ON CORE AND BAR-CORE PARTITIONS

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Abstract. If s and t are relatively prime J. Olsson proved in [7] that the s-core of a t-core partition is again a t-core partition, and that the s-bar-core of a t-bar-core partition is again a t-bar-core partition. Here generalized results are proved for partitions and bar-partitions when the restriction that s and t be relatively prime is removed.

1. Introduction

The basic facts about partitions, hooks and blocks can be found in [3, Chapter 2] or [6, Chapter 1]. We recall a few key definitions here. A partition λ of n is defined as a non-increasing sequence of nonnegative integers $(\lambda_1, \lambda_2, \cdots)$ that sum to n. A partition is represented graphically by its Young diagram $[\lambda]$, which consists of the set of *nodes* $\{(i, j) \,|\, (i, j) \in \mathbb{N}^2, j \leq \lambda_i\}$. The node (i, j) is in the ith row and jth column of $[\lambda]$. The rows of $[\lambda]$ are labelled from top to bottom, while its columns are labelled from left to right.

To each node (i, j) in $[\lambda]$ we associate the *hook* h_{ij} of λ, which consists of the node (i, j) itself, together with all the nodes $\{(i, k) \,|\, j < k\}$ in $[\lambda]$ (i.e. in the same row as and to the right of (i, j)), and all the nodes $\{(\ell, j) \,|\, i < \ell\}$ (i.e. in the same column as and below (i, j)). The *length* of h_{ij} is the total number of nodes contained in the hook. For any integer $\ell \geq 1$, we call *ℓ-hook* a hook of length ℓ, and *(ℓ)-hook* a hook of length divisible by ℓ. The information about the (ℓ)-hooks in λ is encoded in the *ℓ-quotient* $q_\ell(\lambda) = (\lambda_0, \ldots, \lambda_{\ell-1})$ of λ. The λ_i's are partitions whose sizes sum to the number w of (ℓ)-hooks in λ (called the *ℓ-weight* of λ).

The removal of an ℓ-hook h in λ is obtained by removing the ℓ nodes of $[\lambda]$ in h, and migrating the disconnected nodes in $[\lambda]$ up and to the left. The result is a partition of $n - \ell$ denoted by $\lambda \setminus h$. By removing all the (ℓ)-hooks in λ, one obtains the *ℓ-core* $\gamma_\ell(\lambda)$ of λ. The partition $\gamma_\ell(\lambda)$ contains no (ℓ)-hooks, and is uniquely determined by λ (i.e. doesn't depend on the order in which we remove the ℓ-hooks in λ). The partition λ is entirely determined by its ℓ-core and ℓ-quotient.

It is well-known that the irreducible complex characters of the symmetric group $\mathfrak{S}_n$ are labelled by the partitions of n. If p is a prime, then the distribution of irreducible characters of $\mathfrak{S}_n$ into p-blocks has a combinatorial description known as the Nakayama Conjecture: two characters χ_λ, $\chi_\mu \in \mathrm{Irr}(\mathfrak{S}_n)$ belong to the same p-block if and only if λ and μ have the same p-core (see [3, Theorem 6.1.21]). Hence we define, for each integer $\ell \geq 1$, an *ℓ-block of partitions* of n to be the set of all partitions of n having a common given ℓ-core.

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We now recall the analogous notions and results for bar-partitions, which can be found in [6, Chapter 1]. A *bar-partition* is a partition λ comprised of distinct parts. To each bar-partition we associate a shifted Young diagram $S(\lambda)$ obtained by shifting the ith row of the usual Young diagram $(i-1)$ positions to the right. The j-th node in the i-th row will be called the (i,j)-node. To each node (i,j) in $S(\lambda)$, one can associate a *bar* and *bar-length*. For any odd integer ℓ, a bar-partition λ is entirely determined by its *$\bar{\ell}$-core* $\bar{\gamma}_\ell(\lambda)$ and its *$\bar{\ell}$-quotient* $\bar{q}_\ell(\lambda)$. The *bar-core* $\bar{\gamma}_\ell(\lambda)$ is obtained by removing from λ all the bars of length divisible by ℓ (called (ℓ)-bars). The *bar-quotient* of λ is of the form $\bar{q}_\ell(\lambda) = (\lambda_0, \lambda_1, \ldots \lambda_{(\ell-1)/2})$, where λ_0 is a bar-partition, λ_1, ..., $\lambda_{(\ell-1)/2}$ are partitions, and the sizes of the λ_i's sum to the number of (ℓ)-bars in λ (called *$\bar{\ell}$-weight* of λ).

