Intersecting Families of Permutations

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Abstract

A set of permutations $I \subset S_n$ is said to be *k-intersecting* if any two permutations in I agree on at least k points. We show that for any $k \in \mathbb{N}$, if n is sufficiently large depending on k, then the largest k-intersecting subsets of S_n are cosets of stabilizers of k points, proving a conjecture of Deza and Frankl. We also prove a similar result concerning k-cross-intersecting subsets. Our proofs are based on eigenvalue techniques and the representation theory of the symmetric group.

1 Introduction

The classical Erdős-Ko-Rado theorem states that if $r < n/2$, an intersecting family of r-subsets of $\{1, 2, \ldots, n\}$ has size at most $\binom{n-1}{r-1}$; if equality holds, the family must consist of all r-subsets containing a fixed element. The so-called 'second Erdős-Ko-Rado theorem' states that if n is sufficiently large depending on k and r, then any k-intersecting family of r-subsets of $\{1, 2, \ldots, n\}$ has size at most $\binom{n-k}{r-k}$; if equality holds, the family must consist of all r-subsets containing k fixed elements. We deal with analogues of these results for permutations.

As usual, $[n]$ will denote the set $\{1, 2, \ldots, n\}$, and S_n will denote the symmetric group, the group of all permutations of $[n]$. Two permutations $\sigma, \tau \in S_n$ are said to *intersect* if they agree at some point, i.e. if there exists $i \in [n]$ such that $\sigma(i) = \tau(i)$. Similarly, they are said to *k-intersect* if they

agree on at least k points, i.e., if there exist $i_1, i_2, \ldots, i_k \in [n]$ such that $\sigma(i_t) = \tau(i_t)$ for $t = 1, 2, \ldots, k$. A subset $I \subset S_n$ is said to be k-*intersecting* if any two permutations in I k-intersect. In this paper, we characterize the largest k-intersecting subsets of S_n for n sufficiently large depending on k, proving a conjecture of Deza and Frankl. We also prove a similar result concerning k-cross-intersecting subsets, proving a conjecture of Leader.

Our main tool in this paper is Fourier analysis on the symmetric group, which entails representation theory. Although Fourier analysis has become a central tool in combinatorics and computer science in the last two decades (notably, since the landmark paper of [20]), it has not often been applied to combinatorial problems in a non-Abelian setting. In retrospect, it seems that in this case it fits the task perfectly.

As a bonus, we point out a nice aspect of Boolean (0/1 valued) functions on S_n. One of the recurring themes in the applications of discrete Fourier analysis to combinatorics over the last decade has been showing that Boolean functions on $\{0, 1\}^n$ are 'juntas' (i.e. depend essentially on few coordinates) precisely when their Fourier transform is concentrated mainly on small sets. Here, we study Boolean functions on S_n whose Fourier transform is supported on the irreducible representations of low dimension, and connect them to cosets of subgroups which are the pointwise stabilizers of small subsets of $\{1, 2 \ldots, n\}$. Along the way, we also prove an interesting generalization of Birkhoff's theorem on bistochastic matrices.

1.1 History and related results

Let
$$T_{i \mapsto j} = \{\sigma \in S_n, \ \sigma(i) = j\}.$$

Clearly, $T_{i \mapsto j}$ is 1-intersecting subset of S_n, with size $(n-1)!$. Let

$$T_{i_1 \mapsto j_1, \ldots, i_k \mapsto j_k} = \bigcap_{t=1}^{k} T_{i_t \mapsto j_t} = \{\sigma \in S_n : \ \sigma(i_t) = j_t \ (1 \leq t \leq k)\}.$$

If $i_1, \ldots, i_k$ are distinct and $j_1, \ldots, j_k$ are distinct, then $T_{i_1 \mapsto j_1, \ldots, i_k \mapsto j_k}$ is a coset of the stabilizer of k points; we will refer to it as a k-*coset*.

Clearly, a k-coset is a k-intersecting family of size $(n-k)!$. As observed by Deza and Frankl [10], it is easy to prove that a 1-intersecting subset of S_n is no larger than a 1-coset:

Theorem 1. *[10] For any $n \in \mathbb{N}$, if $I \subset S_n$ is 1-intersecting, then $|I| \leq (n-1)!$.*

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Proof. Let H be the cyclic group generated by the n-cycle $(123\ldots n)$. No two permutations in H intersect, and the same is true for any left coset of H. The $(n-1)!$ left cosets of H partition S_n. If $I \subset S_n$ is 1-intersecting, then I contains at most one permutation from each left coset of H, and therefore $|I| \leq (n-1)!$. $\qquad\square$

Deza and Frankl conjectured in [10] that for any $n \in \mathbb{N}$, the 1-cosets are the only 1-intersecting subsets of S_n with size $(n-1)!$. Perhaps surprisingly, this turned out to be substantially harder to prove; it was first proved by Cameron and Ku [6] and independently by Larose and Malvenuto [21].

What about k-intersecting families? For n small depending on k, the k-cosets need not be the largest k-intersecting subsets of S_n. Indeed,

$$\{\sigma \in S_n : \ \sigma \text{ has at least } k+1 \text{ fixed points in } \{1, 2, \ldots, k+2\}\}$$

is a k-intersecting family with size

$$(k+2)(n-k-1)! - (k+1)(n-k-2)!,$$

which is larger than $(n-k)!$ if $k \geq 4$ and $n \leq 2k$. However, Deza and Frankl conjectured in [10] that if n is large enough depending on k, the k-cosets *are* the largest k-intersecting subsets of S_n:

Conjecture 1 (Deza-Frankl)**.** *For any $k \in \mathbb{N}$ and any n sufficiently large depending on k, if $I \subset S_n$ is k-intersecting, then $|I| \leq (n-k)!$. Equality holds if and only if I is a k-coset of S_n.*

This is our main result. Our proof uses eigenvalue techniques, together with the representation theory of S_n. In fact, it was proved by the first author and the last two authors independently in 2008. Our two proofs of the upper bound are essentially equivalent, hence the joint paper. However, the latter two authors proved the equality statement directly, via their generalization of Birkhoff's Theorem (Theorem 29), whereas the first author deduced it from the following 'stability' result, proved in [8]:

Theorem 2 ([8])**.** *Let $k \in \mathbb{N}$ be fixed, let n be sufficiently large depending on k, and let $I \subset S_n$ be a k-intersecting family which is not contained within a k coset. Then I is no larger than the family*

$$\{\sigma \in S_n : \ \sigma(i) = i \ \ \forall i \leq k, \ \sigma(j) = j \text{ for some } j > k+1\}$$
$$\cup \ \{(1, \ k+1), (2, \ k+1), \ldots, (k, \ k+1)\},$$

which has size $(1 - 1/e + o(1))(n-k)!$.

This may be seen as an analogue of the Hilton-Milner theorem [16] on intersecting families of r-sets; the $k = 1$ case was conjectured by Cameron and Ku in [6].

When there exists a sharply k-transitive subset of S_n, the 'partitioning' argument in the above proof of Theorem 1 can be replaced by a Katona-type averaging argument, proving Conjecture 1 in this case.

[Recall that a subset $T \subset S_n$ is said to be *k-transitive* if for any distinct $i_1, \ldots, i_k \in [n]$ and any distinct $j_1, \ldots, j_k \in [n]$, there exists $\sigma \in T$ such that $\sigma(i_t) = j_t$ for each $t \in [k]$; T it is said to be *sharply k-transitive* if there exists a unique such $\sigma \in T$. Note that a k-transitive subset $T \subset S_n$ is sharply k-transitive if and only if it has size $n(n-1)\ldots(n-k+1)$.]

Indeed, suppose that S_n has a sharply k-transitive subset T. Then any left translate σT of T is also sharply k-transitive, so any two distinct permutations in σT agree in at most $k-1$ places. Let $I \subset S_n$ be k-intersecting; then $|I \cap \sigma T| \leq 1$ for each $\sigma \in S_n$. Averaging over all $\sigma \in S_n$ gives $|I| \leq (n-k)!$.

For $k = 2$ and $n = q$ a prime power, S_n has a sharply 2-transitive *subgroup*: identify the ground set with the finite field $\mathbb{F}_q$ of order q, and take H to be the group of all affine maps $x \mapsto ax + b$ ($a \in \mathbb{F}_q \setminus \{0\}$, $b \in \mathbb{F}_q$). Any two distinct permutations in H agree in at most 1 point, and the same is true for any left coset of H. So if $I \subset S_n$ is 2-intersecting, then I contains at most one permutation from each left coset of H. Since the $(n-2)!$ left cosets of H partition S_n, this implies that $|I| \leq (n-2)!$.

Similarly, for $k = 3$ and $n = q + 1$ (where q is a prime power), S_n has a sharply 3-transitive subgroup: identify the ground set with $\mathbb{F}_q \cup \{\infty\}$, and take H to be the group of all Möbius transformations

$$x \mapsto \frac{ax + b}{cx + d} \quad (a, b, c, d \in \mathbb{F}_q, \ ad - bc \neq 0).$$

However, it is a classical result of C. Jordan [19] that the only sharply k-transitive permutation groups for $k \geq 4$ are S_k (for $k \geq 4$), A_{k-2} (for $k \geq 8$), M_{11} (for $k = 4$) and M_{12} (for $k = 5$), where M_{11}, M_{12} are the Matthieu groups. Moreover, sharply k-transitive subsets of S_n have not been found for any other values of n and k. Thus, it seems unlikely that this approach can work in general. Instead, we will use a different approach.

Recall that if G is a group and $X \subset G$ is inverse-closed, the *Cayley graph on G generated by X* is the graph with vertex-set G and edge-set $\{\{g, h\} : gh^{-1} \in X\}$. Let Γ_1 be the Cayley graph on S_n generated by the set of *fixed-point-free* permutations,

$$\mathrm{FPF} = \{\sigma \in S_n : \sigma(i) \neq i \ \forall i \in [n]\}.$$

Note that a 1-intersecting subset of S_n is precisely an independent set in Γ_1. It turns out that calculating the least eigenvalue of Γ_1 (meaning the least eigenvalue of its adjacency matrix) and applying Hoffman's bound (Theorem 11) yields an alternative proof of Theorem 1. Calculating the least eigenvalue of Γ_1 is non-trivial, requiring use of the representation theory of S_n. It was first done by Renteln [25], using symmetric functions, and independently and slightly later by Friedgut and Pilpel [13], and also by Godsil and Meagher [14]. As observed in [13] and [14], this also leads to an alternative proof that the 1-cosets are the unique largest 1-intersecting subsets.

The obvious generalization of this approach fails for k-intersecting subsets of S_n. Let Γ_k be the Cayley graph on S_n generated by the set

$$\text{FPF}_k = \{\sigma \in S_n : \ \sigma \text{ has less than } k \text{ fixed points}\}.$$

A k-intersecting subset of S_n is precisely an independent set in Γ_k, so our task is to find the largest independent sets in Γ_k. Unfortunately, for k fixed and n large, calculating the least eigenvalue of Γ_k and applying Hoffman's bound only gives an upper bound of $\Theta((n-1)!)$ on the size of a k-intersecting family.

A key idea of our proof is to choose various subgraphs of Γ_k, and to construct a 'pseudo-adjacency matrix' A for Γ_k which is a suitable real linear combination of the adjacency matrices of these subgraphs. We then apply a weighted version of Hoffman's Theorem (Theorem 12) to this linear combination, in order to prove the upper bound in Conjecture 1. The subgraphs chosen will be Cayley graphs generated by various unions of conjugacy-classes of S_n; this will enable use to calculate their eigenvalues using the representation theory of S_n. Most of the work of the proof is in showing that an appropriate linear combination exists.

1.2   Our main results

A remark on terminology: we will often identify a subset A of S_n, its characteristic function $1_A : S_n \to \{0, 1\}$, and its characteristic vector $v_A \in \mathbb{R}^{|S_n|}$; so when we say that a set A is spanned by sets $B_1, \ldots, B_t$, we mean that $v_A \in \text{Span}\{v_{B_1}, \ldots, v_{B_t}\}$.

Our main results in this paper are as follows.

Theorem 3. *For any $k \in \mathbb{N}$, and any n sufficiently large depending on k, if $I \subset S_n$ is k-intersecting, then $|I| \le (n-k)!$. Equality holds if and only if I is a k-coset of S_n.*

We also prove a cross-intersecting version of this theorem:

Definition 1. *Two sets $I, J \subset S_n$ are k-cross-intersecting if every permutation in I k-intersects every permutation in J.*

Theorem 4. *For any $k \in \mathbb{N}$ and any n sufficiently large depending on k, if $I, J \subset S_n$ are k-cross-intersecting, then $|I||J| \leq ((n-k)!)^2$. Equality holds if and only if $I = J$ and I is a k-coset of S_n.*

The $k = 1$ case of the above was a conjecture of Leader [22].

Our argument proceeds in the following steps. (In order to not disrupt the flow of the paper, some of the representation-theoretic terms used will only be defined later.)

First, we bound the size of a k-intersecting family.

