HOMOTOPICAL ALGEBRA FOR C^*-ALGEBRAS

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Abstract. Category of fibrant objects is a convenient framework to do homotopy theory, introduced and developed by Ken Brown. In this paper, we apply it to the category of C^*-algebras. In particular, we get a unified treatment of (ordinary) homotopy theory for C^*-algebras, KK-theory and E-theory, as all of these can be expressed as the homotopy theory of a category of fibrant objects.

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0. Introduction

Basic homotopy theory for C^*-algebras can be developed in an analogues way to the homotopy theory for topological spaces, using the Gelfand-Naimark duality between pointed compact Hausdorff spaces and abelian C^*-algebras. This is carried out by Schochet in [Sch84]. Thus, for instance, we have a version of the Puppe exact sequence, with essentially the same proof (cf. [Sch84, Proposition 2.6]).

There is one big difference: the homotopy theory for C^*-algebras does not admit a Quillen model category structure (cf. [GA97, Remark 3.8]). This is unfortunate, since model categories provide a standard and powerful framework to study various aspects of homotopy theories. However, not

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everything is lost: it turns out that the category of C^*-algebras behave as if it was the subcategory of the fibrant objects in a model category, and this is enough for many purposes, because many proofs in model category theory start by reducing to the case of (co)fibrant objects.

The notion of a "category of fibrant objects" is abstracted and developed by Ken Brown in [Bro74]. In this paper, we apply Brown's theory to the category of C^*-algebras. In Section 1, we review some basic facts about abstract homotopy theory in the setting of category of fibrant objects.

In Section 2, we first apply the abstract theory of Section 1 to the ordinary homotopy theory for C^*-algebras (this essentially recovers [Sch84]). We also show that the Meyer-Nest's UCT category (cf. [MN06]), Kasparov's KK-theory (cf. [Kas80, Kas88]), Thom's stable-homotopy category (cf. [Tho03]) and Connes-Higson's E-theory (cf. [Hig90, CH90]) can be described as the homotopy category of a category of fibrant objects. As a corollary, we get a unified treatment of the triangulated structures on these categories.

We note that Brown's theory of category of fibrant objects is of course not the only way to approach the homotopy theory for C^*-algebras. The main "reason" for the failure for the existence of a model structure on the category of C^*-algebras is that the category is too small, so an alternative approach would be to enlarge the category of C^*-algebras. Joachim and Johnson produced a model category structure for KK-theory by enlarging the category of C^*-algebras to a suitable category of topological algebras (cf. [JJ06]). Paul Østvær developed a powerful homotopy theory by enlarging the category of C^*-algebras to the category of C^*-spaces (cf. [Øst10]).

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1. Abstract Homotopy Theory

For the convenience of the reader we recall some basic notions and results from abstract homotopy theory. See [Qui67][Bro74][KP97][GJ99] for details.

1.1. Categories of Fibrant Objects.

The following is our main definition.

Definition 1.1 (Brown [Bro74])**.** Let $\mathbf{C}$ be category with terminal object $*$ and let $\mathbf{F} \subseteq \mathbf{C}$ and $\mathbf{W} \subseteq \mathbf{C}$ be distinguished subcategories. We say that $\mathbf{C}$ is a *category of fibrant objects* if the following conditions (F0) - (FW2) hold.

- (F0) The class $\mathbf{F}$ is closed under composition.
- (F1) Isomorphisms of $\mathbf{C}$ are in $\mathbf{F}$.
- (F2) The pullback in $\mathbf{C}$ of a morphism in $\mathbf{F}$ exists and is in $\mathbf{F}$.
- (F3) For any object B of $\mathbf{C}$, the morphism $B \to *$ is in $\mathbf{F}$.

Morphisms of $\mathbf{F}$ are called *fibrations* and denoted $\twoheadrightarrow$.

- (W1) Isomorphisms of $\mathbf{C}$ are in $\mathbf{W}$.
- (W2) If two of f, g and gf are in $\mathbf{W}$, then so is the third.

Morphisms of $\mathbf{W}$ are called *weak equivalences* and denoted $\xrightarrow{\sim}$.

(FW1) The pullback in $\mathbf{C}$ of a morphism in $\mathbf{W} \cap \mathbf{F}$ is in $\mathbf{W} \cap \mathbf{F}$.

Morphisms of $\mathbf{W} \cap \mathbf{F}$ are called acyclic fibrations and denoted $\overset{\sim}{\twoheadrightarrow}$.

(FW2) For any object B of $\mathbf{C}$, the diagonal map $B \to B \times B$ admits a factorization

$$B \underset{\sim}{\xrightarrow{s}} B^I \overset{d}{\twoheadrightarrow} B \times B, \tag{1.1}$$

where $s \in \mathbf{W}$ is a weak equivalence, $d = (d_0, d_1) \in \mathbf{F}$ is a fibration.

The object B^I or more precisely the tuple (B^I, s, d_0, d_1) is called a path-object of B.

If there is no risk for confusion, we simply say that $\mathbf{C}$ is a category of fibrant objects. If the terminal object is also an *initial* object, we say that $\mathbf{C}$ is a *pointed* category of fibrant objects.

Remark 1.2.
(1) The condition (F0) is superfluous since $\mathbf{F}$ is assumed to be a subcategory. But it is convenient to have a notation for this property.
(2) The conditions (F1) and (W1) imply that $\mathbf{F}$ and $\mathbf{W}$ contain all objects of $\mathbf{C}$.
(3) The conditions (F2) and (F3) imply that $\mathbf{C}$ is has finite products.

The following is the motivating example.

Example 1.3. For any model category $\mathbf{M}$, the full subcategory $\mathbf{M}_b$ consisting of the fibrant objects in $\mathbf{M}$ is naturally a category of fibrant objects. In particular, if **Top** denote the category of compactly generated weakly Hausdorff topological spaces and continuous maps,

(1) **Top**, homotopy equivalences, Hurewicz fibrations;
(2) **Top**, weak homotopy equivalences, Serre fibrations;

are examples of categories of fibrant objects.

Occasionally, we find it convenient to isolate the notions of weak equivalences and fibrations.

Definition 1.4. Let $\mathbf{C}$ be a category. A *subcategory of weak equivalences* is a subcategory $\mathbf{W} \subseteq \mathbf{C}$ satisfying (W1) and (W2). If $\mathbf{C}$ has a terminal object, a *subcategory of fibrations* is a subcategory $\mathbf{F} \subseteq \mathbf{C}$ satisfying (F0) - (F3).

