of all $B_s(X)$ as $s$ varies. Let

$$T(X) = \mathbb{C} \oplus \Omega^1(X) \oplus \left[\Omega^1(X) \otimes \Omega^1(X)\right] \oplus \dots$$

be the tensorial algebra over the space $\Omega^1(X)$ of smooth 1-forms. The map $\omega_1 \otimes \cdots \otimes \omega_r \mapsto \int_p \omega_1 \cdots \omega_r$ is a linear map from $T(X)$ to $B(X)$. This map has a non-trivial kernel which has been determined by Chen in [2].

We denote by $B_s(X)_{x_0}$ the set of restrictions of elements of $B_s(X)$ to $PX_{x_0}$ and the space $B_s(X)_{x_0,x}$ is defined analogously.

**Lemma 1.1** (a) *If $\varphi$ is an orientation preserving diffeomorphism on $[0,1]$, then $\int_p \omega_1 \cdots \omega_r = \int_{p\circ\varphi} \omega_1 \cdots \omega_r$.*

(b) *If $F$ is a diffeomorphism on $X$, then $\int_{F\circ p} \omega_1 \dots \omega_r = \int_p (F^*\omega_1) \dots (F^*\omega_r)$.*

(c) *If $p$ and $q$ are composable paths, then*

$$\int_{pq} \omega_1 \cdots \omega_r = \sum_{j=0}^{r} \int_p \omega_1 \cdots \omega_j \int_q \omega_{j+1} \cdots \omega_r.$$

(d) *One has*

$$\int_p \omega_1 \cdots \omega_r \int_p \omega_{r+1} \cdots \omega_{r+s} = \sum_\sigma \int_p \omega_{\sigma(1)} \cdots \omega_{\sigma(r+s)},$$

*where the sum runs over all $(r,s)$-shuffles, i.e., permutations $\sigma$ on $r+s$ letters with $\sigma^{-1}(1) < \cdots < \sigma^{-1}(r)$ and $\sigma^{-1}(r+1) < \cdots < \sigma^{-1}(r+s)$.*

(e) *$\int_{p^{-1}} \omega_1 \cdots \omega_r = (-1)^r \int_p \omega_r \cdots \omega_1$, where $p^{-1}(t) = p(1-t)$.*

(f) *For given $\omega \in B(X)$, we extend the map $p \mapsto \int_p \omega$ to the free abelian group $\mathbb{Z}[PX]$ generated by $PX$. For given $\alpha_1, \dots, \alpha_s \in PX_{x_0,x_0}$ let $\eta = (\alpha_1 - 1)(\alpha_2 - 1) \dots (\alpha_s - 1) \in \mathbb{Z}[PX]$. For 1-forms $\omega_1, \dots, \omega_r$, $r \leq s$ we have*

$$\int_\eta \omega_1 \cdots \omega_r = \begin{cases} \prod_{i=1}^s \int_{\alpha_i} \omega_i & r = s, \\ 0 & r < s. \end{cases}$$

**Proof:** (a)-(e) are easy exercises. (f) is a result of Hain, see [5], Proposition 2.13. □

If we replace the tensor product on $T(X)$ by the *shuffle product* $*$ given by

$$\omega_1 \cdots \omega_r * \omega_{r+1} \cdots \omega_{r+s} = \sum_\sigma \omega_{\sigma(1)} \cdots \omega_{\sigma(r+s)},$$

where the sum runs over all $(r,s)$-shuffles, we obtain the *shuffle algebra* $\mathrm{Sh}(X)$. We have shown that the iterated integrals form an algebra homomorphism

$$\mathrm{Sh}(X) \to B(X),$$

where the latter is an algebra under pointwise multiplication.

Let $B_s(X)^{\mathrm{hom}}$ denote the space of all elements of $B_s(X)$ which are invariant under homotopies with fixed end-points. Similarly define $B_s(X)^{\mathrm{hom}}_{x_0}$ and $B_s(X)^{\mathrm{hom}}_{x_0,x}$.

