JACOB'S LADDERS, BESSEL'S FUNCTIONS AND THE ASYMPTOTIC SOLUTIONS OF A NEW CLASS OF NONLINEAR INTEGRAL EQUATIONS

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Abstract. It is shown in this paper that there is a connection between the Riemann zeta-function $\zeta\left(\frac{1}{2}+it\right)$ and the Bessel's functions. In this direction, a new class of the nonlinear integral equations is introduced.

1. The first result

1.1. We obtain some new properties of the signal

$$Z(t)=e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right)$$

that is generated by the Riemann zeta-function, where

$$\vartheta(t)=-\frac{t}{2}\ln\pi+\operatorname{Im}\ln\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right)=\frac{t}{2}\ln\frac{t}{2\pi}-\frac{t}{2}-\frac{\pi}{8}+\mathcal{O}\left(\frac{1}{t}\right),$$

namely, the properties connected with the interaction of the function $\zeta\left(\frac{1}{2}+it\right)$ with the Bessel's functions

$$J_\nu(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(\nu+r+1)}\left(\frac{x}{2}\right)^{\nu+2r}$$

where $x>0,\ \nu>-1$ (this is the sufficient case for our pourpose). Let us remind that

$$\tilde{Z}^2(t)=\frac{\mathrm{d}\varphi_1(t)}{\mathrm{d}t},\ \varphi_1(t)=\frac{1}{2}\varphi(t)$$

where

$$\tilde{Z}^2(t)=\frac{Z^2(t)}{2\Phi'_\varphi[\varphi(t)]}=\frac{Z^2(t)}{\left\{1+\mathcal{O}\left(\frac{\ln\ln t}{\ln t}\right)\right\}\ln t},\tag{1.1}$$

(see [1], (3.9); [3], (1.3); [7], (1.1), (3.1), (3.2)), and $\varphi(t)$ is the Jacob's ladder, i.e. a solution to the nonlinear integral equation (see [1])

$$\int_0^{\mu[x(T)]}Z^2(t)e^{-\frac{2}{x(T)}t}\mathrm{d}t=\int_0^T Z^2(t)\mathrm{d}t.$$

Key words and phrases. Riemann zeta-function.

1.2. The system of the Bessel's functions

$$\{J_\nu(\mu_n^{(\nu)}x)\}_{n=1}^\infty,\ x\in[0,1],\ J_\nu(\mu_n^{(\nu)})=0$$

is the orthogonal system on the segment $[0,1]$ with the weight x, i.e. the following formulae hold true

$$\begin{aligned}&\int_0^1 J_\nu(\mu_m^{(\nu)}x)J_\nu(\mu_n^{(\nu)}x)x\mathrm{d}x=0,\ m\neq n,\\ &\int_0^1\left[J_\nu(\mu_n^{(\nu)}x)\right]^2x\mathrm{d}x=\frac{1}{2}J_{\nu+1}(\mu_n^{(\nu)}).\end{aligned}\tag{1.2}$$

It is shown in this paper that the $\tilde{Z}^2$-transformation of the Bessel's functions generates a new orthogonal system of functions that is connected with $\left|\zeta\left(\frac{1}{2}+it\right)\right|^2$. Namely, the following theorem holds true.

Theorem 1. *Let* $x=t-T,\ t\in[T,T+1]$ *and*

$$\varphi_1\{[\mathring{T},\overset{\circ}{\widehat{T+1}}]\}=[T,T+1],\ T\geq T_0[\varphi_1].$$

Then the system of functions

$$J_\nu[\mu_n^{(\nu)}(\varphi_1(t)-T)],\ t\in[\mathring{T},\overset{\circ}{\widehat{T+1}}],\ n=1,2,\dots$$

is orthogonal on $[\mathring{T},\overset{\circ}{\widehat{T+1}}]$ *with the weight*

$$(\varphi_1(t)-T)\tilde{Z}^2(t),$$

i.e. the following system of the new-type integrals

$$\begin{aligned}&\int_{\mathring{T}}^{\overset{\circ}{\widehat{T+1}}}J_\nu[\mu_m^{(\nu)}(\varphi_1(t)-T)]J_\nu[\mu_n^{(\nu)}(\varphi_1(t)-T)]\cdot\\ &\cdot(\varphi_1(t)-T)\tilde{Z}^2(t)\mathrm{d}t=0,\ m\neq n,\\ &\int_{\mathring{T}}^{\overset{\circ}{\widehat{T+1}}}\{J_\nu[\mu_n^{(\nu)}(\varphi_1(t)-T)]\}^2(\varphi_1(t)-T)\tilde{Z}^2(t)\mathrm{d}t=\frac{1}{2}[J_{\nu+1}(\mu_n^{(\nu)})]^2\end{aligned}\tag{1.3}$$

is obtained, where $\varphi_1(t)-T\in[0,1]$, *and*

$$\rho\{[0,1];[\mathring{T},\overset{\circ}{\widehat{T+1}}]\}\sim T,\ T\to\infty,\tag{1.4}$$