1.2. **Fibre and Homotopy Fibre.**

Lemma 1.5 (Factorization Lemma)**.** *Let $f : A \to B$ be a morphism in a category of fibrant objects. Consider the diagram*

$$\begin{CD} Nf @>p>> B \\ @| @AA{d_1}A \\ Nf @>{d_0^*(f)}>> B^I \\ @A{i}AA \quad @V{s \uparrow}VV{d_0} \\ A @>f>> B \end{CD} \quad , \tag{1.2}$$

where (B^I, s, d_0, d_1) is a path-object for B and Nf is the pullback $A \times_B B^I$ and p is the composition $d_1 \circ d_0^*(f)$ and i is the map determined by the section s.

Then p is a fibration and i is a right inverse to an acyclic fibration (in particular, a weak equivalence) and $f = p \circ i$.

Proof. [Bro74, Factorization Lemma]. □

Definition 1.6. We call Nf a *mapping path-object* of f.

Now we consider pointed categories.

Definition 1.7. Let p be a fibration in a pointed category of fibrant objects. The *fibre* F of f is the pullback

$$\begin{CD} F @>i>> E \\ @VVV @VV{p}V \\ * @>>> B \end{CD} \quad . \tag{1.3}$$

We express this situation by the diagram

$$F \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} B \, . \tag{1.4}$$

Definition 1.8. Let $f : A \to B$ be a morphism in a pointed category of fibrant objects. The *homotopy fibre* Ff of f is the fibre of $Nf \overset{p}{\twoheadrightarrow} B$, where p is as in the Factorization Lemma (Lemma 1.5).

Lemma 1.9. *Let p be a fibration in a pointed category of fibrant objects with fibre F. Then the natural map*

$$F \to Fp \tag{1.5}$$

is a weak equivalence.

Proof. Apply [Bro74, Lemma 4.3] to

$$\begin{CD} F @>>> E @>p>> B \\ @VVV @VV{\wr}V @| \\ Fp @>>> Np @>>> B \end{CD} \quad . \tag{1.6}$$

□

1.3. **Homotopy Category.**

Definition 1.10. The *homotopy category* of a category $\mathbf{C}$ of fibrant objects with weak equivalences $\mathbf{W}$ is the localization

$$\mathbf{Ho}(\mathbf{C}) \coloneqq \mathbf{C}[\mathbf{W}^{-1}]. \tag{1.7}$$

In other words, there is given a functor $\gamma : \mathbf{C} \to \mathbf{Ho}(\mathbf{C})$, called the *localization functor*, with the property that for any functor $k : \mathbf{C} \to \mathbf{D}$ such that $k(t)$ is invertible in $\mathbf{D}$ for all $t \in \mathbf{W}$, there exist a unique functor $\mathbf{Ho}(\mathbf{C}) \to \mathbf{D}$ making the diagram

$$\begin{array}{ccc} & \mathbf{Ho}(\mathbf{C}) & \\ {\scriptstyle\gamma}\nearrow & & \searrow \\ \mathbf{C} & \xrightarrow{\quad k \quad} & \mathbf{D} \end{array} \tag{1.8}$$

commute.

Definition 1.11. Let $\mathbf{C}$ be a category of fibrant objects. Two morphisms

$$f_0, f_1 : A \rightrightarrows B \tag{1.9}$$

are said to be *right-homotopic* if for some path-object (B^I, s, d_0, d_1) of B, there is a morphism $h : A \to B^I$ such that $f_0 = d_0 h$ and $f_1 = d_1 h$.

The two are said to be *homotopic* if there is a weak equivalence $t : A' \to A$ such that $f_0 t, f_1 t : A' \rightrightarrows B$ are right-homotopic.

Right-homotopy and homotopy are equivalence relations, and moreover, homotopy is compatible with the composition in $\mathbf{C}$ (c.f. [Bro74, Section 2]).

Definition 1.12. Let $\mathbf{C}$ be a category of fibrant objects. We denote the category of *homotopy classes* in $\mathbf{C}$ by $\pi\mathbf{C}$ and let $\pi : \mathbf{C} \to \pi\mathbf{C}$ denote the quotient functor.

The following is the fundamental result of Brown. If $\mathbf{C}$ is a category, we write $A \in\in \mathbf{C}$ to mean that A is an object of $\mathbf{C}$ and write $\mathrm{Mor}_{\mathbf{C}}(A, B)$ for the space of morphisms from A to B for A, $B \in\in \mathbf{C}$.

Theorem 1.13 (Brown [Bro74, Theorem 2.1])**.** *Let $\mathbf{C}$ be a category of fibrant objects. Then $\pi\mathbf{W} \subseteq \pi\mathbf{C}$ admits a* calculus of right fractions*.*

It follows that, for A, $B \in\in \mathbf{C}$

$$\mathrm{Mor}_{\mathbf{Ho}(\mathbf{C})}(A, B) \cong \operatorname*{colim}_{A' \xrightarrow{\sim} A} \mathrm{Mor}_{\pi\mathbf{C}}(A', B) \tag{1.10}$$

and hence if $\gamma : \mathbf{C} \to \mathbf{Ho}(\mathbf{C})$ is the localization functor, then

(1) any morphism in $\mathrm{Mor}_{\mathbf{Ho}(\mathbf{C})}(A, B)$ can be written as a right-fraction

$$A \xleftarrow{\gamma(t)^{-1}} A' \xrightarrow{\gamma(f)} B \tag{1.11}$$

where $t \in \mathbf{W}$ is a weak equivalence, and

(2) if f_0, f_1 are morphisms in $\mathrm{Mor}_{\mathbf{C}}(A, B)$, then $\gamma(f_0) = \gamma(f_1)$ if and only if f_0 and f_1 are homotopic i.e. $\pi(f_0) = \pi(f_1)$.

□

Corollary 1.14. *Let* $\mathbf{C}$ *be a category of fibrant objects and let* A *be an object in* $\mathbf{C}$*. Suppose that the category* $\mathbf{W}_A$ *of weak equivalences over* A *is "coinitially small" i.e there exists a* set S_A *of objects in* $\mathbf{C}$ *such that for any* $A' \xrightarrow{\sim} A$*, there is a* $A'' \xrightarrow{\sim} A'$ *such that* $A'' \in S_A$*, then* $\mathrm{Mor}_{\mathbf{Ho}(\mathbf{C})}(A, B)$ *is a small set for every* $B \in\in \mathbf{C}$*.* □

Proof. See [GZ67, Proposition 2.4]. □

Now we consider pointed categories.

Definition 1.15. Let B be an object of a pointed category of fibrant objects. A *loop-object* of B is the fibre ΩB of $(d_0, d_1) : B^I \to B \times B$, where (B^I, s, d_0, d_1) is a path-object of B.