Theorem 5. *For any $k \in \mathbb{N}$ and any n sufficiently large depending on k, if $I \subset S_n$ is k-intersecting, then $|I| \leq (n-k)!$. Moreover, if $I, J \subset S_n$ are k-cross-intersecting, then $|I||J| \leq ((n-k)!)^2$.*

Next, we describe the Fourier transform of the characteristic functions of the families which achieve this bound. Let V_k be the linear subspace of real-valued functions on S_n whose Fourier transform is supported on irreducible representations corresponding to partitions μ of n such that $\mu \geq (n-k, 1^k)$, where $\geq$ denotes the lexicographic order (see section 3).

Theorem 6. *For k fixed and n sufficiently large depending on k, if $I \subset S_n$ is a k-intersecting family of size $(n-k)!$, then $1_I \in V_k$. Furthermore if $I, J \subset S_n$ are k-cross-intersecting and $|I||J| = (n-k)!^2$, then $1_I, 1_J \in V_k$.*

We then prove the following:

Theorem 7. *V_k is spanned by the characteristic functions of the k-cosets of S_n.*

Finally, we complete the proof of our main theorem using the following combinatorial result:

Theorem 8. *For any $k \in \mathbb{N}$, if f is a Boolean function on S_n which is spanned by the k-cosets, then f is the characteristic function of a disjoint union of k-cosets.*

Clearly, the last four theorems immediately imply our main result, Theorem 3, and its cross-intersecting version, Theorem 4.

1.3 Structure of the paper

In section 2 we provide the background that we will use from general representation theory and graph theory. In section 3 we prove all necessary results and lemmas that pertain to representation theory of S_n. Section 4 ties together the results of the previous two sections in order to bound the size and provide a Fourier characterization of the largest k-intersecting families. Finally, in section 5 we show that this characterization holds only for k-cosets.

2 Background

2.1 General representation theory

In this section, we recall the basic notions and results we need from general representation theory. For more background, the reader may consult [18].

Let G be a finite group, and let F be a field. A *representation of G over F* is a pair (ρ, V), where V is a finite-dimensional vector space over F, and $\rho : G \to GL(V)$ is a group homomorphism from G to the group of all invertible linear endomorphisms of V. The vector space V, together with the linear action of G defined by $gv = \rho(g)(v)$, is sometimes called an *FG-module*. A *homomorphism* between two representations (ρ, V) and (ρ', V') is a linear map $\phi : V \to V'$ such that such that $\phi(\rho(g)(v)) = \rho'(g)(\phi(v))$ for all $g \in G$ and $v \in V$. If ϕ is a linear isomorphism, the two representations are said to be *equivalent*, and we write $(\rho, V) \cong (\rho', V')$. If $\dim(V) = n$, we say that ρ *has dimension n*. If $V = F^n$, then we call ρ a *matrix representation*; choosing an F-basis for a general V, one sees that any representation is equivalent to some matrix representation.

The representation (ρ, V) is said to be *irreducible* if it has no proper sub-representation, i.e. there is no proper subspace of V which is $\rho(g)$-invariant for all $g \in G$.

The *group algebra* $F[G]$ denotes the F-vector space with basis G and multiplication defined by extending the group multiplication linearly. In other words,

$$F[G] = \left\{ \sum_{g \in G} x_g g : \ x_g \in F \ \forall g \in G \right\},$$

and

$$\left(\sum_{g \in G} x_g g \right) \left(\sum_{h \in G} y_h h \right) = \sum_{g,h \in G} x_g y_h (gh).$$

Idenfifying $\sum_{g \in G} x_g g$ with the function $g \mapsto x_g$, we can view the vector space $F[G]$ as the space of all F-valued functions on G. The representation defined by

$$\rho(g)(x) = gx \quad (g \in G, \ x \in F[G])$$

is called the *left regular representation* of G; the corresponding FG-module is called the *group module*. This will be useful when we describe irreducible representations of S_n.

When $F = \mathbb{R}$ or $\mathbb{C}$, it turns out that there are only finitely many equivalence classes of irreducible representations of G, and *any* representation of G is isomorphic to a direct sum of irreducible representations of G. Hence, we may choose a set of representatives R for the equivalence classes of complex irreducible representations of G. For the rest of section 2, R will be fixed, and will consist of matrix representations.

If (ρ, V) is a representation of G, and α is a linear endomorphism of V, we say that α *commutes* with ρ if $\alpha \circ (\rho(g)) = \rho(g) \circ \alpha$ for every $g \in G$. (So an isomorphism of (ρ, V) is simply an invertible linear endomorphism which commutes with ρ.) We will make use of the following:

Lemma 1 (Schur's Lemma)**.** *If G is a finite group, and (ρ, V) is a complex irreducible representation of G, then the only linear endomorphisms of V which commute with ρ are scalar multiples of the identity.*

If $F = \mathbb{R}$ or $\mathbb{C}$, we may define an inner product on $F[G]$ as follows:

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)}.$$

If (ρ, V) is a complex representation of V, the *character* χ_ρ of ρ is the map defined by

$$\chi_\rho : G \ \to \ \mathbb{C};$$
$$\chi_\rho(g) \ = \ \mathrm{Tr}(\rho(g)),$$

where $\mathrm{Tr}(\alpha)$ denotes the trace of the linear map α (i.e. the trace of any matrix of α). Note that $\chi_\rho(\mathrm{Id}) = \dim(\rho)$, and that χ_ρ is a *class function* on G (meaning that it is constant on each conjugacy-class of G.)

The usefulness of characters lies in the following

Fact. *Two complex representations are isomorphic if and only if they have the same character; the set of complex irreducible characters is an orthonormal basis for the space of class functions in $\mathbb{C}[G]$.*

If ρ is any complex representation of G, its character satisfies $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$ for every $g \in G$. In our case, $G = S_n$, so g^{-1} is conjugate to g for every g, and therefore $\overline{\chi_\rho(g)} = \chi_\rho(g)$, i.e. all the characters are real-valued. The irreducible characters of S_n are therefore an orthonormal basis for the space of class functions on $\mathbb{R}[S_n]$.

Given two representations (ρ, V) and (ρ', V') of G, we can form their direct sum, the representation $(\rho \oplus \rho', V \oplus V')$, and their tensor product, the representation $(\rho \otimes \rho', V \otimes V')$. We have $\chi_{\rho \oplus \rho'} = \chi_\rho + \chi_{\rho'}$, and $\chi_{\rho \otimes \rho'} = \chi_\rho \cdot \chi_{\rho'}$.

2.2 Fourier transforms and convolutions

In this section, we recall the basic notions of Fourier analysis on finite non-Abelian groups. For more background, see for example [28]. Noting that the normalization chosen differs in various texts, we set out our convention below.

Definition 2. *Let G be a finite group, and let $f, g : G \to \mathbb{R}$ be two real-valued functions on G. Their* convolution $f * g$ *is the real-valued function defined by*

$$f * g(t) = \frac{1}{|G|} \sum_{s \in G} f(ts^{-1})g(s) \quad (t \in G). \tag{1}$$

Definition 3. *The* Fourier transform *of a real-valued function $f : G \to \mathbb{R}$ is a matrix-valued function on irreducible representations; its value at the irreducible representation ρ is the matrix*

$$\widehat{f}(\rho) = \frac{1}{|G|} \sum_{s \in G} f(s)\rho(s). \tag{2}$$

We now recall two related formulas we will need: the Fourier transform of a convolution, and the Fourier inversion formula. If $f, g : G \to \mathbb{R}$, and ρ is an irreducible representation of G, then

$$\widehat{f * g}(\rho) = \widehat{f}(\rho)\widehat{g}(\rho). \tag{3}$$

The Fourier transform is invertible; we have:

$$f(s) = \sum_{\rho \in R} \dim(\rho) \mathrm{Tr}\left[\widehat{f}(\rho)\rho(s^{-1})\right]. \tag{4}$$

In other words, the Fourier transform contains all the information about a function f.

2.3 Cayley Graphs

Recall that if G is a group, and $X \subset G$ is inverse-closed, the *Cayley graph on G generated by X* is the graph with vertex-set G and edge-set $\{\{u, v\} \in G^{(2)} : uv^{-1} \in X\}$; it is sometimes denoted by $\text{Cay}(G, X)$. In fact, we will only be considering cases where $G = S_n$ and X is a union of conjugacy classes (i.e., X is conjugation-invariant).

The relevance of this notion to our problem stems from the following observation. Consider the Cayley graph Γ_1 on S_n generated by the set of *fixed-point free* permutations,

$$\text{FPF} = \{\sigma \in S_n : \sigma(i) \neq i \ \forall i \in [n]\}.$$

As observed in section 1.1, a 1-intersecting family of permutations is precisely an independent set in this graph. More generally, a k-intersecting family of permutations is precisely an independent set in the Cayley graph Γ_k on S_n with generating set

$$\text{FPF}_k = \{\sigma \in S_n : \sigma \text{ has at most } k \text{ fixed points}\}.$$

For any real matrix $A \in \mathbb{R}[G \times G]$, the *left action* of A on $\mathbb{R}[G]$ is defined as follows:
$$(Af)(\sigma) = \sum_{\tau \in G} A_{\sigma, \tau} f(\tau).$$

The main observation of this subsection is that the adjacency matrix of a Cayley graph operates on functions in $\mathbb{R}[G]$ by convolution with the characteristic function of the generating set.

Theorem 9. *Let G be a finite group, let $X \subset G$ be inverse-closed, let $Cay(G, X)$ be the Cayley graph on G generated by X, and let A be the adjacency matrix of $Cay(G, X)$. Then for any function $f : G \to \mathbb{R}$,*

$$Af = |G|(1_X * f). \tag{5}$$

Proof. For any $f \in \mathbb{R}[G]$, and any $\sigma \in G$, we have

$$(Af)(\sigma) = \sum_{\tau \in G} A_{\sigma, \tau} f(\tau) = \sum_{\tau \in G} 1_X(\sigma\tau^{-1})f(\tau) = |G|(1_X * f)(\sigma),$$

as required. $\qquad\square$

Taking the Fourier transform of both sides of (5), we obtain:

$$\widehat{Af} = |G| \cdot \widehat{1_X}\widehat{f}. \tag{6}$$

We will see shortly that if X is conjugation-invariant, as in our case, then $\widehat{1_X}$ is a scalar matrix, so (6) essentially reveals all eigenfunctions of the operator A: it is well known that for any finite group G, the entries of the matrices of a complete set of complex irreducible representations of G form an orthogonal basis for the space of all complex-valued functions on G. So in our case, these are also a complete set of eigenfunctions of A.

Theorem 10 (Schur; Babai; Diaconis, Shahshahani; Roichman; others [3, 7, 26])**.** *Let G be a finite group, and let R be a complete set of complex irreducible matrix representations of G, as above. Let $X \subset G$ be inverse-closed and conjugation-invariant, and let $Cay(G, X)$ be the Cayley graph on G with generating set X. Let A be the adjacency matrix of $Cay(G, X)$. For any $\rho \in R$, and any $i, j \leq \dim(\rho)$, consider the function $\rho_{ij}(\sigma) = \rho(\sigma)_{ij}$. Then $\{\rho_{ij}\}_{\rho,i,j}$ is a complete set of eigenvectors of A. Furthermore, the eigenvalue of ρ_{ij} is*

$$\lambda_\rho = \frac{1}{\dim(\rho)} \sum_{\tau \in X} \chi_\rho(\tau) = \frac{|G|\langle \chi_\rho, 1_X \rangle}{\dim(\rho)}, \tag{7}$$

which depends only on ρ.

Proof. Note that, due to (6), the claim regarding eigenvectors will follow immediately once we have shown that $\widehat{1_X}(\rho)$ is a scalar matrix for every irreducible representation ρ. To do this, we will show that $\widehat{1_X}(\rho)$ commutes with every $\rho(\sigma)$, which will imply the result (by Schur's Lemma.) Indeed, for every $\sigma \in G$,

$$
\begin{aligned}
\rho(\sigma)^{-1}\widehat{1_X}(\rho)\rho(\sigma) &= \rho(\sigma^{-1})\widehat{1_X}(\rho)\rho(\sigma) \\
&= \frac{1}{|G|} \sum_{\tau \in G} \rho(\sigma^{-1})\rho(\tau)1_X(\tau)\rho(\sigma) \\
&= \frac{1}{|G|} \sum_{\tau \in X} \rho(\sigma^{-1}\tau\sigma) \\
&= \frac{1}{|G|} \sum_{\tau \in X} \rho(\tau) \\
&= \widehat{1_X}(\rho),
\end{aligned}
$$

where we have used the fact that X is conjugation-invariant for the fourth equality. Hence, by Schur's Lemma, $\widehat{1_X}(\rho)$ is indeed a scalar matrix; write $\widehat{1_X}(\rho) = c_\rho I$. To calculate c_ρ, note that for any $i \leq \dim(\rho)$,

$$c_\rho = \frac{1}{|G|} \sum_{\tau \in X} 1_X(\tau) \rho_{i,i}(\tau).$$

Summing over i, we obtain:

$$c_\rho \dim(\rho) = \frac{1}{|G|} \sum_{\tau \in X} \text{Tr}[\rho(\tau)] = \frac{1}{|G|} \sum_{\tau \in X} \chi_\rho(\tau).$$

From (6), $\lambda_\rho = |G| c_\rho$, completing the proof. $\qquad \square$

2.4 Hoffman's bound

First, a word regarding normalization, as this is always a potential source of confusion when doing Fourier analysis. Given a graph G, we use the uniform probability measure on the vertex-set V of G, *not* the counting measure. The uniform measure induces the following inner product on $\mathbb{R}[V]$:

$$\langle f, g \rangle = \frac{1}{|V|} \sum_{v \in V} f(v) g(v);$$

this induces the Euclean norm

$$||f||_2 = \sqrt{\langle f, f \rangle}.$$

If $G = (V, E)$ is a graph, the *adjacency matrix A* of G is defined by

$$A_{v,w} = 1_{\{vw \in E(G)\}} \quad (v, w \in V(G)).$$

This is a real, symmetric, $n \times n$ matrix, so there exists an orthonormal system of n eigenvectors of A, which forms a basis for $\mathbb{R}[V]$. (Note that the eigenvalues of A are often referred to as the eigenvalues of G.)