(where ρ *stands for the distance of corresponding segments).*

Remark 1. Theorem 1 gives the contact point between the functions $\zeta\left(\frac{1}{2}+it\right)$, $\varphi_1(t)$ and the Bessel's functions $J_\nu(x)$.

This paper is the continuation of the series [1]-[18].

2. The second result: new class of nonlinear integral equations

2.1. Let us remind that

$$t-\varphi_1(t)\sim(1-c)\pi(t)\ \Rightarrow\ \mathring{T}\sim T,\ T\to\infty \tag{2.1}$$

where c is the Euler's constant and $\pi(t)$ is the prime-counting function. Then the second formula in (1.3) via the mean-value theorem (comp. (1.1) leads to

Corollary 1.

$$\begin{aligned}&\int_{\varphi_1^{-1}(T)}^{\varphi_1^{-1}(T+1)}\{J_\nu[\mu_n^{(\nu)}(\varphi_1(t)-T)]\}^2(\varphi_1(t)-T)\left|\zeta\left(\frac12+it\right)\right|^2\mathrm{d}t\sim\\&\sim\frac12[J_{\nu+1}(\mu_n^{(\nu)})]^2\ln T,\ T\to\infty,\ n=1,2,\dots\ .\end{aligned} \tag{2.2}$$

Remark 2. Let the primary oscillations

$$\left|\zeta\left(\frac12+it\right)\right|,\ t\in[\varphi_1^{-1}(T),\varphi_1^{-1}(T+1)]$$

interact with the complicated modulated oscillations

$$|J_\nu[\mu_n^{(\nu)}(\varphi_1(t)-T)]|\sqrt{\varphi_1(t)-T}. \tag{2.3}$$

Then the integral (2.2) expresses the energy of the resulting oscillations. Let us note that the oscillations (2.3) comers to the point t with the big retardation (see (2.1))

$$t-\{\varphi_1(t)-T\}=t-\varphi_1(t)+T\sim(1-c)\pi(t)+T,\ T\to\infty.$$

2.2. In this direction the following theorem holds true.

Theorem 2. *Every Jacob's ladder $\varphi_1(t)=\frac12\varphi(t)$ where $\varphi(t)$ is the (exact) solution to the nonlinear integral equation*

$$\int_0^{\mu[x(T)]}Z^2(t)e^{-\frac{2}{x(T)}t}\mathrm{d}t=\int_0^T Z^2(t)\mathrm{d}t$$

is the asymptotic solution of the new-type nonlinear integral equation

$$\begin{aligned}&\int_{x^{-1}(T)}^{x^{-1}(T+1)}\{J_\nu[\mu_n^{(\nu)}(x(t)-T)]\}^2(x(t)-T)\left|\zeta\left(\frac12+it\right)\right|^2\mathrm{d}t=\\&=\frac12[J_{\nu+1}(\mu_n^{(\nu)})]^2\ln T,\ T\to\infty,\ n=1,2,\dots\ .\end{aligned} \tag{2.4}$$

At the same time the Jacob's ladder $\varphi_1(t)$ is the asymptotic solution to the following nonlinear integral equations (comp. (2.2) with [18]*, (1.5), (2.2), (3.2), (3.3), (3.5), (3.6))*

(2.5)

$$\begin{aligned}&\int_{x^{-1}(T)}^{x^{-1}(T+2)}[P_n^{\alpha,\beta}(x(t)-T-1)]^2(T+2-x(t))^\alpha(x(t)-T)^\beta\left|\zeta\left(\frac12+it\right)\right|^2\mathrm{d}t=\\&=\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}\ln T,\ n=1,2,\dots,\end{aligned}$$

$$\int_{x^{-1}(T)}^{x^{-1}(T+2)}[P_n(x(t)-T-1)]^2\left|\zeta\left(\frac12+it\right)\right|^2\mathrm{d}t=\frac{2}{2n+1}\ln T,\ n=1,2,\dots, \tag{2.6}$$