Lemma 1.16. *Let* $\mathbf{C}$ *be a pointed category of fibrant objects. Then* Ω *defines a functor*

$$\Omega : \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{C}), \tag{1.12}$$

called the loop-object *functor.*

(1) *For any* $B \in\in \mathbf{C}$*, the object* ΩB *is naturally a group object in* $\mathbf{Ho}(\mathbf{C})$ *and* $\Omega^2 B$ *is naturally an abelian group object in* $\mathbf{Ho}(\mathbf{C})$*.*
(2) *For any fibration* $p : E \twoheadrightarrow B$ *with fibre* F*, there is a natural right-action* $F \times \Omega B \to F$ *in* $\mathbf{Ho}(\mathbf{C})$*. In particular, we have a natural map* $\Omega B \to F$ *in* $\mathbf{Ho}(\mathbf{C})$*.*

Proof. See [Bro74, Section 4]. □

Note that while $\mathbf{Ho}(\mathbf{C})$ depends only on the weak equivalences, the loop-object functor Ω depends also on the fibrations.

Definition 1.17. Let $\mathbf{C}$ be a pointed category of fibrant objects. We define the *Spanier-Whitehead category* of $\mathbf{C}$ as the category

$$\mathbf{SW}(\mathbf{C}) \coloneqq \mathbf{Ho}(\mathbf{C})[\Omega^{-1}], \tag{1.13}$$

obtained from $\mathbf{Ho}(\mathbf{C})$ by inverting the endofunctor Ω.

Objects of $\mathbf{SW}(\mathbf{C})$ are (A, n) with $A \in\in \mathbf{Ho}(\mathbf{C})$ and $n \in \mathbb{Z}$ and the morphisms are given by

$$\mathrm{Mor}_{\mathbf{SW}(\mathbf{C})}((A, n), (B, m)) \coloneqq \operatorname*{colim}_{k \to \infty} \mathrm{Mor}_{\mathbf{Ho}(\mathbf{C})}(\Omega^{n+k} A, \Omega^{m+k} B). \tag{1.14}$$

Theorem 1.18. *Let* $\mathbf{C}$ *be a pointed category of fibrant objects. Then* $\mathbf{SW}(\mathbf{C})$ *is a* triangulated category *with the* shift

$$\Sigma = \Omega^{-1} : \mathbf{SW}(\mathbf{C}) \to \mathbf{SW}(\mathbf{C}) \tag{1.15}$$

given by $(A, n) \mapsto (A, n-1)$ *and the* distinguished triangles *given by triangles isomorphic to triangles of the form*

$$(\Omega B, n) \longrightarrow (F, n) \longrightarrow (E, n) \longrightarrow (B, n)\ , \tag{1.16}$$

where $n \in \mathbb{Z}$ *and* $E \to B$ *is a fibration,* $F \to E$ *is the fibre inclusion and* $\Omega B \to F$ *is the morphism obtained from Lemma 1.16.*

Proof. See [Hov99] or [May01]. □

Definition 1.19. We say that a pointed category of fibrant objects $\mathbf{C}$ is *stable*, if the loop functor $\Omega : \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{C})$ is invertible.

Remark 1.20. If $\mathbf{C}$ is a stable pointed category of fibrant objects, then the natural functor

$$\mathbf{Ho}(\mathbf{C}) \to \mathbf{SW}(\mathbf{C}), \quad A \mapsto (A, 0) \tag{1.17}$$

is an equivalence of categories. In particular, $\mathbf{Ho}(\mathbf{C})$ is naturally a triangulated category with shift $\Sigma = \Omega^{-1} : \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{C})$.

1.4. **Example of π_0-Top.** Now we consider a simple, but very useful, example of a category of fibrant objects. Let **Top** denote the category of compactly generated weakly Hausdorff topological spaces and continuous maps.

Definition 1.21. A map $p : E \to B$ is called a π_0-*fibration* if it satisfies the following path-lifting property:

$$\begin{array}{ccc} \{0\} & \longrightarrow & E \\ \Big\downarrow & \nearrow & \Big\downarrow{\scriptstyle p} \\ [0,1] & \longrightarrow & B \end{array} \quad . \tag{1.18}$$

A map $t : A \to B$ is called a π_0-*equivalence* if

$$t_* : \pi_0(A) \to \pi_0(B) \tag{1.19}$$

is a bijection.

A π_0-*acyclic fibration* is a π_0-fibration which is also a π_0-equivalence.

Lemma 1.22. *All π_0-acyclic fibrations are surjective.*

Proof. Let $p : E \to B$ be a π_0-acyclic fibration let $b \in B$. Then there is a diagram

$$\begin{array}{ccc} \{0\} & \longrightarrow & E \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ [0,1] & \xrightarrow{h} & B \end{array}$$

with $h(1) = b$. Lifting h to a path in E, and evaluating at 1, we get $e \in E$ such that $p(e) = b$. Hence p is surjective. □

Proposition 1.23. *The π_0-fibrations and π_0-equivalences give the structure of a category of fibrant objects on* **Top**.

Proof. The π_0-fibrations form a subcategory of fibrations essentially because the path-lifting property is a right-lifting property:

(F0) If $E \twoheadrightarrow D$ and $D \twoheadrightarrow B$ are π_0-fibrations, so is their composition:

$$\begin{array}{ccc} \{0\} & \longrightarrow & E \\ \Big\downarrow & \nearrow & \Big\downarrow \\ & & D \\ \Big\downarrow & \nearrow & \Big\downarrow \\ [0,1] & \longrightarrow & B \end{array} \quad . \tag{1.20}$$

(F1) If $A \to B$ is a homeomorphisms, then it is a π_0-fibration:

$$\begin{array}{ccc} \{0\} & \longrightarrow & A \\ \Big\downarrow & \nearrow & \Big\downarrow \cong \\ [0,1] & \longrightarrow & B \end{array} \quad . \tag{1.21}$$

(F2) If $E \twoheadrightarrow B$ is a π_0-fibration, then for any $A \to B$, the map $A \times_B E \to A$ is a π_0-fibration:

$$\begin{array}{ccccc} \{0\} & \longrightarrow & A \times_B E & \longrightarrow & E \\ \Big\downarrow & \nearrow & \Big\downarrow & \nearrow & \Big\downarrow \\ [0,1] & \longrightarrow & A & \longrightarrow & B \end{array} \quad . \tag{1.22}$$

(F3) For any B, the map $B \to *$ is a π_0-fibration:

$$\begin{array}{ccc} \{0\} & \longrightarrow & B \\ \Big\downarrow & \nearrow & \Big\downarrow \\ [0,1] & \longrightarrow & * \end{array} \quad . \tag{1.23}$$

The properties (W1) and (W2) are obvious.

