Density of rational points on elliptic curves and small transcendence degree

Aleksander Momot
ETH Zürich

November 16, 2010

Table of contents

Chapter 1

Introduction

Let F be a subfield of $\mathbb{C}$ and let E be an elliptic curve over F. A general problem in Diophantine approximation is to find, for a given infinite subgroup $\Gamma \subset E(F)$, conditions when Γ is not dense in $E(\mathbb{C})$ with respect to the Euclidean topology. Then the Euclidean closure $C \subset E(\mathbb{C})$ of Γ is a proper real-analytic submanifold of $E(\mathbb{C})$ and a natural reason for this phenomenon is the existence of an elliptic curve E' over $K = F \cap \mathbb{R}$ and an isogeny $v : E \longrightarrow E' \times_{\mathrm{spec} K} \mathrm{spec}\, F$ such that $v(\Gamma) \subset E'(\mathbb{R})$.
The purpose of this paper is to obtain criterions for the existence of an isogeny v to a curve E' over K such that $v(C) \subset E'(\mathbb{R})$. We formulate conditions in terms of transcendence properties of elliptic logarithms for the case where F is a field with transcendence degree ≤ 1 such that if $c \in F$ then $Re\, c, Im\, c \in F$. Our criterions are application motivated: they allow to improve in certain situations a number of classical transcendence results concerning elliptic functions. The scope are applications in small transcendence degree. Nevertheless, the range of our idea, namely to introduce Weil restrictions onto the stage of transcendence theory, is wider and would also cover results in large transcendence degree.

Before we state our results, we need some preparations. For a smooth variety V over F the set of complex points $V(\mathbb{C})$ is endowed with the structure of a complex analytic manifold. The complex-analytic structure on $V(\mathbb{C})$ implies the structure of a real-analytic manifold on $V(\mathbb{C})$ in an unique manner. If F is contained in $\mathbb{R}$ then $V(\mathbb{R})$ is a proper real-analytic submanifold of $V(\mathbb{C})$. Finally, for a morphism $v : V \longrightarrow W$ of varieties over F we will denote by $v_{\mathbb{R}}$ the respective real-analytic map between the two real-analytic manifolds $V(\mathbb{C})$ and $W(\mathbb{C})$.
Let now G be a commutative algebraic group variety over F. We let $e_G \in G(F)$ be the neutral element and define $Lie\, G$ to be the Lie algebra of invariant vector fields on G. The vector space $Lie\, G$ over F and can be canonically identified with the tangent space of G at the unit element. For a field extension k of F we write $(Lie\, G)(k) = Lie\, G \otimes_F k$. We define $exp_G : (Lie\, G)(\mathbb{C}) \longrightarrow G(\mathbb{C})$ to be the holomorphic exponential map and Λ to be the kernel of exp_G.
In the special case where F is contained in $\mathbb{R}$ the Galois group $Gal(\mathbb{C}|\mathbb{R})$ acts in a natural way on $G(\mathbb{C})$ and $(Lie\, G)(\mathbb{C})$. The actions commute with the exponential map and $(Lie\, \mathbb{G})(\mathbb{R})$ is mapped onto the component of unity $G^o(\mathbb{R})$ of $G(\mathbb{R})$.

Let now $\mathbb{G}$ be a commutative group variety over $K = F \cap \mathbb{R}$ and let $\psi : \mathbb{G}(\mathbb{R}) \longrightarrow G(\mathbb{C})$ be a homomorphism of real-analytic Lie groups. There is an unique $\mathbb{R}$-linear homomorphism

$$\psi_* : (Lie\,\mathbb{G})(\mathbb{R}) \longrightarrow (Lie\,G)(\mathbb{C})$$

such that $\psi \circ \big(exp_{\mathbb{G}}\big)_{\mathbb{R}} = \big(exp_G\big)_{\mathbb{R}} \circ \psi_*$. The homomorphism ψ is **defined over F** if $\psi_*(Lie\,\mathbb{G}) \subset Lie\,G$. Moreover, we say that ψ **descents to** K if there is an algebraic group G' over K, an isogeny $v : G \longrightarrow G' \times_{\mathrm{spec}K} \mathrm{spec}\,F$ of algebraic groups over F and a commutative diagram of real-analytic maps

$$\begin{array}{ccc} \mathbb{G}(\mathbb{R}) & \xrightarrow{\psi} & G(\mathbb{C}) \\ \downarrow{\scriptstyle v_{\mathbb{R}}\circ\psi} & & \downarrow{\scriptstyle v_{\mathbb{R}}} \\ G'(\mathbb{R}) & \hookrightarrow & G'(\mathbb{C}). \end{array}$$

Such a diagram does not exist in general. To get a rough picture of this phenomenon, consider a similar situation in the very easy case when $F = \mathbb{C}$ and $G = G' = \mathbb{G} = \mathbb{G}_a$. Then $\mathbb{G}_a(\mathbb{R})$ and $\mathbb{G}_a(\mathbb{C})$ can be identified with $\mathbb{R}$ and $\mathbb{C}$ respectively. An algebraic morphism $v : G \longrightarrow G'$ is nothing but a polynomial map, and if ψ is a real-analytic map between $\mathbb{R}$ and $\mathbb{C}$ then in general there will be no polynomial $v(z) \in \mathbb{C}[z]$ such that $v_{\mathbb{R}} \circ \psi$ takes values in $G'(\mathbb{R}) = \mathbb{R}$.

Our first main theorem deals with the special case where $F = \overline{\mathbb{Q}}$ and G is an elliptic curve E. We let $h \in Gal(\mathbb{C}|\mathbb{R})$ be the complex conjugation. For a $\mathbb{Z}$-module $I \subset \Lambda$ we define

$$(\Lambda : I) = \{c \in \mathbb{C}; cI \subset \Lambda\}.$$

Theorem 1.0.1. *Consider a non-constant and real-analytic homomorphism* $\psi : \mathbb{G}_a(\mathbb{R}) \longrightarrow E(\mathbb{C})$*. The following four assertions are equivalent.*

1. ψ *descents to* K.
2. $Lie\,E \cap im\,\psi_* \neq \{0\}$ *and* $E\big(\overline{\mathbb{Q}}\big) \cap im\,\psi \neq \{e_E\}$.
3. $Re\,(\Lambda : I_\psi) \subset \mathbb{Q}$ *for* $I_\psi = im\,\psi_* \cap \Lambda$.
4. *The set of points* $r \in \mathbb{G}_a(\mathbb{R})$*, such that* $h(F_r) = F_r$ *for* $F_r = \overline{\mathbb{Q}\big(\psi(r)\big)}$*, is not countable.*

In the situation of the theorem, G' is an elliptic curve E' over K. More generally, Lemma 3.5.3, Theorem 3.7.1 and Lemma 4.1.1 below imply that if a complex elliptic curve E is isogenous to the complexification of an elliptic curve E' over $\mathbb{R}$ then one can distinguish between two disjoint cases.

- E has complex multiplication.
- There is an isogeny $v : E \longrightarrow E^h$ to the complex conjugate E^h (explained in Ch. 3), such that $\deg v$ is a square number, and E is isogenous to a second curve E'' defined over $\mathbb{R}$ with the property that $\mathbf{Hom}_{\mathbb{R}}(E', E'') = 0$.

We also wish to point out (without proof) that the equivalence of Statements 1., 3. and 4. holds more generally if the base field $F = \overline{\mathbb{Q}}$ is replaced by a countable subfield of $\mathbb{C}$ which is stable with respect to complex conjugation.

For the next results we weaken the assumption that F is algebraic over $\mathbb{Q}$. The letter F shall now denote a subfield of $\mathbb{C}$ such that $trdeg_{\mathbb{Q}} F \leq 1$ and, as above, such that if $c \in F$ then $Re\, c, Im\, c \in F$. We define $\mathbb{G} = \mathbb{G}_a \times E$ and for a real-analytic homomorphism $\psi : \mathbb{G}_a(\mathbb{R}) \longrightarrow E(\mathbb{C})$ we let Ψ be the monomorphism $id_{\mathbb{G}_a} \times \psi$ to $\mathbb{G}(\mathbb{C})$.

Theorem 1.0.2. *Let $\psi : \mathbb{G}_a(\mathbb{R}) \longrightarrow E(\mathbb{C})$ be a non-constant and real-analytic homomorphism such that $ker\, \psi \neq 0$. Suppose that one of the following assertions is true.*

1. *ψ is defined over F and either $rank_{\mathbb{Z}}\, im\, \psi \cap E(F) \geq 3$ or $rank_{\mathbb{Z}}\, im\, \Psi \cap \mathbb{G}(F) \geq 3$ and $\Psi^{-1}\big(\mathbb{G}(F)\big) \cap ker\, \psi \neq \{0\}$.*
2. *$F = \overline{\mathbb{Q}}$ and $rank_{\mathbb{Z}}\, im\, \psi \cap E(F) \geq 3$.*

Then ψ descents to K.

Our third main result treats the situation where $\mathbb{G} = \mathbb{G}_a \times \mathbb{G}_m \times E$ is a trivial extension of an elliptic curve E by $\mathbb{G}_a \times \mathbb{G}_m$ over F. We denote by $\pi : \mathbb{G} \longrightarrow E$ the projection to E and let $\Psi : \mathbb{G}_a(\mathbb{R}) \longrightarrow \mathbb{G}(\mathbb{C})$ be a real-analytic homomorphism. We set $\Psi_\pi = \pi \circ \Psi$.

Theorem 1.0.3. *Let $\Psi : \mathbb{G}_a(\mathbb{R}) \longrightarrow \mathbb{G}(\mathbb{C})$ be a real-analytic homomorphism which is defined over F and with Zariski-dense image. Suppose that $rank_{\mathbb{Z}}\, im\, \Psi \cap \mathbb{G}(F) \geq 2$ and $\psi^{-1}\big(\mathbb{G}(F)\big) \cap ker\, \Psi_\pi \neq \{0\}$. Then Ψ_π descents to K.*

Finally, we consider general extensions $\pi : G \longrightarrow E$ over F of an elliptic curve E by a linear algebraic group variety L with dimension $\dim L = 1$. We suppose that π is not isotrivial over $\overline{F}$ and let $\psi_\pi = \pi \circ \psi$. We again define $\Psi = id_{\mathbb{G}_a} \times \psi$.

Theorem 1.0.4. *Let $\psi : \mathbb{G}_a(\mathbb{R}) \longrightarrow G(\mathbb{C})$ be a real-analytic homomorphism which is defined over F and such that ψ_π is non-constant. Suppose that one of the following assertions is true.*

1. *$rank_{\mathbb{Z}}\, \psi^{-1}\big(G(F)\big) \geq 2$ and $ker\, \psi \neq \{0\}$.*
2. *$rank_{\mathbb{Z}}\, im\, \Psi \cap \mathbb{G}(F) \geq 3$.*
3. *$rank_{\mathbb{Z}}\, im\, \psi \cap G(F) \geq 4$.*

Then ψ_π descents to K.

Remark 1.0.5. If Statement 1. in the theorem holds then ψ descents to K as soon as ψ has a Zariski-dense image, even if π is isotrivial.

Organization of the paper. In the next section we state some applications of our theorems which also serve as motivation. Chapter 3 contains a description of Weil restrictions and their basic properties. We already mentioned above that Weil restrictions can not only be applied in the framework of small transcendence degrees. Therefore, we elaborated a rather detailed presentation containing results useful for further applications to transcendence. The content is immediate from the subtitles in the above table. The proof of the theorems is performed in the last section. It relies on the theory of Weil restrictions, transcendence results from Tubbs [4] and our transcendence results from [2]. The input from transcendence theory is summarized in the appendix.

