Equivalence between Extendibility and Factor-Criticality *

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Abstract

In this paper, we show that if $k \geq (\nu+2)/4$, where ν denotes the order of a graph, a non-bipartite graph G is k-extendable if and only if it is $2k$-factor-critical. If $k \geq (\nu-3)/4$, a graph G is $k\frac{1}{2}$-extendable if and only if it is $(2k+1)$-factor-critical. We also give examples to show that the two bounds are best possible. Our results are answers to a problem posted by Favaron [3] and Yu [11].
Key words: n-factor-critical, n-critical, k-extendable, $k\frac{1}{2}$-extendable

1 Introduction, terminologies and preliminary results

All graphs considered in this paper are finite, connected, undirected and simple. Let G be a graph, vertex set and edge set of G are denoted by $V(G)$ and $E(G)$. Let $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S and $G - S$ to denote the subgraph $G[V(G)\backslash S]$. Let G_1 and G_2 be two disjoint graphs. The *union* $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *join* $G_1 \vee G_2$ is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2. The complete graph on n vertices and its complement are denoted

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by K_n and I_n. Let X and Y be two disjoint subsets of $V(G)$, the number of edges of G from X to Y is denoted by $e(X,Y)$. For other terminologies and notations not defined in this paper, we refer the readers to [1].

A *matching* M of G is a subset of $E(G)$ in which no two edges have a common end-vertex. M is said to be a *perfect matching* if it covers all vertices of G. A graph G is said to be *k-extendable* for $0 \le k \le (\nu-2)/2$ if it is connected, contains a matching of size k and any matching in G of size k is contained in a perfect matching of G. G is said to be *minimal k-extendable* if G is k-extendable and $G-e$ is not k-extendable for each $e \in E(G)$. The concept of k-extendable graphs was introduced by Plummer in [8]. In [10], Yu generalized the idea of k-extendibility to $k\frac{1}{2}$-extendibility for graph of odd order. A graph G is said to be *$k\frac{1}{2}$-extendable* if (1) for any vertex v of G there exists a matching of size k in $G-v$, and (2) for every vertex v of G, every matching of size k in $G-v$ is contained in a perfect matching of $G-v$.

A graph G is said to be *n-factor-critical*, or *n-critical*, for $0 \le n \le \nu-2$, if $G-S$ has a perfect matching for any $S \subseteq V(G)$ with $|S| = n$. For $n = 1$, 2, that is *factor-critical* and *bicritical*. G is called *minimal n-factor-critical* if G is n-factor-critical but $G-e$ is not n-factor-critical for any $e \in E(G)$. The concept of n-factor-critical graphs was introduced by Favaron [2] and Yu [10], independently.

It is easy to verify the following theorem.

Theorem 1.1. *For $0 \le k \le \nu/2-1$, a $2k$-factor-critical graph is k-extendable, and a $(2k+1)$-factor-critical graph is $k\frac{1}{2}$-extendable.*

The reverse of Theorem 1.1 does not hold in general. For example, a k-extendable bipartite graphs can not be n-factor-critical for any $n > 0$. However, there has been lots of research on the relationship between k-extendable non-bipartite graphs and n-factor-critical graphs. Most of the results can be viewed as answers to the following problem, which has been posted by Favaron [3] and Yu [11], in slightly different forms.

Problem 1. *Does there exist a non-null function $f(k)$ such that every k-extendable non-bipartite graph of even order $\nu \ge 2k+2$ is $f(k)$-factor-critical?*

The following two results of Plummer [8] are answers to $k = 2$, 3.

Theorem 1.2. *Let G be 2-extendable and non-bipartite with $\nu \ge 6$, then G is bicritical.*

Theorem 1.3. *Let G be 3-extendable and bicritical with $\nu \ge 8$, then $G-e$ is again bicritical for any $e \in E(G)$.*

And they have been generalized for all $k \ge 0$, as below.

Theorem 1.4. (*Favaron [3], Liu and Yu [5]*) *For even integer $k \geq 0$, every connected, non-bipartite, k-extendable graph of even order $\nu > 2k$ is k-factor-critical.*

Theorem 1.5. (*Favaron [3]*) *For even integer $k \geq 0$, every connected non-bipartite, $(k+1)$-extendable graph G of even order $\nu \geq 2k+4$ is k-factor-critical. Moreover, $G - e$ is k-factor-critical for every edge e of G.*

In light of Theorem 1.1, if under some conditions k-extendable graphs are $2k$-factor-critical, then the two classes of graphs are equal. The following results show that this happens when k is large relative to ν.