(FW1) Let $p : E \overset{\sim}{\twoheadrightarrow} B$ be a π_0-acyclic fibration and let $f : A \to B$ be an arbitrary map. Consider the pullback

$$\begin{array}{ccc} A \times_B E & \longrightarrow & E \\ \Big\downarrow f^*(p) & & \wr \Big\downarrow p \\ A & \xrightarrow{f} & B \end{array} \quad . \tag{1.24}$$

Then we need to show that $f^*(p)$ is a π_0-equivalence. The injectivity of $\pi_0(f^*(p))$ follows from the fact that it is detected by the right-lifting property with respect to $\{0,1\} \hookrightarrow [0,1]$. The surjectivity of $\pi_0(f^*(p))$ follows from Lemma 1.22, since the pullback of a surjection is again a surjection and surjections are surjective on π_0.

(FW2) Let $[a,b]$ be a compact interval, $a < b$, and let

$$B^{[a,b]} := \mathrm{Mor}_{\mathbf{Top}}([a,b], B) \tag{1.25}$$

denote the space of continuous maps $[a, b] \to B$. Then the constant-path map $s : B \to B^{[a,b]}$ is a π_0-equivalence (in fact, a homotopy equivalence).

Let $e_c : B^{[a,b]} \to B$ denote the evaluation at $c \in [a, b]$. Then the map $(e_a, e_b) : B^{[a,b]} \to B \times B$ is a π_0-fibration, since the rectangle $[0, 1] \times [a, b]$ retracts to the union of its three sides $\sqsubset$.

Thus $(B^{[a,b]}, s, e_a, e_b)$ is a path-object for B. For fixed a and b, this is functorial.

□

2. Applications to the Category of C^*-algebras

Let $\mathbf{C}^*$ denote the category of C^**-algebras* and $*$*-homomorphisms*. It is complete and cocomplete and pointed – the zero object is the zero algebra 0 – symmetric monoidal category with respect to the maximal tensor product. We refer to [Mey08] for the details.

The category $\mathbf{C}^*$ is naturally enriched over **Top**, the Cartesian closed category of compactly generated weakly Hausdorff topological spaces. Indeed, since C^*-algebras are normed, they are compactly generated and weakly Hausdorff as spaces, hence there is a forgetgul functor $\mathbf{C}^* \to \mathbf{Top}$. For C^*-algebras A and B, we give $\mathrm{Mor}_{\mathbf{C}^*}(A, B)$ the subspace topology from $\mathrm{Mor}_{\mathbf{Top}}(A, B)$ via the forgetful functor. It is easy to see that $\mathrm{Mor}_{\mathbf{C}^*}(A, B)$ is a closed subspace of $\mathrm{Mor}_{\mathbf{Top}}(A, B)$, hence itself a compactly generated weakly Hausdorff space.

Let $\mathbf{A}^* \subset \mathbf{C}^*$ denote the full subcategory of *abelian* C^*-algebras. By the Gelfand-Naimark duality, $\mathbf{A}^*$ is equivalent to the opposite category of the category $\mathbf{CH}_*$ of pointed, compact Hausdorff topological spaces and pointed continuous maps. If X is a compact Hausdorff space, we write $C(X)$ for the (unital) C^*-algebra of continuous functions on X. If in addition X has a base point, we write $C_0(X)$ for the C^*-algebra of continuous functions on X vanishing at the base point.

If we enrich $\mathbf{CH}_*$ over **Top** by the inclusion $\mathbf{CH}_* \subset \mathbf{Top}$, the Gelfand-Naimark duality becomes an equivalence of enriched categories.

Remark 2.1. The category $\mathbf{C}^*$ of C^*-algebras is also enriched over the category of Hausdorff spaces, using the compact-open topology on morphism spaces. However, in order to facilitate the connection to algebraic topology, we use the compactly generated compact-open topology. Note that if A is separable, then the compact-open topology on $\mathrm{Mor}_{\mathbf{C}^*}(A, B)$ is metrizable, hence compactly generated.

Lemma 2.2. *Let B be a C^*-algebra and let X be a compact Hausdorff space. Then $\mathrm{Mor}_{\mathbf{Top}}(X, B)$ is naturally a C^*-algebra isomorphic to $C(X) \otimes B$.*

Proof. By [Str, Proposition 2.13] the topology on $\mathrm{Mor}_{\mathbf{Top}}(X, B)$ coincides with the topology given by the norm $||f|| := \sup_{x \in X} ||f(x)||_B$. The rest is standard (cf. [WO93, Corollary T.6.17]). □

The following is the main property of the enrichment that we use. See also [JJ06, Proposition 3.4] and [Mey08, Proposition 24].

Lemma 2.3. *Let A and B be C^*-algebras and let X be a compact Hausdorff space. Then there is an identification*

$$\mathrm{Mor}_{\mathbf{Top}}(X, \mathrm{Mor}_{\mathbf{C}^*}(A, B)) \cong \mathrm{Mor}_{\mathbf{C}^*}(A, C(X) \otimes B) \tag{2.1}$$

natural in A, B and X.

Proof. Since A and B are compactly generated weakly Hausdorff spaces, we have a natural identification

$$\mathrm{Mor}_{\mathbf{Top}}(X, \mathrm{Mor}_{\mathbf{Top}}(A, B)) \cong \mathrm{Mor}_{\mathbf{Top}}(A, \mathrm{Mor}_{\mathbf{Top}}(X, B)), \tag{2.2}$$

by [Str, Proposition 2.12]. Hence by Lemma 2.2

$$\mathrm{Mor}_{\mathbf{Top}}(X, \mathrm{Mor}_{\mathbf{Top}}(A, B)) \cong \mathrm{Mor}_{\mathbf{Top}}(A, C(X) \otimes B). \tag{2.3}$$

Now it is easy to check that this restricts to the identification in (2.1). □

Often we will make this identification implicitly.

Remark 2.4. Note that there are pointed analogues of Lemma 2.2 and Lemma 2.3.

Corollary 2.5. *For any $D \in\in \mathbf{C}^*$, the functor $\mathrm{Mor}_{\mathbf{C}^*}(D, -) : \mathbf{C}^* \to \mathbf{Top}$ preserves pullbacks.*

Proof. Let D be fixed and let $F := \mathrm{Mor}_{\mathbf{C}^*}(D, -)$.

