SUBSPACES OF A PARA-QUATERNIONIC HERMITIAN VECTOR SPACE

MASSIMO VACCARO

Dipartimento dell'Ingegneria di Informazione e Matematica Applicata
Università di Salerno, 84084 - Fisciano (SA) , Italy
massimo_vaccaro@libero.it

Abstract: Let $(\tilde{Q}, g)$ be a para-quaternionic Hermitian structure on the real vector space V. By referring to the tensorial presentation $(V, \tilde{Q}, g) \simeq (H^2 \otimes E^{2n}, \mathfrak{sl}(H), \omega^H \otimes \omega^E)$, we give an explicit description, from an affine and metric point of view, of main classes of subspaces of V which are invariantly defined with respect to the structure group of $\tilde{Q}$ and $(\tilde{Q}, g)$ respectively.

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1. Introduction

In the last years, quaternionic-like structures have captured an increasing interest both in mathematics and physics. In particular, many recent researches have focused on para-quaternionic structures which are the object of this article from a basic geometrical point of view.

Para-geometries play an important role in physics in some supersymmetric theories. For instance in [1], [2], it was shown that the target space for scalar fields in 4-dimensional Euclidean N = 2 supersymmetry carries a special para-Kaehler structure similar to the special Kaehler structure which arises on the target space of scalar fields for N = 2 Lorentzian 4-dimensional supersymmetry. Also, in [3], where the role of special geometry in the theory of supersymmetric black holes is explained, the target metric is (Riemannian) quaternionic Kaehler or (neutral) para-quaternionic Kaehler according if the space-time signature of the metric is Lorentzian or Euclidean respectively.

In this article, which is the first part of a research project about submanifolds of a para-quaternionic Kaehler manifold, we deal with special subspaces of a para-quaternionic Hermitian vector space. A brief description of the results is given below.

Let V be a real vector space endowed with a para-quaternionic structure $\tilde{Q} \subset End(V)$, i.e. $\tilde{Q}$ is isomorphic to $Im\tilde{\mathbb{H}}$ where $\tilde{\mathbb{H}}$ is the Clifford algebra of para-quaternions. It is known that $\dim V = 2n$ and one has an isomorphism $(V, \tilde{Q}) \simeq (H^2 \otimes E^n, \mathfrak{sl}(H))$, where H and E are real vector spaces and $\mathfrak{sl}(H)$ is the Lie algebra of the special linear group $SL(H)$.

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Here we consider a *para-quaternionic Hermitian* vector space $(V, \tilde{Q}, g)$, where g ia a $\tilde{Q}$-hermitian metric on V. In this case the compatibility conditions imply that $\dim V = 4n$ and that g is pseudo-Euclidean of (neutral) signature $(2n, 2n)$. Moreover one has an isomorphism $(V^{4n}, \tilde{Q}, g) \simeq (H^2 \otimes E^{2n}, \mathfrak{sl}(H)), \omega^H \otimes \omega^E)$ where ω^H and ω^E are two symplectic forms on H and E respectively.

The para-quaternionic Hermitian structure naturally defines some classes of *special subspaces* of V in terms of their behaviour with respect to the endomorphisms of $\tilde{Q}$ and to the metric g, which are interesting to consider: **para-quaternionic**, **complex**, **totally complex**, **weakly para-complex**, **totally para-complex**, **nilpotent**, **real**, **totally real**.

Also by referring to the tensorial presentation of a para-quaternionic Hermitian vector space there are some classes of subspaces of V which is natural to consider: first of all the **product** subspaces and among them the **decomposable** subspaces, furthermore some $\mathbf{U^{F,T}}$ subspaces which we defined as depending on a symplectic basis of H, on a subspace $F \subset E$ and on a linear map $T : F \to E$ (see def.(3.1)). Indeed a generic subspace in V is not a $U^{F,T}$ subspace, but it turns out that any $U \subset V$ admits a decomposition into a pair of such subspaces (prop. 3.4).

The main purpose of this article is to give an explicit description of the special subspaces of the para-quaternionic Hermitian space $(V, \tilde{Q}, g)$ in terms of the tensorial presentation $(H^2 \otimes E^{2n}, \mathfrak{sl}(H)), \omega^H \otimes \omega^E)$. After proving that the *para-quaternionic subspaces coincide with the products* $H \otimes E'$, $E' \subseteq E$ (prop. 3.8), the basic tool consists in restricting to **pure** subspaces, not containing any non trivial para-quaternionic subspace, and by showing that *pure special subspaces are* $U^{F,T}$ *subspaces.* Viceversa we also give the precise *conditions for a* $U^{F,T}$ *subspace to be a special subspace of any given type.*

This presentation is also useful from the metrical point of view to determine, for each subspace, the signature of the induced metric. We give then the conditions for the above special subspaces to be g-non degenerate.