Theorem 1.6. (*Favaron and Shi [4]*) *Every $((\nu/2)-2)$-extendable non-bipartite graph with $\nu \geq 14$ is $(\nu-4)$-factor-critical.*

Theorem 1.7. (*Yu [11]*) *Let G be a non-bipartite graph of even order and k an integer. If G is k-extendable and $k \geq 2(\nu+1)/3$, then G is $2k$-factor-critical.*

Following this direction, we give a better lower bound of k and show that it is the best possible. Furthermore, we show a similar equivalent relationship between $(2k+1)$-factor-critical graphs and $k\frac{1}{2}$-extendable graphs.

The following lemmas will be used in the proofs of the main results.

Lemma 1.8. (*Plummer [8]*) *If G is a k-extendable graph on $\nu \geq 2k+2$ vertices where $k \geq 1$, then G is also $(k-1)$-extendable.*

Lemma 1.9. (*Yu [10]*) *If G is a $k\frac{1}{2}$-extendable graph, then G is also $(k-1)\frac{1}{2}$-extendable.*

Lemma 1.10. (*Lou and Yu [7]*) *If G is a k-extendable graph with $k \geq \nu/4$, then either G is bipartite or $\kappa(G) \geq 2k$.*

Lemma 1.11. *If G is a k-extendable graph, then G is also m-extendable for all integers $0 \leq m \leq k$. If G is a $k\frac{1}{2}$-extendable graph, then G is also $m\frac{1}{2}$-extendable for all integers $0 \leq m \leq k$.*

Proof. Apply repeatedly Lemma 1.8 and Lemma 1.9. ∎

2 Equivalence between extendibility and factor-criticality

Theorem 2.1. *If $k \geq (\nu+2)/4$, then a non-bipartite graph G is k-extendable if and only if it is $2k$-factor-critical .*

Proof. We only need to prove that if $k \geq (\nu+2)/4$, a k-extendable non-bipartite graph is $2k$-factor-critical.

Let G be a k-extendable non-bipartite graph satisfying $k \geq (\nu+2)/4$ but not $2k$-factor-critical. Then, there exists a vertex set $S \subseteq V(G)$ with order $2k$, such that $G-S$ has no perfect matching. Moreover, we choose S so that the size of the maximum matching of $G[S]$ has the maximum value r_0. Clearly, $r_0 \leq k-1$.

Let M_S be a maximum matching of $G[S]$, then there exists two vertices u_1 and u_2 in $G[S]$ that are not covered by M_S. By Lemma 1.11, M_S is contained in a perfect matching M of G. Let $u_iv_i \in M$, where $v_i \in V(G-S)$, $i=1$, 2. Let $S' = (S \backslash \{u_2\}) \cup \{v_1\}$. Then $M_S \cup \{u_1v_1\}$ is a matching of $G[S']$ of size r_0+1. By the choice of S, $G-S'$ has a perfect matching $M_{\bar{S}'}$, and $|M_{\bar{S}'}| \leq k-1$. By Lemma 1.11, $M_{\bar{S}'}$ is contained in a perfect matching M' of G. Clearly, $M' \cap E(G[S'])$ is a perfect matching of $G[S']$ and $M' \cap E(G[S])$ is a matching of $G[S]$ of size $k-1$. Therefore, $r_0 = k-1$. Then $M_{\bar{S}} = M \cap E(G-S)$ is a maximum matching of $G-S$ of size $r = |V(G-S)|/2 - 1 \leq k-2$, and $v_1v_2 \notin E(G)$.

Let $M_S = \{x_1x_2, \ldots, x_{2k-3}x_{2k-2}\}$, $M_{\bar{S}} = \{y_1y_2, \ldots, y_{2r-1}y_{2r}\}$. If v_1y_1, $v_2y_2 \in E(G)$, then $M_{\bar{S}} \cup \{v_1y_1, v_2y_2\} \backslash y_1y_2$ is a perfect matching of $G-S$, contradicting our assumption. Hence $|\{v_1y_1, v_2y_2\} \cap E(G)| \leq 1$. Similarly, $|\{v_1y_2, v_2y_1\} \cap E(G)| \leq 1$. So $e(\{v_1,v_2\},\{y_1,y_2\}) \leq 2$. Similarly, $e(\{v_1,v_2\},\{y_{2i-1},y_{2i}\}) \leq 2$ for $1 \leq i \leq r$.