# 2 A conjecture

**Conjecture 2.1** *The restriction map $B_s(X)^{\mathrm{hom}}_{x_0} \to B_s(X)^{\mathrm{hom}}_{x_0,x_0}$ is surjective.*

**Proposition 2.2** (a) *The Conjecture holds for $s = 0, 1, 2$.*

(b) *The conjecture holds if the cup product*

$$H^1(X) \times H^1(X) \ \to \ H^2(X)$$

*is zero.*

**Proof:** (a) For $s = 0$ it is trivially true. We show it in the case $s = 1$. Let $\omega \in B_1(X)^{\mathrm{hom}}_{x_0,x_0}$. WLOG we can assume that $\omega$ is a 1-form. Let $p$ and $\tilde{p}$ be homotopic paths from $x_0$ to, say, $x \in X$. Then $\tilde{p}p^{-1}$ is a loop at $x_0$ which is homotopic to the constant path, hence $\int_{\tilde{p}} \omega - \int_p \omega = \int_{\tilde{p}p^{-1}} \omega = 0$, as $\omega$ is a single 1-form. Now to the case $s = 2$. Proposition 3.11 of [2] says that a given $\omega \in B_2(X)_{x_0,x_0}$ belongs to $B_2(X)^{\mathrm{hom}}_{x_0,x_0}$ if and only if it can be written as

$$\omega = \sum_{i=1}^n \alpha_i \beta_i + \gamma + c,$$

where $c \in \mathbb{C}$ and $\alpha_i, \beta_i$ are closed 1-forms with

$$\sum_{i=1}^{n} \alpha_1 \wedge \beta_i + d\gamma = 0.$$

By Proposition 3.3 of [2] an iterated integral $\eta = \sum_{i=1}^{n} \alpha_i \beta_i + \gamma + c \in B_2(X)_{x_0}$ is homotopy-invariant if $\sum_{i=1}^{n} \alpha_1 \wedge \beta_i + d\gamma = 0$ holds. So $\eta$ lies in $B_2(X)_{x_0}^{\text{hom}}$ and restricts to $\omega$.

(b) Let $\omega \in B_s(X)_{x_0,x_0}^{\text{hom}}$ and write

$$\omega = \sum_{j=1}^{n} \omega_1^j \dots \omega_s^j + R$$

for some 1-forms $\omega_i^j$ and $R \in B_{s-1}(X)_{x_0,x_0}$. By Proposition 8.30 (2) of [7] it follows that we can assume the forms $\omega_i^j$ to be closed for all $i, j$. By our assumption, the forms $\omega_i^j \wedge \omega_{i+1}^j$ are exact, so there are 1-forms $\omega_{i,i+1}^j$ such that

$$\omega_i^j \wedge \omega_{i+1}^j + d\omega_{i,i+1}^j = 0.$$

In the next step one finds 1-forms $\omega_{i,i+1,i+2}^j$ such that

$$\omega_i^j \wedge \omega_{i+1,i+2}^j + \omega_{i,i+1}^j \wedge \omega_{i+2} + d\omega_{i,i+1,i+2}^j = 0$$

and so on until

$$\omega_1^j \wedge \omega_{2,\dots,s}^j + \dots + d\omega_{1,\dots,s}^j = 0.$$

In other words, the forms $\omega_{i,\dots,i+k}^j$ constitute an extended defining system for Massey products as in [2], section 3.1. By Theorem 3.1 in [2], there exists $u \in B_s(X)_{x_0}^{\text{hom}}$ such that $u \equiv \omega \mod B_{s-1}(X)_{x_0}$. By induction we get the claim. □

# 3 The fundamental group

For a group $\Gamma$ we write its group ring as $\mathbb{Z}\Gamma$ and $J \subset \mathbb{Z}\Gamma$ the augmentation ideal, i.e., the span of all elements of the form $(\gamma - 1)$, where $\gamma \in \Gamma$. For any $\mathbb{Z}\Gamma$-module $V$ we write $\mathrm{H}_s^0(\Gamma, V)$ for the $\mathbb{Z}$-module of all $v \in V$ with $J^s v = 0$. This space can be identified with $\mathrm{Hom}_{\mathbb{Z}\Gamma}(\mathbb{Z}\Gamma/J^s, V)$. The elements of $\mathrm{H}^0_s(\Gamma, V)$ for varying $s$ are called *higher order invariants*. If $v$ is in $\mathrm{H}^0_s(\Gamma, V)$, but not in $\mathrm{H}^0_{s-1}(\Gamma, V)$, then $s$ is called the *order* of $v$.