Finally we notice that the results obtained with regard to the tensorial presentation of the geometry of a para-quaternionic Hermitian vector space can be extended to the quaternionic case due to the fact that the complexification of a quaternionic Hermitian vector space V^{4n} has a natural identification with the tensor product $H \otimes E$ of two complex vector spaces of dimension 2 and 2n respectively ([4],[5]).

This work develops and completes a research undertaken during the Ph.D. thesis whose advisor was professor Dmitri Alekseevsky.

2. Preliminaries

Definition 2.1. *Let* V *be a real vector space of dimension* n *and* $K \in End(V)$ *such that* $K^2 = Id$. *Let denote* V_K^+ *and* V_K^- *the +1 and -1 eigenspaces of* K. *Then* K *is called a* ***product structure on*** V *if* $\dim V_K^+, \dim V_K^- > 0$. *A* ***para-complex structure on*** V *is a product structure with* $\dim V_K^+ = \dim V_K^-$.

A triple (J_1, J_2, J_3) *of anticommuting endomorphisms of* V *satisfying the relations:*

$$-J_1^2 = J_2^2 = J_3^2 = Id, \quad J_1 J_2 = J_3 \tag{1}$$

is called a ***para-hypercomplex structure*** *on* V [a]. *A Lie subalgebra* $\widetilde{Q} \subset \mathfrak{gl}(V)$ *is called a* ***para-quaternionic structure*** *on* V *if it admits a basis* (J_1, J_2, J_3) *satisfying the relations (1). Such a para-hypercomplex structure is called an admissible basis of* $\widetilde{Q}$.

A para-hypercomplex structure (J_1, J_2, J_3) defines on V the structure of a left module over the real algebra $\widetilde{\mathbb{H}}$ of para-quaternions which is the real algebra generated by unity 1 and i, j, k satisfying

$$-i^2 = j^2 = k^2 = 1, \quad ij = -ji = k. \tag{2}$$

$\widetilde{\mathbb{H}}$ is isomorphic, as real pseudo-normed algebra, to the algebra $Mat_2(\mathbb{R})$ of real (2×2)-matrices, the isomorphism being given by

$$\Phi : \mathbf{q} = q_0 + q_1 i + q_2 j + q_3 k \mapsto \begin{pmatrix} q_0 - q_3 & q_2 - q_1 \\ q_2 + q_1 & q_0 + q_3 \end{pmatrix} \tag{3}$$

where $\mathcal{N}(q) := q\bar{q} = q_0^2 + q_1^2 - q_2^2 - q_3^2 = det(\Phi(\mathbf{q}))$.

The **standard para-hypercomplex structure** $(\mathcal{I}, \mathcal{J}, \mathcal{K})$ **of** $V = \mathbb{R}^2$ is represented, in the canonical basis, by

$$\mathcal{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{K} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4}$$

Observe that $< \mathcal{I}, \mathcal{J}, \mathcal{K} >_{\mathbb{R}} \simeq \mathfrak{sl}_2(\mathbb{R})$ the matrix Lie algebra of zero trace real (2×2)-matrices of the unimodular Lie group $SL_2(R)$.

Generalizing, we define the **standard para-hypercomplex structure** $\widetilde{H} = (I, J, K)$ **of** $R^{2n} = \mathbb{R}^2 \oplus \ldots \oplus \mathbb{R}^2$ represented, in the canonical basis, by

$$I = \mathcal{I} \oplus \mathcal{I} \oplus \ldots \oplus \mathcal{I}; \quad J = \mathcal{J} \oplus \mathcal{J} \oplus \ldots \oplus \mathcal{J}; \quad K = \mathcal{K} \oplus \mathcal{K} \oplus \ldots \oplus \mathcal{K}. \tag{5}$$

By the identification $\widetilde{\mathbb{H}} \cong Mat_2(\mathbb{R})$ of (3) and from Wedderburn theorem, stating that every representation of a unitary, associative, semisimple algebras is direct sum of standard representations, it results the following

Proposition 2.2.

- *There exists a unique, up to isomorphism, irreducible* $\widetilde{\mathbb{H}}$*-module* $H^2 \simeq \mathbb{R}^2$.
- *Every* $\widetilde{\mathbb{H}}$*-module* V^{2n} *is reducible as a direct sum* $V = H^2 \oplus \ldots \oplus H^2$.

Note that to have a direct sum decomposition of the $\widetilde{\mathbb{H}}$-module $(V^{2n}, (I, J, K))$, into invariant 2-dimensional subspaces $U_1, \ldots, U_n$, one considers a basis e_i^+ of V_J^+, eigenspace of the para-complex structure J associated to the eigenvalue 1 (then $Ke_1^+, \ldots, Ke_n^+$ is a basis of V_J^-). The 2-dimensional subspaces

$$U_i =< e_i^+, Ke_i^+ >, \qquad i = 1, \ldots, n \tag{6}$$

are clearly $\widetilde{H}$-invariant, irreducible and isomorphic as $\widetilde{\mathbb{H}}$-modules. Choosing the basis $< e_i^+ - Ke_i^+, e_i^+ + Ke_i^+ >$ in each U_i, $\widetilde{H}$ corresponds to the standard para-hypercomplex structure of R^{2n} given in (5).

