Mean field limit of a continuous time finite state game

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Abstract

Mean field games is a recent area of study introduced by Lions and Lasry in a series of seminal papers in 2006. Mean field games model situations of competition between large number of rational agents that play non-cooperative dynamic games under certain symmetry assumptions. They key step is to develop a mean field model, in a similar way that what is done in statistical physics in order to construct a mathematically tractable model. A main question that arises in the study of such mean field problems is the rigorous justification of the mean field models by a limiting procedure.

In this paper we consider the mean field limit of two-state Markov decision problem as the number of players $N \to \infty$. First we establish the existence and uniqueness of a symmetric partial information Markov perfect equilibrium. Then we derive a mean field model and characterize its main properties. This mean field limit is a system of coupled ordinary differential equations with initial-terminal data. Our main result is the convergence as $N \to \infty$ of the N player game to the mean field model and an estimate of the rate of convergence.

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1 Introduction

Mean field games is a recent area of research started by Pierre Louis Lions and Jean Michel Lasry [LL06a, LL06b, LL07a, LL07b] which attempts to understand the limiting behavior of systems involving very large numbers of rational agents which play dynamic games under partial information and symmetry assumptions. Inspired by ideas in statistical physics, Lions and Lasry introduced a class of models in which the individual player contribution is encoded in a mean field that contains only statistical properties about the ensemble. A key question is how to derive such effective or mean field equations that drive the system as well as to show convergence as the number of agents increases to infinity. The literature on mean field games and its applications is growing fast, for a recent survey see [LLG10b] and reference therein. Applications of mean field games arise in the study of growth theory in economics [LLG10a] or environmental policy [ALT], for instance, and it is likely that in the future they will play an important rôle in economics and population models. There is also a growing interest in numerical methods for these problems [ALT], [AD10]. The authors [GMS10] have also considered the discrete time, finite state problem.

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In this paper we consider the mean field limit of games between a large number of players that are allowed to switch between two states. We are particularly interested in understanding the limit as the number of players increases to infinity. We should stress the the fact that we are considering only two states plays no special rôle and we could easily generalize our results to any finite number of states.

A typical application of our models concerns the adoption or change of a technology or services. For instance, a single agent faced with different social networks will have a incentive to move to the network with more potential contacts, however other effects play a role in this player decision, such as the level of services, trouble of changing network, loss of contacts and so on. Another similar example concerns switching between cell phone companies.

We start in Section 2 to model the $N+1$ player problem as a Markov decision process. We assume that N of the players have a fixed Markov switching strategy β and then look at a reference player which looks to minimize a certain performance criterion by choosing a suitable switching strategy $\alpha(\beta)$. This is a well know Markov decision problem. The key novelty in this section consists in showing the existence of a Nash equilibrium such that $\alpha(\beta) = \beta$ and its characterization through a non-linear ordinary differential equation. In fact, this is a continuous time, partial information, symmetric version of the Markov perfect equilibrium notion that has been studied (mostly in discrete time or stationary setting) in [PS09, Liv02, MT01, Str93], and references therein. In [PM01, Sle01] symmetric Markov perfect equilibrium are also considered, and in the last paper the case with an infinite number of players is studied. In [Kap95] the passage from discrete time to continuous time is considered for N players in a war of attrition problem.

In Section 3 we derive a mean field model for the optimal switching policy of a reference player given the fraction $\theta(t)$ of players in one of the states. This model turns out to be a coupled system of ordinary differential equations, where one equation governs the evolution of θ, and is subjected to initial conditions, whereas the other equation models the evolution of the value function and has terminal data. We call this problem the initial-terminal value problem. Initial terminal value problems are in fact a general feature in many mean field game problems, see for instance [LL06a, LL06b, LL07a]. Of course, existence and uniqueness of solutions is not immediate from the general ODE theory but, adapting the methods of Lions and Lasry we were successful in establishing both.

Our main result, theorem 4, is discussed in Section 4 where we prove the convergence as the number of players $N \to \infty$ to a mean field model.

2 The $N+1$ player game

In this section we consider symmetric games between $N+1$ players under a symmetric partial information pattern. We start by discussing the framework of this problem, namely controlled Markov Dynamics, §2.1, admissible controls §2.3, and the individual player problem §2.4. Then in §2.5 we discuss the main assumptions on running and terminal cost that allow us to use Hamilton-Jacobi ODE methods, in §2.6 to solve the $N+1$ player problem. Maximum principle type estimates are considered in §2.7 which are then applied to establishing the existence of Nash equilibrium solutions, §2.8. This section ends with an example §2.9.

2.1 Controlled Markov Dynamics

We consider a dynamic game between $N+1$ players that are allowed to switch between two states denoted by 0 and 1. We suppose that all players are identical and so the game is symmetric with respect to permutation of the players. To describe the game we will use a reference player, which could be chosen as any one of the players.

If we fix any player as the reference player, we will suppose that he knows his own state at time t, given by $i(t)$, and also knows the number $n(t)$ of remaining players that are in state 0. $i(t)$ and $n(t)$ are stochastic processes that we will describe in the following. No further information is available to the reference player. Because the game is symmetric, the identity of the reference player is not important, and all other players have access to the same kind of information, i.e., its own state and the fraction of other players in state 0.

We suppose the process $(n(t), i(t))$ is a continuous time Markov process: the reference player follows a controlled Markov process $i(t)$ with transition rates from state i to the other state $1-i$ given by $\beta = \beta(i, n, t)$. More precisely we have

$$\mathbb{P}\Big(i(t+h) = 1-i \| n(t) = n, i(t) = i\Big) = \beta(i, n, t).h + o(h)\,,$$

where $\lim \frac{o(h)}{h} = 0$ when $h \to 0$. Because of the symmetry of the game, all other players follow their own Markov process controlled by the same transition rate function $\beta : \{0,1\} \times \{0, ..., N\} \times [0, +\infty) \to [0, +\infty)$. Note that the rate function β is a deterministic time-dependent function, which makes $(n(t), i(t))$ a non-time homogeneous Markov process. We will suppose that β is bounded and continuous as a function of time. We will refer to any Markov control with rate function which is bounded and continuous on time, as an *admissible control*.

The transition rates of the process $n(t)$ are given by

$$\begin{aligned} \gamma^+_\beta(i, n, t) &= (N-n)\beta(1, n+1-i, t)\,, \\ \gamma^-_\beta(i, n, t) &= n\beta(0, n-i, t)\,, \end{aligned} \tag{1}$$

where γ^+_β stands for the transition rate from n to $n+1$, and γ^-_β is the transition rate from n to $n-1$. Note that $n+1-i$ is the total number of players in state 0, as seen by a player (distinct from the reference player) in state 1 whereas $n-i$ is the number of players in state 0 as seen by a player (distinct from the reference player) in state 0.

More precisely, we have

$$\begin{aligned} \mathbb{P}\Big(n(t+h) = n+1 \| n(t) = n, i(t) = i\Big) &= \gamma^+_\beta(i, n, t).h + o(h)\,, \\ \mathbb{P}\Big(n(t+h) = n-1 \| n(t) = n, i(t) = i\Big) &= \gamma^-_\beta(i, n, t).h + o(h)\,, \end{aligned}$$

where $\lim \frac{o(h)}{h} = 0$ when $h \to 0$.

We assume further that the state transitions of the different players are independent, conditioned on i and n. Note that no information is available to any player concerning the state of any other individual player. All each player knows is its position and the number of other players in state 0, which mean, the fraction of other players in each one of the states 0 and 1.

2.2 A control problem

Let now $T > 0$, and let $c : \{0,1\} \times [0,1] \times \mathbb{R}^+_0 \to \mathbb{R}$ and $\psi : \{0,1\} \times [0,1] \to \mathbb{R}$ be two (non-negative) functions. We will discuss the precise hypothesis on c and ψ in section 2.5. We suppose $c\left(i, \frac{n}{N}, \beta\right)$ represents a running cost incurred by the reference player when he is in state i, n of the remaining N players are in state 0 and this player has a transition rate β from i to $1-i$. We also suppose $\psi\left(i, \frac{n}{N}\right)$ represents a terminal cost incurred by the reference player at the terminal time T, if he ends up at time T in state i and at that time n of the other players are in state 0.

If $A_t(i, n)$ denotes the event $i(t) = i$ and $n(t) = n$, the expected total cost of the reference player, giving the control β and conditioned on the event $A_t(i, n)$, will be

$$V^\beta(i, n, t) = \mathbb{E}^\beta_{A_t(i,n)}\left[\int_t^T c\left(i(s), \frac{n(s)}{N}, \beta(s)\right) ds + \psi\left(i(T), \frac{n(T)}{N}\right)\right]\,.$$

We could be interested in finding an admissible control β that minimizes, for each (i, n, t), the function V defined above. This however would require a cooperative behavior between players and it would be an usual stochastic optimal control problem. Instead, we are interested in finding an admissible control β that is a symmetric Nash equilibria for the game which we will soon describe.