Consider a pullback diagram

$$\begin{array}{ccc} A \times_B E & \longrightarrow & E \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array} \tag{2.4}$$

in $\mathbf{C}^*$. We need to prove that the natural map

$$\Phi : F(A \times_B E) \to F(A) \times_{F(B)} F(E) \tag{2.5}$$

is a homeomorphism. It is clear that Φ is a continuous bijection. Hence it suffices to prove that for any X compact Hausdorff, a map $X \to F(A \times_B E)$ is continuous if the compositions $X \to F(A)$ and $X \to F(E)$ are continuous. However, this follows from Lemma 2.3 and its proof. □

2.1. **Ordinary Homotopy Theory.** The (ordinary) homotopy category of C^*-algebras is the category of C^*-algebras and homotopy classes of $*$-homomorphisms, denoted $\pi_0\mathbf{C}^*$ for the time being:

$$\mathrm{Mor}_{\pi_0\mathbf{C}^*}(A, B) := \pi_0(\mathrm{Mor}_{\mathbf{C}^*}(A, B)). \tag{2.6}$$

We now give $\mathbf{C}^*$ the structure of a category of fibrant objects, whose homotopy category is $\pi_0\mathbf{C}^*$.

Definition 2.6. A $*$-homomorphism $t : A \to B$ is called a *homotopy equivalence* if the induced map

$$t_* : \mathrm{Mor}_{\mathbf{C}^*}(D, A) \to \mathrm{Mor}_{\mathbf{C}^*}(D, B) \tag{2.7}$$

is a π_0-equivalence, in the sense of Definition 1.21, for all $D \in\in \mathbf{C}^*$.

Remark 2.7. By Yoneda's Lemma, $t \in \mathbf{C}^*$ is a homotopy equivalence if and only if $\pi_0(t) \in \pi_0\mathbf{C}^*$ is invertible.

Definition 2.8. A $*$-homomorphism $p : E \to B$ is called a *Schochet fibration* if the induced map

$$p_* : \mathrm{Mor}_{\mathbf{C}^*}(D, E) \to \mathrm{Mor}_{\mathbf{C}^*}(D, B) \tag{2.8}$$

is a π_0-fibration, in the sense of Definition 1.21, for all $D \in\in \mathbf{C}^*$.

Remark 2.9. Schochet called these maps *cofibrations* in [Sch84], because, under the Galfand-Naimark duality, the condition in Definition 2.8 for a $*$-homomorphism of abelian algebras corresponds to the homotopy extension property for the corresponding map of (pointed compact Hausdorff) spaces.

In a similar way, it is customary that $\mathrm{Mor}_{\mathbf{Top}_*}(S^1, B) \cong C_0(S^1) \otimes B$ is called the *suspension* of B, since $C_0(S^1) \otimes C_0(X) \cong C_0(S^1 \wedge X)$ for $B = C_0(X)$. Here X is a pointed compact Hausdorff space and $C_0(X)$ is the continuous functions vanishing at the base point. See also Remark 2.13.

However, for the sake of consistency, in this paper we will keep our notations and terminologies compatible with that of Section 1.

The following proposition is contained in [Sch84].

Proposition 2.10. *The category of C^*-algebras $\mathbf{C}^*$ is a pointed category of fibrant objects with weak equivalences the homotopy equivalences and fibrations the Schochet fibrations, whose homotopy category is the ordinary homotopy category i.e. $\mathbf{Ho}(\mathbf{C}^*) = \pi_0\mathbf{C}^*$.*

Proof. Everything follows from Proposition 1.23: we need to use Corollary 2.5 for (F2) and (FW1), and Lemma 2.3 with $X = [a, b]$ for (FW2).

It follows from the construction of the path-object in $\mathbf{C}^*$ that two $*$-homomorphisms f_0, $f_1 \in \mathbf{C}^*$ are right-homotopic if and only if $\pi_0(f_0) = \pi_0(f_1)$ in $\pi_0\mathbf{C}^*$ and by Remark 2.7 this happens if and only if f_0, f_1 are homotopic in the sense of Definition 1.11. Hence

$$\mathbf{Ho}(\mathbf{C}^*) = \pi\mathbf{C}^* = \pi_0\mathbf{C}^*. \tag{2.9}$$

□

Note that $\mathbf{C}^*$ has a functorial path-object, given by $C[0, 1] \otimes B$, hence also a functorial loop-object[1] $\Omega B \coloneqq C_0(0, 1) \otimes B$.

The Spanier-Whitehead category $\mathbf{SW}(\mathbf{C}^*)$ is the *suspension-stable homotopy category of C^*-algebras* studied by Rosenberg [Ros82] and Schochet [Sch84].

[1] usually called *suspension* in the C^*-context

Remark 2.11. Let $\mathbf{S}^*$ denote the category of *separable* C^*-algebras. Then considering only D separable in Definitions 2.6 and 2.8, we get a structure of a category of fibrant objects on $\mathbf{S}^*$.

Remark 2.12. The following are well-known and/or easy to see.

(1) The localization $\mathbf{C}^* \to \mathbf{Ho}(\mathbf{C}^*)$ preserves arbitrary coproducts and arbitrary products:

$$\mathrm{Mor}_{\mathbf{Ho}(\mathbf{C}^*)}(\coprod_{i\in I} A_i, B) = \prod_{i\in I} \mathrm{Mor}_{\mathbf{Ho}(\mathbf{C}^*)}(A_i, B), \tag{2.10}$$

$$\mathrm{Mor}_{\mathbf{Ho}(\mathbf{C}^*)}(A, \prod_{i\in I} B_i) = \prod_{i\in I} \mathrm{Mor}_{\mathbf{Ho}(\mathbf{C}^*)}(A, B_i). \tag{2.11}$$

(2) The loop $\Omega : \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{C})$ preserves finite products:

$$\Omega(B_1 \times B_2) \cong \Omega B_1 \times \Omega B_2, \tag{2.12}$$

but not infinite products (for example, the algebras $\Omega \prod_{\aleph_0} \mathbb{C}$ and $\prod_{\aleph_0} \Omega\mathbb{C}$ have different K_1 groups), nor coproducts (for example, the natural map $\Omega\mathbb{C} \coprod \Omega\mathbb{C} \to \Omega(\mathbb{C} \coprod \mathbb{C})$ is not a homotopy equivalence).

(3) The "Spanier-Whitehead functor" $\mathbf{Ho}(\mathbf{C}^*) \to \mathbf{SW}(\mathbf{C}^*)$ preserves finite products, but not finite coproducts.