It is well-known that the bar-partitions of n label the faithful irreducible complex characters of the 2-fold covering group $\tilde{\mathfrak{S}}_n$ of $\mathfrak{S}_n$. These correspond to irreducible projective representations of $\mathfrak{S}_n$, and are known as *spin-characters*. If p is an odd prime, then the distribution of spin-characters of $\tilde{\mathfrak{S}}_n$ of positive defect into p-blocks has a combinatorial description known as the Morris Conjecture: two spin-characters of $\tilde{\mathfrak{S}}_n$ of positive defect belong to the same p-block if and only if the bar-partitions labelling them have the same $\bar{p}$-core (see [6, Theorem 13.1]).

In analogy with this, we define, for each odd integer $\ell \geq 1$, an *$\bar{\ell}$-block of partitions* of n to be the set of all bar-partitions of n having a common given $\bar{\ell}$-core.

2. Some new results on cores and bar-cores

In this section, we generalize to arbitrary integers s and t the results on cores and bar-cores proved by J. B. Olsson in [7] when s and t are coprime. Note that Olsson's result ([7, Theorem 1]) was interpreted by M. Fayers through alcove geometry and actions of the affine symmetric group (see [2]). It was also used by F. Garvan and A. Berkovich to bound the number of distinct values their partition statistic (the GBG-rank) can take on a t-core (mod s) (see [1, Theorem 1.2]).

We keep the notation as in Section 1.

Theorem 2.1. *For any two positive integers s and t, the s-core of a t-core partition is again a t-core partition.*

Remark 2.2. *This result was proved by J. B. Olsson in* [7]*, under the extra hypothesis that s and t are relatively prime. R. Nath then gave in* [5] *a proof of the result in general. We give here another proof which, unlike the one given by Nath, uses Olsson's result, and provides the framework for the proof for bar-partitions.*

Proof. Consider a t-core partition λ. Let $g = \gcd(s,t)$, and write $s_0 = s/g$ and $t_0 = t/g$. It's a well-known fact (see e.g. [6, Theorem 3.3]) that there is a canonical bijection φ between the set of hooks of length divisible by g in λ and the set of hooks in $q_g(\lambda) = (\lambda_0, \ldots, \lambda_{g-1})$ (i.e. hooks in each of the λ_i's). For each positive integer k and hook h of length kg in λ, the hook $\varphi(h)$ has length k. Furthermore, we have $q_g(\lambda \setminus h) = q_g(\lambda) \setminus \varphi(h)$.

In particular, since λ is an t-core, and since $t = t_0 g$, we see that $q_g(\lambda)$ contains no t_0-hook, so that each λ_i is an t_0-core.

Now, the s-hooks in λ are in bijection with the s_0-hooks in $q_g(\lambda)$. When we remove them all, we obtain that the s-core $\gamma_s(\lambda)$ has g-core $\gamma_g(\gamma_s(\lambda)) = \gamma_g(\lambda)$ and g-quotient $q_g(\gamma_s(\lambda)) = (\gamma_{s_0}(\lambda_0), \ldots \gamma_{s_0}(\lambda_{g-1}))$. But, since s_0 and t_0 are coprime, the s_0-core of each t_0-core λ_i is again a t_0-core ([7, Theorem 1]). This shows that

$q_g(\gamma_s(\lambda))$ has no t_0-hook, which in turn implies that $\gamma_s(\lambda)$ contains no t-hook, whence is an t-core.

□

As we mentionned in Section 1, when p is a prime, the study of p-cores is linked to that of the p-modular representation theory of the symmetric group $\mathfrak{S}_n$ (as they label the p-blocks of irreducible characters). When $\ell \geq 2$ is an arbitrary integer, it turns out that it is still possible to describe an ℓ-modular representation theory of $\mathfrak{S}_n$ (see [4]). The theory of ℓ-blocks obtained in this way is in fact related to the ordinary representation theory of an Iwahori-Hecke algebra of type $\mathfrak{S}_n$, when specialized at an ℓ-root of unity. Külshammer, Olsson and Robinson proved in [4] the following analogue of the Nakayama Conjecture: two characters χ_λ, $\chi_\mu \in \mathrm{Irr}(\mathfrak{S}_n)$ belong to the same ℓ-block if and only if λ and μ have the same ℓ-core.

It is therefore legimitate to study ℓ-cores and ℓ-blocks of partitions. In particular, we obtain from Theorem 2.1 a generalization of [7, Corollary 3]. We call *principal* ℓ-block of n the ℓ-block of partitions of n which contains the partition (n) (i.e. the set of partitions labelling the characters of the principal ℓ-block of $\mathfrak{S}_n$).