2.3 The Dynkin formula

Given two admissible controls β and α, we can define a non-time homogeneous Markov process $(n(t), i(t))$ where the transition rates for n are given by (1) and the transition rate for i is given by α as

$$\mathbb{P}\Big(i(t+h) = 1-i \| n(t) = n, i(t) = i\Big) = \alpha(i,n,t).h + o(h)\,,$$

where $\lim \frac{o(h)}{h} = 0$ when $h \to 0$. The idea here is that, while other players use the control β, the reference player can choose another control α.

Furthermore, we have that, for any function $\varphi : \{0,1\} \times \{0,1,2,...,N\} \times [0,+\infty) \to \mathbb{R}$, smooth in the last variable, and any $s > t$,

$$\mathbb{E}^{\beta,\alpha}_{A_t(i,n)}\left[\varphi(i(s),n(s),s) - \varphi(i,n,t)\right] = \mathbb{E}^{\beta,\alpha}_{A_t(i,n)}\left[\int_t^s \frac{d\varphi}{dt}(i,n,r) + A^{\beta,\alpha}\varphi(i,n,r)dr\right]\,, \tag{2}$$

where $A_t(i,n)$ still denotes the event $i(t) = i$ and $n(t) = n$, and

$$\begin{aligned} A^{\beta,\alpha}\varphi(i,n,r) &= \alpha(i,n,r)(\bar{\varphi} - \varphi)(i,n,r) + \\ &+ \gamma^+_\beta(i,n,r)(\varphi(i,n+1,r) - \varphi(i,n,r)) + \gamma^-_\beta(i,n,r)(\varphi(i,n-1,r) - \varphi(i,n,r))\,, \end{aligned} \tag{3}$$

where γ^+_β and γ^-_β are defined by (1), and $\bar{\varphi}(i,n,t) = \varphi(1-i,n,t)$.

We call $A^{\alpha,\beta}$ the generator of the process and (2) the Dynkin's formula in analogy to the Dynkin's formula in stochastic calculus.

2.4 Individual player point of view - introducing the game

Now we suppose the reference player decides unilaterally to use a different control, trying to improve its value function.

We will suppose the other players continue to follow the Markov Chain with transition rate $\beta(i,n,t)$, bounded and continuous on time. Therefore $n(t)$, the number of such players that are in state 0, is a process to which correspond transition rates γ^+_β and γ^-_β as in (1).

The reference player looks for an admissible control α, possibly different from β, that minimizes

$$u(i,n,t,\beta,\alpha) = \mathbb{E}^{\beta,\alpha}_{A_t(i,n)}\left[\int_t^T c\left(i(s), \frac{n(s)}{N}, \alpha(s)\right)ds + \psi\left(i(T), \frac{n(T)}{N}\right)\right]\,.$$

That is, reference player looks for the control α which is a solution to the minimization problem

$$u(i,n,t;\beta) = \inf_\alpha u(i,n,t,\beta,\alpha),$$

where the minimization is performed over the set of all admissible controls α. We will call the function $u(i,n,t;\beta)$ above the value function for the reference player associated to the strategy β of the remaining N players. The control α that attains the minimum above can be called the best response of any player to a control β.

2.5 Assumptions on running and terminal cost

We discuss now the main hypothesis used in this paper concerning the running and terminal costs. We suppose that both the running cost $c = c(i,\theta,\alpha) : \{0,1\} \times [0,1] \times \mathbb{R}^+_0 \to \mathbb{R}$ and the terminal cost $\psi = \psi(i,\theta) : \{0,1\} \times [0,1] \to \mathbb{R}$ are non-negative functions, as mentioned in the previous section, and also that they are Lipschitz continuous in θ. Of course, our results would still be valid without any change if c and ψ are simply bounded below, instead of being non-negative.

We assume that $c(i,\theta,\alpha)$ is uniformly convex on $\alpha \geq 0$ and superlinear. We assume further that c is differentiable, and $c'(\theta,\alpha)$ is Lipschitz in the variable θ.

For $p \in \mathbb{R}$ we define

$$h(p,\theta,i) = \min_{\alpha \geq 0} \left[c(i,\theta,\alpha) + \alpha p\right].$$

Note that h is an increasing concave function of p, Lipschitz in θ, and, hence, bounded below by

$$\min_{\theta \in [0,1], i \in \{0,1\}} h(0,\theta,i).$$

Because of the uniform convexity the minimum is achieved at a single point, and the function

$$\alpha^*(p,\theta,i) = \operatorname{argmin}_{\alpha \geq 0} \left[c(i,\theta,\alpha) + \alpha p\right].$$

is well defined. Furthermore we have

Proposition 1. *The function α^* is locally Lipschitz in p, uniformly in $\theta \in [0,1]$. Furthermore it is uniformly Lipschitz in θ.*

Proof. We will use the following inequalities, which are consequence of the uniform convexity of c: for all $\theta, \alpha', \alpha, p$ and p', we have

$$c(\theta,\alpha') + \alpha' p' \geq c(\theta,\alpha) + \alpha p' + (c'(\theta,\alpha) + p')(\alpha' - \alpha) + \gamma|\alpha' - \alpha|^2, \tag{4}$$

and because $\alpha^*(p,\theta)$ is a minimizer,

$$(c'(\theta,\alpha^*(p,\theta)) + p)(\alpha' - \alpha^*(p)) \geq 0\,. \tag{5}$$

We will first prove that α^* is uniformly Lipschitz in p : for that, we suppose that θ is fixed. By the definition of α^* and equation (4) we have

$$c(\alpha^*(p)) + \alpha^*(p)p' \geq c(\alpha^*(p')) + \alpha^*(p')p' \geq$$

$$\geq c(\alpha^*(p)) + \alpha^*(p)p' + (c'(\alpha^*(p)) + p')(\alpha^*(p') - \alpha^*(p)) + \gamma|\alpha^*(p') - \alpha^*(p)|^2,$$

hence

$$0 \geq (c'(\alpha^*(p)) + p)(\alpha^*(p') - \alpha^*(p)) + (p' - p)(\alpha^*(p') - \alpha^*(p)) + \gamma|\alpha^*(p') - \alpha^*(p)|^2.$$

Now using equation (5) we obtain

$$0 \geq (p' - p)(\alpha^*(p') - \alpha^*(p)) + \gamma|\alpha^*(p') - \alpha^*(p)|^2.$$

Therefore

$$|p' - p|\,|\alpha^*(p') - \alpha^*(p)| \geq \gamma|\alpha^*(p') - \alpha^*(p)|^2\,,$$

which implies

$$|\alpha^*(p') - \alpha^*(p)| \leq \frac{1}{\gamma}|p' - p\,|.$$

This shows that α^* is uniformly Lipschitz in p.

Now we prove that α^* is Lipschitz in θ: for that, we suppose that p is fixed. Again by the definition of α^* and by equation (4) we have

$$c(\theta',\alpha^*(\theta)) + \alpha^*(\theta)p \geq c(\theta',\alpha^*(\theta')) + \alpha^*(\theta')p$$

$$\geq c(\theta',\alpha^*(\theta)) + \alpha^*(\theta)p + c'(\theta',\alpha^*(\theta))(\alpha^*(\theta') - \alpha^*(\theta)) + \gamma|\alpha^*(\theta') - \alpha^*(\theta)|^2,$$

and then

$$0 \geq c'(\theta', \alpha^*(\theta))(\alpha^*(\theta') - \alpha^*(\theta)) + \gamma|\alpha^*(\theta') - \alpha^*(\theta)|^2.$$

Using equation (5) we get

$$0 \geq [c'(\theta', \alpha^*(\theta)) - c'(\theta, \alpha^*(\theta))](\alpha^*(\theta') - \alpha^*(\theta)) + \gamma|\alpha^*(\theta') - \alpha^*(\theta)|^2.$$

As $c'(\theta, \alpha)$ is Lipschitz in the variable θ we have

$$0 \geq -K|\theta' - \theta|\,|\alpha^*(\theta) - \alpha^*(\theta')| + \gamma|\alpha^*(\theta') - \alpha^*(\theta)|^2.$$

Therefore

$$|\alpha^*(\theta) - \alpha^*(\theta')| \leq \frac{K}{\gamma}|\theta - \theta'|\,,$$

which implies that α^* is Lipschitz in θ. □

2.6 The Hamilton-Jacobi ODE

Fix a admissible control β. Consider the system of ODE´s indexed by i and n given by

$$\begin{aligned} -\frac{d\varphi}{dt}(i,n,t) =& \gamma^+_\beta(i,n,t)(\varphi(i,n+1,t) - \varphi(i,n,t)) + \gamma^-_\beta(i,n,t)(\varphi(i,n-1,t) - \varphi(i,n,t)) \\ &+ h\left(\bar{\varphi}(i,n,t) - \varphi(i,n,t), \frac{n}{N}, i\right)\,, \end{aligned}$$

where $\bar{\varphi}_\beta(i,n,t) = \varphi_\beta(1-i,n,t)$, and γ^+_β and γ^-_β are given by (1). Since $\gamma^-_\beta(i,0,t) = 0$ and $\gamma^+_\beta(i,N,t) = 0$, the evaluation of φ at $n+1$ and $n-1$ does not cause problems outside the range, resp. when $n = N$ or $n = 0$). By setting $\varphi_n(i,t) = \varphi(i,n,t)$ we write the previous ODE in compact notation:

$$-\frac{d\varphi_n}{dt} = \gamma^+_\beta(\varphi_{n+1} - \varphi_n) + \gamma^-_\beta(\varphi_{n-1} - \varphi_n) + h\left(\bar{\varphi}_n - \varphi_n, \frac{n}{N}, i\right)\,. \tag{6}$$

This system of ODE is called the Hamilton-Jacobi (HJ) ODE for player $N+1$ associated to the strategy β of the remaining N players. We start by proving a verification theorem, which is completely analogous to the optimal control verification theorem, see [FS06] for instance.