Let $X$ be a connected smooth manifold, $x_0 \in X$ a base-point, and $\Gamma = \pi_1(X, x_0)$ the corresponding fundamental group. We consider $\Gamma$ as group of deck transformations on the universal covering $\tilde{X}$ of $X$. We also fix a pre-image $\tilde{x}_0$ of $x_0$ in $\tilde{X}$.

As $\tilde{X}$ is simply connected, the iterated integral $\int_p \omega$ for $\omega \in B_s(\tilde{X})^{\mathrm{hom}}$ only depends on the endpoints $x, y$ of the path $p$. We therefore write $\int_x^y \omega = \int_p \omega$.

Every $\gamma \in \Gamma$ can be viewed as a homotopy class of a loop based at $x_0 \in X$. In this way we get a map $B_s(X)^{\mathrm{hom}}_{x_0,x_0} \to \mathrm{Map}(\Gamma, \mathbb{C})$ that maps $\omega \in B_s(X)^{\mathrm{hom}}_{x_0,x_0}$ to the map $\gamma \mapsto \int_\gamma \omega$. The latter map induces a $\mathbb{Z}$-linear map from the group ring $\mathbb{Z}\Gamma$ to $\mathbb{C}$. It is the content of Chen's de Rham Theorem for fundamental groups (see [3], Corollar 1 to Theorem 2.6.1, see also [7]) that this map induces a bijection

$$B_s(X)^{\mathrm{hom}}_{x_0,x_0} \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma/J^{s+1}, \mathbb{C}).$$

**Theorem 3.1** *If $\omega \in B_s(X)^{\mathrm{hom}}_{x_0}$, then the function $\int_{x_0}^x \omega$ is an invariant of order at most $s+1$ in the $\Gamma$-module $C^\infty(\tilde{X})$. This defines an injective linear map*

$$\Psi : B_s(X)^{\mathrm{hom}}_{x_0} \hookrightarrow \mathrm{H}^0_{s+1}(\Gamma, C^\infty(\tilde{X})).$$

The case when $X$ is the hyperbolic plane is in the paper [8].

**Proof:** Let $\omega \in B_s(X)^{\mathrm{hom}}$ and set $f_\omega(x) = \int_{x_0}^x \omega$. We have to show

$$[(\gamma_1 - 1) \cdots (\gamma_{s+1} - 1)]^* f_\omega = 0$$

for any $\gamma_1, \ldots, \gamma_{s+1} \in \Gamma$. For given $x \in X$ and $\gamma \in \Gamma$, we choose a path $\gamma_x$ from $x$ to $\gamma x$. The map $\gamma \mapsto \gamma_x$ is extended linearly to a map $\mathbb{Z}\Gamma \to \mathbb{Z}[PX]$. For every $x \in X$ we also fix a smooth path $p_x$ from $x_0$ to $x$. Let $\omega \in B_s(X)^{\mathrm{hom}}$ and let $\eta = \sum_\gamma c_\gamma \gamma$ be an arbitrary element of the group ring $\mathbb{Z}\Gamma$. We have

$$\begin{aligned} \eta^* f_\omega(x) &= \sum_\gamma c_\gamma \gamma^* f_\omega(x) = \sum_\gamma c_\gamma \int_{x_0}^{\gamma x} \omega \\ &= \sum_\gamma c_\gamma \int_{p_x \gamma_x} \omega = \int_{p_x \sum_\gamma c_\gamma \gamma_x} \omega = \int_{p_x \eta_x} \omega. \end{aligned}$$

We apply this to the element $(\gamma_1 - 1)\cdots(\gamma_{s+1}-1)$ of the group ring and we look at any monomial $\omega_1\cdots\omega_r$ in $\omega$, where the $\omega_j$ are 1-forms on $X$. We then have

$$\int_{p_x[(\gamma_1-1)\dots(\gamma_{s+1}-1)]_x} \omega_1\dots\omega_r = \sum_{k=0}^r \int_{p_x}\omega_1\dots\omega_k \int_{[(\gamma_1-1)\dots(\gamma_{s+1}-1)]_x} \omega_{k+1}\dots\omega_r.$$