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[a] Observe that J_2 and J_3 are para-complex structures on V. In fact, since the complex structure J_1 anti-commute with J_2, $J_1(V_{J_2}^+) \subseteq V_{J_2}^-$ and $J_1(V_{J_2}^-) \subseteq V_{J_2}^+$, which implies $\dim V_{J_2}^+ = \dim V_{J_2}^-$, and analogously for J_3.

Let H^2 and E^n be real vector spaces. For any fixed basis (h_1, h_2) of H, one has the identification $H \simeq \mathbb{R}^2$: we define a corresponding **standard para-hypercomplex structure on** $H^2 \otimes E^n$ by

$$(7)\quad I = I(h \otimes e) = \mathcal{I}h \otimes e, \quad J = J(h \otimes e) = \mathcal{J}h \otimes e, \quad K = K(h \otimes e) = \mathcal{K}h \otimes e$$

with $\mathcal{I}, \mathcal{J}, \mathcal{K}$ given in (4) and the **standard para-quaternionic structure** $\mathfrak{sl}_2(\mathbb{R}) \otimes Id$ **on** $H^2 \otimes E^n$ generated by any standard para-hypercomplex structure.

Since $\mathfrak{sl}_2(\mathbb{R}) \simeq \mathfrak{sl}(H)$, the Lie algebra of the Lie group $SL(H)$ of unimodular transformations of H, we will use the equivalent notations

$$\mathfrak{sl}_2(\mathbb{R}) \otimes Id \simeq \mathfrak{sl}_2(\mathbb{R}) \simeq \mathfrak{sl}(H).$$

From prop. (2.2) it follows immediately

Proposition 2.3. *Any vector space V^{2n} with a para-hypercomplex structure $\tilde{H}$ is isomorphic to $H^2 \otimes E^n$ with a standard para-hypercomplex structure. Consequently any para-quaternionic vector space $(V^{2n}, \tilde{Q})$ is isomorphic to $(H^2 \otimes E^n, \mathfrak{sl}(H))$.*

Definition 2.4. *Let (V, K) be a $2n$-dimensional para-complex vector space. A pseudo-Euclidean scalar product $g =< \cdot, \cdot >$ on (V, K) is called $K-$**Hermitian** if K is a skew-symmetric endomorphism of $(V, < \cdot, \cdot >)$.*

*A vector space V endowed with a para-complex structure K and a K-Hermitian scalar product g is called a **para-Hermitian vector space** (V, K, g)*[b].

*A para-hypercomplex structure (J_1, J_2, J_3) on V is called **para-hypercomplex Hermitian structure** with respect to the pseudo-Euclidean scalar product g if its endomorphisms are skew-symmetric with respect to g.*

*A para-quaternionic structure $\widetilde{Q}$ on V is called a **para-quaternionic Hermitian structure** with respect to g if some (and hence any) admissible basis is Hermitian with respect to g.*

The eigenspaces associated to any para-complex structure are clearly totally isotropic, then a *para-hermitian metric has neutral signature* and this leads to the

Proposition 2.5. *The dimension of a vector space V^{2n}, endowed with a para-hypercomplex (resp. para-quaternionic) Hermitian structure $(\widetilde{H}, g)$ (resp.$(\widetilde{Q}, g)$), is a multiple of 4.*

Let consider now the standard para-quaternionic vector space $(H^2 \otimes E^{2n}, \mathfrak{sl}(H))$. Let ω^E be a symplectic form on E and $\omega^H = h_1^* \wedge h_2^*$ a volume form on H.

The 2-form $g_0 = \omega^H \otimes \omega^E$ is a pseudo-Euclidean $\tilde{Q}$-Hermitian metric on $H^2 \otimes E^{2n}$. It is trivial to verify that g_0 is bilinear, symmetric and non degenerate. Moreover, computing on decomposable vectors, $\forall A \in \mathfrak{sl}(H)$ one has

$$g(Ah' \otimes e, \tilde{h} \otimes e') = \omega^H(Ah', \tilde{h})\omega^E(e, e') = -\omega^H(h', A\tilde{h})\omega^E(e, e') = -g(h' \otimes e, A\tilde{h} \otimes e').$$

Definition 2.6. *The $4n$-dimensional space $(H^2 \otimes E^{2n}, \mathfrak{sl}(H), \omega^H \otimes \omega^E)$ is a **standard para-quaternionic Hermitian space**.*

[b] The reason why we do not consider n-dimensional vector spaces endowed with a product structure not para-complex is that the metric on such spaces, direct sum of a pair of totally isotropic eigenspaces of different dimensions, is always degenerate.