Theorem 1. *Let φ_β be a solution to* (6) *satisfying the terminal condition $\varphi_\beta(i,n,T) = \psi\left(i, \frac{n}{N}\right)$. Then*

$$u(i,n,t;\beta) = \varphi_\beta(i,n,t)\,.$$

Also, the control

$$\bar{\alpha}(\beta)(i,n,t) \equiv \alpha^*\left(\bar{\varphi}_\beta(i,n,t) - \varphi_\beta(i,n,t), \frac{n}{N}, i\right), \tag{7}$$

is admissible and satisfies

$$u(i,n,t;\beta) = u(i,n,t,\beta,\bar{\alpha}(\beta))\,.$$

Thus a classical solution to the HJ equation associated to β is the value function corresponding to β and determines an optimal admissible control $\bar{\alpha}(\beta)$, for the reference player.

Proof. Let α be an admissible control. By (2) we have

$$\mathbb{E}^{\beta,\alpha}_{A_t(i,n)}\left[\varphi_\beta(i(T),n(T),T)\right] - \varphi_\beta(i,n,t) = \mathbb{E}^{\beta,\alpha}_{A_t(i,n)}\left[\int_t^T \frac{d\varphi_\beta}{dt}(i,n,r) + A^{\beta,\alpha}\varphi_\beta(i,n,r)dr\right],$$

where $A^{\beta,\alpha}$ is given by (3). Adding

$$\mathbb{E}^{\beta,\alpha}_{A_t(i,n)}\left[\int_t^T c\left(i(r),\frac{n(r)}{N},\alpha(r)\right)dr\right]+\varphi_\beta(i,n,t)\,,$$

to both sides of the previous identity, where $\alpha(r)=\alpha(i(r),n(r),r)$, and using the definition of $A^{\beta,\alpha}\varphi_\beta(i,n,r)$, we have

$$\begin{aligned}
&u(i,n,t;\beta,\alpha)=\\
&=\varphi_\beta(i,n,t)+\mathbb{E}^{\beta,\alpha}_{A_t(i,n)}\left[\int_t^T \frac{d\varphi_\beta}{dt}(i,n,r)+\gamma^+_\beta(i,n,r)(\varphi_\beta(i,n+1,r)-\varphi_\beta(i,n,r))\right.\\
&\left.+\gamma^-_\beta(i,n,r)(\varphi_\beta(i,n-1,r)-\varphi_\beta(i,n,r))+c\left(i,\frac{n}{N},\alpha\right)+\alpha(r)(\bar{\varphi}_\beta-\varphi_\beta)(i,n,r)dr\right].
\end{aligned}$$

The equation above is valid for all admissible controls α. Now we can define

$$\bar{\alpha}(\beta)(i,n,r)=\alpha^*\left(\bar{\varphi}_\beta(i,n,r)-\varphi_\beta(i,n,r),\frac{n}{N},i\right),$$

which is a bounded continuous Markov control and therefore admissible. We have

$$\begin{aligned}
u(i,n,t;\beta)\le& u(i,n,t,\beta,\alpha^*)=\varphi_\beta(i,n,t)\\
&+\mathbb{E}^{\beta,\alpha^*}_{A_t(i,n)}\left[\int_t^T \frac{d\varphi_\beta}{dt}(i,n,r)+\gamma^+_\beta(n,r)(\varphi_\beta(i,n+1,r)-\varphi_\beta(i,n,r))\right.\\
&\left.+\gamma^-_\beta(n,r)(\varphi_\beta(i,n-1,r)-\varphi_\beta(i,n,r))+h\left(\bar{\varphi}_\beta(i,n,r)-\varphi_\beta(i,n,r),\frac{n}{N},i\right)dr\right].
\end{aligned}$$

Now, we see that the integrand vanishes since φ_β is a solution to HJ, and therefore we have $u(i,n,t;\beta)\le\varphi_\beta(i,n,t)$.

Now we prove the other inequality:

$$\begin{aligned}
&u(i,n,t;\beta)=\inf_\alpha u(i,n,t,\beta,\alpha)=\varphi_\beta(i,n,t)\\
&+\inf_\alpha\mathbb{E}^{\beta,\alpha}_{A_t(i,n)}\left[\int_t^T \frac{d\varphi_\beta}{dt}(i,n,r)+\gamma^+_\beta(i,n,r)(\varphi_\beta(i,n+1,r)-\varphi_\beta(i,n,r))\right.\\
&\left.+\gamma^-_\beta(i,n,r)(\varphi_\beta(i,n-1,r)-\varphi_\beta(i,n,r))+c\left(i,\frac{n}{N},\alpha\right)+\alpha(r)(\bar{\varphi}_\beta-\varphi_\beta)(i,n,r)dr\right]\\
&\ge\varphi_\beta(i,n,t)+\mathbb{E}^{\beta}_{A_t(i,n)}\left[\int_t^T \frac{d\varphi_\beta}{dt}(i,n,r)+\gamma^+_\beta(i,n,r)(\varphi_\beta(i,n+1,r)-\varphi_\beta(i,n,r))\right.\\
&\left.+\gamma^-_\beta(i,n,r)(\varphi_\beta(i,n-1,r)-\varphi_\beta(i,n,r))+h\left(\bar{\varphi}_\beta(i,n,r)-\varphi_\beta(i,n,r),\frac{n}{N},i\right)dr\right]\\
&=\varphi_\beta(i,n,t)\,,
\end{aligned}$$

where the last equation holds because the integrand vanishes since φ is a solution to HJ.

Thus we have proved that $u(i,n,t;\beta)=\varphi_\beta(i,n,t)$. □

2.7 Maximum principle

Here we prove that the solutions to the Hamilton-Jacobi equations are uniformly bounded independently on the control β. We denote by

$$\|u(t)\|_\infty = \max_{n,i} |u_n(i,t)|,$$

and

$$M = \max_{(i,\theta)\in\{0,1\}\times[0,1]} |h(0,\theta,i)|.$$

Proposition 2. *Let u be a solution to* (6)*. For all $0 \le t \le T$ we have*

$$\|u(t)\|_\infty \le \|u(T)\|_\infty + 2M(T-t).$$

Proof. Let u be a solution to (6). Let $\tilde{u} = u + \rho(T-t)$. Then

$$-\frac{d\tilde{u}_n}{dt} = \rho + \gamma_\beta^+(\tilde{u}_{n+1} - \tilde{u}_n) + \gamma_\beta^-(\tilde{u}_{n-1} - \tilde{u}_n) + h\left(\bar{\tilde{u}}_n - \tilde{u}_n, \frac{n}{N}, i\right) .$$

Let (i,n,t) be a minimum point of $\tilde{u}$ on $\{0,1\}\times\{0,1,\cdots,N\}\times[0,T]$. We have $\tilde{u}_n(i,t) \le \tilde{u}_{n-1}(i,t)$ and $u_n(i,t) \le u_{n+1}(i,t)$. This implies $\gamma_\beta^-(\tilde{u}_{n-1} - \tilde{u}_n) \ge 0$ and $\gamma_\beta^+(\tilde{u}_{n+1} - \tilde{u}_n) \ge 0$. We also have $\tilde{u}_n(i,t) \le \tilde{u}_n(1-i,t) = \bar{\tilde{u}}_n(i,t)$, which implies $(\bar{\tilde{u}}_n - \tilde{u}_n)(i,t) \ge 0$. Hence

$$-\frac{d\tilde{u}_n}{dt}(i,t) \ge h\left(\bar{\tilde{u}}_n - \tilde{u}_n, \frac{n}{N}, i\right) + \rho \ge h\left(0, \frac{n}{N}, i\right) + \rho\,,$$

because $h(p,\theta,i)$ is monotone increasing in p. Furthermore, if we take $M < \rho < 2M$ we get

$$-\frac{d\tilde{u}_n}{dt}(i,t) > 0.$$

This shows that the minimum of $\tilde{u}$ is achieved at T hence

$$u_n(t,i) \ge -\|u(T)\|_\infty - 2M(T-t).$$

Similarly, let (i,n,t) be a maximum point of $\tilde{u}$ on $\{0,1\}\times\{0,1,\cdots,N\}\times[0,T]$. We have $\tilde{u}_n(i,t) \ge \tilde{u}_{n-1}(i,t)$ and $u_n(i,t) \ge u_{n+1}(i,t)$, and this implies $\gamma_\beta^-(\tilde{u}_{n-1} - \tilde{u}_n) \le 0$ and $\gamma_\beta^+(\tilde{u}_{n+1} - \tilde{u}_n) \le 0$. We also have $\tilde{u}_n(i,t) \ge \tilde{u}_n(1-i,t) = \bar{\tilde{u}}_n(i,t)$, which implies $(\bar{\tilde{u}}_n - \tilde{u}_n)(i,t) \le 0$. Hence

$$-\frac{d\tilde{u}_n}{dt}(i,t) \le h\left(\bar{\tilde{u}}_n - \tilde{u}_n, \frac{n}{N}, i\right) + \rho \le h\left(0, \frac{n}{N}, i\right) + \rho\,,$$

because $h(p,\theta,i)$ is monotone increasing in p. Furthermore, if we take $-2M < \rho < -M$ we get

$$-\frac{d\tilde{u}_n}{dt}(i,t) < 0.$$

This shows that the maximum of $\tilde{u}$ is achieved at T hence

$$u_n(t,i) \le \|u(T)\|_\infty + 2M(T-t).$$

□

2.8 Equilibrium solutions

We now consider the equilibrium situation in which the best response of any player to a control β is β itself.