Chapter 2

Applications

Notation. We let $\mathcal{L} = e^{-1}\left(\overline{\mathbb{Q}}\right) \setminus \{0\}$ and $i = \sqrt{-1}$. For a lattice $\Lambda \subset \mathbb{C}$ we define $\wp_\Lambda$ to be the associated Weierstraß function with invariants $g_3(\Lambda), g_3(\Lambda)$, whereas E_Λ shall mean the respective elliptic curve. By abuse of notation, we will identify the exponential map $exp_{E_\Lambda}(z) = [\wp_\Lambda(z), \wp'_\Lambda(z) : 1]$ of E_Λ with $\wp_\Lambda$. $\mathbb{H}$ is the upper half plane and $j : \mathbb{H} \longrightarrow \mathbb{C}$ is the classical j-function. For a $\tau \in \mathbb{H}$ we set $\wp_\tau = \wp_{\mathbb{Z}+\mathbb{Z}\tau}$, $E_\tau = E_{\mathbb{Z}+\mathbb{Z}\tau}$ and, for $l = 2, 3$, $g_l(\tau) = g_l(\mathbb{Z} + \mathbb{Z}\tau)$. Finally, we define $\mathcal{L} = e^{-1}\big(\overline{\mathbb{Q}}\big)$.

Let $\Lambda \subset \mathbb{C}$ be a lattice with algebraic invariants. One of the most famous transcendence results concerning elliptic functions is Th. Schneider's theorem on the transcendence of algebraic logarithms $\omega \neq 0 \in \wp_\Lambda^{-1}\big(\overline{\mathbb{Q}} \cup \{\infty\}\big)$ from 1936. With the help of our first main theorem we are able to generalize the former result in an essential manner. This is the content of the first three corollaries.

Corollary 2.0.6. *Let $\Lambda \subset \mathbb{C}$ be a lattice with algebraic invariants and let $\lambda \in \Lambda \setminus \{0\}$ satisfy $Re\, \lambda^{-1}\Lambda \not\subseteq \mathbb{Q}$. Then $\mathbb{R}\lambda \cap \overline{\mathbb{Q}} = \{0\}$.*

Proof. Let $\psi(z) = exp_{E_\Lambda}(\lambda z)$. Statement 2. in Thm. 1.0.1 is equivalent to the assertion that $\mathbb{R}\lambda \cap \overline{\mathbb{Q}} = \{0\}$. Statement 3. in Thm. 1.0.1 is equivalent to the assertion that $Re\, \lambda^{-1}\Lambda \not\subseteq \mathbb{Q}$. Thm. 1.0.1 thus yields the claim. □

Corollary 2.0.7. *Let $\Lambda \subset \mathbb{C}$ be a lattice with algebraic invariants and let $\omega \in \wp_\Lambda^{-1}(\overline{\mathbb{Q}})$. Suppose that $\mathbb{Z}\omega + \Lambda$ is dense in $\mathbb{C}$ with respect to the Euclidean topology. Then $\mathbb{R}\omega \cap \overline{\mathbb{Q}} = \{0\}$.*

Proof. Let $\psi(z) = exp_{E_\Lambda}(\omega z)$ and $\xi = exp_{E_\Lambda}(\omega)$. The group $\mathbb{Z}\xi \in E(\overline{\mathbb{Q}})$ is dense in $E(\mathbb{C})$ with respect to the Euclidean topology, so that ψ does not descent to $\mathbb{R}$. The assertion follows thus from the first main theorem.

□

Corollary 2.0.8. *Let $\Lambda \subset \mathbb{C}$ be a lattice with algebraic invariants and associated curve E over $\overline{\mathbb{Q}}$. Then E is isogenous to a curve over $K = \overline{\mathbb{Q}} \cap \mathbb{R}$ precisely when $\omega/|\omega| \in \overline{\mathbb{Q}}$ for some algebraic logarithm $\omega \in \wp_\Lambda^{-1}\big(\overline{\mathbb{Q}}\big)$.*

A further application of the first main theorem is

Corollary 2.0.9. *Let $\Lambda \subset \mathbb{C}$ be a lattice with algebraic invariants and let $u \in \mathbb{C} \setminus \overline{\mathbb{Q}}$ such that $|u| = 1$. Then, for $r \in \mathbb{R}$ outside a countable set, $\wp_\Lambda(ru)$ and its complex conjugate are algebraically independent.*

Proof. We have $\mathbb{R}u \cap \overline{\mathbb{Q}} = \{0\}$, so that $\psi(z) = \wp_\Lambda(zu)$ cannot be of real type. We apply Thm. 1.0.1 and derive the assertion.

□

In his book "*Einführung in die transzendenten Zahlen*" from 1957, p. 138, Schneider posted eight meanwhile popular problems. The eighth was to show that $\{e^e, e^{e^2}\} \not\subseteq \overline{\mathbb{Q}}$. This was proved by Brownawell and Waldschmidt. The following two corollaries, which result from an application of Thm. 1.0.1 and Thm.1.0.2, are inspired by this result.

Corollary 2.0.10. *Let $\tau \in j^{-1}\big(\overline{\mathbb{Q}}\big)$ be such that $Re\,\tau \notin \mathbb{Q}$, $g_2(\tau) \neq 0$ and let F be the algebraically closed field generated by $g_2(\tau)$ and $|g_2(\tau)|$. Fix a number $\xi \in F$ which is real and with degree ≥ 4 over $\mathbb{Q}$. Then*

1. *$g_2(\tau)$ and $|g_2(\tau)|$ are algebraically independent, or*
2. *$\big\{\wp_\tau(\xi), \wp_\tau\big(\xi^2\big)\big\} \not\subseteq F$.*

Proof. The field $F = \overline{\mathbb{Q}\big(g_2(\tau), |g_2(\tau)|\big)}$ is closed with respect to complex conjugation and E_τ is defined over F. The assertion thus results from Thm. 1.0.1 and Thm.1.0.2 by setting $\psi(z) = \wp_\tau(z)$. □

Corollary 2.0.11. *Let $\Lambda \subset \mathbb{C}$ be a lattice with algebraic invariants and let $\lambda \in \Lambda \setminus \{0\}$ such that $Re\,\lambda^{-1}\Lambda \not\subseteq \mathbb{Q}$. Set $f(z) = \wp_\Lambda(\lambda z)$. Then the values $f(e), f\big(e^2\big), f\big(e^3\big)$ are all defined and not all of them are algebraic.*

It is reasonable to conjecture that under the above hypotheses, in fact, all values $f(\xi), \xi \notin \mathbb{Q}$, are transcendental. However, a proof of this assertion seems out of scope.

Proof. The subgroup $\mathbb{Z} + \mathbb{Z}e + \mathbb{Z}e^2 + \mathbb{Z}e^3$ of $\mathbb{C}$ has rank 4. We apply 2. in Thm. 1.0.2 and Thm. 1.0.1 to receive the claim. □

A classical theorem of Gelfond from 1947 asserts that, for an algebraic $\alpha \neq 0, 1$ and a cubic β, the two numbers $\alpha^\beta, \alpha^{\beta^2}$ are algebraically independent. The next consequence of the second main theorem is, to some effect, an elliptic analogon of this. It amends Masser-Wüstholz [1, Cor. 1], where a similar result for elliptic curves with complex multiplication is given.

Corollary 2.0.12. *Let $\Lambda \subset \mathbb{C}$ be a lattice with algebraic invariants and let $\lambda \in \Lambda \setminus \{0\}$. Let $\beta \in \mathbb{R} \setminus \mathbb{Q}$ be algebraic and fix a $\xi \in \mathbb{R}\lambda$ such that $\xi/\lambda \notin \mathbb{Q}[\beta]$. Then*

$$\big\{\wp_\Lambda\big(\beta\xi\big), \wp_\Lambda\big(\beta^2\xi\big)\big\} \not\subseteq \overline{\mathbb{Q}\,(\lambda\ |\lambda|)}.$$

Proof. For brevity we will supress the lattice index of the Weierstraß functions. Assume first that $u = \frac{\lambda}{|\lambda|} \notin \overline{\mathbb{Q}}$. Set $\psi(z) = exp_E\,(uz)$ and consider $u^{-1}\big(\mathbb{Z}\lambda + \mathbb{Z}\beta\xi + \mathbb{Z}\beta^2\xi\big)$.

Thm. 1.0.1 and Thm. 1.0.2 show that $\operatorname{trdeg}_{\mathbb{Q}} K_1 \geq 2$ where K_1 is the algebraically field generated by

$$K_2 = \mathbb{Q}\left(u, \wp\big(\beta\xi\big), \wp\big(\beta^2\xi\big)\right)$$

and the complex conjugate of K_2. However, since $F = \overline{\mathbb{Q}(u)}$ is closed under complex conjugation, it cannot be that $\operatorname{trdeg}_{\mathbb{Q}} K_1 \geq 2$ and $\operatorname{trdeg}_{\mathbb{Q}} K_2 \leq 1$. This proves the claim in the case where $u \notin \overline{\mathbb{Q}}$. Next, assume that $u \in \overline{\mathbb{Q}}$ and that the assertion is wrong for some λ, ξ, β. Then there is a number field K such that $\frac{d}{dz}$ acts on

$$K\big(\wp(\beta^2 z), \wp(\beta z), \wp'(\beta^2 z), \wp'(\beta z)\big),$$

and the theorem of Schneider-Lang implies that $\wp(\beta^2 z), \wp(\beta z)$ are algebraically dependent. Since $\beta \in \mathbb{R} \setminus \mathbb{Q}$, $\beta \notin \mathbf{End}(\Lambda) \otimes \mathbb{Q}$. We receive a contradiction. The corollary is proved. □

Next we state an application of Thm. 1.0.3.

Corollary 2.0.13. *Let $\tau \in j^{-1}\big(\overline{\mathbb{Q}}\big)$ be such that $Re\,\tau \notin \mathbb{Q}$, $g_2(\tau) \neq 0$. Let F be the algebraically closed field generated by $g_2(\tau)$ and $|g_2(\tau)|$. Let $\omega \in \mathcal{L} \cap \mathbb{R}$ and let $\alpha \in F$ be real, but not rational. Then*

$$\mathit{trdeg}_{\mathbb{Q}} F\left(\omega, e^{\alpha\omega}, \wp_\tau(\alpha)\right) \geq 2$$

and

$$\mathit{trdeg}_{\mathbb{Q}} F\left(\omega, e, \wp_\tau(\omega)\right) \geq 2.$$

Proof. For the first assertion take

$$\Psi(z) = \left(z, e^{\omega z}, \wp_\tau(z)\right) \text{ and } \mathbb{Z} + \mathbb{Z}\alpha,$$

and for the second use

$$\Psi(z) = \left(z, e^{z}, \wp_\tau(z)\right) \text{ and } \mathbb{Z} + \mathbb{Z}\omega.$$

□

Thm. 1.0.4 and Rem. 1.0.5 yield similarly

Corollary 2.0.14. *Let $\Lambda \subset \mathbb{C}$ be a lattice with algebraic invariants and suppose that $Re\lambda^{-1}\Lambda \nsubseteq \mathbb{Q}$ for some $\lambda \in \Lambda \setminus \{0\}$. Then*

1. *$\frac{\pi}{\lambda}$ and $\frac{\pi}{|\lambda|}$ are algebraically independent, or*
2. *$\frac{\lambda}{|\lambda|}$ and $\wp_\Lambda\left(\frac{\lambda\omega}{i\pi}\right)$ are algebraically independent for all $\omega \in \mathcal{L} \cap i\mathbb{R} \setminus i\pi\mathbb{Q}$.*

Proof. Let $\psi(z) = \left(e^{iz}, \wp_\Lambda\left(\lambda z/\pi\right)\right)$ and consider $\mathbb{Z}\pi + \mathbb{Z}i\omega$. Thm. 1.0.4 and the remark yield

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}\left(\frac{\pi}{\lambda}, \frac{\pi}{|\lambda|}, \wp_\Lambda\left(\frac{\lambda\omega}{i\pi}\right)\right) \geq 2.$$

The corollary is readily inferred from this. □

Between 1974 and 1981 Chudnovsky proved striking results of algebraic independence related to elliptic functions. One among these results was the algebraic independence of the two numbers

$$\zeta_\Lambda(\omega) - \frac{\eta_\Lambda(\lambda)}{\lambda}\omega, \frac{\eta_\Lambda(\lambda)}{\lambda}$$

for a lattice $\Lambda \subset \mathbb{C}$ with algebraic invariants, an algebraic logarithm $\omega \in \wp_\Lambda^{-1}(\overline{\mathbb{Q}}) \setminus \mathbb{Q}\Lambda$ and associated values $\eta_\Lambda(\lambda)$, $\zeta_\Lambda(\omega)$. The application below is connected to this classical theorem from 1981. In the formulation, next to $\wp_\Lambda$ we also use the Weierstraß functions $\sigma_\Lambda, \eta_\Lambda$ and ζ_Λ.