Proposition 2.7. *Let V^{4n} be a vector space with a para-quaternionic Hermitian structure $(\widetilde{Q}, g)$. Then the para-quaternionic Hermitian space $(V, \widetilde{Q}, g)$ is isomorphic to a standard para-quaternionic Hermitian space.*

Proof. By proposition (2.3) we identify $(V^{4n}, \widetilde{Q}) \simeq (H^2 \otimes E^{2n}, \mathfrak{sl}(H))$.

Then the given para-quaternionic Hermitian metric g on $H^2 \otimes E^{2n}$ can be written $g = \omega^H \otimes \omega^E$ where $\omega^H = h_1^* \wedge h_2^*$ is the standard volume form on H and ω^E is defined by

$$\omega^E(e, e') := \frac{g(h \otimes e, h' \otimes e')}{\omega^H(h, h')},$$

for one (and hence any) pair of linearly independent vectors h, h'. It is straightforward to prove that the right member is well defined by observing that, by Hermitianicity, decomposable vectors are always isotropic and $g(h_1 \otimes e, h_2 \otimes e') + g(h_2 \otimes e, h_1 \otimes e') = 0$. Moreover ω^E is clearly symplectic. □

Finally, let make the following remark that we will use in next section. As an $\widetilde{\mathbb{H}}$-module, on a para-hypercomplex Hermitian vector space $(V^{4n}, \{I, J, K\}, g)$ we define the ($\widetilde{\mathbb{H}}$-valued)-Hermitian product $(\cdot) = (\cdot)_{\{I,J,K\}}$ by:

(8)
$$\begin{array}{rccl} (\cdot): & V \times V & \to & \widetilde{\mathbb{H}} \\ & (X, Y) & \mapsto & X \cdot Y = g(X, Y) + ig(X, IY) - jg(X, JY) - kg(X, KY). \end{array}$$

When considering a para-quaternionic Hermitian vector space $(V, \widetilde{Q}, g)$, we observe that the ($\widetilde{\mathbb{H}}$-valued)-Hermitian product defined in (8) depends on the chosen admissible basis $\{I, J, K\} \in \widetilde{Q}$. Two Hermitian products $(\,\cdot\,)_{\{I,J,K\}}$, $(\,\cdot\,)_{\{I',J',K'\}}$, referred to different admissible basis, are related by an inner automorphism of $\widetilde{\mathbb{H}}$. This implies that

$$\mathcal{N}(Im(X \cdot Y)_{\{I,J,K\}}) = \mathcal{N}(Im(X \cdot Y)_{\{I',J',K'\}}), \quad \forall X, Y \in V$$

since the real part of the norm $\mathcal{N}((X \cdot Y))$ is independent on the basis $\{I, J, K\}$.

3. Subspaces of a para-quaternionic Hermitian vector space

From now on, the para-quaternionic Hermitian vector space $(V^{4n}, \widetilde{Q}, g)$ will be the standard para-quaternionic Hermitian vector space $(H^2 \otimes E^{2n}, \mathfrak{sl}(H), \omega^H \otimes \omega^E)$. We recall that, in this case, any para-hypercomplex admissible basis (I, J, K) of $\widetilde{Q} = \mathfrak{sl}(H)$ is a standard para-hypercomplex Hermitian structure which corresponds to a symplectic basis (h_1, h_2) of H such that $\omega^H = h_1^* \wedge h_2^*$ and one has

$$\begin{array}{ll} I(h_1 \otimes e) = h_2 \otimes e; & I(h_2 \otimes e) = -h_1 \otimes e; \\ J(h_1 \otimes e) = h_2 \otimes e; & J(h_2 \otimes e) = h_1 \otimes e; \\ K(h_1 \otimes e) = -h_1 \otimes e; & K(h_2 \otimes e) = h_2 \otimes e. \end{array} \tag{9}$$

If A and B are subspaces in E, in the following we will denote $\omega^E(A \times B)$ the restriction of the symplectic form ω^E to the subspace $A \times B$ of $E \times E$. Moreover, by saying that $\omega^E(A \times B)$ is degenerate, we will mean that there exists $0 \neq b_0 \in B$ such that $\omega^E(a, b_0) = 0$, $\forall a \in A$ i.e.

$$\ker\, \omega^E(A \times B) = \{b \in B \mid \omega^E(a, b) = 0,\ \forall a \in A\} \neq \{0\}.$$

In the following we will give definitions and explicit descriptions of some relevant classes of subspaces in V. The definitions of the subspaces that we consider involve only the para-quaternionic structure $\widetilde{Q}$, except for totally complex, totally para-complex and totally real subspaces for which we take into account also the Hermitian metric g.