Hoffman [17] observed the following useful bound on the measure of an independent set in a regular graph, in terms of the eigenvalues of the graph:

Theorem 11. *[17] Let $G = (V, E)$ be a d-regular, n-vertex graph. Let A be the adjacency matrix of G. Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal system of eigenvectors of A, with corresponding eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n =$*

$\lambda_{\min}$ *(so that v_1 is the all-1's vector). If $I \subset V$ is an independent set in G, then*

$$\frac{|I|}{|V|} \le \frac{-\lambda_{\min}}{\lambda_1 - \lambda_{\min}}. \tag{8}$$

If equality holds, then

$$1_I \in Span\left(\{v_1\} \cup \{v_i : \lambda_i = \lambda_{\min}\}\right).$$

Proof. Let $f = 1_I$, and let $\alpha = \frac{|I|}{|V|}$. Observe that

$$f^t A f = \sum_{v,w \in I} A_{v,w} = 2e(G[I]) = 0,$$

since I is independent. Write f as a linear combination of the eigenvectors:

$$f = \sum_{i=1}^{n} a_i v_i.$$

Then $\alpha = \langle f, v_1 \rangle = a_1$. Moreover, by Parseval's identity, we have $\sum_i a_i^2 = ||f||_2^2 = \alpha$. Now,

$$0 = f^t A f = \sum_{i=1}^{n} a_i^2 \lambda_i \ge a_1^2 \lambda_1 + \sum_{i=2}^{n} a_i^2 \lambda_{\min} = \alpha^2 \lambda_1 + (\alpha - \alpha^2) \lambda_{\min}.$$

Rearranging gives (8). If equality holds, then $a_i \ne 0$ implies that $i = 1$ or $\lambda_i = \lambda_{\min}$, completing the proof. $\square$

A variant of Hoffman's theorem, which will be crucial for us, comes from weighting the edges of the graph G with real (possibly negative) weights.

Theorem 12. *Let $G = (V, E)$ be an n-vertex graph. Let $G_1, \ldots, G_t$ be regular, spanning subgraphs of G, all having $\{v_1, v_2 \ldots, v_n\}$ as an orthonormal system of eigenvectors (where v_1 is the all-1's vector). Let $\lambda_i^{(j)}$ be the eigenvalue of v_i in G_j. Let $\beta_1, \ldots, \beta_t \in \mathbb{R}$, and let $\lambda_i = \sum_j \beta_j \lambda_i^{(j)}$, and let $\lambda_{\min} = \min_i \lambda_i$. If $I \subset V$ is an independent set in G, then*

$$\frac{|I|}{|V|} \le \frac{-\lambda_{\min}}{\lambda_1 - \lambda_{\min}}. \tag{9}$$

If equality holds, then

$$1_I \in Span\left(\{v_1\} \cup \{v_i : \lambda_i = \lambda_{\min}\}\right).$$

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Proof. The proof is a simple generalization of that of Theorem 11. For each j, let A_j be the adjacency matrix of G_j, and let $A = \sum_j \beta_j A_j$. We have

$$
\begin{aligned}
0 &= f^t A f = \sum_{j=1}^t \beta_j f^t A_j f = \sum_{j=1}^t \beta_j \sum_{i=1}^n a_i^2 \lambda_i^{(j)} = \sum_{i=1}^n a_i^2 \lambda_i \\
&\geq \lambda_1 a_1^2 + \sum_i a_i^2 \lambda_{\min} = \alpha^2 \lambda_1 + (\alpha - \alpha^2) \lambda_{\min}.
\end{aligned}
$$

Rearranging gives (9). If equality holds, then $a_i \neq 0$ implies that $i = 1$ or $\lambda_i = \lambda_{\min}$, completing the proof. $\qquad\square$

We may think of the λ_i's above as the eigenvalues of the linear combination of graphs

$$
Y = \sum_{j=1}^t \beta_j G_j.
$$

The corresponding linear combination of adjacency matrices

$$
A = \sum_{j=1}^t \beta_j A_j
$$

is a so-called *pseudo-adjacency matrix* for G, meaning a symmetric matrix such that $A_{v,w} = 0$ whenever $vw \notin E(G)$; the λ_i's are the eigenvalues of A.

Finally, we will need the following cross-independent version of Hoffman's Theorem. Variants of this theorem can be found in several sources, e.g. [2].

Theorem 13. *Let $G = (V, E)$ be a d-regular, n-vertex graph, and let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal system of eigenvectors of G, with corresponding eigenvalues $d = \lambda_1, \lambda_2, \ldots, \lambda_n$ ordered by descending absolute value (so that v_1 is again the all-1's vector). Let $I, J \subset V$ be (not necessarily disjoint) sets of vertices in G such that there are no edges of G between I and J. Then*

$$
\sqrt{\frac{|I|}{|V|} \cdot \frac{|J|}{|V|}} \leq \frac{|\lambda_2|}{\lambda_1 + |\lambda_2|}. \tag{10}
$$

If equality holds, then

$$
1_I, 1_J \in Span\left(\{v_1\} \cup \{v_i : |\lambda_i| = |\lambda_2|\}\right).
$$

Proof. Let f and g be the characteristic functions of I and J respectively. As in the proof of Hoffman's Theorem, write

$$f = \sum_{i=1}^{n} a_i v_i, \ \ g = \sum_{i=1}^{n} b_i v_i.$$

Let $\alpha = \frac{|I|}{|V|}, \beta = \frac{|J|}{|V|}$. We have

$$0 = 2e(I,J) = f^t A g = \sum_{i=1}^{n} a_i b_i \lambda_i = \alpha \beta \lambda_1 + \sum_{i=2}^{n} a_i b_i \lambda_i. \qquad (11)$$

Hence, by the Cauchy-Schwarz inequality,

$$\alpha \beta \lambda_1 = \left| \sum_{i=2}^{n} a_i b_i \lambda_i \right| \leq \sum_{i=2}^{n} |a_i b_i \lambda_i| \leq |\lambda_2| \sqrt{\alpha - \alpha^2} \sqrt{\beta - \beta^2}.$$

Rearranging gives

$$\sqrt{\frac{\alpha \beta}{(1-\alpha)(1-\beta)}} \leq \frac{|\lambda_2|}{\lambda_1}.$$

Recall the AM-GM inequality for two real variables (see for example [15]):

$$(\alpha + \beta)/2 \geq \sqrt{\alpha \beta}.$$

This implies that

$$(1 - \alpha)(1 - \beta) = 1 - \alpha - \beta + \alpha\beta \leq 1 - 2\sqrt{\alpha\beta} + \alpha\beta = (1 - \sqrt{\alpha\beta})^2,$$

and therefore

$$\sqrt{(1-\alpha)(1-\beta)} \leq 1 - \sqrt{\alpha\beta}.$$

Hence,

$$\frac{\sqrt{\alpha\beta}}{1 - \sqrt{\alpha\beta}} \leq \frac{|\lambda_2|}{\lambda_1}.$$

Rearranging gives

$$\sqrt{\alpha\beta} \leq \frac{|\lambda_2|}{\lambda_1 + |\lambda_2|},$$

as required. The case of equality is dealt with as in the original Hoffman theorem. $\qquad \square$

Note that the generalization of Hoffman's theorem that we mention above holds, with the obvious modifications, in the cross-independent case.

3   Representation Theory of S_n

In this section we gather all the necessary background and results regarding the representation theory of S_n. Readers familiar with the basics of this theory are invited to skip the following subsection.

3.1   Basics

Our treatment follows Sagan [27].

A *partition* of n is a non-increasing sequence of integers summing to n, i.e. a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ and $\sum_{i=1}^{k} \lambda_i = n$; we write $\lambda \vdash n$. For example, $(3, 2, 2) \vdash 7$; we sometimes use the shorthand $(3, 2, 2) = (3, 2^2)$. The following two orders on partitions of n will be useful.

Definition 4. *(Dominance order) Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$ be partitions of n. We say that $\lambda \trianglerighteq \mu$ (λ dominates μ) if $\sum_{j=1}^{i} \lambda_i \geq \sum_{j=1}^{i} \mu_i$ for all i (where we define $\lambda_i = 0$ for all $i > r$, and $\mu_j = 0$ for all $j > s$).*

It is easy to see that this is a partial order.

Definition 5. *(Lexicographic order) Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$ be partitions of n. We say that $\lambda > \mu$ if $\lambda_j > \mu_j$, where $j = \min\{i \in [n] : \lambda_i \neq \mu_i\}$.*

It is easy to see that this is a total order which extends the dominance order.

The *cycle-type* of a permutation $\sigma \in S_n$ is the partition of n obtained by expressing σ as a product of disjoint cycles and listing its cycle-lengths in non-increasing order. The conjugacy-classes of S_n are precisely

$$\{\sigma \in S_n : \text{ cycle-type}(\sigma) = \lambda\}_{\lambda \vdash n}.$$

Moreover, there is an explicit one-to-one correspondence between irreducible representations of S_n (up to isomorphism) and partitions of n, which we now describe.

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partiton of n. The *Young diagram* of λ is an array of n boxes, or *cells*, having k left-justified rows, where row i contains λ_i cells. For example, the Young diagram of the partition $(3, 2^2)$ is:
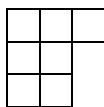
If the array contains the numbers $\{1, 2, \ldots, n\}$ inside the cells, we call it a λ-*tableau*, or a tableau of shape λ; for example:

$$\begin{array}{|c|c|c|}\hline 6 & 1 & 7 \\\hline 5 & 4 \\\cline{1-2} 3 & 2 \\\cline{1-2}\end{array}$$

is a $(3, 2^2)$-tableau. Two λ-tableaux are said to be *row-equivalent* if they have the same numbers in each row; if a λ-tableau t has rows $R_1, \ldots, R_l \subset [n]$ and columns $C_1, \ldots, C_l \subset [n]$, then we let $R_t = S_{R_1} \times S_{R_2} \times \ldots \times S_{R_l}$ be the row-stablizer of t and $C_t = S_{C_1} \times S_{C_2} \times \ldots \times S_{C_k}$ be the column-stabilizer.

A λ-*tabloid* is a λ-tableau with unordered row entries (or formally, a row-equivalence class of λ-tableaux); given a tableau t, we write $[t]$ for the tabloid it produces. For example, the $(3, 2^2)$-tableau above produces the following $(3, 2^2)$-tabloid:

$$\begin{array}{lll} \{1 & 6 & 7\} \\ \{4 & 5\} \\ \{2 & 3\} \end{array}$$

Consider the natural permutation action of S_n on the set X^λ of all λ-tabloids; let $M^\lambda = \mathbb{R}[X^\lambda]$ be the corresponding permutation module, i.e. the real vector space with basis X^λ and S_n action given by extending the permutation action linearly. In general, M^λ is reducible. However, we can describe a complete set of real irreducible representations, as follows.

If t is a tableau, let $\kappa_t = \sum_{\pi \in C_t} \text{sgn}(\pi)\pi$; this is an element of the group module $\mathbb{R}[S_n]$. Let $e_t = \kappa_t\{t\}$. This is a (± 1)-linear combination of tabloids, so is an element of M^λ; we call the e_t's *polytabloids*.

Definition 6. *Let μ be a partition of n. The* Specht module S^μ *is the submodule of M^μ spanned by the μ-polytabloids:*

$$S^\mu = Span\{e_t : \ t \text{ is a } \mu\text{-tabloid}\}.$$

Theorem 14. *The Specht modules are a complete set of pairwise inequivalent, irreducible representations of S_n.*

Hence, any irreducible representation ρ of S_n is isomorphic to some S^λ; in this case, we say that ρ has Young diagram λ. For example, $S^{(n)} = M^{(n)}$ is the trivial representation; $M^{(1^n)}$ is the left-regular representation, and $S^{(1^n)}$ is the sign representation *sgn*.

From now on we will write $[\lambda]$ for the equivalence class of the irreducible representation S^λ, χ_λ for the character χ_{S^λ}, and ξ_λ for the character of M^λ. Notice that the set of λ-tabloids forms a basis for M^λ, and therefore $\xi_\lambda(\sigma)$, the trace of the corresponding permutation representation, is precisely the number of λ-tableaux fixed by σ.