Remark 2.13 (Abelian Algebras)**.** Let $\mathbf{A}^* \subseteq \mathbf{C}^*$ denote the full subcategory consisting of abelian C^*-algebras. Then $\mathbf{A}^*$ is a coreflexive monoidal subcategory of $\mathbf{C}^*$ – the left-adjoint of the inclusion $i : \mathbf{A}^* \to \mathbf{C}^*$ is the abelianization $-^{\mathrm{ab}} : \mathbf{C}^* \to \mathbf{A}^*$:

$$\mathrm{Mor}_{\mathbf{C}^*}(D, iB) = \mathrm{Mor}_{\mathbf{A}^*}(D^{\mathrm{ab}}, B), \tag{2.13}$$

for $D \in\in \mathbf{C}^*$, $B \in\in \mathbf{A}^*$. It follows that $\mathbf{A}^*$ is a category of fibrant objects and $\mathbf{Ho}(\mathbf{A}^*)$ and $\mathbf{SW}(\mathbf{A}^*)$ are full subcategories of $\mathbf{Ho}(\mathbf{C}^*)$ and $\mathbf{SW}(\mathbf{C}^*)$, respectively.

2.2. **C^*-Invariant Homotopy Theory.** Let $\mathcal{K}$ be the algebra of compact operators on a separable Hilbert space.

Proposition 2.14. *Defining the weak equivalences to be*

$$\{t \in \mathbf{C}^* \mid t \otimes \mathrm{id}_{\mathcal{K}} \textit{ is a homotopy equivalence}\} \tag{2.14}$$

and the fibrations to be

$$\{p \in \mathbf{C}^* \mid p \otimes \mathrm{id}_{\mathcal{K}} \textit{ is a Schochet fibration}\} \tag{2.15}$$

defines a category of fibrant objects on $\mathbf{C}^*$*, denoted* $\mathbf{M}$*.*

Proof. This is clear since $- \otimes \mathrm{id}_{\mathcal{K}}$ preserves pullbacks. □

Let $e_{11} : \mathbb{C} \to \mathcal{K}$ denote a rank-one projection. Then for any $B \in\in \mathbf{M}$, the morphism $\mathrm{id}_B \otimes e_{11}$ is a weak equivalence in $\mathbf{M}$.

It follows that $\mathbf{Ho}(\mathbf{M})$ is the "monoidal" localization $\mathbf{Ho}(\mathbf{C}^*)[\otimes e_{11}^{-1}]$:

$$\begin{aligned} \mathrm{Mor}_{\mathbf{Ho}(\mathbf{M})}(A,B) &\cong \mathrm{Mor}_{\mathbf{Ho}(\mathbf{C}^*)}(A, B\otimes\mathcal{K}) && (2.16)\\ &\cong \mathrm{Mor}_{\mathbf{Ho}(\mathbf{C}^*)}(A\otimes\mathcal{K}, B\otimes\mathcal{K}). && (2.17)\end{aligned}$$

In the notation of [Hig90], the categories $\mathbf{Ho}(\mathbf{M})$ and $\mathbf{SW}(\mathbf{M})$ are the not necessarily separable analogues of $\mathbf{TH}$ and $\mathbf{TS}$ respectively.

2.3. Topological K-Theory.

Definition 2.15 ([Sch84])**.** A *fibre homology theory* on $\mathbf{C}^*$ is a homology theory of the triangulated category $\mathbf{SW}(\mathbf{C}^*)$.

Proposition 2.16. *Let $\mathcal{H}$ be a fibre homology theory on $\mathbf{C}^*$. Then $\mathcal{H}$-equivalences and Schochet fibrations define a category of fibrant objects on $\mathbf{C}^*$ with loop object $\Omega B = C_0(0,1)\otimes B$, denoted $R_{\mathcal{H}}\mathbf{C}^*$.*

Proof. It is clear that $\mathcal{H}$-equivalences form a subcategory of weak equivalences. Since homology theories are homotopy invariant by definition, homotopy equivalences are $\mathcal{H}$-equivalences, and thus path-objects exist.

It remains to show that a pullback $f^*(p)$ of a Schochet fibration $p : E \twoheadrightarrow B$ that is an $\mathcal{H}$-equivalence is again an $\mathcal{H}$-equivalence, where $f : A \to B$ is arbitrary. But this follows from the long exact sequence applied to the diagram

$$\begin{array}{ccccc} F & \rightarrowtail & E\times_B A & \xrightarrow{f^*(p)} & A \\ \| & & \downarrow & & \downarrow{\scriptstyle f} \\ F & \rightarrowtail & E & \xrightarrow{p} & B \end{array} \quad , \qquad (2.18)$$

where $F = \ker(p)$ is the kernel of p. □

Taking $\mathcal{H}$ to be topological K-theory in Proposition 2.16, we get a category $\mathbf{K} = R_K\mathbf{C}^*$ of fibrant objects whose weak equivalences are K-equivalences and fibrations are Schochet fibrations. Compare [JJ06] and [MN06]. It follows from Theorem 2.17, that $\mathbf{Ho}(\mathbf{K})$ has small hom sets.

Let $\mathcal{K}$ be the algebra of compact operators on a separable Hilbert space and let $e_{11} : \mathbb{C} \to \mathcal{K}$ denote a rank-one projection. Then

$$\mathrm{id}_A \otimes e_{11} : A \to A\otimes\mathcal{K} \qquad (2.19)$$

is a K-equivalence. We also have a natural isomorphism $\Omega^2 A \to A\otimes\mathcal{K}$ in $\mathbf{Ho}(\mathbf{K})$, since Bott periodicity can be implemented by a boundary map associated to a Toeplitz type extension. It follows that

$$\Omega : \mathrm{Mor}_{\mathbf{Ho}(\mathbf{K})}(A,B) \to \mathrm{Mor}_{\mathbf{Ho}(\mathbf{K})}(\Omega A, \Omega B) \qquad (2.20)$$

is invertible. Hence $\mathbf{K}$ is stable and the natural functor $\mathbf{Ho}(\mathbf{K}) \to \mathbf{SW}(\mathbf{K})$ is an equivalence of categories. In particular, $\mathbf{Ho}(\mathbf{K})$ is a triangulated category in a natural way, and $\mathbf{SW}(\mathbf{C}^*) \to \mathbf{Ho}(\mathbf{K})$ is a triangulated functor.

The following is a version of the Universal Coefficient Theorem of Rosenberg and Schochet (cf. [RS87]). It can be deduced from results in [MN06], but we give a self-contained proof.