Definition 1. *Let β be an admissible control. This control β is a Nash equilibrium if $\bar{\alpha}(\beta) = \beta$.*

Theorem 2. *There exists a Nash equilibrium, i.e, an admissible Markov control β_*, which satisfies $\bar{\alpha}(\beta_*) = \beta_*$. Moreover, the Nash equilibrium is unique.*

Proof. It suffices to observe that, by (7)

$$\beta_*(i,n,t) = \alpha^* \left(\bar{\varphi}_{\beta^*} - \varphi_{\beta^*}, \frac{n}{N}, i\right),$$

and hence the Markov control can be obtained by solving the system of nonlinear differential equations

$$-\frac{du_n}{dt} = \gamma_n^+(u_{n+1} - u_n) + \gamma_n^-(u_{n-1} - u_n) + h\left(\bar{u}_n - u_n, \frac{n}{N}, i\right), \tag{8}$$

with terminal condition $u(i,n,T) = \psi\left(i, \frac{n}{N}\right)$, where $\gamma_n^\pm$ are given by

$$\gamma_n^+(i,t) = (N-n)\alpha^* \left(\bar{u}_{n+1-i} - u_{n+1-i}, \frac{n+1-i}{N}, 1\right) \tag{9}$$

$$\gamma_n^-(i,t) = n\alpha^* \left(\bar{u}_{n-i} - u_{n-i}, \frac{n-i}{N}, 0\right).$$

Note that (8) is well posed because u_n is bounded and the righthand side is Lipschitz. Hence it follows the existence and uniqueness of a Nash equilibrium. □

For the record we give here some properties of $\gamma_n^\pm$:

$$|\gamma_n^\pm| \le CN,$$

and

$$|\gamma_{n+1}^\pm - \gamma_n^\pm| \le C + CN\|u_{n+1} - u_n\|_\infty.$$

2.9 An example

Let $f : \{0,1\} \times [0,1] \to \mathbb{R}$ and $g : \{0,1\} \times [0,1] \to \mathbb{R}$ be two continuous function. We take

$$c(i,\theta,\alpha) = f(i,\theta) + \frac{\alpha^2}{2} - \alpha g(i,\theta).$$

This example could model, for instance, the marketshare of cellular companies where there are only two competitors and N individual costumers. If the state of the player represents the company he uses, we can think of $g(i,\theta)$ as a bonus the company i offers customers of company $1-i$ in case they decide to switch. If there are no such bonus, we set $g = 0$.

Then

$$h(p,\theta,i) = \min_{\alpha \ge 0}\left[c(i,\theta,\alpha) + \alpha p\right] = f(i,\theta) - \frac{((g(i,\theta) - p)^+)^2}{2},$$

and

$$\alpha^*(p,\theta,i) = \operatorname{argmin}_{\alpha \ge 0}\left[c(i,\theta,\alpha) + \alpha p)\right] = (g(i,\theta) - p)^+.$$

Therefore (8) becomes

$$-\frac{du}{dt} = f - \frac{((u - \bar{u} + g)^+)^2}{2} + (N-n)(u - \bar{u} + g)^+_{1,n+1-i}(u_{n+1} - u_n) + n(u - \bar{u} + g)^+_{0,n-i}(u_{n-1} - u_n). \tag{10}$$

By the results of section 2.7 we know that any solution to (10) is bounded a-priori. Hence, if f and g are Lipschitz, (10) has a unique solution u. Therefore, there exists a unique Nash equilibrium.

3 A mean field model

This section is dedicated to a mean field model which, as we will see in the next section, corresponds to the limit as the number of players $N+1 \to \infty$. We start in §3.1 by discussing the model and its derivation under the mean field hypothesis. Then, in §3.2 we address existence of solutions. Uniqueness of solutions (under a monotonicity hypothesis similar to the ones in [LL06a, LL06b]) is established in §3.3. Finally, in §3.4, we continue the study of the model problem from §2.9.

3.1 The control problem in the mean field model and Nash equilibria

If the number of players is very large, we expect their distribution between the two states to be a deterministic function of the time t, as it would happen if we could somehow apply the law of large numbers. So, we suppose the fraction of players in state 0 is given by a deterministic function $\theta(t)$. If all players use the same Markovian control $\beta = \beta(i,t)$, which now only depends on i and t, then θ is a solution to

$$\frac{d\theta}{dt} = (1-\theta)\beta_1 - \theta\beta_0 \qquad\qquad \theta(0) = \bar{\theta}\,, \tag{11}$$

where β_i denotes the function $t \to \beta(i,t)$, and $0 \leq \bar{\theta} \leq 1$ is given and represents the initial distribution. We suppose here that β_i are continuous and bounded, for $i=0$ and $i=1$, and call such controls admissible controls.

We can now consider the optimization problem from a single player point of view. As before, we fix an individual player as the reference player and assume he can choose any admissible control α, while other players have a probability distribution among states determined by (11). Let

$$u(i,t,\alpha) = \mathbb{E}^{\alpha}_{i(t)=i}\left[\int_t^T c(i(s),\theta(s),\alpha(i(s),s))ds + \psi(i(T),\theta(T))\right],$$

where $i(t)$ is a controlled Markov chain switching between state 0 and 1 with rate α. We assume this player looks for an admissible control α which solves

$$u(i,t) = \inf_{\alpha} u(i,t,\alpha).$$

Note that the situation is now simpler than in the $N+1$-player game, because θ is deterministic and the only stochastic process is $i(t)$ whose switching rate is controlled by α. We call $u(i,t)$ the value function associated to the mean field distribution θ.

Consider the following HJ equation:

$$-\frac{du}{dt} = h(\bar{u}-u,\theta,i)\,. \tag{12}$$

As in the verification theorem of §2.6, any solution u to the equation above, with the terminal condition $u(i,T) = \psi(i,\theta(T))$, is the value function associated to θ. Furthermore, the optimal control is $\alpha^*(\bar{u}-u,\theta,i)$.

Under the symmetry hypothesis, all players must use the same control when the Nash equilibria is attained. In other words, Nash equilibria is the fixed point to the operator described above, i.e., the operator that uses the control β to calculate θ as a solution to (11), and after that determines the control $\alpha^*(\bar{u}-u,\theta,i)$ where u is the solution to the HJ equation (12) determined by θ, making the control $\alpha^*(\bar{u}-u,\theta,i)$ the image of β under this operator.

This leads then to the following system of ordinary differential equations

$$\begin{cases} -\frac{du}{dt} = h(\bar{u}-u,\theta,i) \\ \frac{d\theta}{dt} = (1-\theta)\alpha^*(u(0,t)-u(1,t),\theta,1) - \theta\alpha^*(u(1,t)-u(0,t),\theta,0), \end{cases} \tag{13}$$

with the boundary data

$$\begin{cases} u(i,T) = \psi(i, \theta(T)) \\ \theta(0) = \bar{\theta}\,. \end{cases} \tag{14}$$

Note that from the ODE point of view this problem is somewhat non-standard as some of the variables have initial conditions whereas other variables have prescribed terminal data. We call this the initial-terminal value problem.

3.2 Existence of Nash Equilibria in the MFG

We now address the existence of solutions to (13) satisfying the initial-terminal conditions (14). The proof of existence will be based upon a fixed point argument, using the operator ξ described in the following, which is the analogous of the operator acting on the controls described in the last section, but now acting on distributions.

Proposition 3. *There exists a solution to* (13) *satisfying the initial-terminal conditions* (14).

Proof. We need to solve (13) and (14) which can be rewritten as

$$\frac{d\theta}{dt} = (1-\theta)\alpha_1 - \theta\alpha_0 \qquad\qquad \theta(0) = \bar{\theta} \tag{15}$$

$$-\frac{du}{dt} = h(\bar{u} - u, \theta, i) \qquad u(i,T) = \psi(i, \theta(T)) \tag{16}$$

where

$$\alpha = \alpha^*(\bar{u} - u, \theta, i).$$

Let $\mathcal{F}$ be the set of continuous functions defined on $[0,T]$ and taking values in $[0,1]$, with the C^0 norm. Consider the function $\xi : \mathcal{F} \to \mathcal{F}$ that is obtained in the following way: given $\theta \in \mathcal{F}$, let u^θ be the solution of equation (16). Let $\beta^\theta = \alpha^*(\bar{u}^\theta - u^\theta, \theta, i)$, and then let $\xi(\theta)$ be the solution to $\frac{d\theta}{dt} = (1-\theta)\beta_1^\theta - \theta\beta_0^\theta$ and $\theta(0) = \bar{\theta}$.