3.1. **Special subspaces of $\mathbf{V} = \mathbf{H} \otimes \mathbf{E}$.** Fixed a symplectic basis (h_1, h_2) of H, any $X \in H \otimes E$ can be written in a unique way

$$X = h_1 \otimes e + h_2 \otimes e', \qquad e, e' \in E. \tag{10}$$

Let denote by $p_i : H \otimes E \to E,\ i = 1, 2$, the natural linear projections defined by

$$p_1(X) = e; \quad p_2(X) = e'. \tag{11}$$

If U is a subspace then $p_1(U) = E_1$, $p_2(U) = E_2$ are subspaces of E, depending on the chosen symplectic basis (h_1, h_2) in H. Notice that the sum $p_1(U) + p_2(U)$ is invariant.

With respect to the tensor product structure, the following subspaces of V can be defined. First of all there are the **product subspaces** $H' \otimes E'$, with $H' \subseteq H$ and $E' \subseteq E$ any given subspaces. Referring to the dimension of the non trivial factor in H, only two classes of such subspaces are to be considered.

A non zero product subspace $U = h \otimes E' \subset H \otimes E$ where h is a fixed element in H and $E' \subset E$ a subspace, will be called a **decomposable subspace** (meaning that all elements in U are decomposable vectors; subspaces $H \otimes e', e' \in E$ will not be considered under such terminology). W.r.t. the metric g, any decomposable subspace is totally isotropic.

We introduce another important family of subspaces that we denote by $U^{F,T}$.

Definition 3.1. *Let (h_1, h_2) be a symplectic basis of H, $F \subseteq E$ a subspace, $T : F \to E$ a linear map. We define the following subspace of $H \otimes E$*

$$U^{F,T} := \{h_1 \otimes f + h_2 \otimes Tf,\ f \in F\}. \tag{12}$$

Note that the map

$$\begin{array}{rrl} \phi : & F \to & U^{F,T} \\ & f \mapsto & h_1 \otimes f + h_2 \otimes Tf \end{array} \tag{13}$$

is an isomorphism of real vector spaces. By saying that a subspace $U \subset H \otimes E$ is a $\mathbf{U^{F,T}}$ ***subspace****, we will mean that it admits the form (12) w.r.t. some symplectic basis (h_1, h_2) of H.*

As a first example of subspaces admitting the $U^{F,T}$ form we have the decomposable subspaces $U = h \otimes E',\ h \in H,\ E' \subseteq E$: in any basis $(h_1 = h, h_2)$ let $F = E'$ and $T \equiv 0$. Also, in a basis (h_1, h_2) with $h_1, h_2 \neq h$, one has $F = p_1(U) = p_2(U) = E'$ and $T = \lambda Id$ where $\lambda = \frac{\beta}{\alpha}$ for $h = \alpha h_1 + \beta h_2$. On the other hand, U does not admit the form (3.1) w.r.t. any basis $(h_1, h_2 \equiv \alpha h),\ \alpha \in \mathbb{R}$.

It is immediate to prove the following

Proposition 3.2.

a) *A subspace U is a $U^{F,T}$ subspace iff there exists $h \neq 0$ in H such that $(h \otimes E) \cap U = \{0\}$.*

b) *W.r.t. the symplectic basis (h_1, h_2), the map T for the subspace $U = U^{F,T}$ is injective iff $(h_i \otimes E) \cap U = \{0\},\ i = 1, 2$.*

Observe that if $U = U^{F,T}$ w.r.t. h_1, h_2 and also $U = U^{F',T'}$ w.r.t. h_1', h_2' where

$$h_1 = \alpha h_1' + \beta h_2', \qquad \text{and} \qquad h_2 = \gamma h_1' + \delta h_2',$$

then

$$F' = (\alpha Id + \gamma T)F \qquad \text{and} \qquad T' = (\alpha Id + \gamma T)^{-1}(\beta Id + \delta T).$$

Proposition 3.3. *A $U^{F,T}$ subspace can always be written as $U^{F',T'}$ with T' injective by performing a suitable change of basis in H.*

Proof. A subspace $U = U^{F,T} = \{h_1 \otimes f + h_2 \otimes Tf,\ f \in F\}$ of dimension m contains at most m distinct non zero decomposable vectors $k_i \otimes f_i,\ i = 1, \ldots, t$, if the $k_i \in H$ are pairwise independent. In fact, $k_i \otimes f_i \in U,\ i = 1, \ldots, t$, with $k_i = a_i h_1 + b_i h_2$, iff $f_i \in F$ and $Tf_i = b_i/a_i\ f_i = \lambda_i f_i$. Considering the restriction of T to the subspace $< f_1, \ldots, f_t >$ the conclusion follows. □

Remark that, from the isomorphism (13), the decomposable subspaces contained in a $U = U^{F.T}$ subspace are direct addends in U.

