

STABLE HIGGS BUNDLES ON COMPACT GAUDUCHON MANIFOLDS

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ABSTRACT. Let M be a compact complex manifold equipped with a Gauduchon metric. If TM is holomorphically trivial, and (V, θ) is a stable $SL(r, \mathbb{C})$ -Higgs bundle on M , then we show that $\theta = 0$. We show that the correspondence between Higgs bundles and representations of the fundamental group for a compact Kähler manifold does not extend to compact Gauduchon manifolds. This is done by applying the above result to $\Gamma \backslash G$, where Γ is a discrete torsionfree cocompact subgroup of a complex semisimple group G .

RÉSUMÉ. **Les fibrés de Higgs stables sur les variétés de Gauduchon.** Soit M une variété complexe compacte muni d'une métrique de Gauduchon. Si TM est holomorphiquement trivial, et (V, θ) est un fibré $SL(r, \mathbb{C})$ -Higgs stable, alors on démontre que $\theta = 0$. On démontre que la correspondance entre les fibrés de Higgs et les représentations du groupe fondamental pour une variété kählérienne compacte ne s'étend pas aux variétés de Gauduchon. Ceci est accompli en appliquant le résultat ci-dessus à $\Gamma \backslash G$, où Γ est un sous-groupe discret, sans torsion et co-compact d'un groupe semi-simple complexe G .

1. INTRODUCTION

Let (M, g) be a compact connected Kähler manifold. A theorem due to Uhlenbeck and Yau says that the isomorphism classes of stable vector bundles on M are in bijective correspondence with the solutions of the Hermitian–Yang–Mills equation on M [12]. This theorem was later extended to compact complex manifold equipped with a Gauduchon metric by Li–Yau and Buchdahl [8], [2] (the result of [2] is for complex surfaces). Gauduchon metrics are a generalization of Kähler metrics; their definition is recalled in Section 2.

Hitchin, Simpson, Donaldson and Corlette established a bijective correspondence between the isomorphism classes of stable $SL(r, \mathbb{C})$ -Higgs bundles on M , with vanishing rational Chern classes, and the equivalence classes of irreducible homomorphisms from $\pi_1(M)$ to $SL(r, \mathbb{C})$ [6], [10], [4], [3]. It is natural to ask whether this correspondence extends to compact complex Gauduchon manifolds. We show that the correspondence does not extend in general by constructing explicit examples of compact complex Gauduchon manifolds for which this correspondence fails.

Let (M, g) be a compact connected complex Gauduchon manifold. We prove the following theorem:

Theorem 1.1. *If TM is holomorphically trivial, and (V, θ) is a stable $SL(r, \mathbb{C})$ -Higgs bundle on M , then $\theta = 0$.*

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The above mentioned correspondence valid for Kähler manifolds implies that if (M, g) is a compact connected Kähler manifold such that $\theta = 0$ for any stable $\mathrm{SL}(r, \mathbb{C})$ -Higgs bundle (V, θ) on M , then any irreducible representation of $\pi_1(M)$ into $\mathrm{SL}(r, \mathbb{C})$ is unitarizable (see Remark 2.3).

Let G be a connected complex semisimple group defined over \mathbb{C} . Let

$$\Gamma \subset G$$

be a torsionfree discrete subgroup such that the quotient $\Gamma \backslash G$ is compact. This compact complex manifold $\Gamma \backslash G$ is not Kähler, but it has explicit Gauduchon metrics. Also, $T(\Gamma \backslash G)$ is trivial, so Theorem 1.1 applies to it. It turns out that the restriction of each nontrivial irreducible representation of G is a nonunitarizable irreducible representation of Γ .

2. STABLE HIGGS BUNDLES

Let M be a compact connected complex manifold of complex dimension d . Let g be a C^∞ Hermitian structure on the holomorphic tangent bundle TM . Let ω_g be the positive $(1, 1)$ -form on M given by g . We recall that g is called a *Gauduchon metric* if

$$\partial\bar{\partial}\omega_g^{d-1} = 0.$$

A theorem due to P. Gauduchon says that given any C^∞ Hermitian structure g_0 on TM , there is a positive smooth function f on M such that fg_0 is a Gauduchon metric; furthermore, if $n \geq 2$, then f is unique up to a positive constant. (See [5, p. 502].)