Let $\bar{x}$ be the image of $x$ in $X$ and let $\gamma_{\bar{x}}$ be the image of $\gamma_x$ in $X$. Then $\gamma_{\bar{x}}$ is a loop based at $\bar{x}$. As the forms $\omega_j$ are $\Gamma$-invariant, we have

$$\begin{aligned} \int_{[(\gamma_1-1)\dots(\gamma_{s+1}-1)]_x} \omega_{k+1}\dots\omega_r &= \int_{[(\gamma_1-1)\dots(\gamma_{s+1}-1)]_{\bar{x}}} \omega_{k+1}\dots\omega_r \\ &= \int_{(\gamma_{1,\bar{x}}-1)\dots(\gamma_{s+1,\bar{x}}-1)} \omega_{k+1}\dots\omega_r \\ &= 0 \end{aligned}$$

by Lemma 1.1 (e). This proves the first claim. For the injectivity of the induced map let $\omega \in B_s(\tilde{X})^{\rm hom}_{x_0}$ with $\int_{x_0}^x \omega = 0$. This just means that $\omega = 0$ in $B_s(\tilde{X})^{\rm hom}_{x_0}$. $\square$

**Questions.**

- Ist $B(\tilde{X})^\Gamma = B(X)$?
- Let $\omega \in H^0_q(\Gamma, B_s(\tilde{X})^{\rm hom})$. Is it true that the function $\int_{x_0}^x \omega$ lies in $H^0_{q+1}(\Gamma, C^\infty(\tilde{X}))$?

We formally set $H^0_0 = 0$ and $B_{-1} = 0$.

**Theorem 3.2** *Suppose that Conjecture 2.1 holds. Let* $\bar{B}_s = B_s/B_{s-1}$ *and let* $K_s$ *be the kernel of the map*

$$B_s(X)^{\rm hom}_{x_0} \xrightarrow{\rm res} B_s(X)^{\rm hom}_{x_0,x_0} \to \bar{B}_s(X)^{\rm hom}_{x_0,x_0}.$$

*Then*

$$\Psi(K_s) \subset H^0_s(\Gamma, C^\infty(\tilde{X}))$$

*and* $\Psi$ *induces an isomorphism of* $C^\infty(X)$*-modules*

$$C^\infty(X)\otimes\left(B_s(X)^{\rm hom}_{x_0}/K_s\right) \xrightarrow{\cong} \bar{H}^0_{s+1}(\Gamma, C^\infty(\tilde{X})).$$

**Proof:** By Chen's de Rham Theorem for fundamental groups (see [3], Corollar 1 to Theorem 2.6.1, see also [7]) the evaluation of iterated integrals gives an isomorphism

$$\bar{B}_s(X)^{\mathrm{hom}}_{x_0,x_0} \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(J^s/J^{s+1},\mathbb{C}).$$

The right hand side can also be viewed as $\mathrm{Hom}_{\mathbb{Z}\Gamma}(J^s/J^{s+1},\mathbb{C})$ and as such be embedded into

$$\mathrm{Hom}_A(J^s/J^{s+1}, C^\infty(X)) \cong \mathrm{Hom}_A(J^s/J^{s+1}, C^\infty(\tilde{X})),$$

where we have written $A = \mathbb{Z}\Gamma$. More precisely, the image in

$$\mathrm{Hom}_A(J^s/J^{s+1}, C^\infty(X)) \cong C^\infty(X) \otimes \mathrm{Hom}_A(J^s/J^{s+1},\mathbb{C})$$

is a basis of this $C^\infty(X)$-module, which means that we have an isomorphism of $C^\infty(X)$-modules,

$$C^\infty(X) \otimes \bar{B}_s(X)^{\mathrm{hom}}_{x_0,x_0} \xrightarrow{\cong} \mathrm{Hom}_A(J^s/J^{s+1}, C^\infty(\tilde{X})).$$

**Lemma 3.3** *We have* $H^1(\Gamma, C^\infty(\tilde{X})) = 0$.

**Proof:** A 1-cocycle is a map $\alpha : \Gamma \to C^\infty(\tilde{X})$ such that $\alpha(\gamma\tau) = \gamma\alpha(\tau)+\alpha(\gamma)$ holds for all $\gamma, \tau \in \Gamma$. We have to show that for any given such map $\alpha$ there exists $f \in C^\infty(\tilde{X})$ such that $\alpha(\tau) = \tau f - f$.