We now explain how the permutation modules M^μ decompose into irreducibles.

Definition 7. *Let λ, μ be partitions of n. A λ-tableau is produced by placing a number between 1 and n in each cell of the Young diagram of λ; if it has μ_i i's $(1 \le i \le n)$ it is said to have* content μ. *A generalized λ-tableau is said to be* semistandard *if the numbers are non-decreasing along each row and strictly increasing down each column.*

Definition 8. *Let λ, μ be partitions of n. The Kostka number $K_{\lambda,\mu}$ is the number of semistandard generalized λ-tableaux with content μ.*

Theorem 15. *(Young's rule) Let μ be a partition of n. Then the permutation module M^μ decomposes as*

$$M^\mu \cong \oplus_{\lambda \vdash n} K_{\lambda,\mu} S^\lambda.$$

Hence,

$$\xi_\mu = \sum_{\lambda \vdash n} K_{\lambda,\mu} \chi_\lambda.$$

For example, $M^{(n-1,1)}$, which corresponds to the natural permutation action of S_n on $[n]$, decomposes as

$$M^{(n-1,1)} \cong S^{(n-1,1)} \oplus S^{(n)},$$

and therefore $\xi_{(n-1,1)} = \chi_{(n-1,1)} + 1$.

The restriction of an irreducible representation of S_n to the subgroup $\{\sigma \in S_n : \sigma(i) = i \; \forall i > n - k\} = S_{n-k}$ can be decomposed into irreducible representations of S_{n-k} as follows:

Theorem 16. *(The branching rule.) Let α be a partition of $n - k$, and λ a partition of n. We write $\alpha \subset^k \lambda$ if the Young diagram of α can be produced from that of λ by sequentially removing k corners (so that after removing the ith corner, we have the Young diagram of a partition of $n - i$.) Let $a_{\alpha,\lambda}$ be the number of ways of doing this; then we have*

$$[\lambda] \downarrow S_{n-k} = \sum_{\alpha \vdash n-k: \; \alpha \subset^k \lambda} a_{\alpha,\lambda} [\alpha],$$

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and therefore

$$\chi_\lambda \downarrow S_{n-k} = \sum_{\alpha \vdash n-k:\ \alpha \subset^k \lambda} a_{\alpha,\lambda} \chi_\alpha.$$

Definition 9. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n; if its Young diagram has columns of lengths $\lambda'_1 \geq \lambda'_2 \geq \ldots \geq \lambda'_l \geq 1$, then the partition $\lambda^t = (\lambda'_1 \ldots, \lambda'_l)$ is called the* transpose *of λ, as its Young diagram is the transpose of that of λ.*

Theorem 17. *Let λ be a partition of $[n]$; then $[\lambda] \otimes [1^n] = [\lambda^t]$. Hence, $\chi_{\lambda^t} = \chi_\lambda \cdot sgn$, and $\dim[\lambda] = \dim[\lambda^t]$.*

Definition 10. *The* hook *of a cell (i,j) in the Young diagram of a partition μ is $H_{i,j} = \{(i,j') : j' \geq j\} \cup \{(i',j) : i' \geq i\}$. The* hook length *of (i,j) is $h_{i,j} = |H_{i,j}|$.*

Theorem 18 (Frame, Robinson, Thrall [9])**.** *If λ is a partition of n with hook lengths $(h_{i,j})$, then*

$$\dim[\lambda] = \frac{n!}{\prod_{i,j} h_{i,j}}. \tag{12}$$

3.2 Lemmas regarding representations and characters

In this subsection we state and prove several lemmas regarding representations of S_n and their characters; these will be instrumental in proving our main theorem.

3.2.1 Dimensions of irreducible representations

Lemma 2. *Let $k \in \mathbb{N}$. Then there exists $E_k > 0$ depending on k alone such that for any irreducible representation $[\lambda]$ of S_n with all rows and columns of length greater than $n - k$, $\dim[\lambda] \geq E_k n^{k+1}$.*

To prove this lemma, we need two simple claims dealing with irreducible representations with a relatively long row or column, and a separate result dealing with the rest.

Claim 1. *Let $[\lambda]$ be an irreducible representation whose first row or column is of length $n - t$. Then*

$$\dim[\lambda] \geq \binom{n}{t} e^{-t}. \tag{13}$$

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Proof. Note that if λ has first column of length $n - t$, then λ^t has first row of length $n - t$. Since $\dim[\lambda] = \dim[\lambda^t]$, we may assume that λ has first row of length $n - t$.

By the hook formula (12), it suffices to prove that

$$\prod_{i,j} h_{i,j} \leq t!(n-t)!e^t.$$

Delete the first row R_1 of the Young diagram of λ; the resulting Young diagram D corresponds to a partition of t, and therefore a representation of S_t, which has dimension

$$\frac{t!}{\prod_{(i,j)\in D} h_{i,j}} \geq 1.$$

Hence,

$$\prod_{(i,j)\in D} h_{i,j} \leq t!.$$

We now bound the product of the hook lengths of the cells in the first row; this is of the form

$$\prod_{(i,j)\in R_1} h_{i,j} = \prod_{j=1}^{n-t} (j + c_j),$$

where $\sum_{j=1}^{n-t} c_j = t$. Using the AM/GM inequality, we obtain:

$$
\begin{aligned}
\prod_{j=1}^{n-t} \frac{j + c_j}{j} &= \prod_{j=1}^{n-t} \left(1 + \frac{c_j}{j}\right) \\
&\leq \left(\sum_{j=1}^{n-t} \frac{1 + \frac{c_j}{j}}{n - t}\right)^{n-t} \\
&\leq \left(\frac{n - t + \sum_{j=1}^{n-t} c_j}{n - t}\right)^{n-t} \\
&= \left(1 + \frac{t}{n - t}\right)^{n-t} \\
&< e^t.
\end{aligned}
$$

Hence,

$$\prod_{i,j} h_{i,j} \leq t!(n-t)!e^t,$$

as desired. $\qquad\square$

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Note that, if t is suffiently small depending on n, $\binom{n}{t}e^{-t}$ is an increasing function of t:

Claim 2. *Let $L(n,t) = \binom{n}{t}e^{-t}$. Then $L(n,t) \leq L(n,t+1)$ for all $t \leq (n-e)/(e+1)$.*

Proof. Observe that
$$\frac{L(n,t)}{L(n,t+1)} = \frac{e(t+1)}{n-t}.$$
Solving for when this expression is at most 1 proves the claim. $\qquad\square$

For the representations not covered by Claim 1, we use the following.

Theorem 19 ([24]). *If $\alpha, \epsilon > 0$, then there exists $N(\alpha, \epsilon) \in \mathbb{N}$ such that for all $n > N(\alpha, \epsilon)$, any irreducible representation $[\lambda]$ of S_n which has all rows and columns of length at most n/α has*

$$\dim[\lambda] \geq (\alpha - \epsilon)^n.$$

The proof of Lemma 2 is now immediate.

Proof of Lemma 2: If the partition λ contains a row or column of length between $n - (n-e)/(e+1)$ and $n - k - 1$, then by Claims 1 and 2,

$$\dim[\lambda] \geq e^{-(k+1)}\binom{n}{k+1} \geq E_k n^{k+1},$$

provided we choose $E_k > 0$ sufficiently small. Otherwise, the conditions of Theorem 19 hold with $\alpha = (e+1)/e - \epsilon'$ for some small $\epsilon' > 0$, and therefore, by Theorem 19,

$$\dim[\lambda] \geq n^{k+1},$$

provided n is sufficiently large depending on k, completing the proof. $\qquad\square$

3.2.2 Character tables and their minors

We will be working with certain minors of the character table of S_n. The following lemmas imply that certain related matrices are upper-triangular with 1's all along the diagonal.

Lemma 3. *If λ, μ are partitions of n, let $K_{\lambda,\mu}$ denote the Kostka number, the number of semistandard λ-tableaux of content μ. If $K_{\lambda,\mu} \geq 1$, then $\lambda \geq \mu$. Moreover, $K_{\lambda,\lambda} = 1$ for all λ.*

Proof. Observe that if there exists a semistandard generalized λ-tableau of content μ, then all μ_i i's must appear in the first i rows, so $\sum_{j=0}^{i} \mu_j \leq \sum_{j=0}^{i} \lambda_j$ for each i. Hence, $\lambda \trianglerighteq \mu$. Since the lexicographic order extends the dominance order, it follows that $\lambda \geq \mu$. Observe that the generalized λ-tableau with λ_i i's in the ith row is the unique semistandard generalized λ-tableau of content λ, so $K_{\lambda,\lambda} = 1$ for every partition λ. $\square$

Lemma 4. *Let λ be a partition of n, and let ξ_λ be the character of the permutation module M^λ. Let $\sigma \in S_n$. If $\xi_\lambda(\sigma) \neq 0$, then cycle-type$(\sigma) \trianglelefteq \lambda$. Moreover, if cycle-type$(\sigma) = \lambda$, then $\xi_\lambda(\sigma) = 1$.*

Proof. The set of λ-tabloids is a basis for the permutation module M^λ. Thus, $\xi_\lambda(\sigma)$, which is the trace of the corresponding representation on the permutation σ, is simply the number of λ-tabloids fixed by σ. If $\xi_\lambda(\sigma) \neq 0$, then σ fixes some λ-tabloid $[t]$. Hence, every row of length l in $[t]$ is a union of the sets of numbers in a collection of disjoint cycles of total length l in σ. Thus, the cycle-type of σ is a refinement of λ, and therefore cycle-type$(\sigma) \trianglelefteq \lambda$, as required. If σ has cycle-type λ, then it fixes just one λ-tabloid, the one whose rows correspond to the cycles of σ, so $\xi_\lambda(\sigma) = 1$. $\square$

Theorem 20. *Let C be the character table of S_n, with rows and columns indexed by partitions / conjugacy classes in decreasing lexicographic order (so $C_{\lambda,\mu} = \chi_\lambda(\sigma_\mu)$ where σ_μ is a permutation with cycle-type μ, and the top-left corner of C is $\chi_{(n)}(\sigma_{(n)})$.) Then the contiguous square minor $\tilde{C}$ of C with rows and columns $\psi : \psi > (n-k, 1^k)$ is invertible and does not depend on n, provided $n > 2k$.*

Proof. Let K be the Kostka matrix, and let D be the matrix of permutation characters,

$$D_{\lambda,\mu} = \xi_\lambda(\sigma_\mu), \tag{14}$$

where σ_μ denotes a permutation with cycle-type μ. Let $\tilde{K}$ and $\tilde{D}$ denote the top-left minor of K and D respectively (i.e. the minor with rows and columns $\psi : \psi > (n-k, 1^k)$).

Recall that by Young's rule (Theorem 15), we have

$$M^\mu \cong \oplus_\lambda K_{\lambda,\mu} S^\lambda, \tag{15}$$

and therefore

$$\xi_\mu = \sum_\lambda K_{\lambda,\mu} \chi_\lambda. \tag{16}$$

Hence,

$$(K^t C)_{\lambda,\mu} = \sum_\tau K_{\tau,\lambda} C_{\tau,\mu} = \sum_\tau K_{\tau,\lambda} \chi_\tau(\sigma_\mu) = \xi_\lambda(\sigma_\mu) = D_{\lambda,\mu}, \qquad (17)$$

and therefore

$$K^t C = D. \qquad (18)$$

Since the rows and columns of K are sorted in decreasing lexicographic order, K is upper-triangular with 1's all along the diagonal, by Lemma 3. Therefore, K^t is lower-triangular with 1's all along the diagonal.

Since K^t is lower-triangular, in addition to (18), we also have

$$\tilde{K}^t \tilde{C} = \tilde{D}. \qquad (19)$$

Since $\tilde{K}^t$ is lower-triangular with 1's all along the diagonal, it is invertible, and therefore

$$\tilde{C} = (\tilde{K}^t)^{-1} \tilde{D}.$$

By Lemma 4, $\tilde{D}$ is upper-triangular with 1's all along the diagonal, and is therefore invertible. It follows that $\tilde{C}$ is invertible also.

We will now show that $\tilde{K}$ and $\tilde{D}$ are independent of n, provided $n > 2k$; this will prove that $\tilde{C}$ is also independent of n.

Let $\lambda > (n - k, 1^k)$ be a partition. Then $\lambda_1 \geq n - k$. Write $\lambda' = (\lambda_1 - (n - k), \lambda_2, \ldots)$. (Note that this may not be a *bona fide* partition, as it may not be in non-increasing order.) Now the mapping $\lambda \mapsto \lambda'$ has the same image over $\{\lambda : \lambda > (n - k, 1^k)\}$ for all $n \geq 2k$: namely, 'partitions' of k where the first row is not necessarily the longest.