From standard ODE theory we know ξ is a continuous function from $\mathcal{F}$ to $\mathcal{F}$. Moreover, as β is bounded, $\xi(\theta)$ is Lipschitz, with Lipschitz constant Λ independent of θ.

Now consider the set $\mathcal{C}$ of all Lipschitz continuous function in $\mathcal{F}$ with Lipschitz constant bounded by Λ. This is a set of uniformly bounded and equicontinuous functions. Thus, by Arzela-Ascoli, it is a relatively compact set. It is also clear that it is a convex set. Hence, by Brouwer fixed point theorem, ξ has a fixed point in $\mathcal{C}$. □

3.3 Uniqueness of Equilibria

To establish uniqueness we need to use the monotonicity method of [LL06a, LL06b]. From the concavity of h in p we have, for all p, q, θ and i

$$h(q, \theta, i) - h(p, \theta, i) - \alpha^*(p, \theta, i)(q - p) \leq 0, \tag{17}$$

because $\alpha^*(p, \theta, i) \in \partial_p^+ h(p, \theta, i)$. We suppose the additional monotonicity property

$$\begin{gathered} \theta\Big(h(q,\tilde{\theta},0) - h(q,\theta,0)\Big) + \tilde{\theta}\Big(h(p,\theta,0) - h(p,\tilde{\theta},0)\Big) \\ +(1-\theta)\Big(h(-q,\tilde{\theta},1) - h(-q,\theta,1)\Big) + (1-\tilde{\theta})\Big(h(-p,\theta,1) - h(-p,\tilde{\theta},1)\Big) \leq -\gamma|\theta - \tilde{\theta}|^2, \end{gathered} \tag{18}$$

for all $p, q \in \mathbb{R}$, for some $\gamma > 0$. This property will hold, for instance, if

$$h(p, \theta, i) = h_0(p) + f(i, \theta), \tag{19}$$

with f satisfying

$$(\theta-\tilde{\theta})(f(0,\tilde{\theta})-f(0,\theta))+(\tilde{\theta}-\theta)(f(1,\tilde{\theta})-f(1,\theta))\leq-\gamma|\theta-\tilde{\theta}|^2. \tag{20}$$

Note that the example of section 2.9 easily fits the previous conditions (19) and (20) provided we suppose g is a constant function and the functions $\theta\mapsto f(0,\theta)$ and $\theta\mapsto f(1,\theta)$ satisfy

$$(\tilde{\theta}-\theta)(f(0,\tilde{\theta})-f(0,\theta))\geq\frac{\gamma}{2}|\theta-\tilde{\theta}|^2$$

and

$$(\tilde{\theta}-\theta)(f(1,\tilde{\theta})-f(1,\theta))\leq-\frac{\gamma}{2}|\theta-\tilde{\theta}|^2,$$

which could be seen as a consequence of the fact that the running cost is greater when the reference player is in the more crowded state (i.e. when $\theta=1$ if $i=0$ and when $\theta=0$ if $i=1$).

Then , using (17) and (18) we obtain

$$\begin{aligned}&\theta\Big(h(q,\tilde{\theta},0)-h(p,\theta,0)-\alpha^*(p,\theta,0)(q-p)\Big)+\tilde{\theta}\Big(h(p,\theta,0)-h(q,\tilde{\theta},0)-\alpha^*(q,\tilde{\theta},0)(p-q)\Big)\\&+(1-\theta)\Big(h(-q,\tilde{\theta},1)-h(-p,\theta,1)-\alpha^*(-p,\theta,1)(p-q)\Big)\\&+(1-\tilde{\theta})\Big(h(-p,\theta,1)-h(-q,\tilde{\theta},1)-\alpha^*(-q,\tilde{\theta},1)(q-p)\Big)\leq-\gamma|\theta-\tilde{\theta}|^2.\end{aligned} \tag{21}$$

Theorem 3. *Under the monotonicity hypothesis* (18)*, the system (15) and (16) has a unique solution* (θ,u)*.*

Proof. To establish uniqueness we will use monotonicity argument from [LL06a, LL06b].

Suppose (θ,u) and $(\tilde{\theta},\tilde{u})$ are solutions of (15) and (16). At the initial and terminal points, respectively $t=0$ and $t=T$ we have that $(\theta-\tilde{\theta})(u-\tilde{u})=0$ and $((1-\theta)-(1-\tilde{\theta}))(\bar{u}-\bar{\tilde{u}})=0$, where $u(t)=u(0,t)$ and $\bar{u}=u(1,t)$, and similarly for $\tilde{u}$. Then

$$(\theta-\tilde{\theta})(u-\tilde{u})_t=(\theta-\tilde{\theta})[-h(\bar{u}-u,\theta,0)+h(\bar{\tilde{u}}-\tilde{u},\tilde{\theta},0)],$$

and

$$((1-\theta)-(1-\tilde{\theta}))(\bar{u}-\bar{\tilde{u}})_t=((1-\theta)-(1-\tilde{\theta}))[-h(u-\bar{u},\theta,1)+h(\tilde{u}-\bar{\tilde{u}},\tilde{\theta},1)].$$

Furthermore,

$$\begin{aligned}(u-\tilde{u})(\theta-\tilde{\theta})_t=&(u-\tilde{u})[(1-\theta)\alpha^*(u-\bar{u},\theta,1)-\theta\alpha^*(\bar{u}-u,\theta,0)\\&-(1-\tilde{\theta})\alpha^*(\tilde{u}-\bar{\tilde{u}},\tilde{\theta},1)+\tilde{\theta}\alpha^*(\bar{\tilde{u}}-\tilde{u},\tilde{\theta},0)],\end{aligned}$$

and

$$\begin{aligned}(\bar{u}-\bar{\tilde{u}})((1-\theta)-1+\tilde{\theta})_t=&(\bar{u}-\bar{\tilde{u}})[\theta\alpha^*(\bar{u}-u),\theta,0)-(1-\theta)\alpha^*(u-\bar{u},\theta,1)\\&-\tilde{\theta}\alpha^*(\bar{\tilde{u}}-\tilde{u},\tilde{\theta},0)+(1-\tilde{\theta})\alpha^*(\tilde{u}-\bar{\tilde{u}},\tilde{\theta},1)].\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{d}{dt}\Big((\theta-\tilde{\theta})(u-\tilde{u})+((1-\theta)-(1-\tilde{\theta}))(\bar{u}-\bar{\tilde{u}})\Big)=\\=&\theta\Big(-h(\bar{u}-u,\theta,0)+h(\bar{\tilde{u}}-\tilde{u},\tilde{\theta},0)+[(\bar{u}-\bar{\tilde{u}})-(u-\tilde{u})]\alpha^*(\bar{u}-u,\theta,0)\Big)\\&+\tilde{\theta}\Big(h(\bar{u}-u,\theta,0)-h(\bar{\tilde{u}}-\tilde{u},\tilde{\theta},0)+[-(\bar{u}-\bar{\tilde{u}})+(u-\tilde{u})]\alpha^*(\bar{\tilde{u}}-\tilde{u},\tilde{\theta},0)\Big)\\&+(1-\theta)\Big(-h(u-\bar{u},\theta,1)+h(\tilde{u}-\bar{\tilde{u}},\tilde{\theta},1)+[(u-\tilde{u})-(\bar{u}-\bar{\tilde{u}})]\alpha^*(u-\bar{u},\theta,1)\Big)\\&+(1-\tilde{\theta})\Big(h(u-\bar{u},\theta,1)-h(\tilde{u}-\bar{\tilde{u}},\tilde{\theta},1)+[-(u-\tilde{u})+(\bar{u}-\bar{\tilde{u}})]\alpha^*(\tilde{u}-\bar{\tilde{u}},\tilde{\theta},1)\Big).\end{aligned}$$

Then, by using (21), with $p = \bar{u} - u$ and $q = \bar{\tilde{u}} - \tilde{u}$, we obtain

$$\frac{d}{dt}\Big((\theta - \tilde{\theta})(u - \tilde{u}) + ((1-\theta) - (1-\tilde{\theta}))(\bar{u} - \bar{\tilde{u}})\Big) \leq -\gamma|\theta - \tilde{\theta}|^2. \tag{22}$$

Integrating the previous equation between 0 and T we obtain the uniqueness for θ. Then once θ is known to be unique, we obtain by a standard ODE argument that $u = \tilde{u}$. □

3.4 Back to the example

Just to illustrate, equations (13), in the special case of the example of section 2.9, and supposing g is a constant function, becomes

$$\frac{d\theta}{dt} = (1-\theta)(u - \bar{u})_1^+ - \theta(u - \bar{u})_0^+ ,$$

and

$$-\frac{du}{dt} = f(i, \theta) - \frac{((u - \bar{u} + g)^+)^2}{2} .$$

As we have already seen, provided the condition (20) holds and given the initial-terminal condition

$$\theta(0) = \bar{\theta}, \quad u(i, T) = \psi(i, \theta(T))$$

the system above has a unique solution.