Theorem 2.17. *For $B \in\in \mathbf{K}$, we have*

$$\mathrm{Mor}_{\mathbf{Ho}(\mathbf{K})}(\mathbb{C}, B) \cong K_0(B). \tag{2.21}$$

More generally, for A, $B \in\in \mathbf{K}$, there is a natural short exact sequence

$$\mathrm{Ext}(K_{*+1}(A), K_*(B)) \rightarrowtail \mathrm{Mor}_{\mathbf{Ho}(\mathbf{K})}(A, B) \twoheadrightarrow \mathrm{Hom}(K_*(A), K_*(B)) \ , \tag{2.22}$$

where

$$\mathrm{Hom}(K_*(A), K_*(B)) := \bigoplus_{i=0,1} \mathrm{Hom}_{\mathbb{Z}}(K_i(A), K_i(B)) \quad \textit{and} \tag{2.23}$$

$$\mathrm{Ext}(K_{*-1}(A), K_*(B)) := \bigoplus_{i=0,1} \mathrm{Ext}^1_{\mathbb{Z}}(K_{i-1}(A), K_i(B)). \tag{2.24}$$

Proof. We have a natural (additive) map

$$\mathrm{Mor}_{\mathbf{Ho}(\mathbf{K})}(A, B) \to \mathrm{Hom}_{\mathbb{Z}}(K_*(A), K_*(B)). \tag{2.25}$$

We claim that this is an isomorphism if $K_*(A)$ is free — for $A = \mathbb{C}$ we get (2.21).

Indeed, suppose that $K_*(A)$ is free. First note that we have natural isomorphisms

$$K_0(D) = \mathrm{Mor}_{\mathbf{Ho}(\mathbf{C}^*)}(q\mathbb{C}, D \otimes \mathcal{K}), \tag{2.26}$$

$$K_1(D) = \mathrm{Mor}_{\mathbf{Ho}(\mathbf{C}^*)}(\Omega\mathbb{C}, D \otimes \mathcal{K}), \tag{2.27}$$

where $q\mathbb{C}$ is the kernel of the folding map ($\mathbb{C} \coprod \mathbb{C} \to \mathbb{C}$). We have a K-equivalence $q\mathbb{C} \xrightarrow{\sim} \mathbb{C}$.

Then it is clear that any map $K_*(A) \to K_*(B)$ can be implemented by an element of the form

$$\begin{array}{ccc} (\coprod_I q\mathbb{C}) \coprod (\coprod_J \Omega\mathbb{C}) & \longrightarrow & B \otimes \mathcal{K} \\ \downarrow \wr & & \uparrow \\ A \otimes \mathcal{K} & & \wr \\ \uparrow \wr & & \\ A & & B \end{array} \tag{2.28}$$

in $\mathbf{Ho}(\mathbf{K})$. Hence (2.25) is surjective. To see injectivity of (2.25), let

$$A \xleftarrow{\sim} A' \longrightarrow B \tag{2.29}$$

be a morphism in $\mathrm{Mor}_{\mathbf{Ho}(\mathbf{K})}(A, B)$ that maps to $0 \in \mathrm{Hom}(K_*(A), K_*(B))$. Then we can complete (2.29) to a homotopy-commutative diagram

$$\begin{array}{ccccc} (\coprod_I q\mathbb{C}) \coprod (\coprod_J \Omega\mathbb{C}) & \xrightarrow{\sim} & A' \otimes \mathcal{K} & \longrightarrow & B \otimes \mathcal{K} \\ \downarrow \wr & & & & \\ A \otimes \mathcal{K} & & \uparrow \wr & & \uparrow \wr \\ \uparrow \wr & & & & \\ A & \xleftarrow{\sim} & A' & \longrightarrow & B \end{array} \tag{2.30}$$

in $\mathbf{Ho}(\mathbf{C}^*)$. Then the top horizontal map is null-homotopic, i.e. zero in $\mathbf{Ho}(\mathbf{C}^*)$, hence zero in $\mathbf{Ho}(\mathbf{K})$. In other words, (2.25) is injective if $K_*(A)$ is free.

The general case follows using a geometric resolution of $K_*(A)$. See for instance [Uuy04]. □

2.4. *KK*-**Theory.** In the next two subsections, we will concentrate on the category $\mathbf{S}^*$ of *separable* C^*-algebras.

Definition 2.18. A $*$-homomorphisms $t : A \to B$ in $\mathbf{S}^*$ is called a *KK*-equivalence if

$$t_* : \mathrm{Mor}_{\mathbf{C}^*}(qD, A \otimes \mathcal{K}) \to \mathrm{Mor}_{\mathbf{C}^*}(qD, B \otimes \mathcal{K}) \tag{2.31}$$

is a π_0-equivalence for all $D \in\in \mathbf{S}^*$, where qD is the kernel of the map $\mathrm{id}_D \coprod \mathrm{id}_D : D \coprod D \to D$.

Theorem 2.19. *The category of* separable C^**-algebras form a category of fibrant objects with weak equivalences the KK-equivalences and fibrations the Schochet fibrations, denoted* **KK***, whose homotopy category* $\mathbf{Ho}(\mathbf{KK})$ *is equivalent to the KK-category of Kasparov. It follows that Kasparov's KK-category is a stable triangulated category.*

Proof. The proof of Proposition 2.10 works in this case as well and proves that **KK** is a category of fibrant objects.

We use the Cuntz type picture of *KK*-theory:

$$KK(A, B) := \mathrm{Mor}_{\mathbf{Ho}(\mathbf{C}^*)}(qA \otimes \mathcal{K}, qB \otimes \mathcal{K}). \tag{2.32}$$

Consider the functor $\Phi : \mathbf{KK} \to KK$ that send $f : A \to B$ to the composition

$$q(f) \otimes \mathrm{id}_{\mathcal{K}} : qA \otimes \mathcal{K} \to qB \otimes \mathcal{K}. \tag{2.33}$$

Then Φ is indeed functor which is additive and $\Phi(f)$ is a homotopy equivalence if and only if f is a *KK*-equivalence. Hence the induced functor $\mathbf{Ho}(\Phi) : \mathbf{Ho}(\mathbf{KK}) \to KK$ is faithful. Moreover, by [Cun87, Theorem 1.6], $\Phi(A)$ is homotopy equivalent to $\Phi(\Phi(A))$, hence $\mathbf{Ho}(\Phi)$ is full. It follows that $\mathbf{Ho}(\Phi)$ is an equivalence of categories.

Stability follows from Bott Periodicity. □

Remark 2.20. Note that in Theorem 2.19, we can take the semi-split surjections, i.e. surjections with a completely positive contractive splitting, to be the fibrations. Indeed, the only nontrivial part is (FW1): if $p : E \to B$ is a semi-split surjection which is also a KK-equivalence and $f : A \to B$ is arbitrary, then the pullback $f^*(p)$ is also a KK-equivalence. However, this is clear since if p is a semi-split surjection with kernel F, then $F \to Fp$ is a KK-equivalence (see [Bla98, Theorem 19.5.5]), hence p is a KK-equivalence if and only F is KK-contractible if and only if $f^*(p)$ is a KK-equivalence (see Diagram (2.18)).

Note also that Schochet fibrations and semi-split surjections give rise to the same class of distinguished triangles on $\mathbf{Ho}(\mathbf{KK}) \cong \mathbf{SW}(\mathbf{KK})$.