If $v_1x_1, v_2x_2 \in E(G)$, then $M_{\bar{S}} \cup \{v_1x_1, v_2x_2\}$ is a matching of G of size no more than k. By Lemma 1.11, $M_{\bar{S}} \cup \{v_1x_1, v_2x_2\}$ is contained in a perfect matching M'' of G. But then $(M'' \cap E(G[S])) \cup \{x_1x_2\}$ is a perfect matching of $G[S]$, a contradiction. Hence, $|\{v_1x_1, v_2x_2\} \cap E(G)| \leq 1$. Similarly, $|\{v_1x_2, v_2x_1\} \cap E(G)| \leq 1$. So $e(\{v_1,v_2\},\{x_1,x_2\}) \leq 2$. Similarly, $e(\{v_1,v_2\},\{x_{2i-1},x_{2i}\}) \leq 2$ for $1 \leq i \leq k-1$.

Then we have

$$d(v_1) + d(v_2) \leq 2(k-1) + 2r + 4 \leq 2(k-1) + 2(k-2) + 4 = 4k-2.$$

But by Lemma 1.10, $\delta(G) \geq \kappa(G) \geq 2k$. So $d(v_1)+d(v_2) \geq 2k+2k = 4k$, a contradiction. ∎

To show that the lower bound in Theorem 2.1 is best possible, we consider the following class of graphs. Let $G^{(k)} = (K_{2k-1} \cup K_1) \vee (K_{2k-1} \cup K_1)$, $k \geq 2$. Then $\nu(G^{(k)}) = 4k$ and $G^{(k)}$ is non-bipartite.

Theorem 2.2. *$G^{(k)}$ is k-extendable but not $2k$-factor-critical.*

Proof. Let G_1 and G_2 be two copies of $K_{2k-1} \cup K_1$ and $G^{(k)} = G_1 \vee G_2$. $G^{(k)}$ is not $2k$-factor-critical, since $G_2 = G^{(k)} - V(G_1)$ does not have a perfect matching. Now we prove that $G^{(k)}$ is k-extendable.

Let M be a matching of size k in G, we show that $G^{(k)} - V(M)$ has a perfect matching. Let $|M \cap E(G_i)| = k_i$, $i = 1, 2$, then $k_1, k_2 \le k-1$. Without lose of generality we suppose $k_1 \ge k_2$. The size of the maximum matching in $G_2 - V(M)$ is no less than $\lfloor (2k-1-(k-k_1-k_2)-2k_2)/2 \rfloor = \lfloor ((k-1)+(k_1+k_2)-2k_2)/2 \rfloor \ge \lfloor (2k_1-2k_2)/2 \rfloor = k_1 - k_2$. Therefore we can find a matching M' of size $k_1 - k_2$ in $G_2 - V(M)$.

In $G^{(k)} - V(M) - V(M')$, half of the vertices are from G_1 and the other half are from G_2, hence we can find nonadjacent edges from G_1 to G_2 covering all vertices in it. So we get a perfect matching in $G^{(k)} - V(M)$ and $G^{(k)}$ is k-extendable. ∎

Now we divert our attention to $k\frac{1}{2}$-extendable graphs and $(2k+1)$-factor-critical graphs. Note that by definition a $k\frac{1}{2}$-extendable graph can never be bipartite.

Theorem 2.3. *If $k \ge (\nu-3)/4$, then a graph G is $k\frac{1}{2}$-extendable if and only if it is $(2k+1)$-factor-critical.*

Proof. We only need to prove that for $k \ge (\nu-3)/4$, a $k\frac{1}{2}$-extendable graph G is $(2k+1)$-factor-critical.

Suppose that G is a $k\frac{1}{2}$-extendable graph with $k \ge (\nu-3)/4$, but not $(2k+1)$-factor-critical. Then, there exists a set $S \subseteq V(G)$ of order $2k+1$, such that there is no perfect matching in $G - S$. Denote by r the size of the maximum matching in $G[S]$. Clearly, $r \le k-1$.

Let M_S be a maximum matching of $G[S]$, and v_1 be a vertex of $G[S]$ not covered by M_S. Then by Lemma 1.11, M_S is contained in a perfect matching M of $G - v_1$. Then $M \cap E(G-S)$ is a matching of $G - S$ of size at most $|V(G-S)|/2 - 1 \le k$.

Let v be a vertex in $G-S$ not covered by $M \cap E(G-S)$, then $M \cap E(G-S)$ is contained in a perfect matching M' of $G - v$. But $M' \cap E(G[S])$ is a matching of $G[S]$ of size at least $r+1$, a contradiction. ∎

We present a class of graphs below to show that the bound in Theorem 2.3 is best possible. Let $H^{(k)} = I_{k+2} \vee (K_{k+3} \cup K_{2k})$. Then $\nu(H^{(k)}) = 4k+5$.