Fix a Gauduchon metric g on M . As before, the corresponding $(1, 1)$ -form on M will be denoted by ω_g .

Let F be a coherent analytic sheaf on F . Consider the determinant line bundle $\det F$ on M ; see [7, Ch. V, § 6] for the construction of $\det F$. Fix a Hermitian structure h_F on $\det F$. Define the *degree* of F to be

$$\mathrm{degree}(F) := \int_M c_1(\det F, h_F) \wedge \omega_g^{d-1} \in \mathbb{R},$$

where $c_1(\det F, h_F)$ is the Chern form of the Hermitian connection on $\det F$. It should be clarified that $\mathrm{degree}(F)$ is independent of the choice of h_F , but it depends on g ; see [2, p. 626], [8, p. 563].

A holomorphic vector bundle V on M is called *stable* if for every coherent analytic subsheaf $F \subset V$ with $0 < \mathrm{rank}(F) < \mathrm{rank}(V)$, the inequality

$$\frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} < \frac{\mathrm{degree}(V)}{\mathrm{rank}(V)}$$

holds.

Let Ω_M^1 be the holomorphic cotangent bundle of M . A *Higgs field* on a holomorphic vector bundle V on M is a section

$$\theta \in H^0(M, \mathrm{End}(V) \otimes \Omega_M^1)$$

such that $\theta \wedge \theta = 0$. A Higgs vector bundle is a pair (V, θ) , where V is a holomorphic vector bundle, and θ is a Higgs field on V ; see [6], [10].

A Higgs vector bundle (V, θ) is called *stable* if for every coherent analytic subsheaf $F \subset V$ satisfying the two conditions that $\theta(F) \subset F \otimes \Omega_M^1$ and $0 < \text{rank}(F) < \text{rank}(V)$, the inequality

$$\frac{\text{degree}(F)}{\text{rank}(F)} < \frac{\text{degree}(V)}{\text{rank}(V)}$$

holds.

Let $\text{trace} : \text{End}(V) \otimes \Omega_M^1 \rightarrow \Omega_M^1$ be the homomorphism defined by $\text{trace} \otimes \text{Id}_{\Omega_M^1}$. So for a Higgs vector bundle (V, θ) ,

$$\text{trace}(\theta) \in H^0(M, \Omega_M^1).$$

A $\text{SL}(r, \mathbb{C})$ -Higgs bundle is a Higgs vector bundle (V, θ) of rank r such that $\det V = \mathcal{O}_M$ (the trivial line bundle), and $\text{trace}(\theta) = 0$.

Theorem 2.1. *Assume that the tangent bundle TM is holomorphically trivial. Let (V, θ) be a stable $\text{SL}(r, \mathbb{C})$ -Higgs bundle on M . Then $\theta = 0$.*

Proof. Fix a holomorphic trivialization of Ω_M^1 by choosing d linearly independent sections

$$\beta_i \in H^0(M, \Omega_M^1)$$

$1 \leq i \leq d$. Then

$$\theta = \sum_{i=1}^d \theta_i \otimes \beta_i,$$

where $\theta_i \in H^0(M, \text{End}(V))$. Since $\theta \wedge \theta = 0$,

$$(1) \quad \theta_i \circ \theta_j = \theta_j \circ \theta_i$$

for all $i, j \in [1, d]$.

Assume that $\theta \neq 0$. Choose i_0 such that $\theta_{i_0} \neq 0$.

For any point $x \in M$, let $\lambda_1(x), \dots, \lambda_{n_x}(x)$ be the eigenvalues of $\theta_{i_0}(x) \in \text{End}_{\mathbb{C}}(V_x)$; let m_j^x be the multiplicity of the eigenvalue $\lambda_j(x)$. Since all holomorphic functions on M are constants, the characteristic polynomial of $\theta_{i_0}(x)$ is independent of x . Hence the collection $\{(\lambda_1(x), m_1^x), \dots, (\lambda_{n_x}(x), m_{n_x}^x)\}$ is independent of x . Let $V_1 \subset V$ be the generalized eigenbundle for the eigenvalue $\lambda_1(x)$ of θ_{i_0} . So for each point $x \in M$, the fiber of V_1 over x is generalized eigenspace of $\theta_{i_0}(x)$ for the eigenvalue $\lambda_1(x)$; it is a holomorphic subbundle. Let $V_1^c \subset V$ be the holomorphic subbundle given by the direct sum of the generalized eigenbundles for all the eigenvalues of θ_{i_0} different from $\lambda_1(x)$. Therefore, we have a decomposition