2.5. ***E*-Theory.** We consider $\mathbf{S}^*$ as a category of fibrant objects with weak equivalences the homotopy equivalences and fibrations the Schochet fibrations. As in Definition 2.15, a fibre homology theory on $\mathbf{S}^*$ is a homology theory on the triangulated category $\mathbf{SW}(\mathbf{S}^*)$ to $\mathbf{Ab}$.

Definition 2.21. We say that a fibre homology theory $\mathcal{H}$ on $\mathbf{S}^*$ is *excisive* with respect to a surjection $p : E \to B$, if the inclusion $\ker(p) \to Fp$ is an $\mathcal{H}$-equivalence.

A *homology theory* on $\mathbf{S}^*$ is a fibre homology theory excisive with respect to all surjections.

Definition 2.22. We say that a morphism $t \in \mathbf{S}^*$ is a *weak equivalence* if it is an $\mathcal{H}$-equivalence for all homology theories $\mathcal{H}$ on $\mathbf{S}^*$.

Remark 2.23. Note that a homotopy equivalence is also a weak equivalence.

Theorem 2.24. *The category $\mathbf{S}^*$ is a pointed category of fibrant objects with weak equivalences as in Definition 2.22 and fibrations the surjections, denoted* $\mathbf{AM}$*. The Spanier-Whitehead category* $\mathbf{SW}(\mathbf{AM})$ *is equivalent to the stable homotopy category of A. Thom* [Tho03]*.*

We start with some simple lemmas of independent interest.

Lemma 2.25. *Schochet fibrations are surjections.*

Proof. Let $p : E \to B$ be a Schochet fibration. Consider the universal algebra generated by a positive contraction:

$$C \coloneqq \mathbf{C}^*(x \mid 0 \leq x \leq 1) = C_0(0,1]. \tag{2.34}$$

Then for any $b \in B$, $0 \leq b \leq 1$, there is a path

$$[0,1] \ni r \mapsto (x \mapsto rb) \in \mathrm{Mor}_{\mathbf{C}^*}(C, B), \tag{2.35}$$

which lifts to $0 \in \mathrm{Mor}_{\mathbf{C}^*}(C, E)$ at $r = 0$. Lifting the path to $\mathrm{Mor}_{\mathbf{C}^*}(C, E)$, we get $e \in E$, $0 \leq e \leq 1$, such that $p(e) = b$. It follows that p is surjective. □

Lemma 2.26. *Let $p : E \to B$ be a surjection with kernel F. Then p is a weak equivalence if and only if F is $\mathcal{H}$-acyclic for all homology theories $\mathcal{H}$ on $\mathbf{S}^*$.*

Proof. First note that we have a map of extensions where the vertical maps are all weak equivalences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{i} & E & \xrightarrow{p} & B & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \| & & \\ 0 & \longrightarrow & Fp & \rightarrowtail & Np & \twoheadrightarrow & B & \longrightarrow & 0 \end{array} \quad (2.36)$$

Now the proof is complete by the naturality of the long exact sequence associated to homology theories. ☐

Proof of Theorem 2.24. It follows from Lemma 2.26, that (FW1) holds. The rest of the proof that **AM** is a category of fibrant objects is clear.

Now we prove that the functor $j : \mathbf{S}^* \to (\mathbf{SW}(\mathbf{AM}), \Omega^{-1})$, $A \mapsto (A, 0)$ is a *universal* triangulated homology theory in the sense of [Tho03, Definition 2.3.3]. Then [Tho03, Theorem 3.3.6] finishes the proof.

Clearly $j : \mathbf{S}^* \to (\mathbf{SW}(\mathbf{AM}), \Sigma = \Omega^{-1})$ is a triangulated homology theory. Let $R : \mathbf{S}^* \to (\mathbf{P}, \Omega^{-1})$ be another triangulated homology theory. By [Tho03, Theorem 2.3.8], for any $D \in\in \mathbf{AM}$, the functor $\mathcal{H} : \mathbf{SW}(\mathbf{S}^*) \to \mathbf{Ab}$, $(A, n) \mapsto \mathrm{Hom}_R(D, \Sigma^n A)$ is a homology theory on $\mathbf{S}^*$. It follows that if $t : A \to B \in \mathbf{AM}$ is a weak equivalence then $R(t)$ is invertible (by Yoneda's Lemma). Thus R_* induces a functor $R : \mathbf{Ho}(\mathbf{AM}) \to \mathbf{P}$. By the proof of [Tho03, Proposition 2.3.4], R_* intertwines the Ω's, hence induces a functor $\widehat{R} : \mathbf{SW}(\mathbf{AM}) \to \mathbf{P}$. Since R is a triangulated homology theory, $\widehat{R}$ is a triangulated functor and $R = \widehat{R} \circ j$. The uniqueness of $\widehat{R}$ is clear, and thus j is a universal triangulated homology theory. ☐

Remark 2.27. In Theorem 2.24, we can take the fibrations to be the Schochet fibrations. However, the distinguished triangles on $\mathbf{SW}(\mathbf{AM})$ would be the same (see the diagram (2.36)).

Let $e_{11} : \mathbb{C} \to \mathcal{K}$ be a rank-one projection.

Definition 2.28. A (fibre) homology theory $\mathcal{H}$ on $\mathbf{S}^*$ is said to be *C^*-invariant* if $\mathrm{id}_B \otimes e_{11}$ is an $\mathcal{H}$-equivalence for all $B \in\in \mathbf{S}^*$.

Definition 2.29. A morphism $t \in \mathbf{S}^*$ is said to be an *E-equivalence* if it induces isomorphism on all C^*-invariant homology theories.

Proposition 2.30. *The category $\mathbf{S}^*$ is a category of fibrant objects with weak equivalences the E-equivalences and fibrations the surjections, denoted* **E**. *The homotopy category* $\mathbf{Ho}(\mathbf{E})$ *is a triangulated category, equivalent to the E-theory of Higson.*

Proof. It is clear that **E** is a category of fibrant objects. By Cuntz's Bott periodicity, $\mathbf{Ho}(\mathbf{E})$ satisfies Bott periodicity. Hence $\Omega : \mathbf{Ho}(\mathbf{E}) \to \mathbf{Ho}(\mathbf{E})$ is an equivalence and $\mathbf{Ho}(\mathbf{E})$ is equivalent to $\mathbf{SW}(\mathbf{E})$. Thus $\mathbf{Ho}(\mathbf{E})$ is a triangulated category. Moreover, every homotopy invariant, half-exact, C^*-invariant functor comes from a C^*-invariant homology theory. Universal property of E finishes the proof. ☐

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