Corollary 2.0.15. *Let Λ be a lattice with algebraic invariants, $\lambda \in \Lambda \setminus \{0\}$ and let $\omega \in \mathbb{R}\lambda$ be a number not in $\mathbb{Q}\lambda$. Assume $Re\,\lambda^{-1}\Lambda \not\subseteq \mathbb{Q}$ and define*

$$F = \mathbb{Q}\left(\wp_\Lambda(\omega), \frac{e^{\eta_\Lambda(\lambda)\omega^2/\lambda}}{\sigma_\Lambda^6(\omega)}, \zeta_\Lambda(\omega) - \frac{\eta_\Lambda(\lambda)}{\lambda}\omega, \frac{\lambda}{|\lambda|}\right).$$

Then $trdeg_{\mathbb{Q}} F \geq 2$.

Proof. For brevity we will supress the lattice index of the Weierstraß functions. Define

$$f_\omega(z_1, z_2) = \frac{\sigma^3(z_2 - \omega)e^{3\zeta(\omega)z_2 + z_1}}{\sigma^3(z_2)\sigma^3(\omega)}.$$

As shown in Wüstholz [6], there exists an extension $\pi : G \longrightarrow E$ by $\mathbb{G}_m$ over $Fh(F)$, which is not isotrivial and which admits an uniformization $exp_G : \mathbb{C}^2 \longrightarrow \mathbf{P}^5$ over F given by

$$exp_G(z_1, z_2) = \big[\wp(z_2) : \wp'(z_2) : 1 : \wp(z_2 - \omega)f_\omega(z_1, z_2) : \wp'(z_2 - \omega)f_\omega(z_1, z_2) : f_\omega(z_1, z_2)\big].$$

We let $\xi = 3\zeta(\omega)z - \frac{3\eta(\lambda)\omega z}{\lambda}$ and

$$\psi^*(z) = exp_G(-\xi, z)$$

Since $\sigma(z + \lambda) = \sigma(z)e^{\eta(\lambda)z + a_\lambda}$ for some constant a_λ, it holds that ψ^* is λ-periodic. Moreover,

$$\lim_{z\to 0} \sigma^3(z) \cdot \wp'(z) = -1/2.$$

Therefore,

$$\lim_{z\to\omega} \wp'(z - \omega)f_\omega(z, \xi) = -\frac{e^{\eta(\lambda)\omega^2/\lambda}}{2\sigma^6(\omega)},$$

whereas $\wp(z - \omega)f_\omega(z, \xi), f_\omega(z, \xi)$ tend to zero if $z \to \omega$. We apply Statement 1. in Thm. 1.5 to $\psi(z) = \psi^*(\lambda z/|\lambda|)$ and obtain the claim. ☐

With the same argument one can prove

Corollary 2.0.16. *Let Λ be a lattice with algebraic invariants, let $u \in \mathbb{C}$ with $|u| = 1$ and let $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \mathbb{Z}\omega_3 + \mathbb{Z}\omega_4$ be a subgroup of $\mathbb{R}u$ with rank 4. Then either*

1. $\{\wp_\Lambda(\omega_1), \wp_\Lambda(\omega_2), \wp_\Lambda(\omega_3), \wp_\Lambda(\omega_4)\} \not\subseteq \overline{\mathbb{Q}(u)}$, *or*

2. $\{\zeta_\Lambda(\omega_1), \zeta_\Lambda(\omega_2), \zeta_\Lambda(\omega_3), \zeta_\Lambda(\omega_4)\} \not\subseteq \overline{\mathbb{Q}(u)}$.

Proof. Assume first that $\mathbb{R}u \cap \Lambda = \{0\}$. For some $t \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$ let

$$f_1(z_1, z_2) = \wp(z_2)z_1 + t\big(\wp'(z_2) + 2\zeta(z_2)\wp'(z_2)\big),$$

$$f_2(z_1, z_2) = \wp'(z_2)z_1 + t\big(\wp''(z_2) + 2\zeta(z_2)\wp'(z_2)\big)$$

and

$$f_3(z_1, z_2) = z_1 + 2t\zeta(z_2).$$

According to Wüstholz [6], there is an extension $\pi : G \longrightarrow E$ by $\mathbb{G}_a$ over $\overline{\mathbb{Q}}$, which is not isotrivial and which admits an uniformization $exp_G : \mathbb{C}^2 \longrightarrow \mathbf{P}^5$ over $\overline{\mathbb{Q}}$ given by

$$exp_G(z_1, z_2) = \big[\wp(z_2) : \wp'(z_2) : 1 : f_1(z_1, z_2) : f_2(z_1, z_2) : f_3(z_1, z_2)\big].$$

We apply 3. in Thm. 1.0.5 to $\psi(z) = exp_G(0, uz)$. Noting that ψ is defined over $\overline{\mathbb{Q}(u)}$ and that ψ_π cannot be of real type, we derive the assertion similarly as in the proof of the previous corollary.
Next, suppose that $\mathbb{R}u \cap \Lambda \neq 0$ but $u \notin \overline{\mathbb{Q}}$. Then the corollary follows as above taking into account Statement 2. in Thm. 1.0.1. Finally, if $u \in \overline{\mathbb{Q}}$ then the corollary follows from a well-known theorem of Schneider. □

Chapter 3

Weil restrictions of commutative group varieties

3.1 Conjugate varieties

Let F/K be a finite extension of fields with characteristic 0 and denote by $\mathbf{Var}_F$ (resp. $\mathbf{Var}_K$) the category of quasi-projective varieties over F (resp. K). Let $G \in \mathbf{Var}_F$. The construction of the Weil restriction $\mathcal{N}_{F/K}(G) \in \mathbf{Var}_K$ rests on the properties of conjugate varieties in an essential manner. We will therefore consider the definition and properties of conjugate varieties in more detail. After stating some general properties we present an example for the convenience of the reader.

3.1.1 Definition of conjugate varieties

Fix a homomorphism $h \in Hom_K(F, \overline{F})$ and let $G = (\underline{G}, \mathcal{O}_G) \in \mathbf{Var}_F$ be a quasi-projective variety with underlying topological space $\underline{G}$ and sheaf of rings $\mathcal{O}_G$. Then the **h-conjugate** $G^h \in \mathbf{Var}_{h(F)}$ is defined in the following way. Let G_o be the scheme over K which as locally ringed space coincides with $(\underline{G}, \mathcal{O}_G)$. Define the scalar multiplication of $h(F)$ on the sheaf $\mathcal{O}_G$ by $c * f = h^{-1}(c)f$ for a $c \in h(F)$ and a $f \in \Gamma(U, G)$, where $U \subset G$ is an open subset. This way the structure of a quasi-projective $h(F)$-variety on G_o is defined; it is denoted by $G^h \in \mathbf{Var}_{h(F)}$.

3.1.2 Realization in the affine case

Assume G is an affine subvariety of $\mathbf{A}_F^N$ with associated ideal $J_G \subset F[X_1, ..., X_N]$. Then the affine $h(F)$-algebra

$$\big(F[X_1, ..., X_N]/J_G, *\big)$$

is naturally isomorphic to the affine $h(F)$-algebra

$$\big(h(F)[X_1, ..., X_N]/h(J_G), \cdot\big),$$

where '$\cdot$' denotes the usual multiplication. Therefore, a model of G^h is provided by the affine subvariety of $\mathbf{A}^N_{h(F)}$ with ideal $h(J_G)$. This in turn implies a natural identification

$$G^h = \{\xi^h \in \mathbf{A}^N_{h(F)}; \xi \in G\}.$$

If H is a further affine variety in $\mathbf{A}^M_F$ and if $v : G \longrightarrow H$ is a morphism over F, which is represented by polynomials $\mathbf{Q}_1, ..., \mathbf{Q}_M \in F[X_1, ..., X_N]$, then $\mathbf{Q}^h_1, ..., \mathbf{Q}^h_M \in h(F)[X_1, ..., X_N]$ define a morphism $v^h : G^h \longrightarrow H^h$ such that

$$v^h\big(\xi^h\big) = \big(v(\xi)\big)^h. \tag{3.1.1}$$

These constructions are natural and up to isomorphism not depending on the chosen embedding into affine space.

3.1.3 Conjugate sheaves and line bundles

Let $G \in \mathbf{Var}_F$ and let $\mathcal{L}$ be a coherent locally free sheaf on G. Then $\mathcal{L}$ defines a geometric vector bundle $\pi : \mathbf{Vect}(\mathcal{L}) \longrightarrow G$. Conjugation with h yields a geometric vector bundle $\pi^h : (\mathbf{Vect}(\mathcal{L}))^h \longrightarrow G^h$, which (up to isomorphism) corresponds to a coherent locally free sheaf $\mathcal{L}^h$ on G^h. Moreover, h defines a correspondence between local sections of π and π^h. In the category of sheaves we thus receive, for an open subset $U \subset G$, isomorphisms of groups of sections

$$h : \Gamma\big(U, \mathcal{L}\big) \longrightarrow \Gamma\big(U^h, \mathcal{L}^h\big).$$

In the special case, where $\mathcal{L} \in Pic(G)$ is a line bundle which defines an embedding

$$u_{\mathcal{L}} : G \longrightarrow \mathbf{P}^N_F$$

realized by sections $\sigma_0, ..., \sigma_N \in H^0(G, \mathcal{L})$ over F, the conjugate $\mathcal{L}^h$ is a line bundle in $Pic\big(G^h\big)$, which defines an embedding

$$u_{\mathcal{L}^h} : G^h \longrightarrow \mathbf{P}^N_{h(F)}$$

realized by the sections $h(\sigma_0), ..., h(\sigma_N) \in H^0\big(G^h, \mathcal{L}^h\big)$. Thereby, it holds that $u_{\mathcal{L}^h} = (u_{\mathcal{L}})^h$, and (3.1.1) is satisfied.