We first consider K. Recall once again that $K_{\lambda,\mu}$ is the number of semistandard λ-tableaux of content μ. Let t be a semistandard λ-tableau of content μ; we now count the number of choices for t. Since the numbers in a semistandard tableau are strictly increasing down each column and non-increasing along each row, and $\mu_1 \geq n - k$, we must always place 1's in the first $n - k$ cells of the first row of t. We must now fill the rest of the cells with content μ'. Provided $n \geq 2k$, μ' is independent of n, and the remaining cells in the first row have no cells below them, so the number of ways of doing this is independent of n. Hence, the entire minor $\tilde{K}$ is independent of n.

Now consider D. Recall that $D_{\lambda,\mu} = \xi_\lambda(\sigma_\mu)$ is simply the number of λ-tabloids fixed by σ_μ. To count these, first note that the numbers in the long cycle of σ (which has length at least $n - k$) must all be in the first row of the λ-tabloid (otherwise the long cycle of σ must intersect two or more

rows, as $n - k > k$.) This leaves us with a $(\lambda_1 - \mu_1, \lambda_2, \ldots, \lambda_r)$-'tableau', which we need to fill with the remaining $n - \mu_1$ elements in such a way that σ fixes it. It is easy to see that, again, the number of ways of doing this is independent of n. $\quad\square$

In particular, if $n \geq 2k$, the number of partitions λ of n such that $\lambda \geq (n - k, 1^k)$ is independent of n; we denote it by q_k. Note that

$$q_k = \sum_{t=0}^{k} p_t,$$

where p_t denotes the number of partitions of t.

We need a slightly more general result, which allows us to split some of the partitions.

Definition 11. *Assume that $n > 3k + 1$, and let $\mu = (\mu_1, \ldots, \mu_r)$ be a partition of n with $\mu_1 \geq n - k$. We define*

$$Split(\mu) = (\mu_1 - k - 1, k + 1, \mu_2, \ldots, \mu_r).$$

It is easy to see that $Split(\mu)$ is indeed a partition (i.e. it is in descending order). Further, exactly one of μ and $Split(\mu)$ is even.

Theorem 21. *Let C be as above, and let $n > 3k + 1$. Let $\phi_1, \ldots, \phi_{q_k-1}$ be the partitions $> (n - k, 1^k)$. Let $\mu_1, \ldots, \mu_{q_k-1}$ be partitions such that, for each j, either $\mu_j = \phi_j$ or else $\mu_j = Split(\phi_j)$. Then the square minor $\check{C}$ of C with ith row ϕ_i and jth column μ_j is independent of the choices of the μ_j's, so is always equal to the top-left minor $\tilde{C}$.*

Proof. It is easy to see that if $\lambda, \mu > (n - k, 1^k)$, σ has cycle-type μ and σ' has cycle-type $Split(\mu)$, then in fact, $\xi_\lambda(\sigma) = \xi_\lambda(\sigma')$. (All rows of a λ-tabloid below the first have length at most k. Hence, if a permutation σ' with cycle-type $Split(\mu)$ fixes a λ-tabloid $[t]$, the numbers in $(\mu_1 - k - 1)$-cycle and the $(k+1)$-cycle must all lie in the first row of $[t]$. It follows that a permutation σ produced by merging these two cycles of σ' fixes exactly the same λ-tabloids as σ' does.) Since the Kostka matrix K is upper-triangular with 1's all down the diagonal, $\{\xi_\lambda : \lambda > (n - k, 1^k)\}$ and $\{\chi_\lambda : \lambda > (n - k, 1^k)\}$ are bases for the same linear space, and therefore

$$\chi_\lambda(\sigma) = \chi_\lambda(\sigma')$$

for each $\lambda > (n - k, 1^k)$. Thus,

$$C_{\lambda,\mathrm{Split}(\mu)} = C_{\lambda,\mu} \quad \forall \lambda, \mu > (n - k, 1^k),$$

as required. $\quad\square$

3.2.3 Functions with Fourier transform concentrated on the 'fat' irreducible representations

One of the recurring themes in applications of discrete Fourier analysis to combinatorics is proving that certain functions depend on few coordinates, by showing that their Fourier transform is concentrated on the 'low frequencies', the characters indexed by small sets. Examples of this can be found for example in [5], [11] and [12]. In this paper, we need a non-Abelian analogue. We show that functions on S_n whose Fourier transform is supported on irreducible representations which are large with respect to the lexicographic order, are spanned by the cosets of the pointwise stabilizers of small sets.

Definition 12. *Let V_k be the linear space of functions whose Fourier transform is supported only on representations $\geq (n - k, 1^k)$.*

We are now ready to prove

Theorem 7. *V_k is the span of the k-cosets.*

Proof. First, we show that the characteristic function of any k-coset is indeed in V_k. Let $T = T_{a_1 \mapsto b_1, \ldots, a_k \mapsto b_k}$. It is easy to check that if $f : S_n \to \mathbb{R}$, $\tau, \pi \in S_n$, and

$$
\begin{aligned}
g : S_n &\to \mathbb{R} \\
\sigma &\mapsto f(\pi \sigma \tau),
\end{aligned}
$$

i.e. the function g is a 'double-translate' of f, then $\hat{f}(\rho) = 0 \Rightarrow \hat{g}(\rho) = 0$. Hence, if $f \in V_k$, so is g. (This is saying that V_k is a 'two-sided ideal' of the group algebra $\mathbb{R}[S_n]$.)

Hence, by double-translation, without loss of generality, we may assume that $a_i = b_i = i$ for each $i \in [k]$, i.e. $T = T_{1 \mapsto 1, \ldots, k \mapsto k}$, so $T \cong S_{n-k}$. We use the branching rule (Theorem 16.) If $\rho < (n - k, 1^k)$, then ρ has at least $k + 1$ cells outside the first row. In that case, every $\mu \subset^k \rho$ is a nontrivial irreducible representation of S_{n-k}. We claim that this implies

$$
\sum_{\sigma \in S_{n-k}} \mu(\sigma) = 0.
$$

To see this, observe that the linear map $\sum_{\sigma \in S_{n-k}} \mu(\sigma)$ commutes with μ, and therefore by Schur's Lemma, it must be a scalar multiple of the identity map. Its trace is zero, since χ_μ is orthogonal to the trivial character, so it must be the zero map. Hence, $\widehat{1_T}(\rho) = 0$ for each $\rho < (n - k, 1^k)$.

(Note also that if $\rho = (n - k, 1^k)$, then the only $\mu \subset^k \rho$ with nonzero sum on T is that corresponding to the partition $(n - k)$, i.e. the trivial representation, which has sum $(n-k)!$ on T. Therefore, $\widehat{1_T}((n-k, 1^k)) \neq 0$. This will be useful in the proof of Theorem 22.)

In the other direction, let $f \in V_k$. Using the Fourier inversion formula, we have

$$f(\sigma) = \sum_{\rho \geq (n-k,1^k)} \dim[\rho] \mathrm{Tr}\left[\hat{f}(\rho)\rho(\sigma^{-1})\right]. \tag{20}$$

Consider the permutation module $M^{(n-k,1^k)}$. By Young's Rule (Theorem 15), $M^{(n-k,1^k)}$ must contain at least one copy of every Schur module S^ρ with $\rho \geq (n - k, 1^k)$, and no others. We can therefore rewrite the previous formula as:

$$f(\sigma) = \mathrm{Tr}\left[A\psi(\sigma^{-1})\right], \tag{21}$$

where ψ is a matrix representation corresponding to $M^{(n-k,1^k)}$, and A is a block-diagonal matrix whose blocks correspond to the irreducible modules S^ρ, i.e. it has $K_{\rho,(n-k,1^k)}$ blocks corresponding to ρ, all equal to

$$\frac{\dim[\rho]}{K_{\rho,(n-k,1^k)}} \hat{f}(\rho),$$

where the Kostka number $K_{\rho,(n-k,1^k)}$ is the multiplicity of S^ρ in $M^{(n-k,1^k)}$.

Next, recall that the permutation module $M^{(n-k,1^k)}$ corresponds to the permutation action of S_n on $(n - k, 1^k)$-tabloids, which can be identified with ordered k-tuples of distinct numbers between 1 and n. Since trace is conjugation-invariant, we can perform a change of basis, replacing A and ψ by similar matrices B and ϕ, so that

$$f(\sigma) = \mathrm{Tr}\left[B\phi(\sigma^{-1})\right], \tag{22}$$

where $\phi_{\alpha,\beta}(\tau)$ is 1 if τ takes the ordered k-tuple α to the ordered k-tuple β, and 0 otherwise. But then equation (22) means precisely that

$$f = \sum_{\alpha,\beta} B_{\alpha,\beta} T_{\alpha \mapsto \beta},$$

completing the proof. $\qquad\Box$

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4 Proof of main theorems

4.1 Proof strategy

In this section we finally proceed to eat the pudding. In view of Theorem 7, we would like to prove that whenever I is a maximum-sized k-intersecting family, the Fourier transform of its characteristic function is supported on the irreducible representations corresponding to partitions $\rho \geq (n - k, 1^k)$, i.e. those whose Young diagram has first row of length at least $n - k$. Let us call these representations the 'fat' representations, and their transposes (those whose Young diagram has first column of height at least $n - k$) the 'tall' representations. The rest will be called 'medium' representations.

Given a k-intersecting family $I \subset S_n$, we will first consider it as an independent set in the Cayley graph Γ_k on S_n generated by FPF_k, the set of permutations with less than k fixed points. Since FPF_k is a union of conjugacy classes, by Theorems 10 and 14, the eigenvalues of Γ_k are given by

$$\lambda_\rho^{(k)} = \frac{1}{\dim[\rho]} \sum_{\sigma \in \mathrm{FPF}_k} \chi_\rho(\sigma) \quad (\rho \vdash n). \tag{23}$$

Fixed-point-free permutations are also called *derangements*, and Γ_1 is also called the *derangement graph*. Let $d_n = |\mathrm{FPF}_1(n)|$; it is well-known (and easy to see, using the inclusion-exclusion formula), that

$$d_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n - i)! = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!} = (1/e + o(1))n!.$$

For $n \geq 5$, the eigenvalues of Γ_1 satisfy:

$$
\begin{aligned}
\lambda_{(n)}^{(1)} &= d_n \\
\lambda_{(n-1,1)}^{(1)} &= -d_n/(n - 1) \\
|\lambda_\rho^{(1)}| &< cd_n/n^2 < d_n/(n - 1) \quad \text{for all other } \rho \vdash n
\end{aligned}
$$

(where c is an absolute constant). Hence, the matrix Γ_1 has

$$\lambda_{\min}/\lambda_1 = -1/(n - 1).$$

As observed in [14] and [25], this implies via Hoffman's bound (Theorem 11) that any 1-intersecting family $I \subset S_n$ satisfies $|I| \leq (n - 1)!$. If equality holds, then $1_I \in V_1$, from which we may conclude (e.g. from the $k = 1$ case of Theorems 7 and 8) that I is a 1-coset. Also, we may conclude via

Theorem 13 that any 1-cross-intersecting pair of families $I, J \subset S_n$ safisfy $|I||J| \leq ((n-1)!)^2$. If equality holds, then $1_I, 1_J \in V_1$, which enables us to conclude that $I = J$ is a 1-coset of S_n. This proves Leader's conjecture on 1-cross-intersecting families in S_n for $n \geq 5$ (it can be verified directly for $n = 4$).

However, for k fixed and n large, calculating the least eigenvalue of Γ_k and applying Hoffman's bound only gives an upper bound of $\Theta((n-1)!)$ for the size of a k-intersecting subset of S_n. Indeed,

$$
\begin{aligned}
\lambda^{(k)}_{(n-1,1)} &= \frac{1}{\dim[n-1,1]} \sum_{\sigma \in \mathrm{FPF}_k} \chi_{(n-1,1)}(\sigma) \\
&= \frac{1}{n-1} \sum_{\sigma \in \mathrm{FPF}_k} (\xi_{(n-1,1)}(\sigma) - 1) \\
&= \frac{1}{n-1} \sum_{i=0}^{k-1} \binom{n}{i} d_{n-i}(i-1) \\
&= -\frac{1}{n-1}(1/e + o(1))n!\left(1 - \sum_{i=1}^{k-2} \frac{i}{(i+1)!}\right).
\end{aligned}
$$

Note that

$$
\sum_{i=1}^{\infty} \frac{i}{(i+1)!} = \frac{d}{dx}\frac{e^x - 1}{x}\bigg|_{x=1} = 1,
$$

so for any $k \in \mathbb{N}$,

$$
1 - \sum_{i=1}^{k-2} \frac{i}{(i+1)!} > 0,
$$

and therefore $\lambda^{(k)}_{(n-1,1)} = -\Theta_k((n-1)!)$. It turns out that $|\lambda^{(k)}_\rho| \leq a_k n!/n^2$ for all partitions $\rho \neq (n), (n-1,1)$, where $a_k > 0$ depends on k alone. Hence, if k is fixed and n is large, Γ_k has

$$
\lambda_{\min} = \lambda^{(k)}_{(n-1,1)} = -\Theta_k((n-1)!).
$$

Hence, applying Hoffman's bound only gives

$$
|I| \leq \Theta_k((n-1)!),
$$

for a k-intersecting $I \subset S_n$.