Theorem 2.4. *$H^{(k)}$ is $k\frac{1}{2}$-extendable but not $(2k+1)$-factor-critical.*

Proof. Let $H_1 = I_{k+2}$, $H_2 = K_{k+3}$, $H_3 = K_{2k}$ and $H^{(k)} = H_1 \vee (H_2 \cup H_3)$.

Let S_1 be a subset of $V(H_2)$ of order $k-2$ and $u \in V(H_3)$. Let $S_0 = V(H_1) \cup S_1 \cup \{u\}$. Then $|S_0| = 2k+1$ and $H^{(k)} - S_0$ does not have a perfect matching. Therefore $H^{(k)}$ is not $(2k+1)$-factor-critical.

To prove the $k\frac{1}{2}$-extendibility of $H^{(k)}$, we let $v \in V(H^{(k)})$, M be a matching of size k in $H^{(k)} - v$ and $S = \{v\} \cup V(M)$. We show that $H^{(k)} - S$ has a perfect matching.

Let $V_1 = V(H_1) - S$, $V_2 = V(H_2) - S$ and $V_3 = V(H_3) - S$. The existence of a perfect matching in $H^{(k)} - S$ is equivalent to the existence of a partition of V_1 into two subsets V_1' and V_1'', such that $|V_1'| \le |V_2|$, $|V_1''| \le |V_3|$, $|V_1'| \equiv |V_2| \pmod 2$ and $|V_1''| \equiv |V_3| \pmod 2$. Since $|V_1| + |V_2| + |V_3| = |V(G)| - (2k+1)$ is even, $|V_1|$ and $|V_2| + |V_3|$ have the same parity. And since $|V_2| + |V_3| \ge (k+3) + 2k - (2k+1) = k+2 \ge |V_1| \ge k+2-1-k = 1$, such a partition can always be obtained. Hence we find a perfect matching in $H^{(k)} - S$ and $H^{(k)}$ is $k\frac{1}{2}$-extendable. ∎

3 Final remarks

As we have pointed out earlier, a k-extendable bipartite graph G can not be n-factor-critical for any $n > 0$. This is because we can choose a vertex set S of order n so that $G - S$ is not balanced. However, for $n = 2k$, if we keep the two partitions of $G - S$ balanced when we choose S, then $G - S$ does have a perfect matching. This is a result by Plummer [9].

Theorem 3.1. *Let G be a connected bipartite graph with bipartition (U, W) and suppose k is a positive integer such that $k \le \nu/2 - 1$. Then G is k-extendable if and only if for all $u_1, \ldots, u_k \in U$ and $w_1, \ldots, w_k \in W$, $G' = G - u_1 - \cdots - u_k - w_1 - \cdots - w_k$ has a perfect matching.*

Hence, following the terms in the definition of n-factor-critical graphs, if we define "$2k$-factor-criticality" in a balanced bipartite graph G so that we keep the two partitions of $G - S$ balanced when choosing S, then G is k-extendable if and only if it is "$2k$-factor-critical", for $0 \le k \le \nu/2 - 1$.

Plummer [8] has proved that $\kappa(G) \ge k+1$ for a k-extendable graph G. Hence $\delta(G) \ge \kappa(G) \ge k+1$. For minimal k-extendable bipartite graphs, the following result of Lou [6] shows that the bound can always be reached.

Theorem 3.2. *Every minimal k-extendable bipartite graph G with bipartition (U, W) has at least $2k+2$ vertices of degree $k+1$. Furthermore, both U and W contain at least $k+1$ vertices of degree $k+1$.*

While for minimal k-extendable non-bipartite graphs we have not found such a simple characterization. When $k = 1$, the minimum degree can be 2 or 3. And no result is known for $k \ge 2$. Illuminated by Lemma 1.10, Lou and Yu [7] raised the following conjecture.

Conjecture 1. *Let G be a minimal k-extendable graph on ν vertices with $\nu/2 + 1 \le 2k + 1$. Then $\delta(G) = k+1$, $2k$ or $2k+1$.*

For minimal n-factor-critical graphs, Favaron and Shi [4] raised the following conjecture.

Conjecture 2. *Every minimal n-factor-critical graph G has* $\delta(G) = n+1$.

By the results obtained, we see that except the case that $\nu = 4k$, Conjecture 1 is actually part of Conjecture 2 and the value $2k$ in Conjecture 1 can be excluded.

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