3.1.4 Conjugate Lie algebras

Assume that $G \in \mathbf{Var}_F$ is a commutative group variety with Lie algebra $Lie\, G$. Then G^h is a commutative group variety with a Lie algebra $Lie\, G^h$ and h defines a homomorphism of groups

$$h : G(F) \longrightarrow G^h\big(h(F)\big).$$

The underlying vector space of the Lie algebra $Lie\, G$ (resp. $Lie\, G^h$) is the space of global sections $H^0(G, \mathcal{T}_G)$ (resp. $H^0\big(G^h, \mathcal{T}_{G^h}\big)$) of the sheaf of invariant differentials. It is not hard to verify that the isomorphism from Subsubsect. 3.1.3 induces an isomorphism of Lie algebras over K

$$Lie(h) : Lie\, G \longrightarrow Lie\, G^h \backsimeq (Lie\, G)^h$$

which is defined in the following way: Let $U \subset G^h$ be an affine set over F and let $\Delta \in Lie\,G$. Then $Lie\,(h)\big(\Delta\big)$ acts on functions $\mathbf{F} \in h(F)[U]$ by

$$\big(Lie\,(h)\big(\Delta\big)\mathbf{F}\big)(\xi) = h\left(\Delta\mathbf{F}^{h^{-1}}\big(\xi^{h^{-1}}\big)\right)$$

where $\xi \in U$ is arbitrary.

3.1.5 Conjugate varieties and Galois actions

Assume F/K is a Galois extension. Consider a variety $V \in \mathbf{Var}_F$ which results from a variety $V' \in \mathbf{Var}_K$ by base change. Then $V = V^h$ for all $h \in Gal(F|K)$ and $Gal(F|K)$ acts on V in a natural manner.

3.1.6 An example

Let $F = \mathbb{C}$ and $K = \mathbb{R}$ and let $\Lambda \subset \mathbb{C}$ be a lattice with invariants g_2, g_3. Consider the associated elliptic curve

$$E : Y^2 = 4X^3 - g_2X - g_3.$$

We receive an identification $(Lie\,E)(\mathbb{C}) = \mathbb{C}$ resulting from the commutative diagram of universal coverings

$$\begin{array}{ccc} (Lie\,E)(\mathbb{C}) & \longrightarrow & \mathbb{C} \\ & \searrow^{exp_E} & \downarrow{\scriptstyle [\wp_\Lambda:\wp'_\Lambda:1]} \\ & & E(\mathbb{C}) \end{array}$$

The complex conjugate of E has a Weierstraß form

$$E^h : Y^2 = 4X^3 - h(g_2)X - h(g_3).$$

The lattice corresponding to this Weierstraß form is nothing but $h(\Lambda)$, and we have again an identification $Lie\,E^h = \mathbb{C}$. Moreover, since the complex conjugation $h \in Gal(\mathbb{C}|\mathbb{R})$ is a real-analytic map, we have a commutative diagram

$$\begin{array}{ccc} Lie\,E = \mathbb{C} & \xrightarrow{Lie(h)=h} & Lie\,E^h = \mathbb{C} \\ \downarrow{\scriptstyle [\wp_\Lambda:\wp'_\Lambda:1]} & & \downarrow{\scriptstyle [\wp_{h(\lambda)}:\wp'_{h(\Lambda)}:1]} \\ E(\mathbb{C}) & \xrightarrow{h} & E^h(\mathbb{C}) \end{array}$$

3.2 Classical definition of Weil restrictions

Consider a variety $G \in \mathbf{Var}_F$. The **Weil restriction of G with respect to the extension F/K** is a pair $(\mathcal{N}_{F/K}(G), p_G)$ consisting of a variety $\mathcal{N}_{F/K}(G) \in \mathbf{Var}_K$ and a morphism

$$p_G : \mathcal{N}_{F/K}(G) \times_{\mathrm{spec}\,K} \mathrm{spec}\,F \longrightarrow G$$

over F such that, for all $V' \in \mathbf{Var}_K$ and all morphisms

$$v : V = V' \times_{\operatorname{spec} K} \operatorname{spec} F \longrightarrow G$$

over F, there is an unique morphism $v_{\mathcal{N}} : V' \longrightarrow \mathcal{N}_{F/K}(G)$ with the property that

$$v = p_G \circ \big(v_{\mathcal{N}} \times_{\operatorname{spec} K} \operatorname{spec} F\big).$$

That is, we have a commutative diagram in $\mathbf{Var}_F$

$$\begin{array}{ccc} V & \xrightarrow{v_{\mathcal{N}} \times_{\operatorname{spec} K} \operatorname{spec} F} & \mathcal{N}_{F/K}(G) \times_{\operatorname{spec} K} \operatorname{spec} F \\ & \searrow^{v} & \downarrow{\scriptstyle p_G} \\ & & G \end{array}$$

It follows from the definition that $\mathcal{N}_{F/K}(G)$ is unique up to isomorphism over K. To formulate an alternative definition of the Weil restriction, assume F/K is an extension such that Weil restrictions with respect to F/K exist for every variety $G \in \mathbf{Var}_F$ (as is the case for Galois extensions). Consider then the bi-functors

$$\mathcal{F}_1 : \mathbf{Var}_K \times \mathbf{Var}_F \longrightarrow \mathbf{Sets}, \mathcal{F}_1(V', G) = \mathbf{Mor}_K\big(V', \mathcal{N}_{F/K}(G)\big)$$

and

$$\mathcal{F}_2 : \mathbf{Var}_K \times \mathbf{Var}_F \longrightarrow \mathbf{Sets}, \mathcal{F}_2(V', G) = \mathbf{Mor}_F\big(V' \times_{\operatorname{spec} K} \operatorname{spec} F, G\big)$$

The collection $\{\mathcal{N}_{F/K}(G), p_G\}_{G \in \mathbf{Var}_F}$ defines a pair $\big(\mathcal{N}_{F/K}, \mathcal{F}_{F/K}\big)$ consisting of a covariant functor

$$\mathcal{N}_{F/K} : \mathbf{Var}_F \longrightarrow \mathbf{Var}_K$$

and an equivalence $\mathcal{F}_{F/K} : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ defined by $\mathcal{F}_{F/K}(v) = v_{\mathcal{N}}$. The Weil restriction is thus the left adjoint of '$- \times_{\operatorname{spec} K} \operatorname{spec} F$'.

3.3 Existence and basic properties

If F/K is a finite Galois extension then we have the following existence result.

Lemma 3.3.1. *For all varieties $G \in \mathbf{Var}_F$, a Weil restriction $(\mathcal{N}_{F/K}(G), p_G)$ exists such that*

$$\mathcal{N}_{F/K}(G) \times_{spec K} spec F = \prod_{h \in Gal(F|K)} G^h$$

and with the property that

$$p_G : \mathcal{N}_{F/K}(G) \times_{spec K} spec F \longrightarrow G$$

is the projection to G. For a $\tau \in Gal(F|K)$ the Galois action on $\mathcal{N}_{F/K}(G) \times_{spec K} spec F$ is given by

$$\tau\big(\prod_h \{\xi_h\}\big) = \prod_{\tau h} \{\big(\xi_h^\tau\big)_{\tau h}\}.$$

Proof. For $h, \tau \in Gal(F|K)$ the natural map $h \circ \tau^{-1} : G^h \longrightarrow G^\tau$ is a morphism of schemes. Given a variety $G \in \mathbf{Var}_F$ on the product over $\operatorname{spec} F$

$$\mathcal{N}_F = \prod_{h \in Gal(F|K)} G^h,$$

a descent datum $\{u_h : \mathcal{N}_F \longrightarrow \mathcal{N}_F; h \in Gal(F|K)\}$ is defined by requiring that

$$pr_\tau \circ u_h = \tau^{-1} h \tau^{-1} \circ pr_{\tau^{-1}h}$$

for all $h \in Gal(F|K)$. $\mathcal{N} \in \mathbf{Var}_K$ is then the variety associated to this descent datum, so that $\mathcal{N} \times_{\operatorname{spec} K} \operatorname{spec} F = \mathcal{N}_F$. It was then shown in Weil [5] that $\mathcal{N}$ together with the projection $p_G = pr_{id}$ is a Weil restriction. The idea of the proof is the following: A morphism

$$v : V = V' \times_{\operatorname{spec} K} \operatorname{spec} F \longrightarrow G$$

implies a morphism

$$v'_{\mathcal{N}} = \prod_h v^h : V \longrightarrow \prod_h G^h,$$

such that if $\xi \in V^{Gal(F|K)}$ then

$$v'_{\mathcal{N}}(\xi') \in \left\{ \prod_{h \in Gal(F|K)} \{\xi^h\}; \xi \in G(F) \right\}.$$

It follows that $v'_{\mathcal{N}}$ is defined over K, that is, $v'_{\mathcal{N}} = v_{\mathcal{N}} \times_{\operatorname{spec} K} \operatorname{spec} F$ for a morphism $v_{\mathcal{N}}$ between K-varieties. This yields the lemma.

□

Consider a Weil restriction $\mathcal{N} = \mathcal{N}_{F/K}(G)$, where $G \in \mathbf{Var}_F$ and F/K is Galois. Lemma 3.3.1 implies that we have set-theoretical identifications

$$\mathcal{N} = \left\{ \prod_{h \in Gal(F|K)} \{\xi^h\}; \xi \in G \right\}.$$

and

$$\mathcal{N}(K) = \left\{ \prod_{h \in Gal(F|K)} \{\xi^h\}; \xi \in G(F) \right\}. \tag{3.3.1}$$

The following lemma summarizes properties of Weil restrictions in case of its existence.

Lemma 3.3.2. *Let $G, G_1, G_2 \in \mathbf{Var}_F$ be such that their Weil restrictions exist.*

1. *If F/K is a Galois extension then, for all $G \in \mathbf{Var}_F$ and all $h \in Gal(F|K)$, we have a natural isomorphism $\mathcal{N}_{F/K}(G) \backsimeq \mathcal{N}_{F/K}(G^h)$ and $u_{G^h} = (u_G)^h$.*
2. *If F/K is a Galois extension then $dim\, \mathcal{N}_{F/K}(G) = [F : K] \dim G$.*

3. *$\mathcal{N}_{F/K}$ commutes with products, that is, for all $G_1, G_2 \in \mathbf{Var}_F$ there is a natural isomorphism*

$$\mathcal{N}_{F/K}(G_1 \times_{spec\,F} G_2) \backsimeq \mathcal{N}_{F/K}(G_1) \times_{spec\,K} \mathcal{N}_{F/K}(G_2).$$

4. *If F/K is a Galois extension and if K_o is a subfield of an extension of F such that $[FK_o : KK_o] = [F : K]$ then then there is a natural isomorphism*

$$\mathcal{N}_{F/K}(G) \times_{spec\,K} \; spec\,FK_o \backsimeq \mathcal{N}_{FK_oK/K_o}(G_o)$$

where $G_o = G \times_{spec\,F} spec\,FK_o$.

Proof. To prove Statement 1., let $v : V \longrightarrow G$ be a morphism. v factorizes as

$$V \longrightarrow \mathcal{N}_{F/K}(G) \times_{\text{spec}\,K} \text{ spec}\,F \xrightarrow{p_G} G.$$

Since V is defined over K, we receive a morphism $v^h : V = V^h \longrightarrow G^h$ with a splitting

$$V^h \longrightarrow \big(\mathcal{N}_{F/K}(G) \times_{\text{spec}\,K} \text{ spec}\,F\big)^h \xrightarrow{(p_G)^h} G^h.$$

Noting that

$$\big(\mathcal{N}_{F/K}(G) \times_{\text{spec}\,K} \text{ spec}\,F\big)^h = \mathcal{N}_{F/K}(G) \times_{\text{spec}\,K} \text{ spec}\,F,$$

we get Statement 1. Statement 2. follows from the previous lemma. Assertion 3. follows from the universal properties of products. For 4. we note that $\mathcal{N}_{FK_o/KK_o}(G_K)$ arises from an analogous descent datum

$$\{u_h : \mathcal{N}_{KF} \longrightarrow \mathcal{N}_{KF}; h \in Gal(FK_o|KK_o)\}.$$

Since we have a canonical isomorphism $Gal(FK_o|KK_o) \backsimeq Gal(F|K)$, the latter is isomorphic to the descent datum

$$\{u_h \times_{\text{spec}\,F} \text{ spec}\,KF : \mathcal{N}_F \times_{\text{spec}\,F} \text{ spec}\,KF \longrightarrow \mathcal{N}_F \times_{\text{spec}\,F} \text{ spec}\,KF; h \in Gal(F|K)\},$$

which defines $\mathcal{N}_{F/K}(G) \times_{\text{spec}\,K_o} \text{ spec}\,K_oK$. We receive Statement 4. □

In the next lemma we consider a commutative group variety G over F.