Instead, we will construct a linear combination Y of subgraphs of Γ_k (each a Cayley graph generated by a conjugacy-class within FPF_k) which

has the correct eigenvalues for us to prove Theorems 5 and 6 from Theorem 12. By Theorem 10, if $X_1, \ldots, X_t$ are conjugacy-classes within FPF_k, and $\beta_1, \ldots, \beta_t \in \mathbb{R}$, then the eigenvalues of the linear combination

$$Y = \sum_{j=1}^{t} d_j \mathrm{Cay}(S_n, X_j)$$

are

$$\lambda_\rho = \frac{1}{\dim[\rho]} \sum_{j=1}^{t} d_j \sum_{\sigma \in X_j} \chi_\rho(\sigma) = \frac{1}{\dim[\rho]} \sum_{j=1}^{t} d_j |X_j| \chi_\rho(\tau_j), \quad (\rho \vdash n) \quad (24)$$

where τ_j denotes any permutation in X_j. So the eigenvalues of Y still correspond to partitions of n, and are therefore relatively easy to handle.

The value of $\lambda_{\min}/\lambda_1$ required in Theorem 12 to produce the upper bound in Theorem 5 is as follows:

Corollary 1. *Define*

$$\omega = \omega_{n,k} = -\frac{(n-k)!}{n! - (n-k)!} = -\frac{1}{n(n-1)\ldots(n-k+1) - 1}.$$

Assume the conditions of Theorem 12. If $\lambda_{min}/\lambda_1 = \omega$, and I is an independent set in G, then $|I| \leq (n-k)!$.

Proof. Immediate from Theorem 12. □

Since rescaling the linear combination of graphs makes no difference to the above, our aim will be to construct a linear combination Y with $\lambda_1 = 1$ and $\lambda_{\min} = \omega$. Since equality has to hold in Theorem 12 when the independent set is a k-coset, we know that for any k-coset T, we must have

$$1_T \in \mathrm{Span}\left(\{v_1\} \cup \{v_i : \lambda_i = \omega\}\right).$$

By Theorem 7, it follows that we must have $\lambda_\rho = \omega$ for each fat partition $\rho \neq (n)$.

We will in fact construct two linear combinations, Y_{even} and Y_{odd}, from Cayley graphs generated by respectively even/odd conjugacy-classes within FPF_k. We design these linear combinations so that $\lambda_{(n)} = 1$ and $\lambda_\rho = \omega$ for all fat $\rho \neq (n)$. Recalling for any partition ρ, we have $\chi_{\rho^t} = \chi_\rho \cdot sgn$, where sgn is the sign character, we see from (24) that for any partition ρ, we have:

- $\lambda_{\rho^t} = \lambda_\rho$ in Y_{even}, whereas

	(n)	fat, $\neq (n)$	(1^n)	tall, $\neq (1^n)$	medium
Y_{even}	1	ω	1	ω	$o(|\omega|)$
Y_{odd}	1	ω	-1	$-\omega$	$o(|\omega|)$
Y	1	ω	0	0	$o(|\omega|)$

Table 1: Eigenvalues

- $\lambda_{\rho^t} = -\lambda_\rho$ in Y_{odd}.

Hence, simply taking

$$Y = \tfrac{1}{2}(Y_{\text{even}} + Y_{\text{odd}})$$

will ensure that the eigenvalues of Y corresponding to the tall representations are all zero.

Moreover, we will show that in both Y_{even} and Y_{odd} (and consequently in Y), the eigenvalues corresponding to the medium representations all have absolute value $|\lambda| \leq c_k |\omega| / \sqrt{n} = o(|\omega|)$, where $c_k > 0$ depends on k alone. So provided n is sufficiently large, ω is both the minimal eigenvalue of Y and the second-largest in absolute value, and is attained only on the non-trivial fat irreducible representations. The situation is summarized in Table 4.1; note that the $o(|\omega|)$ function is always the same function.

Applying Theorem 12 to Y will not only prove that $|I| \leq (n-k)!$, but also that if equality holds, then the Fourier transform of the characteristic function of I is totally supported on the fat representations, yielding the proof of our main theorem (pending the results of section 5). Since ω is also the eigenvalue of second-largest absolute value, we will also be able to deduce Theorem 4, concerning k-cross-intersecting families.

In order to carry out our plan, we will identify appropriate conjugacy-classes X_j to use as the generating sets for our Cayley graphs, and appropriate coefficients d_j. The set of linear equations the d_j's must satisfy will always correspond to a specific square minor of the character table of S_n, which turns out to be non-singular and independent of n (for n sufficiently large). The latter statement is precisely the content of Theorems 20 and 21.

As for the medium representations, we will show that their eigenvalues have small absolute value using the lower bound we proved on their dimensions in Lemma 2.

4.2   Calculations of eigenvalues of Cayley graphs

The Cayley graphs we will use to construct our linear combination will be generated by conjugacy-classes of permutations with specific cycle-types.

These cycle-types will fall into two categories: those corresponding to partitions $> (n - k, 1^k)$, and those corresponding to partitions π, where

$$(n - k - 1, k + 1) \geq \pi > (n - 2k - 1, k + 1, 1^k).$$

Note that the second category is exactly the split of the first category (see definition 11). We need to use the split partitions in order to ensure that our conjugacy-classes have the correct sign.

We will need the following bounds on the sizes of the above conjugacy-classes:

Lemma 5. *Let X be a conjugacy class of S_n with a cycle of length $n - t$, where $t < n/2$. Then*

$$\frac{n!}{(n - t)t^t} \leq |X| \leq 2(n - 1)!.$$

Proof. Suppose the cycle-type of X is $(n - t, c_1, \ldots, c_l)$, where $l \leq t$ and $\sum_{i=1}^{l} c_i = t$. Define a mapping

$$F : \ S_n \to X$$

by taking a permutation $\sigma \in S_n$, writing it in sequence-notation

$$\sigma(1), \sigma(2), \ldots, \sigma(n),$$

and then placing parentheses at the appropriate points, producing a permutation in X (written in disjoint cycle notation). This mapping is clearly surjective, and each element of X has the same number of preimages, N say. Note that the preimages of a permutation in X are obtained by rotating each cycle and then permuting the cycles of length j for each j. Let $a_j = |\{i : \ c_i = j\}|$ for each j; then

$$N = (n - t) \prod_j a_j! \prod_{i=1}^{l} c_i.$$

Observe that

$$n/2 < n - t \leq N \leq (n - t)l! \prod_{i=1}^{l} c_i \leq (n - t)l!(t/l)^l \leq (n - t)t^l \leq (n - t)t^t,$$

using the AM/GM inequality. Hence,

$$\frac{n!}{(n - t)t^t} \leq |X| \leq 2(n - 1)!,$$

as required. $\qquad\square$

We now proceed to the key step of the proof.

Theorem 22. *If ρ is a partition of n, let X_ρ denote the conjugacy class of permutations with cycle-type ρ. Assume that $n > 3k + 1$, and let q_k denote the number of partitions $\geq (n - k, 1^k)$. Let $\phi_1, \ldots, \phi_{q_k-1}$ be the partitions of n which are $> (n - k, 1^k)$ (in decreasing lexicographic order). Let $\mu_1, \ldots, \mu_{q_k-1}$ be such that for all j, either $\mu_j = \phi_j$ or else $\mu_j = Split(\phi_j)$. Let $G_j = Cay(S_n, X_{\mu_j})$, and let $(\lambda_i^{(j)})_i$ be the eigenvalues of G_j, in decreasing lexicographic order of their representations (so that $\lambda_1^{(j)} = |X_{\mu_j}|$).*

Then there exist $d_1, \ldots, d_{q_k-1} \in \mathbb{R}$ such that:

$$\sum_j d_j \lambda_i^{(j)} = \begin{cases} 1, & \text{for } i = 1 \\ \omega, & \text{for } 1 < i \leq q_k \end{cases} \tag{25}$$

Moreover, there exists $B_k > 0$ depending only on k such that

$$\max_j |d_j| \leq \frac{B_k}{(n-1)!}.$$

Theorem 22 will enable us to get the correct eigenvalues on the fat representations, but we will also need the following, which ensures that the eigenvalues for the medium representations are smaller in absolute value:

Theorem 23. *Under the assumptions and notation of Theorem 22, let ρ be a medium partition, and let $\lambda_\rho = \sum_{j=1}^{q_k-1} d_j \lambda_\rho^{(j)}$. Then*

$$|\lambda_\rho| \leq c_k |\omega| / \sqrt{n} = o(|\omega|),$$

where $c_k > 0$ depends only on k.

We defer the proof of this until later, and turn first to the proof of Theorem 22.

Proof of Theorem 22: There are three steps in the proof. Step 1 is showing the existence of coefficients d_j such that equation (25) holds for $i < q_k$. Step 2 is showing that with these coefficients, the equation also holds for $i = q_k$. Step 3 is obtaining the bound on $\max_j |d_j|$.

Step 1 ($i < q_k$): Consider the system of linear equations $Md = b$, where $d = (d_1, \ldots, d_{q_k-1})$, $b = (1, \omega, \ldots, \omega)$, and $M_{ji} = \lambda_i^{(j)}$. By Theorem 10,

$$\lambda_i^{(j)} = \frac{1}{\dim[\phi_i]} \sum_{\tau \in X_j} \chi_{\phi_i}(\tau) = \frac{|X_j|}{\dim[\phi_i]} \chi_{\phi_i}(\tau_j), \tag{26}$$

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where τ_j is a representative for the conjugacy-class X_j (recall that characters are class-functions). Equivalently,

$$M = N_1 \check{C}^t N_2,$$

where $\check{C}$ denotes the minor of the character table of S_n with rows $\phi_1, \ldots, \phi_{q_k-1}$ and columns $\mu_1, \ldots, \mu_{q_k-1}$, and N_1 and N_2 are the diagonal row- and column-normalization matrices respectively; explicitly,

$$(N_1)_{i,j} = \frac{\delta_{i,j}}{\dim[\phi_i]}, \quad (N_2)_{i,j} = \delta_{i,j}|X_j|. \tag{27}$$

Recall from Theorem 21 that $\check{C}$ is always equal to the top-left minor $\tilde{C}$, and that $\tilde{C}$ (and therefore $\tilde{C}^t$) is invertible. Therefore, there is a (unique) solution for d, so we can find appropriate values for $d_1, \ldots, d_{q_k-1}$.

Step 2 $(i = q_k)$**:** Write

$$\lambda = \sum_j d_j \lambda_{q_k}^{(j)}.$$

We must show that $\lambda = \omega$. This will follow from analyzing the proof of the generalized Hoffman bound, Theorem 12, when $G = \Gamma_k$ and the independent set I is a k-coset.

Let T be a k-coset of S_n, and let $f = 1_T$ be its characteristic function. Choose the orthonormal basis of eigenvectors $\{v_i\}$ in Theorem 12 to be the normalization of the orthogonal basis formed by the entries of (matrix-equivalents of) the irreducible representations $[\rho]$ (see section 2.3). Observe that

$$\alpha = \mathbb{E}f = \frac{(n-k)!}{n!}.$$

By the argument in the proof of Theorem 7, we see that

$$\widehat{f}(\rho) \neq 0 \Rightarrow \rho \geq (n-k, 1^k), \tag{28}$$

and moreover, $\widehat{f}((n-k, 1^k)) \neq 0$.

Let $W(\rho) = Tr[\hat{f}(\rho)(\hat{f}(\rho))^t]$. This is simply the L^2-weight of the coefficients of eigenvectors v_i which correspond to entries of (the matrix-equivalent of) $[\rho]$, i.e. if we write as usual $f = \sum a_i v_i$, then $W(\rho) = \sum a_i^2$, where the sum is over all i such that v_i is (the normalization of) one of the $(\dim \rho)^2$ eigenvectors which are entries of $[\rho]$. Hence,

$$W(\rho) \neq 0 \Rightarrow \rho \geq (n-k, 1^k),$$

and moreover, $W((n - k, 1^k)) \neq 0$.

Using the coefficients $d_1, \dots, d_{q_k-1}$ in Theorem 12, we have $\lambda_1 = 1$ and $\lambda_t = \omega$ for $1 < t < q_k$. Since T is an independent set in the graph $G = \Gamma_k$, we have

$$
\begin{aligned}
0 &= \sum_{\rho \geq (n-k,1^k)} \lambda_\rho W(\rho) \\
&= \alpha^2 + \omega(\alpha - \alpha^2 - W((n - k, 1^k))) + \lambda W((n - k, 1^k)).
\end{aligned}
$$

By definition of ω, we have $\omega(\alpha - \alpha^2) = -\alpha^2$; combining this with the above yields $\lambda = \omega$, as required.