4 Convergence

This last section addresses the convergence as the number of players tends to infinity to the mean field model derived in the previous section.

We start this section by discussing some preliminary estimates in §4.1. Then, in §4.2 we establish uniform estimates for $|u_{n+1} - u_n|$, which are essential to prove our main result, theorem 4, which is discussed in §4.3. This theorem shows that the model derived in the previous section can be obtained as an appropriate limit of the model with $N+1$ players discussed in section 2.

4.1 Preliminary results

Consider the system of ordinary differential equations

$$-\dot{z}_n = a_n(t)(z_{n+1} - z_n) + b_n(t)(z_{n-1} - z_n) + \mu_n(t)(\bar{z}_n - z_n), \tag{23}$$

with $a_n(t), b_n(t), \mu_n(t) \geq 0$. Here $z_n = (z_n^0, z_n^1)$, $a_n = (a_n^0, a_n^1)$, etc. We assume further that $a_N = 0$ and $b_0 = 0$.

We write (23) in compact form as

$$-\dot{z}(t) = M(t)z(t). \tag{24}$$

The solution to this equation with terminal data $z(T)$ can be written as

$$z(t) = K(t, T)z(T), \tag{25}$$

where $K(t, T)$ is the fundamental solution to (24) with $K(T, T) = I$. Note that equations (24) and (25) imply

$$\frac{d}{dt}K(t, T) = -M(t)K(t, T). \tag{26}$$

Lemma 1. *For $t < T$ we have*

$$\|z(t)\|_\infty \leq \|z(T)\|_\infty.$$

Furthermore, if $z(T) \leq 0$ then $z(t) \leq 0$.

Proof. Let z be a solution of (24), and fix $\epsilon > 0$. We define $\tilde{z} = z + \epsilon(t - T)$. Hence $\tilde{z}$ satisfies

$$-\dot{\tilde{z}}_n = -\epsilon + a_n(t)(\tilde{z}_{n+1} - \tilde{z}_n) + b_n(t)(\tilde{z}_{n-1} - \tilde{z}_n) + \mu_n(t)(\bar{\tilde{z}}_n - \tilde{z}_n).$$

Let (i, n, t) be a maximum point of $\tilde{z}$ on $\{0,1\} \times \{0, 1, \cdots, N\} \times [0, T]$. We have $\tilde{z}_n(i,t) \geq \tilde{z}_{n-1}(i,t)$ and $z_n(i,t) \geq z_{n+1}(i,t)$, also $\tilde{z}_n(i,t) \geq \tilde{z}_n(1-i,t) = \bar{\tilde{z}}_n(i,t)$, this implies $b_n(t)(\tilde{z}_{n-1} - \tilde{z}_n) \leq 0$ and $a_n(t)(\tilde{z}_{n+1} - \tilde{z}_n) \leq 0$ and $\mu_n(t)(\bar{\tilde{z}}_n - \tilde{z}_n)(i,t) \leq 0$. Hence

$$-\frac{d\tilde{z}_n}{dt}(i,t) \leq -\epsilon.$$

This shows that the maximum of $\tilde{z}$ is achieved at T. Therefore, for all (j, m, t),

$$z_m(j,t) + \epsilon(t - T) = \tilde{z}_m(j,t) \leq \tilde{z}_n(i,T) = z_n(i,T)$$

Letting $\epsilon \to 0$, we get

$$z_m(j,t) \leq \max_{n,i} z_n(i,T).$$

From this equation we have the following conclusions:

1. if $z(T) \leq 0$, we then have $z_m(j,t) \leq 0$, for all (j,m,t), and so $z(t) \leq 0$;
2. for all (j, m, t),

$$z_m(j,t) \leq \|z(T)\|_\infty.$$

Now we define $\tilde{z} = z + \epsilon(T - t)$. Hence $\tilde{z}$ satisfies

$$-\dot{\tilde{z}}_n = \epsilon + a_n(t)(\tilde{z}_{n+1} - \tilde{z}_n) + b_n(t)(\tilde{z}_{n-1} - \tilde{z}_n) + \mu_n(t)(\bar{\tilde{z}}_n - \tilde{z}_n).$$

Let (i, n, t) be a minimum point of $\tilde{z}$ on $\{0,1\} \times \{0, 1, \cdots, N\} \times [0, T]$. We have $\tilde{z}_n(i,t) \leq \tilde{z}_{n-1}(i,t)$ and $z_n(i,t) \leq z_{n+1}(i,t)$, also $\tilde{z}_n(i,t) \leq \tilde{z}_n(1-i,t) = \bar{\tilde{z}}_n(i,t)$. This implies $b_n(t)(\tilde{z}_{n-1} - \tilde{z}_n) \geq 0$, and $a_n(t)(\tilde{z}_{n+1} - \tilde{z}_n) \geq 0$ and $\mu_n(t)(\bar{\tilde{z}}_n - \tilde{z}_n)(i,t) \geq 0$. Therefore we have

$$-\frac{d\tilde{z}_n}{dt}(i,t) \geq \epsilon.$$

This shows that the minimum of $\tilde{z}$ is also achieved at T, hence for all (j, m, t)

$$z_m(j,t) + \epsilon(T - t) = \tilde{z}_m(j,t) \geq \tilde{z}_n(i,T) = z_n(i,T).$$

Letting $\epsilon \to 0$, we get

$$z_m(j,t) \geq \min_{n,i} z_n(i,T).$$

Hence

$$z_m(j,t) \geq -\|z(T)\|_\infty.$$

Therefore we have $\|z(t)\|_\infty \leq \|z(T)\|_\infty$. □

Note: let $z(t) = K(t,s)z(s)$ be a solution of (24) with terminal data $z(s) = b$, then lemma 1 implies that $\|z(t)\|_\infty \leq \|z(s)\|_\infty$, and therefore

$$\|K(t,s)b\|_\infty \leq \|b\|_\infty, \forall\, b. \tag{27}$$

From the previous lemma we also conclude

Lemma 2. *If $p_1 \leq p_2$, and $t \leq s$, then we have*

$$K(t,s)p_1 \leq K(t,s)p_2.$$

Proof. Observe that if $p_1 - p_2 \leq 0$ then $K(t,s)(p_1 - p_2) \leq 0$, by lemma 1. □

We note now that if $t \leq s \leq T$ we have $K(t,s)K(s,T) = K(t,T)$, which implies

$$\frac{d}{ds}\Big(K(t,s)K(s,T)\Big) = 0.$$

Hence, using equation (26) we get

$$-K(t,s)M(s)K(s,T) + \left(\frac{d}{ds}K(t,s)\right)K(s,T) = 0,$$

and therefore, by taking $T = s$ we conclude that

$$\frac{d}{ds}K(t,s) = K(t,s)M(s). \tag{28}$$

We now prove the main technical lemma:

Lemma 3. *Suppose z is a solution to*

$$-\dot{z}(s) \leq M(s)z(s) + f(z(s)). \tag{29}$$

Then

$$z(t) \leq \|z(T)\|_\infty + \int_t^T \|f(z(s))\|_\infty ds.$$

Proof. Multiplying (29) by the order preserving operator $K(t,s)$, we have

$$-K(t,s)\dot{z}(s) \leq K(t,s)M(s)z(s) + K(t,s)f(z(s))$$

using the identity

$$\frac{d}{ds}K(t,s)z(s) = K(t,s)\dot{z}(s) + K(t,s)M(s)z(s),$$

which follows from (28), we get

$$-\frac{d}{ds}\Big(K(t,s)z(s)\Big) + K(t,s)M(s)z(s) \leq K(t,s)M(s)z(s) + K(t,s)f(z(s)).$$

Thus, integrating between t and T, we have

$$z(t) - K(t,T)z(T) \leq \int_t^T K(t,s)f(z(s))ds.$$

So, using equation (27),

$$z(t) \leq \|z(T)\|_\infty + \int_t^T \|f(z(s))\|_\infty ds.$$

□

4.2 Uniform estimates

In this section we prove "gradient estimates" for the $N+1$ player game, that is, we assume that the difference $u_{n+1} - u_n$ is of the order $\frac{1}{N}$ at time T and show that it remains so for $0 \le t \le T$, as long as T is sufficiently small.

We start by establishing an auxiliary result:

Lemma 4. *Suppose $v = v(s)$ is a solution to the ODE with terminal condition*

$$\begin{cases} -\frac{dv}{ds} = Cv + CNv^2 + \frac{C}{N} \\ v(T) \le \frac{C}{N}, \end{cases} \tag{30}$$

where N is a natural number, and $C > 0$. Then, there exists $T^\star > 0$, which does not depend on N, such that $T \le T^\star$ implies $v(s) \le \frac{2C}{N}$ for all $0 \le s \le T$.