Lemma 3.3.3. *Let $G \in \mathbf{Var}_F$ be a commutative group variety such that their Weil restrictions exist.*

1. *$\mathcal{N}_{F/K}(G)$ is a commutative group variety, and, for all $V_o \in \mathbf{Var}_K$,*

$$\mathcal{N}_{F/K} : \mathbf{Hom}_F\big(V, G\big) \longrightarrow \mathbf{Hom}_F\big(V_o, \mathcal{N}_{F/K}(G)\big)$$

is a homomorphism of commutative groups.

2. *If F/K is a Galois extension then we have a natural isomorphism*

$$Lie\,\mathcal{N}_{F/K}(G) \backsimeq \mathcal{N}_{F/K}\left(Lie\,G\right).$$

In particular, there is a natural identification

$$Lie\,\mathcal{N}_{F/K}(G) = \left\{\prod_h v^h; v \in Lie\,G\right\}.$$

Proof. To prove Statement 1., let μ and i denote the multiplication and the inversion morphism respectively. Statement 3. in the previous lemma implies that $\mu_{\mathcal{N}}$ and $i_{\mathcal{N}}$ define a multiplication and an inversion on the Weil restriction, so that an algebraic group structure is defined. Finally, to show the second assertion, recall that $(Lie\, G)^h \backsimeq Lie\, G^h$ by Subsect. 3.1.4 and let

$$\mathfrak{V} = \left\{ \prod_h v^h ; v \in Lie\, G \right\}.$$

Let $U \subset G$ is an affine open set. Then $K[\mathcal{N}_{F/K}(U)]$ is spanned by functions

$$\sum_{\tau \in Gal(F|K)} (\otimes_h \mathbf{G}_h^h)^\tau$$

where $\mathbf{G}_h \in F[U]$. Using this, one calculates that always $\mathfrak{V} K[\mathcal{N}_{F/K}(U)] \subset K[\mathcal{N}_{F/K}(U)]$, so that $\mathfrak{V} \subset Lie\, \mathcal{N}_{F/K}(G)$. Moreover, $\dim_F \mathfrak{V} = \dim\, G$, so that

$$\dim_K \mathfrak{V} = [F : K] \dim\, G = \dim\, \mathcal{N}_{F/K}(G) = \dim_K Lie\, \mathcal{N}_{F/K}(G).$$

This implies Statement 2.

□

3.4 Extension to exact sequences of commutative group varieties

We have the following lemma.

Lemma 3.4.1. *Let F/K be a finite Galois extension of fields. Then the functor $\mathcal{N}_{F/K}$ extends to the category of exact sequences of commutative group varieties over F. That is, to an exact sequence $L \xrightarrow{i} G \xrightarrow{\pi} A$ of commutative groups varieties over F there is associated an exact sequence*

$$\mathcal{N}_{F/K}(L) \xrightarrow{i_{\mathcal{N}}} \mathcal{N}_{F/K}(G) \xrightarrow{\pi_{\mathcal{N}}} \mathcal{N}_{F/K}(A)$$

of commutative groups varieties over K in a functorial manner.

Proof. For each $h \in Gal(F|K)$ we get an exact sequence

$$L^h \xrightarrow{i^h} G^h \xrightarrow{\pi^h} A^h.$$

Taking direct products over spec F, we receive an exact sequence

$$\prod_{h \in Gal(F|K)} L^h \xrightarrow{i_{\mathcal{N}}} \prod_{h \in Gal(F|K)} G^h \xrightarrow{\pi_{\mathcal{N}}} \prod_{h \in Gal(F|K)} A^h.$$

It is readily verified that the arising descent data with respect to F/K are compatible with $i_{\mathcal{N}}$ and $\pi_{\mathcal{N}}$ respectively. This implies the lemma. □

3.5 Subvarieties of Weil restrictions

Statement 1. in Lemma 3.3.1 implies the following **characterization of subvarieties of Weil restrictions:**

Let F/K be a field extension, let $G \in \mathbf{Var}_F$ be a variety with a Weil restriction $(\mathcal{N}_{F/K}(G), p_G)$. Then a variety $V' \in \mathbf{Var}_K$ is isomorphic to a subvariety of $\mathcal{N}_{F/K}(G)$ if and only if there exists a morphism $v : V \longrightarrow G$ in $\mathbf{Var}_F$ with the property that whenever $w' : V' \longrightarrow W'$ is a surjective morphism in $\mathbf{Var}_K$ such that v factors through $w = w' \times_{spec\,K} spec\,F$ then w' is an isomorphism.

In two particular cases we will make this observation more precise.

3.5.1 Algebraic subgroups of Weil restrictions of simple abelian varieties

In the remainder of the sequel we assume that $[F : K] = 2$ and let $h \in Gal(F|K)$ be the element generating the Galois group. For two simple abelian varieties A_1, A_2 over F we let $\mathbf{Hom}_F(A_1, A_2)$ be the group of isogenies over F. Statement (3.3.1) and Statement 5. in Lemma 3.3.2 imply

Lemma 3.5.1. *Let A be a simple abelian variety over F and let A' be a proper abelian subvariety of $\mathcal{N} = \mathcal{N}_{F/K}(A)$ with positive dimension. Then A' is a simple abelian variety over K such that $\mathbf{Hom}_F\big(A' \times_{spec\,K} spec\,F, A\big) \neq 0$ and conjugation with h induces an isomorphisms of groups*

$$\mu_h : \mathbf{Hom}_F(A' \times_{spec\,K} spec\,F, A) \longrightarrow \mathbf{Hom}_F(A' \times_{spec\,K} spec\,F, A^h), \mu_h(v) = v^h.$$

Proof. Assume a proper abelian subvariety $A' \subset \mathcal{N}$ over K. Let

$$\mu = p_{A|V} : V = A' \times_{\mathrm{spec}\,K} \mathrm{spec}\,F \longrightarrow A$$

be the restriction. Then $\mu = 0$ precisely when $\mu^h = 0$. Hence $\mu \neq 0$. It follows that μ and μ^h are surjective morphisms. On the other hand, if $V \cap \big(\{e_A\} \times A^h\big)$ has positive dimension then $A^h \subset V$ and $A \subset V$. Since the latter is impossible, we find that $V \cap \big(\{e_A\} \times A^h\big)$ is finite. Hence, μ is an isogeny, and so is μ^h. The assertion is now easy to see.

□

It follows that $\mathcal{N}$ is simple unless $\mathbf{Hom}_F(A, A^h) \neq 0$. On the other hand, there are examples where A and A^h are isogenous over F but $\mathcal{N}_{F/K}(A)$ is simple (cf.[3]). We have the following criterion; it was suggested to us by Wüstholz.

Lemma 3.5.2. *Let A be a simple abelian variety over F. If the Weil restriction $\mathcal{N} = \mathcal{N}_{F/K}(A)$ is not simple (over K) then there exists an isogeny $v \in \mathbf{Hom}_F(A, A^h)$ such that* $\deg v = k^{2\dim A}$ *for some $k \in \mathbb{N}$.*

Proof. Assume a proper abelian subvariety $A' \subset \mathcal{N}$ over K and let μ be as in the previous proof. There exists a dual isogeny $\mu^* : A \longrightarrow V$ with the property that $\mu \circ \mu^* = [k]_A$ for some $k \in \mathbb{N}$. Moreover, we have the conjugate isogeny

$$\mu^h : V^h = V \longrightarrow A^h$$

and $\ker \mu^h = \big(\ker \mu\big)^h$. Hence,

$$k^{2 \dim A} = \deg \big(\mu \circ \mu^*\big) = (\deg \mu)(\deg \mu^*) = (\deg \mu^h)(\deg \mu^*).$$

It follows that $v = \mu^h \circ \mu^*$ is an isogeny satisfying the assertion. □

In the special case where $\mathbf{End}_F(A) \backsimeq \mathbb{Z}$ the last observation can be improved to

Lemma 3.5.3. *Let A be a simple abelian variety over F such that $\mathbf{End}_F(A) \backsimeq \mathbb{Z}$. Then we have three equivalent assertions.*

1. *The Weil restriction $\mathcal{N} = \mathcal{N}_{F/K}(A)$ is not simple.*
2. *There exists an isogeny $v : A \longrightarrow A^h$ with the property that $\deg v = k^{2 \dim A}$ for some $k \in \mathbb{N}$.*
3. *$\mathbf{Hom}_F(A, A^h) \neq 0$ and for all $v \in \mathbf{Hom}_F(A, A^h)$ we have $\deg v = k^{2 \dim A}$ for some $k \in \mathbb{N}$.*

Proof. That Statement 1. implies Statement 2. was shown in the previous lemma. Assume Statement 2. holds and let $v : A \longrightarrow A^h$ be an isogeny with the property $\deg v = k^{2 \dim A}$. We have the conjugate isogeny

$$v^h : A^h \longrightarrow A.$$

Since $\mathbf{End}_F(A) \backsimeq \mathbf{End}_F\big(A^h\big) \backsimeq \mathbb{Z}$, we have $v^h \circ v = [m]_A$ and $v \circ v^h = [m]_{A^h}$ for $m = (\deg v)^{2/2 \dim A} = k^2 \in \mathbb{N}$. We will show that 1. holds true. We consider the isogeny

$$w : \mathcal{N} \times_{\operatorname{spec} K} \operatorname{spec} F \longrightarrow \mathcal{N} \times_{\operatorname{spec} K} \operatorname{spec} F$$

defined by $w(\xi_1, \xi_2) = \big(v^h(\xi_2), v(\xi_1)\big)$ for $(\xi_1, \xi_2) \in A \times A^h = \mathcal{N} \times_{\operatorname{spec} K} \operatorname{spec} F$. Recall (3.1.1) and the identification of sets preceding (3.3.1), and let $(\xi, \xi^h) \in \mathcal{N}$. We calculate

$$w\big(\xi, \xi^h\big) = \big(v^h(\xi^h), v(\xi)\big) = \Big(\big(v(\xi)\big)^h, v(\xi)\Big) \in \mathcal{N}.$$

Hence, w respects the K-structure. It follows that w is actually defined over K, that is, $w \in \mathbf{End}_K(\mathcal{N})$. Moreover, the definition of w implies that $w^2 = [k^2]_{\mathcal{N}}$. Thus,

$$\big(w - [k]_{\mathcal{N}}\big) \circ \big(w + [k]_{\mathcal{N}}\big) = [0]_{\mathcal{N}}.$$

Since obviously $w \neq \big[\pm k\big]_{\mathcal{N}}$, it results that $\mathbf{End}_K(\mathcal{N})$ is no division ring. Therefore, $\mathcal{N}$ is not simple. Hence, Statement 1. follows.