Step 3 (bounding the coefficients d_j) : Recall that $M = N_1 \tilde{C}^t N_2$, and by Theorem 21, $\tilde{C}$ is invertible and independent of n. Hence, the entries of $(\tilde{C}^t)^{-1}$ are uniformly bounded by some function of k alone. Since $Md = b$, we have

$$
d = M^{-1}b = (N_2)^{-1}(\tilde{C}^t)^{-1}(N_1^{-1}b). \tag{29}
$$

We now proceed to bound uniformly the entries of the vector $N_1^{-1}b$. We have

$$
(N_1^{-1})_{ij} = \delta_{ij} \dim[\phi_i],
$$

and therefore

$$
(N_1^{-1}b)_i = \dim[\phi_i]b_i.
$$

For $i = 1$, we have

$$
(N_1^{-1}b)_1 = \dim[\phi_1]b_1 = 1 \cdot 1 = 1.
$$

For $i > 1$, we have

$$
\dim[\phi_i] + 1 \leq \dim(M^{(n-k,1^k)}) = n(n - 1) \dots (n - k + 1),
$$

since both ϕ_i and the trivial representation are constituents of $M^{(n-k,1^k)}$. Hence,

$$
|(N_1^{-1}b)_i| = |\dim[\phi_i]||\omega| \leq 1.
$$

We now bound uniformly the entries of the matrix $(N_2)^{-1}$. Using Lemma 5, we have

$$
(N_2^{-1})_{ij} = \frac{\delta_{ij}}{|X_j|} \leq \frac{(n - 2k - 1)(2k - 1)^{2k-1}}{n!} \leq \frac{b_k}{(n - 1)!},
$$

where $b_k > 0$ depends only on k.

Combining these bounds with (29), we see that there exists $B_k > 0$ depending only on k such that

$$|d_j| \le \frac{B_k}{(n-1)!} \; \forall j < q_k,$$

completing the proof. $\qquad \qquad \square$

Next, we need two more lemmas to assist in the proof of Theorem 23.

Lemma 6. *Let G be a finite group, let $X \subset G$ be inverse-closed and conjugation-invariant, and let $Cay(G, X)$ be the Cayley graph on G with generating set X. Let ρ be an irreducible representation of G with dimension d, and let λ_ρ be the corresponding eigenvalue of $Cay(G, X)$, as in Theorem 10. Then*

$$|\lambda_\rho| \le \frac{\sqrt{|G||X|}}{d}.$$

Proof. Since the irreducible characters of G are orthonormal, we have

$$\langle \chi_\rho, \chi_\rho \rangle = \frac{1}{|G|} \sum_{g \in G} |\chi_\rho(g)|^2 = 1.$$

By the Cauchy-Schwarz inequality,

$$|\langle \chi_\rho, 1_X \rangle| \le \sqrt{\langle \chi_\rho, \chi_\rho \rangle \langle 1_X, 1_X \rangle} = \sqrt{|X|/|G|}.$$

Substituting into Equation 7, we obtain

$$|\lambda_\rho| = \frac{|G||\langle \chi_\rho, 1_X \rangle|}{d} \le \frac{\sqrt{|G||X|}}{d},$$

as required. $\qquad \qquad \square$

Combining this with Lemma 5 yields:

Lemma 7. *Let X be a conjugacy-class of S_n, with cycle-type having a cycle of length at least $n - 2k - 1$. Let $Cay(S_n, X)$ be the Cayley graph on S_n generated by X. For any $C_k > 0$, there exists $D_k > 0$, depending only on k, such that if ρ is an irreducible representation of dimension at least $C_k n^{k+1}$, then the corresponding eigenvalue λ_ρ of $Cay(S_n, X)$ satisfies*

$$|\lambda_\rho| < D_k \frac{(n-1)!|\omega_{n,k}|}{\sqrt{n}}.$$

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Proof. Note that $|\omega_{n,k}| = \Theta(1/n^k)$. By choosing D_k large enough, we may assume that $n > 4k + 2$, so $n - 2k - 1 > n/2$, and therefore the hypotheses of Lemma 5 hold. Assume that $\dim \rho \geq C_k n^{k+1}$. Then

$$
\begin{aligned}
|\lambda_\rho| &\leq \frac{\sqrt{|G||X|}}{C_k n^{k+1}} \\
&\leq \frac{\sqrt{2n!(n-1)!}}{C_k n^{k+1}} \\
&= \frac{\sqrt{2}(n-1)!}{C_k n^{k+1/2}} \\
&\leq D_k \frac{(n-1)!|\omega_{n,k}|}{\sqrt{n}},
\end{aligned}
$$

as required. $\qquad\square$

We can now prove Theorem 23.

Proof of Theorem 23: Assume the hypotheses of Theorem 23. By Lemma 2, we have $\dim[\rho] \geq E_k n^{k+1}$. Hence, we see that

$$
\begin{aligned}
|\lambda_\rho| &= \left| \sum_{j=1}^{q_k-1} d_j \lambda_\rho^{(j)} \right| \\
&\leq (q_k - 1) \max_j |d_j| \max_j |\lambda_\rho^{(j)}| \\
&\leq (q_k - 1) \frac{B_k}{(n-1)!} D_k \frac{(n-1)!|\omega_{n,k}|}{\sqrt{n}} \\
&= c_k |\omega| / \sqrt{n},
\end{aligned}
$$

as required. $\qquad\square$

4.3 The linear combination of Cayley graphs

Finally, we can now construct the linear combination of Cayley graphs we will use to prove our main theorem.

Theorem 24. *There exists a linear combination Y_{even} of Cayley graphs generated by conjugacy-classes of even permutations within FPF_k, such that its eigenvalues are as described in the first line of Table 1, i.e.,*

- $\lambda_{(n)} = 1$,

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- $\lambda_\rho = \omega$ for each fat $\rho \neq (n)$,

- $\lambda_{(1^n)} = 1$,

- $\lambda_\rho = \omega$ for each tall $\rho \neq (1^n)$, and

- $|\lambda_\rho| \leq c_k |\omega|/\sqrt{n} \leq o(|\omega|)$ for each medium ρ, where $c_k > 0$ depends only on k.

Proof. Recall that for each partition $\phi_j > (n-k, 1^k)$, exactly one of ϕ_j and $\mathrm{Split}(\phi_j)$ is even. For each partition $\phi_j > (n-k, 1^k)$, define

$$\mu_j = \begin{cases} \phi_j, & \text{if } X_{\phi_j} \text{ is an even conjugacy class;} \\ \mathrm{Split}(\phi_j), & \text{if } X_{\phi_j} \text{ is an odd conjugacy class.} \end{cases}$$

Then each X_{μ_j} consists of even permutations. Take

$$Y_{\text{even}} = \sum_{j=1}^{q_k - 1} d_j \mathrm{Cay}(S_n, X_{\mu_j}),$$

where the d_j's are as defined in Theorem 22. Then we have $\lambda_{(n)} = 1$, and for each fat $\rho \neq (n)$, we have $\lambda_\rho = \omega$. By Theorem 17, for any partition ρ, we have $\chi_{\rho^t} = \chi_\rho \cdot \mathrm{sgn}$. So by equation (24), for any partition ρ, we have $\lambda_{\rho^t} = \lambda_\rho$, since our Cayley graphs are all generated by conjugacy-classes of even permutations. Therefore, $\lambda_{(1^n)} = 1$, and $\lambda_\rho = \omega$ for each tall $\rho \neq (1^n)$. By Theorem 23, $|\lambda_\rho| \leq c_k |\omega|/\sqrt{n} \leq o(|\omega|)$ for each medium ρ, completing the proof. $\square$

Theorem 25. *There exists a linear combination Y_{odd} of Cayley graphs generated by conjugacy-classes of odd permutations within FPF_k, such that its eigenvalues satisfy*

- $\lambda_{(n)} = 1$,

- $\lambda_\rho = \omega$ for all fat $\rho \neq (n)$,

- $\lambda_{(1^n)} = -1$,

- $\lambda_\rho = -\omega$ for each tall $\rho \neq (1^n)$, and

- $|\lambda_\rho| \leq c_k |\omega|/\sqrt{n} \leq o(|\omega|)$ for each medium ρ, where $c_k > 0$ depends only on k.

Proof. For each partition $\phi_j > (n - k, 1^k)$, define

$$\mu_j = \begin{cases} \phi_j, & \text{if } X_{\phi_j} \text{ is an odd conjugacy class;} \\ \text{Split}(\phi_j), & \text{if } X_{\phi_j} \text{ is an even conjugacy class.} \end{cases}$$

Then each X_{μ_j} consists of odd permutations. Take

$$Y_{\text{odd}} = \sum_{j=1}^{q_k - 1} d_j \text{Cay}(S_n, X_{\mu_j}),$$

where the d_j's are as defined in Theorem 22. Then we have $\lambda_{(n)} = 1$, and for each fat $\rho \neq (n)$, we have $\lambda_\rho = \omega$. This time, our Cayley graphs are all generated by odd permutations, so we have $\lambda_{\rho^t} = -\lambda_\rho$ for every partition ρ. Hence, $\lambda_{(1^n)} = -1$, and $\lambda_\rho = -\omega$ for each tall $\rho \neq (1^n)$. Again, by Theorem 23, $|\lambda_\rho| \leq c_k |\omega| / \sqrt{n} \leq o(|\omega|)$ for each medium ρ, completing the proof. $\square$

Theorem 26. *There exists a linear combination Y of Cayley graphs generated by conjugacy-classes within FPF_k, such that its eigenvalues satisfy:*

- $\lambda_{(n)} = 1$,

- $\lambda_\rho = \omega$ *for each fat $\rho \neq (n)$,*

- $\lambda_\rho = 0$ *for each tall ρ, and*

- $|\lambda_\rho| \leq c_k |\omega| / \sqrt{n} \leq o(|\omega|)$ *for each medium ρ, where $c_k > 0$ depends only on k.*

Proof. Set $Y = \frac{1}{2}(Y_{\text{even}} + Y_{\text{odd}})$. $\square$

Proof of Theorems 5 and 6: If n is sufficiently large depending on k, then $c_k |\omega| / \sqrt{n} < |\omega|$, so ω is both the minimum eigenvalue of Y and the second-largest in absolute value. By our choice of ω, applying the generalized Hoffman Theorem 12 to Y, we see that if $I \subset S_n$ is k-intersecting, then $|I| \leq (n - k)!$, and that if equality holds, then the characteristic function $1_I \in V_k$. Similarly, applying Theorem 13 to Y, we see that if $I, J \subset S_n$ are k-cross-intersecting, then $|I||J| \leq ((n - k)!)^2$, and that if equality holds, then $1_I, 1_J \in V_k$. $\square$

5 Boolean functions

Recall that V_k is the linear space of real-valued functions on S_n whose Fourier transform is supported only on irreducible representations $\geq (n - k, 1^k)$. Theorem 7 states that V_k is spanned by the k-cosets of S_n. We now wish to characterize the Boolean functions in V_k, proving (a strengthened version of) Theorem 8.

We will show that every *non-negative* function in V_k can be written as a linear combination of the characteristic functions of k-cosets *with non-negative coefficients*, and every $0/1$ valued function in V_k can be written with $0/1$ coefficients.

Let $\mathcal{A}_k$ be the set of ordered k-tuples of distinct numbers between 1 and n, and let $(n)_k = |\mathcal{A}_k| = \frac{n!}{(n-k)!} = n(n-1)\ldots(n-k+1)$. We will prove the following

Theorem 27. *Let $f \in V_k$ be non-negative. Then there exist non-negative coefficients $(b_{\alpha,\beta})_{\alpha,\beta \in \mathcal{A}_k}$ such that $f = \sum b_{\alpha,\beta} 1_{T_{\alpha \mapsto \beta}}$. Furthermore, if f is Boolean, then f is the characteristic function of a disjoint union of k-cosets.*

For didactic reasons, we begin by dealing with the case $k = 1$.

Theorem 28. *If $f \in V_1$ is nonnegative, then there exist $b_{i,j} \geq 0$ such that $f = \sum_{i,j} b_{i,j} 1_{T_{i \mapsto j}}$.*

Proof. We say a real $n \times n$ matrix $A = (a_{i,j})_{i,j \in [n]}$ *represents* a function $f \in V_1$ if

$$f = \sum_{i,j} a_{i,j} 1_{T_{i \mapsto j}},$$

or equivalently,

$$f(\sigma) = \sum_{i=1}^{n} a_{i,\sigma(i)} \quad \forall \sigma \in S_n.$$

Let A be a matrix representing f. Our task is to find a *non-negative* matrix B which also represents f. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers such that

$$\sum_{i=1}^{n} x_i + \sum_{j=1}^{n} y_j = 0.$$

Let $B = (b_{i,j})_{i,j \in [n]}$ be the matrix produced from A by adding x_i to row i for each i, and then adding y_j to column j for each j, i.e.

$$b_{i,j} = a_{i,j} + x_i + y_j \quad (i,j \in [n]).$$

Note that B also represents f. We wish to show that there exists a choice of x_i's and y_j's such that the matrix B has all its entries non-negative, i.e. we wish to solve the system of inequalities

$$x_i + y_j \geq -a_{i,j} \ (1 \leq i, j \leq n) \quad \text{subject to} \quad \sum_i x_i + \sum_j y_j = 0. \quad (30)$$

By the strong duality theorem of linear programming (see [23]), this is unsolvable if and only if there exist $c_{i,j} \geq 0$ such that

$$\sum_j c_{i,j} = 1 \ (1 \leq i \leq n), \quad \sum_i c_{i,j} = 1 \ (1 \leq j \leq n), \quad \sum_{i,j} c_{i,j} a_{i,j} < 0.$$

Suppose for a contradiction that this holds. The matrix $C = (c_{i,j})_{i,j \in [n]}$ is bistochastic, and therefore by Birkhoff's theorem (see [4]), it can be written as a convex combination of permutation matrices,

$$C = \sum_{t=1}^{r} s_t P_{\sigma_t},$$

where $s_t \geq 0 \ (1 \leq t \leq r)$, $\sum_{t=1}^{r} s_t = 1$, $\sigma_1, \ldots, \sigma_t \in S_n$, and P_σ denotes the permutation matrix of σ, i.e. $(P_\sigma)_{i,j} = 1_{\{\sigma(i)=j\}}$. But then $\sum_{i,j} c_{i,j} a_{i,j}$ is a convex combination of values of f:

$$\sum_{i,j} c_{i,j} a_{i,j} = \sum_{t=1}^{r} s_t \sum_{i,j} (P_{\sigma_t})_{i,j} a_{i,j} = \sum_{t=1}^{r} s_t \sum_{i=1}^{n} a_{i,\sigma_t(i)} = \sum_{t=1}^{r} s_t f(\sigma_t) \quad (31)$$

and is therefore non-negative, a contradiction. Hence, the system (30) is solvable, proving the theorem. $\qquad \square$

The following corollary is immediate.