In general, a subspace $U \subset V$ does not admit the form $U^{F,T}$: an example is given by any product subspace $H \otimes E',\ E' \subset E$. On the other hand, any subspace can be written as direct sum of some $U^{F,T}$ subspaces. In fact, we have the following

Proposition 3.4. *Any subspace U can be written in the forms*

(1) *$(h \otimes F') \oplus U^{F'',T''}$ for some $h \in H$ and $U^{F'',T''}$ of maximal dimension w.r.t. all subspaces of the form $U^{F,T}$ contained in U.*

(2) *$k_1 \otimes F_1 \oplus \ldots \oplus k_s \otimes F_s \oplus U^{\widetilde{F},\widetilde{T}}$ with the $k_i \in H, i = 1, \ldots, s$, pairwise independent and $U^{\widetilde{F},\widetilde{T}}$ of maximal dimension w.r.t. all subspaces of the form $U^{F,T}$ containing no decomposable subspace.*

Proof. 1) If $U = U^{F,T}$ there is nothing to prove. Otherwise let consider all maximal decomposable subspaces in U and let $U_1 = h \otimes F' \subset U$ of minimum dimension among them. Then any subspace complementary to U_1 in U is clearly a $U^{F,T}$ subspace and of maximal dimension.

2) If U contains no decomposable subspaces, there is nothing to prove; otherwise let $U_1 \subset U$ be of maximal dimension among all $U^{F,T}$ subspaces in U containing no decomposable subspaces. Then $U = U_1 \oplus U_2$ with U_2 any complementary. The subspace U_2 can be decomposed into a direct sum of some maximal decomposable subspaces and a $U_3 = U^{F,T}$ subspace containing no decomposable subspaces i.e $U_2 = \bigoplus_{i=1}^{s}(k_i \otimes F_i) \oplus U_3$ with the k_i pairwise independent. Necessarily $U_1 \oplus U_3$ contains some maximal decomposable subspaces, then $U = U_1 \oplus \bigoplus_{i=1}^{s}(k_i \otimes F_i) \oplus \bigoplus_{j=s+1}^{t}(k_j \otimes F_j) \oplus U_4$ with $U_4 = U^{F,T}$ a subspace containing no decomposable subspaces with $\dim U_4 < \dim U_3$ and the $k_i, i = 1, \ldots, t$ pairwise independent. By carrying on such procedure, the thesis follows. □

Concerning the unicity of the presentation of the form $U^{F,T}$ we state the following lemma whose proof is straightforward (see proof of prop. 3.3)

Lemma 3.5. *Given a subspace $U \subset H \otimes E$ the following conditions are equivalent:*
1) U does not contain any non zero decomposable vectors;
2) $U = U^{F,T}$ w.r.t. ***any*** *symplectic basis $\mathcal{B} = (h_1, h_2)$, $F = F(\mathcal{B}) \subset E$ and $T = T(\mathcal{B})$ injective.*
3) there exists a basis (h_1, h_2), such that $U = U^{F,T}$ for some subspace $F \subset E$ and

some linear injective map T with no invariant line (i. e. if $Tf = \lambda f,\ \lambda \in \mathbb{R},\ f \in F$ then $f = 0$).

A necessary condition for 1),2),3) to hold is that $\dim U \leq \dim E$.

From the metrical point of view we have easily the following

Lemma 3.6. *Let $U = U^{F,T}$ be a subspace and ϕ the isomorphism (13). Let $g_F = \phi^* g_U$ be the pullback of the (possibly degenerate) restriction of g to U. Then*

$$g_F(f, f') = -2(\omega^E \circ T)^{sym}(f, f') = -[\omega^E(Tf, f') + \omega^E(Tf', f)] \tag{14}$$

3.2. **Para-quaternionic subspaces.**

Definition 3.7. *A subspace $U \subset V$ is called* ***para-quaternionic*** *if it is $\widetilde{Q}$-invariant, or equivalently, for one and hence for any para-hypercomplex basis (I, J, K) of $\widetilde{Q}$ one has $IU \subset U,\ JU \subset U,\ KU \subset U$.*

The sum and the intersection of para-quaternionic subspaces is para-quaternionic.

Proposition 3.8. *Let $(E')^k \subset E$ be any subspace. Then*

$$U^{2k} = H \otimes E' \tag{15}$$

is a para-quaternionic subspace of dimension $2k$. Viceversa any para-quaternionic subspace of V has this form. Moreover U is para-quaternionic Hermitian (with neutral metric) iff E' is ω^E-symplectic.

Proof. The subspace $U^{2k} = H^2 \otimes (E')^k \subset H \otimes E$ is clearly $\widetilde{Q}$-invariant. Viceversa, let $U \subset V$ be a para-quaternionic subspace. Let fix a basis (h_1, h_2) in H and let moreover (I, J, K) be the associated standard para-hypercomplex Hermitian structure. The subspaces $E' = p_1(U)$ and $p_2(U)$ coincide since, by the I-invariance, for any $X = h_1 \otimes e + h_2 \otimes e' \in U$, the vector $IX = h_2 \otimes e - h_1 \otimes e'$ is in U. Also, since $KX = -h_1 \otimes e + h_2 \otimes e'$, by the K-invariance the decomposable vectors $h_1 \otimes e$ and $h_2 \otimes e'$ are in U. Therefore $U = H \otimes E'$.