Next, assume Statement 2. again. Since $\mathbf{Hom}_F\big(A, A^h\big)$ is isomorphic to a submodule of $\mathbf{End}_F(A)$, it follows that $\mathbf{Hom}_F\big(A, A^h\big) = \mathbb{Z}v_{\min}$ for some

$$v_{min} \in \mathbf{Hom}_F\big(A, A^h\big).$$

Consequently, if v is as in Statement 2., then $v = [m]_{A^h} \circ v_{\min}$, so that

$$\exists k_1 \in \mathbb{N} : \deg\, v = k_1^{2\dim A} \Longleftrightarrow \exists k_2 \in \mathbb{N} : \deg\, v_{\min} = k_2^{2\dim A}.$$

Thus, 2. implies 3. Finally, that 3. implies 2. is obvious. Everything is proved. □

3.5.2 Algebraic subgroups of Weil restrictions of extensions by linear groups

We again assume that $[F : K] = 2$. Slightly easier as in the previous case of Weil restrictions of simple abelian varieties is the classification of algebraic subgroups of $\mathcal{N}_{F/K}(G)$ where G is an extension of an elliptic curve E by a linear group L with $\dim\, L = 1$. To this end, recall that these extensions are classified by $Ext^1(L, E)$, such that the following assertions are equivalent.

- π defines a torsion element in $Ext^1(E, L)$.
- There exists an elliptic curve E' and a nontrivial homomorphism $v : E' \longrightarrow G$ over $\overline{F}$.
- There exists a nontrivial homomorphism $v : G \longrightarrow L$ over $\overline{F}$.
- π is isotrivial (over $\overline{F}$), that is, it becomes a trivial fibration after an etale base extension.

With this in mind, we show

Lemma 3.5.4. *Let E be an elliptic curve over F and consider an extension $\pi : G \longrightarrow E$ over F by a linear group L with* $\dim\, L = 1$. *Assume that π is not isotrivial. Let $H \subset \mathcal{N} = \mathcal{N}_{F/K}(G)$ be an algebraic subgroup over K with dimension* 3. *Then $E' = \pi_{\mathcal{N}}(H) \subset \mathcal{N}_{F/K}(E)$ is an elliptic curve over K.*

Proof. Over $\operatorname{spec} F$ we have two projections $p_G : H_F \longrightarrow G$ and $p_{G^h} : H_F \longrightarrow G^h$ such that $ker\, p_G = H_F \cap \big(\{e_G\} \times G^h\big)$ and $ker\, p_{G^h} = H_F \cap \big(G \times \{e_{G^h}\}\big)$. We treat all possibilities:

1. $ker\, p_G$ and $ker\, p_{G^h}$ are linear. Then $\mathcal{N}_{F/K}(L) \subset H$, so that E' has dimension 1.
2. $ker\, p_G$ is linear, but $ker\, p_{G^h}$ is not. Since $ker\, p_{G^h} \neq 0$ for dimension reasons and as G^h contains no elliptic curves, we must have $G \times \{e_{G^h}\} \subset H_F$. Thus, $H_F = G \times L^h$, so that again $\mathcal{N}_{F/K}(L) \subset H$.
3. $ker\, p_{G^h}$ is linear, but $ker\, p_G$ is not. This yields the result as in 2.

We see that all possibilities imply the assertion. □

3.6 Weil restrictions to real subfields and real-analytic one-parameter homomorphisms

Let F be a subfield of $\mathbb{C}$ such that if $c \in F$ then also $Re\, c, Im\, c \in F$. Set $K = F \cap \mathbb{R}$. It follows that $F\mathbb{R} = \mathbb{C}$ and $K\mathbb{R} = \mathbb{R}$, and each element in $Gal(F|K)$ extends to a real-analytic map in $Gal(\mathbb{C}|\mathbb{R})$. Let $G \in \mathbf{Var}_F$ be a commutative group variety. We then have a natural identification

$$G^h \times_{\mathrm{spec}\, F} \mathrm{spec}\, \mathbb{C} = \big(G \times_{\mathrm{spec} F} \mathrm{spec}\mathbb{C}\big)^h.$$

This in turn implies that the isomorphism of K-algebras

$$Lie(h) : Lie\, G \longrightarrow Lie\, G^h$$

extends to an isomorphism

$$Lie(h) : (Lie\, G)(\mathbb{C}) \longrightarrow \big(Lie\, G^h\big)(\mathbb{C})$$

of $\mathbb{R}$-algebras.

In accordance with our main results we consider a real-analytic homomorphism $\psi : \mathbb{G}_a(\mathbb{R}) \longrightarrow G(\mathbb{C})$.

Lemma 3.6.1. *Let G be a commutative group variety over F. Then there exists a real-analytic homomorphism $\psi_{\mathcal{N}} : \mathbb{R} \longrightarrow \mathcal{N}(\mathbb{C})$ such that:*

1. *$p_G \circ \mathcal{N}(\psi) = \psi$ and, in particular, $\psi^{-1}\big(G(F)\big) \subset \psi_{\mathcal{N}}^{-1}\big(\mathcal{N}(K)\big)$.*
2. *If ψ is defined over F then $\psi_{\mathcal{N}}$ is defined over K.*

Before we proceed to a proof of the lemma, let us consider a simple example similar to the one from Subsection 3.1.6. Let $\Lambda \subset \mathbb{C}$ be a lattice with algebraic invariants g_2, g_3 and let

$$E : Y^2 = 4X^3 - g_2 X - g_3$$

be the associated elliptic curve. As in Subsection 3.1.6 we have an identification $Lie\, E = \overline{\mathbb{Q}}$ stemming from universal coverings and the complex conjugate of E has a Weierstraß form

$$E^h : Y^2 = 4X^3 - h(g_2)X - h(g_3)$$

with corresponding lattice $h(\Lambda)$. We have $\mathbb{G}_a(\mathbb{R}) = \mathbb{R}$ and $exp_{\mathbb{G}_a}$ is an isomorphism. Let $\frac{d}{dt} = exp_{\mathbb{G}_a}^{-1}(1)$ and consider a real-analytic homomorphism $\psi : \mathbb{G}_a(\mathbb{R}) \longrightarrow E(\mathbb{C})$ such that $\psi_* \left(\frac{d}{dt}\right) \in Lie\, E$. Let $f(z) = \wp_\Lambda(z)$ and $g(z) = \wp_{h(\Lambda)}(z)$. Then

$$\psi(t) = [f(vt) : f'(vt) : 1]$$

for some $v \in \overline{\mathbb{Q}}$ and

$$\psi^h(t) = \big[h\big(f(vt)\big) : h\big(f'(vt)\big) : 1\big] .$$

We use the fundamental property that h is a real-analytic map and receive

$$\psi^h(t) = \big[g\big(h(v)t\big) : g'\big(h(v)t\big) : 1\big] .$$

Therefore, the homomorphism

$$\psi_{\mathcal{N}} : \mathbb{G}_a(\mathbb{R}) \longrightarrow \big(E \times E^h\big)(\mathbb{C}), \psi_{\mathcal{N}} = \psi \times \psi^h$$

has the property that

$$im\, \psi_{\mathcal{N}} \subset \{(\xi, \xi^h)\,; \xi \in E(\mathbb{C})\}$$

and

$$(\psi_{\mathcal{N}})_* \left(\frac{d}{dt}\right) \in \{(v, v^h); v \in Lie\, E\}.$$

Hence $\psi_{\mathcal{N}}$ fulfills the assertion of the lemma for $G = E$ and $F = \overline{\mathbb{Q}}$. The general proof goes along the same lines.

Proof. Recall that $\mathcal{N}_{F/K}(G) \times_{\mathrm{spec}\, K} \mathrm{spec}\, F = G \times G^h$ and that $h : G(\mathbb{C}) \longrightarrow G^h(\mathbb{C})$ is a real-analytic map. Statement (3.1.1) and Lemma 3.3.1 imply Statement 1. For the proof of Statement 2. we fix a basis $\Delta_1, ..., \Delta_g \in Lie\, G$. The basis induces an isomorphism $\delta : (Lie\, G)(\mathbb{C}) \longrightarrow \mathbb{C}^g(\mathbb{C})$ defined over F which sends Δ_i to the i^{th} standard vector $\mathbf{e}_i$, and an isomorphism $\delta^h : \big(Lie\, G^h\big)(\mathbb{C}) \longrightarrow \mathbb{C}^g$ defined over F which sends Δ_i^h to the i^{th} standard vector $\mathbf{e}_i$. One verifies with the definition from Subsubsection 3.1.4 that

$$\delta^h \circ Lie(h) \circ \delta^{-1} = h. \tag{3.6.1}$$

Choose an embedding $u : G \longrightarrow \mathbf{P}^N$ over F and let $\underline{z} = (z_1, ..., z_g)$ be the linear coordinates of $(Lie\, G)(\mathbb{C})$ corresponding to $\Delta_1, ..., \Delta_g$. For a $v_o \in (Lie\, G)(\mathbb{C})$ the map $u \circ exp_G(v + v_o)$ is then locally represented by holomorphic maps

$$f_{jv_o}(\underline{z}) = \sum_I a_{Ijv_o} \underline{z}^I \qquad (j = 0, ..., N)$$

such that if $v_o \in (Lie\, F)(\mathbb{C})$ then $\partial^I f_j(v_o) = a_{Ijv_o} \in F$ for all I and all j. It follows from (3.6.1) that the tuples of functions

$$f^h_{jv_o}(\underline{z}) = \sum_I h(a_{Ijv_o}) \underline{z}^I \qquad (j = 0, ..., N).$$

represent $u^h \circ exp_{G^h}$. We thus receive a commutative diagram

$$\begin{array}{ccc} \mathbb{C}^g & \xrightarrow{\quad h \quad} & \mathbb{C}^g \\ \Big\downarrow {\scriptstyle exp_G \circ \delta^{-1}} & & \Big\downarrow {\scriptstyle exp_{G^h} \circ \left(\delta^h\right)^{-1}} \\ G(\mathbb{C}) & \xrightarrow{\quad h \quad} & G^h(\mathbb{C}) \end{array}$$

Assume now that ψ is defined over F. Then ψ admits a representation $\psi(t) = u \circ exp_G(\underline{v}t)$ for a $\underline{v} \in F^g$ and $t \in \mathbb{R}$. We have then, for $t_o \in \mathbb{Q}$, local representations

$$\psi^h(t_o + t) = (\psi(t_o + t))^h = \left[f^h_{0(t_o\underline{v})}\big(t\underline{v}^h\big) : ... : f^h_{N(t_o\underline{v})}\big(t\underline{v}^h\big)\right].$$

Let $\frac{d}{dt} \in Lie\, \mathbb{G}_a \setminus \{0\}$. With (3.6.1) we infer that if

$$\psi_* \left(\frac{d}{dt}\right) = v \in Lie\, G$$

then

$$(\psi^h)_* \left(\frac{d}{dt}\right) = v^h \in Lie\, G.$$

Using Statement 2. from Lemma 3.3.3, we finally receive

$$(\psi_{\mathcal{N}})_* \left(\frac{d}{dt}\right) \in \left\{ \prod_h v^h; v \in (Lie\, G)(F) \right\} = Lie\, \mathcal{N}.$$