Corollary 2. *If $f \in V_1$ is Boolean, then it is the characteristic function of a disjoint union of 1-cosets.*

Proof. By induction on the number of non-zero values of f. Let $f \in V_1$ be a Boolean function, and suppose the statement is true for all Boolean functions with fewer non-zero values. By Theorem 28, there exist $b_{i,j} \geq 0$ such that

$$f = \sum_{i,j} b_{i,j} 1_{T_{i \mapsto j}}.$$

Choose i, j such that $b_{i,j} > 0$. Then $f > 0$ on $T_{i \mapsto j}$, so $f = 1$ on $T_{i,j}$. Hence, $f - 1_{T_{i \mapsto j}}$ is also Boolean, with fewer non-zero values than f, and so by the induction hypothesis, it is the characteristic function of a disjoint union of 1-cosets. Hence, the same is true of f, as required. $\qquad \square$

We now extend this proof to the case $k > 1$, proving Theorem 27. However, some preliminaries are necessary.

Let A be an $(n)_k \times (n)_k$ matrix with rows and columns indexed by $\mathcal{A}_k$. Choose an ordering of the k coordinates of the k-tuples, or equivalently a permutation $\pi \in S_k$, and consider the natural lexicographic ordering on $\mathcal{A}_k$ induced by this ordering, i.e. $\alpha < \beta$ iff $\alpha_{\pi(m)} < \beta_{\pi(m)}$, where $m = \min\{l : \alpha_{\pi(l)} \neq \beta_{\pi(l)}\}$. This lexicographic ordering on $\mathcal{A}_k$ recursively partitions A into blocks: first it partitions A into n^2 $(n-1)_{k-1} \times (n-1)_{k-1}$ blocks $B_{i,j}$ according to the $\pi(1)$-coordinate of each k-tuple; then it partitions each block $B_{i,j}$ into $(n-1)^2$ $(n-2)_{k-2} \times (n-2)_{k-2}$ sub-blocks according to the $\pi(2)$-coordinate of each k-tuple, and so on.

The following two recursive definitions concerning such matrices will be crucial.

Definition 13. *We define a k-line in an $(n)_k \times (n)_k$ matrix A as follows. If $k = 1$, i.e. A is an $n \times n$ matrix, a 1-line in A is just a row or column of A. If $k > 1$, and A is an $(n)_k \times (n)_k$ matrix, a k-line in A is given by choosing an ordering π of the k-coordinates, partitioning A into n^2 blocks according to the $\pi(1)$-coordinate of each k-tuple as above, choosing a row or column of $(n-1)_{k-1} \times (n-1)_{k-1}$ blocks $B_{i,j}$, and then taking a union of $(k-1)$-lines, one from each block.*

Definition 14. *We say that an $(n)_k \times (n)_k$ matrix A is k-bistochastic if the following holds. If $k = 1$, an $n \times n$ matrix A is 1-bistochastic if it is bistochastic in the usual sense, i.e. all its entries are non-negative, and all its row and column sums are 1. If $k > 1$, an $(n)_k \times (n)_k$ matrix A is k-bistochastic if, for any partition into n^2 blocks $B_{i,j}$ of size $(n-1)_{k-1} \times (n-1)_{k-1}$ according to a lexicographic order on $\mathcal{A}_k$ induced by any of the $k!$ orderings of the coordinates, there exists a bistochastic $n \times n$ matrix $R = (r_{i,j})$ and n^2 $(k-1)$-bistochastic matrices $M_{i,j}$ of order $(n-1)$, such that $B_{i,j} = r_{i,j} M_{i,j}$.*

The following two self-evident observations indicate the relevance of these two notions.

Observation 1. *An $(n)_k \times (n)_k$ matrix with non-negative entries is k-bistochastic if and only if the sum of the entries of every k-line in it (coming from any of the $k!$ recursive partitions into blocks) is 1.*

Observation 2. *Given a k-line ℓ, the corresponding k-cosets $\{T_{\alpha \mapsto \beta} : (\alpha, \beta) \in \ell\}$ partition S_n.*

Any permutation $\sigma \in S_n$ induces a permutation of $\mathcal{A}_k$; we write $P_\sigma^{(k)}$ for the corresponding $(n)_k \times (n)_k$ permutation matrix, i.e.

$$(P_\sigma^{(k)})_{\alpha,\beta} = 1_{\{\sigma(\alpha)=\beta\}}.$$

These $n!$ permutations are only a small fraction of all $((n)_k)!$ permutations of $\mathcal{A}_k$. It is not hard to show that any $P_\sigma^{(k)}$ is k-bistochastic. It is a bit harder to show that any other permutation matrix is not k-bistochastic (for this it is important to note that we demand taking into account lexicographic orderings induced by all possible orderings of the coordinates.)

The authors thank Siavosh Benabbas for suggesting the correct formulation of the following theorem and for help in proving it.

Theorem 29. *(Generalized Birkhoff theorem)[Benabbas, Friedgut, Pilpel] An $(n)_k \times (n)_k$ matrix M is k-bistochastic if and only if it is a convex combination of $(n)_k \times (n)_k$ matrices of permutations of $\mathcal{A}_k$ which are induced by permutations of $[n]$.*

Proof. By induction on k.

For $k = 1$, this is Birkhoff's theorem.

For the induction step, let $M = r_{i,j} \cdot M_{i,j}$ be a block decomposition of M as in the definition of k-bistochasticity, according to the natural ordering of the coordinates, i.e. $\pi = \mathrm{id}$. By Birkhoff's theorem, $R = (r_{i,j})$ is a convex combination of permutation matrices, and therefore it is either a permutation matrix, or else it can be written as a convex combination $R = sP + (1 - s)T$, with P a permutation matrix, and T a bistochastic matrix with more zero entries than R. Treating P separately, and proceeding in this manner by induction, we may assume that R is a permutation matrix, and without loss of generality, we may assume that $R = I$, the identity matrix. So in M, every non-zero entry is indexed by a pair of k-tuples of the form $((i, *, *, \ldots, *), (i, *, *, \ldots, *))$ for some i.

Now, we reorder the rows and columns of M with a lexicographic order on $\mathcal{A}_k$ determined by the transposition $\pi = (1\ 2)$. Consider now an off-diagonal block in this ordering of the form $((*, i, *, *, \ldots, *), (*, j, *, *, \ldots, *))$ with $i \neq j$. It breaks into $(n-1)^2$ sub-blocks of size $(n-2)_{k-2} \times (n-2)_{k-2}$, but only $n-2$ of them have nonzero entries, namely the sub-blocks of the form $((l, i, *, *, \ldots, *), (l, j, *, *, \ldots, *))$ for some $l \neq i, j$. Hence, the block has a row of zero sub-blocks, namely the sub-blocks

$$((j, i, *, *, \ldots, *), (m, j, *, *, \ldots, *))$$

for $m \neq j$, and a column of zero sub-blocks, namely the sub-blocks

$$((m, i, *, *, \ldots, *), (i, j, *, *, \ldots, *))$$

for $m \neq i$. Since the block is $(k-1)$-bistochastic, it must be the zero matrix.

As for the diagonal blocks of M in the new ordering of $\mathcal{A}_k$, blocks of the form $((*, i, *, *, \ldots, *), (*, i, *, *, \ldots, *))$, they can only have nonzero sub-blocks on the diagonal. Since they are $(k-1)$-bistochastic, by the induction hypothesis they are each equal to a convex linear combination of $P_\sigma^{(k-1)}$'s where each σ is a permutation of $[n] \setminus \{i\}$. But if any of these σ's has $\sigma(l) \neq l$ for some l, then the off-diagonal sub-block

$$((l, i, *, *, \ldots, *), (\sigma(l), i, *, *, \ldots, *))$$

is non-zero, a contradiction. Hence, each permutation σ must be the identity, and therefore all the diagonal blocks are copies of the $(n-1)_{k-1} \times (n-1)_{k-1}$ identity matrix. Hence, M is the identity matrix, and we are done. $\qquad\square$

We now can prove Theorem 27.

Proof of Theorem 27: We follow the lines of the proof of Theorem 28. Let $f \in V_k$ be non-negative, and let $M = (M_{\alpha,\beta})_{\alpha,\beta \in \mathcal{A}_k}$ be an $(n)_k \times (n)_k$ matrix (with rows and columns indexed by $\mathcal{A}_k$) which represents f, meaning that

$$f = \sum_{\alpha,\beta \in \mathcal{A}_k} M_{\alpha,\beta} 1_{T_{\alpha \mapsto \beta}},$$

i.e.

$$f(\sigma) = \sum_{\alpha \in \mathcal{A}_k} M_{\alpha, \sigma(\alpha)} \quad \forall \sigma \in S_n.$$

Let $\mathcal{L}_k$ be the set of all k-lines. Recall from Observation 2 that every k-line corresponds to a partition of S_n into k-cosets. Therefore, adding a constant $x \in \mathbb{R}$ to a single k-line in M corresponds to increasing f by x. We therefore associate with every line $\ell \in \mathcal{L}_k$ a variable x_ℓ. We wish to solve the following system of linear inequalities

$$\sum_{\ell:(\alpha,\beta) \in \ell} x_\ell \geq -M_{\alpha,\beta} \ (\ell \in \mathcal{L}_k) \quad \text{subject to} \quad \sum_{\ell \in \mathcal{L}_k} x_\ell = 0.$$

Again, by the strong duality theorem of linear programming, this is possible unless there exists an $(n)_k \times (n)_k$ matrix $C = (c_{\alpha,\beta})_{\alpha,\beta \in \mathcal{A}_k}$ with non-negative entries, such that

$$\sum_{(\alpha,\beta) \in \ell} c_{\alpha,\beta} = 1 \quad \forall \ell \in \mathcal{L}_k,$$

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and
$$\sum_{\alpha,\beta \in \mathcal{A}_k} c_{\alpha,\beta} M_{\alpha,\beta} < 0.$$

By Observation 1, C is k-bistochastic. By the Generalized Birkhoff Theorem, C is in the convex hull of the permutation matrices induced by permutations of $[n]$. This means, as before, that

$$\sum_{\alpha,\beta \in \mathcal{A}_k} c_{\alpha,\beta} M_{\alpha,\beta}$$

is a convex linear combination of values of f, and is therefore non-negative, a contradiction. Therefore, the system of linear inequalities above is solvable, proving the first part of the theorem. The second part follows as in the $k = 1$ case, by induction on the number of non-zero values of f. $\qquad \square$

This completes the proof of Theorems 3 and 4.

6 Open Problems

In their landmark paper [1], Ahlswede and Khachatrian characterized the largest k-intersecting subsets of $[n]^{(r)}$ for every value of k, r and n. In our opinion, the most natural open problem in the area is to characterize the largest k-intersecting subsets of S_n for every value of n and k. We make the following

Conjecture 2. *A k-intersecting family in S_n with maximum size must be a 'double translate' of one of the families*

$$M_i = \{\sigma \in S_n : \ \sigma \ has \ \geq k+i \ fixed \ points \ in \ [k+2i]\} \ (0 \leq i \leq (n-k)/2),$$

i.e. of the form $\sigma M_i \tau$ where $\sigma, \tau \in S_n$.

This would imply that the maximum size is $(n - k)!$ for $n > 2k$. It is natural to ask how large n must be for our method to give this bound, i.e. when there exists a weighted graph Y which is a real linear combination of Cayley graphs on S_n generated by conjugacy-classes in FPF_k, such that Y has maximum eigenvalue 1 and minimum eigenvalue

$$\omega_{n,k} = -\frac{1}{n(n-1)\ldots(n-k+1) - 1}.$$

It turns out that this need not be the case when $n = 2k + 2$; indeed, it fails for $k = 2$ and $n = 6$. We believe that new techniques will be required to

prove Conjecture 2.

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