Corollary 2.3. *Let r, s and t be any positive integers such that $s > r \geq t$, and let $n = as + r$ for some $a \in \mathbb{Z}_{\geq 0}$. Then the principal s-block of n contains no t-core.*

Proof. Suppose the partition λ of n is a t-core. The s-core γ of λ, which is obtained by removing s-hooks, must therefore be a partition of some m which differs from n by a multiple of s, i.e. $m = bs + r$ for some b such that $a \geq b \geq 0$. By Theorem 2.1, γ is also a t-core. Now, if λ was in the principal s-block of n, then its s-core would be the same as that of the cycle (n), hence also a cycle. We would thus have $\gamma = (m)$. But since $m \geq r \geq t$, the cycle (m) contains a t-hook, hence cannot be a t-core. □

In terms of blocks of characters, this means that, if s, t and n are as above, then there is no trivial block inclusion of a t-block in the principal s-block of $\mathfrak{S}_n$ (see [8]).

We now prove the analogue results for bar-cores, which was proved by Olsson when s and t are odd and coprime ([7, Theorem 4]).

Theorem 2.4. *For any two odd positive integers s and t, the $\bar{s}$-core of an $\bar{t}$-core partition is again a $\bar{t}$-core partition.*

Proof. Take any $\bar{t}$-core λ. Let $g = \gcd(s, t)$, and write $s_0 = s/g$ and $t_0 = t/g$. There is a canonical bijection φ between the set of bars of length divisible by g in λ and the set of bars in its $\bar{g}$-quotient $\bar{q}_g(\lambda) = (\lambda_0, , \lambda_1, \ldots, \lambda_{(g-1)/2})$, where a bar in $\bar{q}_g(\lambda)$ is either a bar in the bar-partition λ_0 or a hook in one of the partitions $\lambda_1, \ldots, \lambda_{(g-1)/2}$ (see [6, Theorem 4.3]). For each positive integer k and bar b of length kg in λ, the bar $\varphi(b)$ has length k. Furthermore, we have $\bar{q}_g(\lambda \setminus b) = \bar{q}_g(\lambda) \setminus \varphi(b)$.

The same argument as in the proof of Theorem 2.1 thus proves that λ_0 is an $\bar{t_0}$-core, that each λ_i $(1 \leq i \leq (g-1)/2)$ is an t_0-core, and that the $\bar{s}$-core $\bar{\gamma}_s(\lambda)$ of λ has $\bar{g}$-quotient $\bar{q}_g(\bar{\gamma}_s(\lambda)) = (\bar{\gamma}_{s_0}(\lambda_0), \gamma_{s_0}(\lambda_1), \ldots \gamma_{s_0}(\lambda_{(g-1)/2}))$. And, since s_0 and t_0 are coprime, the s_0-core of each t_0-core λ_i $(1 \leq i \leq (g-1)/2)$ is again a t_0-core ([7, Theorem 1]), and the $\bar{s_0}$-core of the $\bar{t_0}$-core λ_0 is again a $\bar{t_0}$-core ([7, Theorem 4]). This shows that the $\bar{g}$-quotient of $\bar{\gamma}_s(\lambda)$ contains no t_0-bar, which finally implies that $\bar{\gamma}_s(\lambda)$ contains no t-bar, whence is an $\bar{t}$-core.

□

In analogy with the partition case, we call *principal* $\bar{\ell}$-block of bar-partitions of n (for ℓ odd) the $\bar{\ell}$-block containing the bar-partition (n). Then the same argument as for the proof of Corollary 2.3 yields

Corollary 2.5. *Let r, s and t be any positive integers such that s and t are odd and $s > r \geq t$, and let $n = as + r$ for some $a \in \mathbb{Z}_{\geq 0}$. Then the principal $\bar{s}$-block of n contains no $\bar{t}$-core.*

References

[1] A. Berkovich and F. Garvan. The GBG-rank and t-cores I. Counting and 4-cores. *J. Comb. Number Theory*, 1(3):237–252, 2009.

[2] M. Fayers. The t-core of an s-core. *Preprint.*

[3] G. James and A. Kerber. *The representation theory of the symmetric group*, volume 16 of *Encyclopedia of Mathematics and its Applications.* Addison-Wesley Publishing Co., Reading, Mass., 1981.

[4] B. Külshammer, J. Olsson, and G.R. Robinson. Generalized blocks for symmetric groups. *Invent. Math.*, 151(3):513–552, 2003.

[5] R. Nath. On the t-core of an s-core partition. *Integers*, 8:A28, 2008.

[6] J. B. Olsson. Combinatorics and representations of finite groups. *Vorlesungen aus dem Fachbereich Mathematik der Universität GH Essen*, Heft 20, 1993.

[7] J. B. Olsson. A theorem on the cores of partitions. *J. Combinatorial Theory A*, 116:733–740, 2009.

[8] J. B. Olsson and D. Stanton. Block inclusions and cores of partitions. *Aequationes Math.*, 74:90–110, 2007.

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