$$\int_{x^{-1}(T)}^{x^{-1}(T+2)} [T_n(x(t)-T-1)]^2 \frac{\left|\zeta\left(\frac{1}{2}+it\right)\right|^2}{\sqrt{1-(x(t)-T-1)^2}}\mathrm{d}t = \frac{\pi}{2}\ln T,\ n=1,2,\dots, \tag{2.7}$$

$$\int_{x^{-1}(T)}^{x^{-1}(T+2)} \frac{\left|\zeta\left(\frac{1}{2}+it\right)\right|^2}{\sqrt{1-(x(t)-T-1)^2}}\mathrm{d}t = \pi\ln T, \tag{2.8}$$

$$\int_{x^{-1}(T)}^{x^{-1}(T+2)} [U_n(x(t)-T-1)]^2\sqrt{1-(x(t)-T-1)^2}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t = \tag{2.9}$$
$$= \frac{\pi}{2}\ln T,\ n=1,2,\dots,$$

$$\int_{x^{-1}(T)}^{x^{-1}(T+2)} \sqrt{1-(x(t)-T-1)^2}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t = \frac{\pi}{2}\ln T, \tag{2.10}$$

for every fixed $T \geq T_0[\varphi_1]$ *where* $P_n^{\alpha,\beta}(t)$, $P_n(t)$, $T_n(t)$, $U_n(t)$ *denote the polynomials of Jacobi, Legendre and Chebyshev of the first and second kind, respectivelly.*

Remark 3. There are the fixed-point methods and other methods of the functional analysis used to study the nonlinear equations. What can be obtained by using these methods in the case of the nonlinear integral equations (2.4)-(2.10)?

3. Proof of Theorem 1

3.1. Let us remind that the following lemma holds true (see [6], (2.5); [7], (3.3)): for every integrable function (in the Lebesgue sense) $f(x)$, $x \in [\varphi_1(T), \varphi_1(T+U)]$ we have

$$\int_T^{T+U} f[\varphi_1(t)]\tilde{Z}^2(t)\mathrm{d}t = \int_{\varphi_1(T)}^{\varphi_1(T+U)} f(x)\mathrm{d}x,\ U \in \left(0, \frac{T}{\ln T}\right] \tag{3.1}$$

where

$$t - \varphi_1(t) \sim (1-c)\pi(t),$$

c is the Euler's constant and $\pi(t)$ is the prime-counting function. In the case (comp. Theorem 1) $T = \varphi_1(\mathring{T}), T+U = \varphi_1(\widehat{T+U})$ we obtain from (3.1)

$$\int_{\mathring{T}}^{\widehat{T+U}} f[\varphi_1(t)]\tilde{Z}^2(t)\mathrm{d}t = \int_T^{T+U} f(x)\mathrm{d}x. \tag{3.2}$$

3.2. Putting

$$f(t) = J_\nu[\mu_m^{(\nu)}(t-T)]J_\nu[\mu_n^{(\nu)}(t-T)](t-T),\ U=1$$

we have by (3.2) and (1.2) the following $\tilde{Z}^2$-transformation

$$\int_{\mathring{T}}^{\widehat{T+1}} J_\nu[\mu_m^{(\nu)}(\varphi_1(t)-T)]J_\nu[\mu_n^{(\nu)}(\varphi_1(t)-T)]\cdot$$
$$\cdot(\varphi_1(t)-T)\tilde{Z}^2(t)\mathrm{d}t =$$
$$= \int_T^{T+1} J_\nu[\mu_m^{(\nu)}(t-T)]J_\nu[\mu_n^{(\nu)}(t-T)](t-T)\tilde{Z}^2 t\mathrm{d}t =$$
$$= \int_0^1 J_\nu(\mu_m^{(\nu)}x)J_\nu(\mu_n^{(\nu)}x)x\mathrm{d}x = 0,\ m\neq n,$$

where $t = x+T$, i.e. the first formula in (1.3) holds true. Similarly, we obtain the second formula in (1.3).

3.3. Next, for $\xi \in (\mathring{T}, \widehat{T+1})$ we have (see (2.1) and [18], (4.4))

$$\ln\xi = \ln\mathring{T} + \mathcal{O}\left(\frac{1}{\ln\mathring{T}}\right) = \ln T + \mathcal{O}\left(\frac{1}{\ln T}\right). \tag{3.3}$$

The property (3.3) was used in (2.1), ...

I would like to thank Michal Demetrian for helping me with the electronic version of this work.

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