Proof. Note that (30) implies that v is a monotone decreasing function of s and is equivalent to

$$\begin{cases} \frac{ds}{dv} = \frac{-1}{Cv + CNv^2 + \frac{C}{N}} \\ s(\frac{C}{N}) \le T. \end{cases}$$

This implies by direct integration that

$$s\left(\frac{2C}{N}\right) \le T - \int_{\frac{C}{N}}^{\frac{2C}{N}} \frac{dv}{Cv + CNv^2 + \frac{C}{N}}.$$

Now

$$\int_{\frac{C}{N}}^{\frac{2C}{N}} \frac{dv}{Cv + CNv^2 + \frac{C}{N}} \ge \int_{\frac{C}{N}}^{\frac{2C}{N}} \frac{N}{2C^2 + 4C^3 + C} dv = \frac{1}{2C + 4C^2 + 1}.$$

Therefore if we define $T^\star = \frac{1}{2C+4C^2+1}$, we have that $s\left(\frac{2C}{N}\right) \le 0$ if $T \le T^\star$. Hence this implies $v(0) \le \frac{2C}{N}$, which yields the desired result when we take into account that v is a decreasing function of s. ☐

Proposition 4. *Suppose that*

$$\|u_{n+1}(T) - u_n(T)\|_\infty \le \frac{C}{N}. \tag{31}$$

for $C > 0$. Let u be a solution of (8). Then there exists $T^\star > 0$ such that, for $0 < T < T^\star$,we have

$$\|u_{n+1}(t) - u_n(t)\|_\infty \le \frac{2C}{N},$$

for all $0 \le t \le T$.

Proof. Let

$$z_n = u_{n+1} - u_n.$$

Note that, as usual, $z_n = (z_n^0, z_n^1)$. We have

$$-\dot{z}_n = \gamma_{n+1}^+ z_{n+1} - \gamma_n^+ z_n - \gamma_{n+1}^- z_n + \gamma_n^- z_{n-1} + h\left(\frac{n+1}{N}, i, \bar{u}_{n+1} - u_{n+1}\right) - h\left(\frac{n}{N}, i, \bar{u}_n - u_n\right).$$

We can write

$$\begin{aligned} \gamma_{n+1}^+ z_{n+1} - \gamma_n^+ z_n - \gamma_{n+1}^- z_n + \gamma_n^- z_{n-1} \\ = \frac{\gamma_{n+1}^+ + \gamma_n^+}{2}(z_{n+1} - z_n) + \frac{\gamma_{n+1}^+ - \gamma_n^+}{2}(z_{n+1} + z_n) \\ + \frac{\gamma_{n+1}^- + \gamma_n^-}{2}(z_{n-1} - z_n) + \frac{\gamma_n^- - \gamma_{n+1}^-}{2}(z_{n-1} + z_n). \end{aligned}$$

We must now observe that

$$\left|\frac{\gamma_n^- - \gamma_{n+1}^-}{2}\right| \le C + CN\|z\|_\infty,$$

as well as

$$\left|\frac{\gamma_{n+1}^+ - \gamma_n^+}{2}\right| \le C + CN\|z\|_\infty.$$

Furthermore, we have

$$\begin{aligned}
&h\left(\frac{n+1}{N}, i, \bar{u}_{n+1} - u_{n+1}\right) - h\left(\frac{n}{N}, i, \bar{u}_n - u_n\right) \\
&= h\left(\frac{n+1}{N}, i, \bar{u}_{n+1} - u_{n+1}\right) - h\left(\frac{n}{N}, i, \bar{u}_{n+1} - u_{n+1}\right) \\
&+ h\left(\frac{n}{N}, i, \bar{u}_{n+1} - u_{n+1}\right) - h\left(\frac{n}{N}, i, \bar{u}_n - u_n\right) \\
&\le \frac{C}{N} + h_p\left(\frac{n}{N}, i, \bar{u}_n - u_n\right)((\bar{u}_{n+1} - u_{n+1}) - (\bar{u}_n - u_n)) \\
&\le \frac{C}{N} + \mu_n(\bar{z}_n - z_n),
\end{aligned}$$

where $\mu_n = h_p\left(\frac{n}{N}, i, \bar{u}_n - u_n\right) \ge 0$.

At this point we are in position to apply lemma 3 from the previous section. We obtain

$$z_n(t) = (u_{n+1} - u_n)(t) \le \|z(T)\|_\infty + \int_t^T C\|z(s)\|_\infty + C\|z(s)\|_\infty^2 + \frac{C}{N}\, ds\ .$$

We can also use the same argument applied to

$$\tilde{z}_n = u_n - u_{n+1}\ .$$

Finally, if we set $w = \|u_{n+1} - u_n\|_\infty$ we conclude that

$$w(t) \le w(T) + \int_t^T Cw(s) + CNw(s)^2 + \frac{C}{N} ds.$$

Now we define

$$\eta(t) = w(T) + \int_t^T Cw(s) + CNw(s)^2 + \frac{C}{N} ds.$$

We have that

$$w(t) \le \eta(t), \tag{32}$$

and also that

$$\frac{d\eta}{dt}(t) = -g(w(t)),$$

where g is the nondecreasing function $g(w) = Cw + CNw^2 + \frac{C}{N}$. Thus

$$\begin{cases} \frac{d\eta}{dt}(t) \ge -g(\eta(t)) \\ \eta(T) = w(T). \end{cases}$$

A standard argument from the basic theory of differential inequalities can now be used to prove that $\eta(t) \le v(t)$ for $0 \le t \le T$ if $v(t)$ is the solution of

$$\begin{cases} \frac{dv}{dt}(t) = -g(v(t)) \\ v(T) = w(T). \end{cases}$$

This last result can be combined with lemma 4, the hypothesis $w(T) \le \frac{C}{N}$ and the inequality (32), to prove that $w(t) \le \frac{2C}{N}$ for all $0 \le t \le T$, which ends the proof of the proposition. □

4.3 Convergence

In this section we prove theorem 4, which implies the convergence of both distribution and value function of the $N+1$-player game to the mean field game, for small times.

We start by assuming that at the initial time the N players distinct from the reference player distribute themselves between states 0 and 1 according to a Bernoulli distribution with probability $\bar{\theta}$ of being in state 0.

Let

$$\begin{cases} V_N(t) \equiv \mathbb{E}\left[\left(\frac{n(t)}{N} - \theta(t)\right)^2\right], \\ W_N(t) \equiv \mathbb{E}\left[\left(u(0,t) - u_{n(t)}(0,t)\right)^2\right], \\ \bar{W}_N(t) \equiv \mathbb{E}\left[\left(u(1,t) - u_{n(t)}(1,t)\right)^2\right], \\ Q_N(t) \equiv W_N(t) + \bar{W}_N(t), \end{cases} \tag{33}$$

where $\theta(t)$ is the solution of (11), $0 \le n(t) \le N$ is the number of players (distinct from the reference player) which are in state 0 at time t, and $u = u(i,t)$ and $u_n = u_n(i,t)$ are respectively the solution of the HJ equation and terminal conditions for the MFG (13) and $N+1$ player game (8).

We have

$$V_N(0) = \mathrm{Var}\left[\frac{n(0)}{N}\right] = \frac{\bar{\theta}(1-\bar{\theta})}{N},$$

because $n(0)$ is the sum of N iid rv with Bernoulli distribution.

In this section $\alpha = \alpha(i,t)$ is the optimal control for the MFG, while $\alpha^N = \alpha^N(i,n,t)$ is the optimal control for the $N+1$ player game. We know from sections 2.5, 2.6 and 3.1 that $\alpha^N = \alpha^*\left(\bar{u}_n - u_n, \frac{n}{N}, i\right)$ and $\alpha = \alpha^*(\bar{u} - u, \theta, i)$.

Lemma 5. *There exists $C_1 > 0$ such that*

$$V_N(t) \le \int_0^t C_1(V_N(s) + Q_N(s))ds + \frac{C_1}{N}.$$

Proof. Using Dynkin's Formula (2) with $\varphi(i,n,s) = \left(\frac{n(s)}{N} - \theta(s)\right)^2$, we have

$$V_N(t) - \frac{\theta_0(1-\theta_0)}{N} = \mathbb{E}\int_0^t \omega_N(s) + \varsigma_N(s)ds$$

where

$$\omega_N(s) = (N-n)\alpha_1^N\left[\left(\frac{n+1}{N} - \theta\right)^2 - \left(\frac{n}{N} - \theta\right)^2\right] + n\alpha_0^N\left[\left(\frac{n-1}{N} - \theta\right)^2 - \left(\frac{n}{N} - \theta\right)^2\right],$$

$$\alpha_0^N = \alpha^*\left(\bar{u}_{n-i} - u_{n-i}, \frac{n-i}{N}, 0\right),$$

$$\alpha_1^N = \alpha^*\left(\bar{u}_{n+1-i} - u_{n+1-i}, \frac{n+1-i}{N}, 1\right),$$

$$u_n = u_N(i,n,t),$$

and

$$\varsigma_N(s) = \frac{d\varphi}{dt}(i,n,r) = -2\left(\frac{n}{N} - \theta\right)((1-\theta)\alpha_1 - \theta\alpha_0).$$