$$V = V_1 \oplus V_1^c.$$

From (1) it follows immediately that $\theta_j(V_1) \subset V_1$ and $\theta_j(V_1^c) \subset V_1^c$ for all j . Hence $\theta(V_1) \subset V_1 \otimes \Omega^1$ and $\theta(V_1^c) \subset V_1^c \otimes \Omega^1$. Therefore, if both V_1 and V_1^c are nonzero, then the Higgs bundle (V, θ) decomposes. But a stable Higgs bundle is indecomposable. Since

(V, θ) is stable, we conclude that θ_{i_0} has exactly one eigenvalue. On the other hand, $\text{trace}(\theta) = 0$. Hence 0 is the only eigenvalue of θ_{i_0} . So, θ_{i_0} is nilpotent.

Consider the short exact sequence of coherent analytic sheaves on M

$$(2) \quad 0 \longrightarrow \text{kernel}(\theta_{i_0}) \longrightarrow V \longrightarrow \text{image}(\theta_{i_0}) \longrightarrow 0.$$

From (1) it follows that

$$\theta_j(\text{kernel}(\theta_{i_0})) \subset \text{kernel}(\theta_{i_0}) \quad \text{and} \quad \theta_j(\text{image}(\theta_{i_0})) \subset \text{image}(\theta_{i_0})$$

for all j . Hence

$$(3) \quad \theta(\text{kernel}(\theta_{i_0})) \subset \text{kernel}(\theta_{i_0}) \otimes \Omega_M^1 \quad \text{and} \quad \theta(\text{image}(\theta_{i_0})) \subset \text{image}(\theta_{i_0}) \otimes \Omega_M^1.$$

Since θ_{i_0} is nonzero and nilpotent,

$$0 < \text{rank}(\text{kernel}(\theta_{i_0})), \text{rank}(\text{image}(\theta_{i_0})) < r.$$

In view of (3), the stability condition for (V, θ) says that

$$(4) \quad \frac{\text{degree}(\text{kernel}(\theta_{i_0}))}{\text{rank}(\text{kernel}(\theta_{i_0}))}, \frac{\text{degree}(\text{image}(\theta_{i_0}))}{\text{rank}(\text{image}(\theta_{i_0}))} < \frac{\text{degree}(V)}{\text{rank}(V)}.$$

On the other hand, from (2),

$$\text{degree}(\text{kernel}(\theta_{i_0})) + \text{degree}(\text{image}(\theta_{i_0})) = \text{degree}(V)$$

and $\text{rank}(\text{kernel}(\theta_{i_0})) + \text{rank}(\text{image}(\theta_{i_0})) = \text{rank}(V)$. But these contradict (4). Therefore, $\theta = 0$. \square

Theorem 2.1 has the following corollary:

Corollary 2.2. *If TM is holomorphically trivial, and (V, θ) is a stable $\text{SL}(r, \mathbb{C})$ -Higgs bundle on M , then the vector bundle V is stable.*

Let (M, g) be a compact connected Kähler manifold. All the Chern classes will be with rational coefficients. There is a bijective correspondence between the isomorphism classes of stable $\text{SL}(r, \mathbb{C})$ -Higgs bundles (V, θ) on M , with $c_i(V) = 0$ for all $i \geq 1$, and the equivalence classes of irreducible homomorphisms from $\pi_1(M)$ to $\text{SL}(r, \mathbb{C})$ (see [11] for the details of this correspondence). Also, there is a bijective correspondence between the isomorphism classes of stable vector bundles V on M of rank r and trivial determinant, with $c_i(V) = 0$ for all $i \geq 1$, and the equivalence classes of irreducible homomorphisms from $\pi_1(M)$ to $\text{SU}(r)$ (see [11]). The first correspondence is an extension of the second correspondence: The inclusion of $\text{SU}(r)$ in $\text{SL}(r, \mathbb{C})$ gives a map of homomorphisms, and a stable vector bundles V on M of rank r and trivial determinant produces a stable $\text{SL}(r, \mathbb{C})$ -Higgs bundle by assigning the zero Higgs field.