Fix a smooth map $u : \tilde{X} \to [0,1]$ such that

$$\sum_{\tau\in\Gamma} u(\tau^{-1}x) \;\equiv\; 1,$$

where we can assume that the sum is locally finite. Set

$$f(x) \;=\; -\sum_{\tau\in\Gamma} \alpha(\tau)(x)\, u(\tau^{-1}x).$$

Then the function $f$ lies in the space $C^\infty(\tilde{X})$. We now compute for $\gamma \in \Gamma$,

$$\begin{aligned}
\gamma f(x) - f(x) &= f(\gamma^{-1}x) - f(x) \\
&= \sum_{\tau\in\Gamma} \alpha(\tau x) u(\tau^{-1}x) - \alpha(\tau)(\gamma^{-1}x) u(\tau^{-1}\gamma^{-1}x) \\
&= \sum_{\tau\in\Gamma} \alpha(\tau)(x) u(\tau^{-1}x) + \alpha(\gamma)(x) \sum_{\tau\in\Gamma} u((\gamma\tau)^{-1}x) \\
&\quad - \sum_{\tau\in\Gamma} \alpha(\gamma\tau)(x) u((\gamma\tau)^{-1}x)
\end{aligned}$$

The first and the last sum cancel and the middle sum is $\alpha(\gamma)(x)$. Therefore, the lemma is proven. $\square$

Since $H^1(\Gamma, C^\infty(\tilde{X})) = 0$, the exact sequence

$$0 \to J^s/J^{s+1} \to A/J^{s+1} \to A/J^s \to 0,$$

induces an isomorphism

$$\operatorname{Hom}_A(J^s/J^{s+1}, C^\infty(\tilde{X})) \cong \operatorname{Hom}_A(A/J^{s+1}, C^\infty(\tilde{X}))/\operatorname{Hom}_A(A/J^s, C^\infty(\tilde{X})).$$

We write

$$\overline{\operatorname{Hom}}_A(A/J^{s+1}, C^\infty(\tilde{X})) = \operatorname{Hom}_A(A/J^{s+1}, C^\infty(\tilde{X}))/\operatorname{Hom}_A(A/J^s, C^\infty(\tilde{X})).$$

We have to show that the ensuing diagram

$$\begin{array}{ccc}
\bar{B}_s(X)^{\mathrm{hom}}_{x_0} & \xrightarrow{\mathrm{res}} & \bar{B}_s(X)^{\mathrm{hom}}_{x_0,x_0} \\
\downarrow & & \downarrow \\
\overline{\operatorname{Hom}}_A(A/J^{s+1}, C^\infty(\tilde{X})) & \xrightarrow{\cong} & \operatorname{Hom}_A(J^s/J^{s+1}, C^\infty(\tilde{X}))
\end{array}$$

commutes. This will give the claim, as we already have seen that the right vertical arrow becomes an isomorphism after tensoring with $C^\infty(X)$. This commutativity is a direct consequence of formula (f) in Lemma 1.1. $\square$

# References

[1] Kuo-Tsai Chen, *Iterated integrals and exponential homomorphisms*, Proc. London Math. Soc. (3) **4** (1954), 502–512.

[2] Kuo-tsai Chen, *Algebras of iterated path integrals and fundamental groups*, Trans. Amer. Math. Soc. **156** (1971), 359–379.

[3] Kuo Tsai Chen, *Iterated path integrals*, Bull. Amer. Math. Soc. **83** (1977), no. 5, 831–879.

[4] Anton Deitmar, *Higher order group cohomology and the Eichler-Shimura map*, J. Reine Angew. Math. **629** (2009), 221–235.

[5] Richard M. Hain, *The geometry of the mixed Hodge structure on the fundamental group*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 247–282.

[6] Maxim Kontsevich, *Vassiliev's knot invariants*, I. M. Gel′fand Seminar, Adv. Soviet Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, pp. 137–150.

[7] Chris A. M. Peters and Joseph H. M. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 52, Springer-Verlag, Berlin, 2008.

[8] Ramesh Sreekantan, *Higher order modular forms and mixed Hodge theory*, Acta Arith. **139** (2009), no. 4, 321–340.

Mathematisches Institut
Auf der Morgenstelle 10
72076 Tübingen
Germany
deitmar@uni-tuebingen.de

16.11.2010