From the metrical point of view, the subspaces $h_1 \otimes E'$ and $h_2 \otimes E'$ are totally isotropic and the metric on U, with respect to the decomposition $U = h_1 \otimes E' \oplus h_2 \otimes E'$, is given by

$$g|_U = \begin{pmatrix} 0 & \omega^E|_{E'} \\ (\omega^E|_{E'})^t & 0 \end{pmatrix}. \tag{16}$$

Then U is Hermitian para-quaternionic if and only if E' is $\omega^E|_{E'}$-symplectic. □

Remark 3.9. *Referring to the decompositions* 1), 2) *given in prop. (3.4), notice that a para-quaternionic subspace $U = H \otimes E'$ decomposes respectively as*

1) $U = h_1 \otimes E' \oplus h_2 \otimes E'$, 2) $U = h_1 \oplus E' \oplus \{h_1 \otimes e' + h_2 \otimes Te',\ e' \in E'\}$

w.r.t. any basis (h_1, h_2), with T any automorphism of E' with no real eigenvalues. In this case then, the dimensions of the maximal $U^{F,T}$ subspaces in $U = H \otimes E'$ in the decompositions 1) and 2) coincide and equal the dimension of E'.

Any subspace U of V contains a (possibly zero) *maximal para-quaternionic subspace* $U_0 = U \bigcap_{A \in \widetilde{Q}} A(U)$. Equivalently, $U_0 = U \cap IU \cap JU \cap KU$ for any admissible basis (I, J, K) of $\widetilde{Q}$.

Definition 3.10. *A subspace $U \subset V$ is called* ***pure*** *if $U_0 = \{0\}$, i.e. it does not contain any non zero para-quaternionic subspace.*

Clearly, from prop. (3.2a), *any $U^{F,T}$ subspace is pure.*

3.3. Complex subspaces.

Definition 3.11. *A subspace $U \subset V$ is called* ***complex*** *if there exists a compatible complex structure $I \in \widetilde{Q}$ such that U is I-invariant i.e $IU \subset U$. We denote it by (U, I).*

We shall include I into an admissible basis (I, J, K) of $\widetilde{Q}$. Such basis will be called **adapted** to the subspace (U, I). Adapted bases are defined up to a rotation in the real plane spanned by J and K.

Lemma 3.12. *The complex structure I is unique up to sign unless U is para-quaternionic.*

Proof. Let $\tilde{I} = aI + bJ + cK, \quad \tilde{I}^2 = -Id \quad (a^2 - b^2 - c^2 = 1)$, be a compatible complex structure such that $\tilde{I}U = U$. Then for any $X \in U$ one has $aIX + bJX - cJIX \in U$, hence $J(bX - cIX) \in U, \quad \forall X \in U$. If $(b, c) = (0, 0)$ then $\tilde{I} = \pm I$; otherwise the map $X \mapsto (bX - cIX)$, $\forall X \in U$ is an automorphism of U, since I has no real eigenvalues, hence $JU = U$ i.e. U is para-quaternionic. □

Lemma 3.13. *A complex subspace $(U, I) \subset (V, \widetilde{Q})$ is pure if and only if there exists a para-complex structure $J \in \widetilde{Q}$ such that $IJ = -JI$, $JU \cap U = \{0\}$.*

Proof. Let (U, I) be pure complex. Suppose there exists $X \in U$ such that $JX \in U$ with J a compatible para-complex structure and $IJ = -JI = K$. Then $KX = IJX \in U$ and $< X, IX, JX, KX >_{\mathbb{R}} \subset U$ is a para-quaternionic subspace. Hence $X \neq 0$ would give a contradiction. Viceversa is obvious. □

It is also immediate to verify that if (U, I) is pure complex, then, for <u>any</u> $A \in \widetilde{Q}, A \neq \pm aI,\ a \in \mathbb{R}$, one has $AU \cap U = \{0\}$.

Considering now also the metric structure of V we have the following special class of pure complex subspaces.

Definition 3.14. *An Hermitian complex subspace (U, I) of V is called* ***totally complex*** *if there exists an adapted hypercomplex basis (I, J, K) such that $JU \perp U$ ($\Leftrightarrow KU \perp U$) with respect to the (non degenerate) induced metric g.*

Note that the Hermitian complex subspace (U, I) is totally complex iff, with respect to the adapted basis (I, J, K), the restriction to U of the Hermitian product (8) has complex values[c]. Note also that the hypothesis of Hermitianicy is necessary to ensure that any totally complex subspace is pure.