□

3.7 Complex twins

The topic of this subsection has no influence on the proof of our theorems and is therefore only sketched. Let F be an algebraically closed subfield of $\mathbb{C}$ and write $K = F \cap \mathbb{R}$. We assume that $[F : K] = 2$. Let A' be an abelian variety over K and let $A = A' \times_{\operatorname{spec} K} \operatorname{spec} F'$. Then $A = A^h$ for the complex conjugation h. As seen in the proof of the previous lemma, there exists an uniformization $A(\mathbb{C}) = \mathbb{C}^g/\Lambda$ such that $K^g = Lie\, A'$ canonically and $\Lambda = h(\Lambda)$. We note that the set $A(i\mathbb{R}) = i\mathbb{R}^g + \Lambda/\Lambda$ is the fix locus of the real-analytic map $h \circ [-1]_A$. Let $\Lambda_1 = \Lambda \cap \mathbb{R}^g$ and $\Lambda_2 = \Lambda \cap i\mathbb{R}^g$. Then $\Lambda_1 + \Lambda_2$ is a sublattice of Λ. We let $\Lambda_{[i]} = i\Lambda_1 + i\Lambda_2$. $\Lambda_{[i]}$ is stable with respect to complex conjugation and defines an abelian variety $A_{[i]}$ which is isogenous over F to A. The action of h on $(\mathbb{C}^g, \Lambda_{[i]})$ descents to an action on $A_{[i]}$ and defines a K-structure on $A_{[i]}$; we denote by $A'_{[i]} \in \mathbf{Var}_K$ the respective abelian variety. We call it **the complex twin of $\mathbf{A}'$**. The following observation is left to the reader:

Lemma 3.7.1. *Let F be an algebraically closed subfield of $\mathbb{C}$ such that $[F : K] = 2$ for $K = F \cap \mathbb{R}$. Consider a simple abelian variety $A' \in \mathbf{Var}_K$. Then we have the following equivalent assertions.*

1. *The twins A' and $A'_{[i]}$ are isogenous over K.*
2. *$\mathcal{N}_{F/K}(A)$ is isogenous to a power $B \times B$ of an abelian variety $B \in \mathbf{Var}_K$.*
3. *$\mathbf{End}_F(A)$ contains an element μ such that, for each $\xi \in \mu\big(A(K)\big)$, we have $\xi^h = -\xi$ and with the property that $\mu^2 = -[k]_A$ for some $k \in \mathbb{N}$.*
4. *$\mathbf{End}_F(A) \backslash \mathbf{End}_K(A')$ contains an element μ such that $\mu^2 = -[k]_A$ for some $k \in \mathbb{N}$.*

Chapter 4

Proof of the main theorems

4.1 Proof of Thm. 1.1

4.1.1 Two auxiliary lemmas

Let $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ for some $\tau \in \mathbb{H}$ and let E_τ be the associated elliptic curve with exponential map exp_{E_τ} induced by $\wp_\tau$ and its derivative. We start by proving

Lemma 4.1.1. *The following assertions are equivalent.*

1. *There is an elliptic curve E' over $\mathbb{R}$ and an isogeny $\nu : E' \times_{spec\mathbb{R}} spec\,\mathbb{C} \longrightarrow E_\tau$ of elliptic curves with the property that $exp_{E_\tau}(\mathbb{R}) \subset \nu\big(E'(\mathbb{R})\big)$.*
2. *$Re\,\tau \in \mathbb{Q}$.*
3. *$Re\,\big(\Lambda_\tau : \mathbb{Z}\big) \subset \mathbb{Q}$.*
4. *$\Lambda_\tau \cap \Lambda_\tau^h$ is a lattice in $\mathbb{C}$.*

Proof. Let $h \in Gal(\mathbb{C}|\mathbb{R})$ be the complex conjugation. We first show that Statement 2. implies Statement 1. If $Re\,\tau \in \mathbb{Q}$ then Λ_τ contains a sublattice Λ with $\Lambda = h(\Lambda)$. Hence, for all $z \in \mathbb{R} \setminus \Lambda$,

$$\wp_\Lambda(z) = \wp_{h(\Lambda)}(z) = h\big(\wp_\Lambda(z)\big) \in \mathbb{R} \text{ and } \wp'_\Lambda(z) \in \mathbb{R}.$$

This has two consequences: Firstly, Λ has real invariants g_2, g_3 and corresponds to an elliptic curve

$$E' : Y^2 = 4X^3 - g_2 X - g_3$$

over $\mathbb{R}$. Secondly, since $Lie\,E'$ is mapped to the component of unity C' of $E'(\mathbb{R})$, we have an unique commutative diagram of universal coverings

$$\begin{array}{ccc} Lie\,E' & \longrightarrow & \mathbb{R} \\ & \searrow^{exp_{E'}} & \big\downarrow {\scriptstyle [\wp_\Lambda : \wp'_\Lambda : 1]} \\ & & C' \end{array}$$

Therefore, 2. implies 1.

To show that Statement 1. implies Statement 2., note that $C = exp_{E_\tau}(\mathbb{R})$ is a connected real-analytic submanifold of $E(\mathbb{C})$. It thus follows that the unity component of $E'(\mathbb{R})$ is mapped to C. Assume that E' is in Weierstraß form corresponding to a lattice Λ' with $\Lambda' = h(\Lambda')$. As above we have an identification $Lie\, E' = \mathbb{R}$, so that v_* is the multiplication with a number $a \in \mathbb{R}$. Hence, $a\Lambda' \subset \Lambda_\tau$ satisfies $a\Lambda' = h(a\Lambda') \subset \Lambda_{h(\tau)}$. There are thus unique $(a_1, a_2), (b_1, b_2) \in \mathbb{Z}^2$ such that

$$a_1 + a_2\tau = a\tau' = b_1 + b_2 h(\tau).$$

Looking at imaginary parts, we find that $a_2 = -b_2 \neq 0$. Hence, $2a_2 Re\,\tau = b_1 - b_2$, so that $Re\,\tau \in \mathbb{Q}$. Statement 2. follows.

Finally, the equivalences "2. $\Longleftrightarrow$ 3." and "3. $\Longleftrightarrow$ 4." are obvious.

□

The next lemma states that if a complex elliptic curve admits a model over $\mathbb{R}$ and a model over $\mathbb{Q}$ then it admits a model over $K = \overline{\mathbb{Q}} \cap \mathbb{R}$. For a precise statement let E_1 be an elliptic curve over $\overline{\mathbb{Q}}$ and E_2 an elliptic curve over $\mathbb{R}$.

Lemma 4.1.2. *Consider an isomorphism of complex elliptic curves*

$$v : E_2 \times_{spec\,\mathbb{R}} spec\,\mathbb{C} \longrightarrow E_1 \times_{spec\,\overline{\mathbb{Q}}} spec\,\mathbb{C}.$$

Then there is an elliptic curve E over K and an isomorphism

$$w : E \times_{spec\,K} spec\,\overline{\mathbb{Q}} \longrightarrow E_1$$

such that v factors through the complex isomorphism $w_\mathbb{C} = w \times_{spec\,\overline{\mathbb{Q}}} spec\,\mathbb{C}$ and such that $w_\mathbb{C}^{-1} \circ v$ is defined over $\mathbb{R}$.

Observe that Lemma 4.1.2 is wrong for general varieties.

Proof. Let j be the j-invariant of E_1 and assume that $j \neq 1728$. Then E_1 is isomorphic over $\overline{\mathbb{Q}}$ to a curve with equation $Y^2 = 4X^3 - X - g_3$ where $g_3 \in \overline{\mathbb{Q}}$. Similarly, E_2 is isomorphic over $\mathbb{R}$ to a curve E_3 with equation $Y^2 = 4X^3 - k_2 X - k_3$ and $k_2, k_3 \in \mathbb{R}$. Since $j \neq 1728$, $k_2 \neq 0$.

- Assume $k_2 > 0$ and let $\alpha \in \mathbb{R}$ be such that $\alpha^4 = k_2$. We find that E_2 is isomorphic over $\mathbb{R}$ to a curve with equation $Y^2 = 4X^3 - X - \overline{g}_3$ where $\overline{g}_3 = k_3/\alpha^6$. It follows that $\beta g_3 = \overline{g}_3 \in K$ for some fourth root of unity $\beta \in \mu_4$.
- If $k_2 < 0$ then E_3 is isomorphic to $Y^2 = 4X^3 + X - \overline{g}_3$ over $\mathbb{R}$ where $\overline{g}_3 = k_3/\alpha^6$. It follows that $\beta g_3 = \overline{g}_3 \in K$ for some fourth root of -1.

Since E_3 is isomorphic to E_1 over $\overline{\mathbb{Q}}$, this proves the lemma in the case where $j \neq 1728$. Finally, if $j = 1728$ then it is well known that E admits a model over $\mathbb{Q}$, and we argue *mutatis mutandis* as in the first case.

□

4.1.2 '1. $\iff$ 2'

Let $K = F \cap \mathbb{R} = \overline{\mathbb{Q}} \cap \mathbb{R}$. Since $exp_{\mathbb{G}_a}$ is an isomorphism we can define $\frac{d}{dt} = (exp_{\mathbb{G}_a})^{-1}(1)$. We will identify $(Lie\,\mathbb{G}_a)(\mathbb{R})$ with $\mathbb{R}$ by sending $\frac{d}{dt}$ to 1. The sets $\mathbb{G}_a(\mathbb{C})$ and $\mathbb{G}_a(\mathbb{R})$ are identified with $\mathbb{C}$ and $\mathbb{R}$. Under this identifications $exp_{\mathbb{G}_a}$ becomes the identity map.

In the first part of the proof we show that Statement 1. is equivalent to Statement 2. Assume Statement 1. holds and write $G' = E'$ to keep in mind that G' is an elliptic curve. Let $\psi' = v_{\mathbb{R}} \circ \psi$ and consider $\psi' : \mathbb{R} \longrightarrow E'(\mathbb{R})$. As $E'(\mathbb{R})$ is a compact one-dimensional real-analytic manifold, it follows that ψ' is not injective. Hence, $(\psi')_*(\mathbb{R})$ contains an element $\lambda \in (Lie\,E')(\mathbb{R})$ of the period group Λ' of E'. Hence, $(\psi')_*(s) \in Lie\,E'$ for some $s \in \mathbb{R}^*$. As v_* is represented by multiplication with an algebraic number, we receive that $Lie\,E \cap im\,\psi_* \neq \{0\}$. Moreover, since $im\,\psi_*$ hits a kernel element of exp_E, it follows that $im\,\phi \cap E(F) \neq \{e_E\}$. We obtain Statement 2.
Conversely, assume Statement 2. After a reparametrization the homomorphism ψ is defined over $\overline{\mathbb{Q}}$, and then $\psi_*\left(\frac{d}{dt}\right) \in Lie\,E$. Let $\mathcal{N} = \mathcal{N}_{\overline{\mathbb{Q}}/K}(E)$. Lemma 3.6.1 and our hypotheses imply that

$$(\psi_{\mathcal{N}})_*\left(\frac{d}{dt}\right) \in Lie\,\mathcal{N}$$

and, for some $r \in \mathbb{R}$,

$$\psi_{\mathcal{N}}(r) \in \mathcal{N}(K) \setminus \{e_{\mathcal{N}}\}.$$

Let $\phi_{\mathcal{N}}$ be the complexification of $\psi_{\mathcal{N}}$. That is,

$$\phi_{\mathcal{N}}(z) = exp_{\mathcal{N}}\left(z \cdot (\psi_{\mathcal{N}})_*\left(\frac{d}{dt}\right)\right),$$

and $\phi_{\mathcal{N}|\mathbb{R}} = \psi_{\mathcal{N}}$. By the above

- $\phi_{\mathcal{N}}$ is defined over $\overline{\mathbb{Q}}$ and
- $im\,\phi_{\mathcal{N}} \cap \mathcal{N}(\overline{\mathbb{Q}}) \neq \{e_G\}$.