We have

$$\begin{aligned}\omega_N(s) =& \left(1-\frac{n}{N}\right)\alpha_1^N\left(\frac{2n+1}{N}-2\theta\right)-\frac{n}{N}\alpha_0^N\left(\frac{2n-1}{N}-2\theta\right)\\ =&2\alpha_1^N\left(1-\frac{n}{N}\right)\left(\frac{n}{N}-\theta\right)-2\alpha_0^N\frac{n}{N}\left(\frac{n}{N}-\theta\right)+\tau_N(s),\end{aligned}$$

where $\tau_N(s)=\frac{\alpha_1^N}{N}+\frac{n}{N^2}(\alpha_0^N-\alpha_1^N)$. Now

$$\begin{aligned}\omega_N(s)+\varsigma_N(s) =& 2\left(\frac{n}{N}-\theta\right)\left[\alpha_1^N\left(1-\frac{n}{N}\right)-\alpha_0^N\frac{n}{N}-((1-\theta)\alpha_1-\theta\alpha_0)\right]+\tau_N(s)\\ =&2\left(\frac{n}{N}-\theta\right)\left[(\alpha_1^N+\alpha_0^N)\left(-\frac{n}{N}\right)+(\alpha_1+\alpha_0)\theta+(\alpha_1^N-\alpha_1)\right]+\tau_N(s)\\ =&2\left(\frac{n}{N}-\theta\right)\left[(\alpha_1^N+\alpha_0^N)\left(\theta-\frac{n}{N}\right)+(\alpha_1-\alpha_1^N+\alpha_0-\alpha_0^N)\theta+(\alpha_1^N-\alpha_1)\right]+\tau_N(s)\\ =&-2(\alpha_0^N+\alpha_1^N)\left(\frac{n}{N}-\theta\right)^2+2\left(\frac{n}{N}-\theta\right)\left((\alpha_1-\alpha_1^N+\alpha_0-\alpha_0^N)\theta+(\alpha_1^N-\alpha_1)\right)+\tau_N(s).\end{aligned}$$

Then

$$\begin{aligned}V_N(t)-\frac{\theta_0(1-\theta_0)}{N} =& -2\mathbb{E}\left[\int_0^t(\alpha_0^N+\alpha_1^N)\left(\frac{n}{N}-\theta\right)^2ds\right]\\ &+\mathbb{E}\left[\int_0^t 2\left(\frac{n}{N}-\theta\right)\left((\alpha_1-\alpha_1^N+\alpha_0-\alpha_0^N)\theta+(\alpha_1^N-\alpha_1)\right)ds\right]\\ &+\mathbb{E}\left[\int_0^t\tau_N(s)ds\right].\end{aligned}$$

Now we see that

$$\begin{aligned}|\alpha_0-\alpha_0^N| =& \left|\alpha^*(\bar{u}-u,\theta,0)-\alpha^*\left(\bar{u}_{n-i}-u_{n-i},\frac{n-i}{N},0\right)\right|\\ <& K\left(\left|\theta-\frac{n-i}{N}\right|+|\bar{u}-\bar{u}_{n-i}|+|u-u_{n-i}|\right)\\ <& K\left(\left|\theta-\frac{n}{N}\right|+|\bar{u}-\bar{u}_n|+|\bar{u}_{n-i}-\bar{u}_n|+|u-u_n|+|u_{n-i}-u_n|+\frac{1}{N}\right)\\ <& K\left(\left|\theta-\frac{n}{N}\right|+|\bar{u}-\bar{u}_n|+|u-u_n|+\frac{3}{N}\right),\end{aligned}$$

where we used that α^* is Lipschitz in both variables, and u and u^N are bounded, and the uniform bounds on $|u_{n+1}-u_n|$ obtained in proposition 4 of §4.2. Similarly

$$|\alpha_1-\alpha_1^N|<K\left(\left|\theta-\frac{n}{N}\right|+|\bar{u}-\bar{u}_n|+|u-u_n|+\frac{3}{N}\right).$$

Thus

$$\begin{aligned}V_N(t) \le& K_1\int_0^t V_N(s)ds+2\mathbb{E}\int_0^t\left(\frac{n}{N}-\theta\right)K\left(\left|\theta-\frac{n}{N}\right|+|\bar{u}-\bar{u}_n|+|u-u_n|+\frac{3}{N}\right)ds+\frac{K_2}{N}\\ \le&(K_1+2K)\int_0^t V_N(s)ds+2\mathbb{E}\int_0^t\left(\frac{n}{N}-\theta\right)K\left(|\bar{u}-\bar{u}_n|+|u-u_n|+\frac{3}{N}\right)ds+\frac{K_2}{N}\\ \le&(K_1+2K)\int_0^t V_N(s)ds+2K\int_0^t 2V_N(s)+(W_N(s)+\bar{W}_N(s))ds+\frac{K_2+6T}{N}\\ =&\int_0^t K_3V_N(s)+2KQ_N(s)ds+\frac{K_2+6T}{N}\\ \le&\int_0^t C_1(V_N(s)+Q_N(s))ds+\frac{C_1}{N}.\end{aligned}$$

□

Lemma 6. *There exists $C_2 > 0$ such that*

$$Q_N(t) \leq \int_t^T C_2(V_N(s) + Q_N(s))ds + \frac{C_2}{N}.$$

Proof. In this proof, $u_n(s)$ or simply u_n will denote the expected minimum cost of player $N+1$ conditioned on its state being equal to 0 at time s, i.e., $u_{n(s)}(0, s)$. We will also use, here, $u(s)$ or simply u to denote $u(0, s)$.

Using Dynkin formula (2) with $\varphi(i, n, s) = \left(u_{n(s)}(0, s) - u(0, s)\right)^2$, and equations (8) and (12), we have

$$\begin{aligned}
&W_N(t) - W_N(T) = -\mathbb{E}[(u_n(t) - u(t))^2] + \mathbb{E}[(u_n(T) - u(T))^2] \\
=&\mathbb{E}\int_t^T 2(u_n - u)\frac{d}{ds}(u_n - u)ds \\
&\quad + \mathbb{E}\int_t^T \gamma_n^+ \left[(u_{n+1} - u)^2 - (u_n - u)^2\right] + \gamma_n^- \left[(u_{n-1} - u)^2 - (u_n - u)^2\right] ds \\
=&\mathbb{E}\int_t^T 2(u_n - u)\left(-\gamma_n^+(u_{n+1} - u_n) - \gamma_n^-(u_{n-1} - u_n) - h\left(\bar{u}_n - u_n, \frac{n}{N}, 0\right) + h(\bar{u} - u, \theta, 0)\right) ds \\
&\quad + \mathbb{E}\int_t^T \gamma_n^+ \left[(u_{n+1} - u)^2 - (u_n - u)^2\right] + \gamma_n^- \left[(u_{n-1} - u)^2 - (u_n - u)^2\right] ds \\
=&\mathbb{E}\int_t^T \gamma_n^+(u_{n+1} - u_n)^2 + \gamma_n^-(u_{n-1} - u_n)^2 - 2\left(h\left(\bar{u}_n - u_n, \frac{n}{N}, 0\right) - h(\bar{u} - u, \theta, 0)\right)(u_n - u)ds,
\end{aligned}$$

where $\gamma_n^\pm = \gamma_n^\pm(0, n(s), s)$. In the last equation we used the fact that

$$-2(u_n - u)\gamma_n^+(u_{n+1} - u_n) + \gamma_n^+ \left[(u_{n+1} - u)^2 - (u_n - u)^2\right] = \gamma_n^+ \left(u_{n+1} - u_n\right)^2 ,$$

and a similar calculation for γ_n^-.

Now, using results from §4.2, proposition 4, we have that $\gamma_n^+(u_{n+1} - u_n)^2$, $\gamma_n^-(u_{n-1} - u_n)^2$ and $W_N(T)$ are bounded by $\frac{K_5}{N}$, which implies

$$W_N(t) \leq \frac{K_6}{N} + 2\mathbb{E}\int_t^T \left(h\left(\bar{u}_n - u_n, \frac{n}{N}, 0\right) - h\left(\bar{u} - u, \theta, 0\right)\right)(u_n - u)ds\,.$$

Using the fact that h is Lipschitz in both variables, we have

$$\left|h\left(\bar{u}_n - u_n, \frac{n}{N}, 0\right) - h(\bar{u} - u, \theta, 0)\right| < K\left(\left|\theta - \frac{n}{N}\right| + |\bar{u} - \bar{u}_n| + |u - u_n|\right).$$

Thus

$$W_N(t) \leq \frac{K_6}{N} + K_7\int_t^T V_N(s) + W_N(s) + \bar{W}_N(s)ds\,.$$

With a similar calculation we have a analogous inequality for $\bar{W}_N(t)$, which ends the proof. □

Now we can state and prove our main result that establishes the convergence of the $N+1$ player game to the mean field model as $N \to \infty$.

Theorem 4. *If $\rho = TC < 1$, where $C = \max\{C_1, C_2\}$, and $Q_N(t) + V_N(t)$ is given in (33) then*

$$Q_N(t) + V_N(t) \leq \frac{C}{1-\rho}\frac{1}{N} \quad \forall t \in [0, T]\,.$$

Proof. Adding both inequalities given in the two last lemmas, we have

$$Q_N(t) + V_N(t) \leq C \int_0^T (V_N(s) + Q_N(s))ds + \frac{C}{N}\,.$$

Now suppose $\rho = TC < 1$. Defining

$$Q_N + V_N = \max_{0 \leq t \leq T} Q_N(t) + V_N(t),$$

we have

$$Q_N + V_N \leq \rho(Q_N + V_N) + \frac{C}{N},$$

which proves the theorem. □

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