Remark 2.3. Assume that the Kähler manifold (M, g) has the following property: If (V, θ) is a stable $\text{SL}(r, \mathbb{C})$ -Higgs bundle, then $\theta = 0$. Take any irreducible homomorphism

$$\rho : \pi_1(M) \longrightarrow \text{SL}(r, \mathbb{C}).$$

Let (V_ρ, θ_ρ) be the stable Higgs bundle corresponding to ρ . We have $\theta_\rho = 0$ by the assumption on M . Hence ρ is conjugate to a unitary representation, meaning there is an element $A \in \text{SL}(r, \mathbb{C})$ such that $A^{-1}(\rho(z))A \in \text{SU}(r)$ for all $z \in \pi_1(M)$.

3. AN EXAMPLE

Let G be a connected semisimple affine algebraic group defined over \mathbb{C} ; we assume that $G \neq e$. Let d be the (complex) dimension of G . Let

$$\Gamma \subset G$$

be a torsionfree discrete subgroup such that $\Gamma \backslash G$ is compact. Since $\Gamma \backslash G$ is compact, the subgroup Γ is Zariski dense in G [1]. (See [9] for such manifolds.)

We note that there are explicit Gauduchon metrics on the complex manifold $\Gamma \backslash G$. Indeed, take any Hermitian metric h on $\Gamma \backslash G$ given by some left translation invariant Hermitian metric \tilde{h} on G . Let ω_h and $\omega_{\tilde{h}}$ be the corresponding $(1, 1)$ -forms on $\Gamma \backslash G$ and G respectively. Since \tilde{h} is left translation invariant, the top degree form $\partial\bar{\partial}\omega_{\tilde{h}}^{d-1}$ is also left translation invariant. Hence $\partial\bar{\partial}\omega_h^{d-1} = c_0 \cdot \mu_G$, where $c_0 \in \mathbb{R}$, and μ_G is the Haar measure form on G . The form $\partial\bar{\partial}\omega_h^{d-1}$ is closed because $\partial\bar{\partial}\omega_h^{d-1} = d\bar{\partial}\omega_h^{d-1}$. Hence using Stokes' theorem,

$$0 = \int_{\Gamma \backslash G} \partial\bar{\partial}\omega_h^{d-1} = \int_{\Gamma \backslash G} c_0 \mu_G = c_0 \text{Vol}_{\mu_G}(\Gamma \backslash G).$$

Therefore, $c_0 = 0$. Hence h is a Gauduchon metric.

The holomorphic tangent bundle $T(\Gamma \backslash G)$ is trivial (a trivialization is given by any left translation invariant trivialization of TG).

It can be shown that $\Gamma \backslash G$ does not admit any Kähler metric. Indeed, any compact connected Kähler manifold with trivial tangent bundle is isomorphic to a complex torus, implying that its fundamental group is abelian. But the fundamental group Γ of $\Gamma \backslash G$ is not abelian. (Since Γ is Zariski dense in G , if Γ is abelian, then G is abelian.)

Take any nontrivial irreducible representation

$$\rho' : G \longrightarrow \text{SL}(V_0).$$

Let

$$(5) \quad \rho := \rho'|_{\Gamma}$$

be the restriction of ρ' to the subgroup Γ .

We have $\pi_1(\Gamma \backslash G) = \Gamma$. Since Γ is Zariski dense in G , the restriction ρ in (5) remains irreducible. Since

$$\rho(\Gamma) \subset \text{SL}(V_0)$$

is an infinite, closed and discrete subgroup, it cannot be conjugated, by some element of $\text{SL}(V_0)$, to a subgroup of a maximal compact subgroup of $\text{SL}(V_0)$ (every closed infinite subgroup of a compact group has a limit point, hence it is not discrete). Compare this with Remark 2.3.

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