Wüstholz' analytic subgroup theorem literally implies that $im\,\phi_{\mathcal{N}}$ contains an algebraic subgroup $E' \subset \mathcal{N} \times_{\mathrm{spec}\,K} \mathrm{spec}\,F$ with positive dimension. For dimension reasons we have $E'(\mathbb{C}) = im\,\phi_{\mathcal{N}}$, so that E' is an elliptic curve. Moreover, since

$$\phi_{\mathcal{N}}(\mathbb{Q}r) \subset \mathcal{N}(\overline{\mathbb{Q}}) \cap \mathcal{N}(\mathbb{R}) = \mathcal{N}(K),$$

E' is the closure of infinitely many algebraic points in $\mathcal{N}(K)$. Consequently, E' is defined over K. Letting $w = p_{E|E'}$, $G' = E'$, $G = E$ and $\mathbb{G} = \mathbb{G}_a$, we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{G}(\mathbb{R}) & \longrightarrow & G'(\mathbb{R}) \\ & \searrow^{\psi} & \downarrow w_{\mathbb{R}} \\ & & G(\mathbb{C}) \end{array}$$

The isogeny w implies a dual isogeny v as required for Statement 1.

4.1.3 '1. $\Longleftrightarrow$ 3.'

That Statement 1. implies Statement 3. results directly from Lemma 4.1.1. On the other hand, if Statement 3. holds then one first verifies that $ker\,\psi \neq 0$, so that $I_\psi \neq 0$. It follows from Lemma 4.1.1 and the existence of dual isogenies that ψ descents to $\mathbb{R}$. With Lemma 4.1.2 one then obtains Statement 1.

4.1.4 '1. $\Longleftrightarrow$ 4.'

Assume Statement 1. and let $\mathcal{N} = \mathcal{N}_{F/K}(E)$. Then $\psi_\mathcal{N}(\mathbb{R}) \subset \mathcal{N}(\mathbb{R})$ is contained in a curve E' over K. Hence, each point

$$\left(\xi, \xi^h\right) \in \mathcal{N} \times_{\mathrm{spec}\,K} \mathrm{spec}\,F = E \times E^h, \xi = \psi(r),$$

is contained in an algebraic curve over K, and Statement 4. follows readily. Conversely, suppose Statement 4. Then there is an uncountable set $\mathfrak{S} \subset \mathbb{R}$ such that $\overline{F\left(\xi^h\right)} = \overline{F\left(\xi\right)}$ for $\xi = \psi(r)$ and all $r \in \mathfrak{S}$. There exists, for all $r \in \mathfrak{S}$, a curve $C_r \subset E \times E^h$ over F with the property that $\psi_\mathcal{N}(r) \in C_r$. However, the number of all distinct curves C_r is at most countable. Consequently, there is an uncountable subset $\mathfrak{S}_o \subset \mathfrak{S}$ and a curve $C \subset E \times E^h$ over F with the property that $\psi_\mathcal{N}(\mathfrak{S}_o) \subset C$. It is an easy exercise to check that $\mathfrak{S}_o$ admits a point of culmination r_o such that $\xi_o = \psi_\mathcal{N}(r_o) \in C$. Let U be an affine neighborhood of ξ_o together with a function $f \in F[U]$ defining $C \cap U$. It follows that $f \circ \psi_\mathcal{N} \equiv 0$. Hence, $\psi_\mathcal{N}(\mathbb{R}) \subset C$. This in turn implies that C is an elliptic curve E' which is the algebraic closure of infinitely many points in $\mathcal{N}(K) \cap \psi_\mathcal{N}(\mathbb{R})$. Therefore, E' is defined over K and $\psi_\mathcal{N}(\mathbb{R}) \subset E'(\mathbb{R})$. Letting $v = p_{G|E'}$ we infer Statement 1. Everything is proved.

4.2 Proof of Thm. 1.0.2

We start by proving Statement 1. of Thm. 1.0.2. With notations as in the previous section, let $\phi_\mathcal{N}$ be the complexification of $\psi_\mathcal{N}$. Assume first that

$$rank_\mathbb{Z} im\,\psi \cap E(F) \geq 3.$$

Then it follows from Lemma 3.6.1, the fact that $F = h(F)$ and Thm. 5.0.1 below that the Zariski-closure of $\phi_\mathcal{N}$ in $\mathcal{N} = \mathcal{N}_{F/K}(E)$ is an elliptic curve E' which is defined over K. If on the other hand

$$rank_\mathbb{Z} im\,\Psi \cap \mathbb{G}(F) \geq 3,$$

then Statement 2. in Thm. 5.0.2 implies that $\phi_\mathcal{N}(\mathbb{C})$ is an elliptic curve E' over K. In both cases we define $w = p_{E|E'}$, and infer Statement 1. in Thm. 1.0.2 using dual isogenies.

For the proof of 2. in Thm. 1.0.2 let $\phi_\mathcal{N}$ be as above and let $\Delta_1, \Delta_2 \in Lie\,\mathcal{N}$ be a basis over K. After eventually reparametrizing $\phi_\mathcal{N}$ and eventually renumbering Δ_1, Δ_2, one can arrange that $\left(\phi_\mathcal{N}\right)_* \left(\frac{d}{dt}\right) = \Delta_1 + t\Delta_2$ for some $t \in \mathbb{C}$. Then $\phi_\mathcal{N}$ is defined over $K(t)$ and we infer from Thm. 5.0.1 that $im\,\phi_\mathcal{N}$ is not Zariski-dense in $\mathcal{N}$. The Zariski-closure is thus an elliptic curve E' over K.

4.3 Proof of Thm. 1.0.3

The proof relies on Statement 1. in Thm. 5.0.2. and is left to the reader.

4.4 Proof of Thm. 1.0.4

We start by proving Statement 1. and Rem. 1.0.5. To this end, assume ψ has a non-zero kernel and a Zariski-dense image. Suppose the existence of an additive subgroup $\Sigma \subset \mathbb{R} \cap \psi^{-1}\big(G(F)\big)$ with rank 2 and which contains the kernel of ψ. Observe thereby that Σ lies in $\mathbb{R}$ and therefore cannot be contained in a kernel of a homomorphism to an algebraic group. Let $\mathcal{N} = \mathcal{N}_{F/K}(G)$. We notations as in the previous proofs we then consider

$$\phi_{\mathcal{N}} : \mathbb{C} \longrightarrow \mathcal{N}(\mathbb{C}).$$

The fact that $\psi(\Sigma)$ is infinite together with Thm. 5.0.1 below imply that the Zariski-closure of $\phi_{\mathcal{N}}(\mathbb{C})$ is an algebraic subgroup H of $\mathcal{N}$ over K with dimension ≤ 2. Since ψ has Zariski-dense image, Statement (3.1.1) yields $p_G(H_F) = G$. Thus, $p_{G|H_F}$ is an isogeny over F and H is an extension of the elliptic curve $E' = \pi_{\mathcal{N}}(H)$ over K. We see that ψ and ψ_π descent do K. Next, we observe that ψ has a Zariski-dense image as soon as π is not isotrivial and ψ_π non-constant. Altogether Statement 1. and the remark follow.

Suppose Statement 2. in Thm. 1.0.4 holds. We then choose a subgroup $\Sigma \subset K$ with rank 3 such that $\Psi(\Sigma) \subset \big(\mathbb{G}_a \times G\big)(F)$ and let Φ be the complexification of $id_{\mathbb{G}_a} \times \psi_{\mathcal{N}}$. Thm. 5.0.1 implies that $\overline{\Phi(\mathbb{C})}^{Zar}$ is an algebraic subgroup Y of $\mathbb{G}_a \times \mathcal{N}$ over K with dimension ≤ 4. By assumption the algebraic groups $\mathbb{G}_a$ and $\mathcal{N}$ are disjoint, so that

$$H = Y \cap (\{0\} \times \mathcal{N})$$

has dimension ≤ 3. We infer the assertion from Lemma 3.5.4.

The proof in case of Statement 3. follows the same way from Thm. 5.0.1 and Lemma 3.5.4. Everything is shown.

Chapter 5

Appendix: Two transcendence results

For convenience of the reader, we restate the transcendence results used in the proof of our main theorems. The first is shown in Tubbs [4]. Thm. 5.0.2 follows from [2]. In the formulation, $\mathbb{G} = \mathbb{G}_a^{d_a} \times \mathbb{G}_m^{d_m} \times G$ shall denote a commutative algebraic group over a subfield F of $\mathbb{C}$ with transcendence degree ≤ 1 over $\mathbb{Q}$ and $d_a \in \{0, 1\}$. We let $g = \dim G$ and suppose that $d_a + d_m + g \geq 2$. The symbol δ_{ij} means Kronecker's Delta and $\pi : \mathbb{G} \longrightarrow G$ is the projection to G. If $\phi : \mathbb{G}_a(\mathbb{C}) \longrightarrow \mathbb{G}(\mathbb{C})$ is a complex-analytic homomorphism we define $\phi_\pi = \pi \circ \phi$.

Theorem 5.0.1. *Let $\phi : \mathbb{G}_a(\mathbb{C}) \longrightarrow \mathbb{G}(\mathbb{C})$ be a complex analytic homomorphism which is defined over F. If either*

1. $(d_a + d_m + g - 2) \cdot rank_{\mathbb{Z}}\, \phi^{-1}\big(\mathbb{G}(F)\big) \geq 2g + d_m + \delta_{d_a 1}$, *or*
2. $ker\, \phi \neq 0$ *and* $\big(2(d_a + d_m + g - 1) - 1\big) \cdot rank_{\mathbb{Z}}\, \phi^{-1}\big(\mathbb{G}(F)\big) \geq 2g + d_m$,

then $im\, \phi$ is not Zariski-dense in $\mathbb{G}$.

In the next theorem we let $pr : \mathbb{G} \longrightarrow \mathbb{G}_m^{d_m} \times \mathbb{G}$ be the projection and set $\phi_{pr} = \phi \circ pr$.

Theorem 5.0.2. *Let $\phi : \mathbb{G}_a(\mathbb{C}) \longrightarrow \mathbb{G}(\mathbb{C})$ be a complex-analytic homomorphism which is defined over F. Let $\Sigma \subset \mathbb{G}_a(\mathbb{C})$ be an additive subgroup with $rank_{\mathbb{Z}}\, \Sigma \geq 2$. Suppose $\phi(\Sigma) \subset \mathbb{G}(F)$.*

1. *If $d_a = d_m = 1$, $g \geq 2$ and $\Sigma \cap ker\, \phi_\pi \neq \{0\}$ then the Zariski-closure of $im\, \phi_{pr}$ in $\mathbb{G}_m \times G$ has dimension ≤ 2.*
2. *If $d_a = 1$, $\Sigma \cap \ker\, \phi_\pi \neq 0$ and $rank_{\mathbb{Z}}\, \Sigma \geq 3$ then $im\, \phi_\pi$ is an algebraic subgroup of $\mathbb{G}$ over F.*

Bibliography

[1] Masser, D. W.; Wüstholz, G, *Algebraic independence of values of elliptic functions*, Math. Ann. 276, no. 1, 1-17 (1986)

[2] Momot, A, *Algebraic groups and some new results in small transcendence degree*, will be published online soon

[3] Momot, A, *On the classification of complex tori arising from real abelian surfaces*, Abh. Math. Semin. Univ. Hambg. 79, no. 2, 283-298 (2009)

[4] Tubbs, R., *Algebraic groups and small transcendence degree I*, J. Number Theory 25, no. 3, 279-307 (1987)

[5] Weil, A., *The field of definition of a variety*, Am. J. of Math. 78, 509-524 (1956).

[6] Wüstholz, G., *Transzendenzeigenschaften von Perioden elliptischer Integrale*, J. Reine Angew. Math. 354, 164-174 (1984)