Let (U, I) be a totally complex subspace and (I, J, K) an adapted basis such that $JU \perp U$. Any $A = aI + bJ + cK \in \widetilde{Q}$, satisfies $AU \perp U$ if and only if $a = 0$. Then again, adapted bases are defined up to a rotation in the plane $< J, K >$.

By taking into account that U_0 is I-complex and from known facts about complex structures, one has the following

[c]In fact, following terminology of [6], a totally complex subspace could be called a *subspace with complex Hermitian product.*

Proposition 3.15. *Any complex subspace (U, I) is a direct sum of the maximal para-quaternionic subspace U_0 and a pure I-complex subspace (U', I) i.e. $U = U_0 \oplus U'$. If (U'', I) is another I-pure complex subspace complementary to U_0, then U' and U'' are isomorphic as I-complex spaces.*

Hence *the description of complex subspaces reduces to the description of pure complex subspaces.*

Let I be a compatible complex structure and (h_1, h_2) a symplectic basis of H such that $I \equiv \mathcal{I} \otimes Id$ using the notations introduced for proposition (2.3), i.e. I as in (9).

Theorem 3.16. *With respect to (h_1, h_2), a subspace $U \subseteq V$ is I-pure complex iff $U = U^{F,T}$ with T a complex structure on $F = p_1(U)$. Then the map*

$$\begin{array}{cccc} \phi: & (F, T^{-1}) & \to & (U^{F,T}, I) \\ & f & \mapsto & h_1 \otimes f + h_2 \otimes Tf \end{array}$$

is an isomorphism of complex vector spaces. The signature of the metric on U is of type $(2p, 2s, 2q)$, $2s = \dim \ker g|_U$, and U is Hermitian if and only if F is g_F-non degenerate. In this case $\phi : (F, T^{-1}, g_F) \to (U, I, g|_U)$ is an isomorphism of Hermitian spaces. In particular T^{-1} is g_F-skew symmetric. The Kaehler form of (U, I) is given by

$$\phi^*(g|_U \circ I) = g_F \circ T^{-1} = -(\omega^E|_F + \omega^E|_F(T\,\cdot, T\,\cdot)).$$

The subspace (U, I) is totally complex if and only F is ω^E-symplectic and T is $\omega^E|_F$-skew-symmetric ($\iff$ if T preserves the form $\omega^E|_F$ i.e.

$$\omega^E|_F(f, f') = \omega^E|_F(Tf, Tf') \qquad \forall f, f' \in F) \tag{17}$$

or equivalently $g_F = -2\omega^E|_F \circ T$.

Proof. Let (U, I) be a pure complex subspace in $H \otimes E$, $(h_1, h_2 = \mathcal{I}h_1)$ a symplectic basis of H s.t. $I \equiv \mathcal{I} \otimes Id$ and (I, J, K) an adapted basis. Observe that there is no non-zero decomposable element $X = h \otimes e$ in U; in fact since $IX \in U$, it would follows that $h_1 \otimes e$ and $h_2 \otimes e$ are both in U, hence $H \otimes \mathbb{R}e \subset E$ which is a contradiction. From lemma (3.5), $U = U^{F,T}$ w.r.t. the symplectic basis $(h_1, h_2 = \mathcal{I}h_1)$; then it is straightforward to verify that $T^2 = -Id$. Viceversa, it is immediate to verify that the pure subspace $U = U^{F,T} = \{h_1 \otimes f + h_2 \otimes Tf,\ T^2 = -Id\}$ is I-invariant. The statements about the isomorphism ϕ are straightforward to verify. The expression of the Kaehler form follows from a direct computation.

Any pure I-complex subspace admits a decomposition into pure I-complex 2-planes; each one of them is whether totally isotropic or with definite metric. This implies that the signature of the metric on U is $(2p, 2s, 2q)$, $2s = \dim \ker g|_U$ and clearly equals the signature of g_F on F. Consequently U is Hermitian pure complex if and only if F is g_F-non degenerate.

We now prove the last statement. Observe that, since w.r.t. (h_1, h_2) T is a complex structure on F, the claimed equivalence $\iff$ is straightforward. For any $X = h_1 \otimes f + h_2 \otimes Tf, \in U$, we have $JX = h_2 \otimes f + h_1 \otimes Tf$, hence $JU = \{Y = h_1 \otimes f - h_2 \otimes Tf, \quad f \in F\}$. (Observe that for any pure complex subspace (U, I), the subspace JU is pure I-complex). Then $U \perp JU$ if and only if, for any $f, f' \in F$,

$$0 = g(h_1 \otimes f + h_2 \otimes Tf, h_1 \otimes f' - h_2 \otimes Tf') = -\omega^E(f, Tf') - \omega^E(Tf, f')$$

that is $\omega^E(f, Tf') = -\omega^E(Tf, f')$ which is equivalent to (17). The metric on U verifies

$$g(h_1 \otimes f + h_2 \otimes Tf, h_1 \otimes f' + h_2 \otimes Tf') = g_F(f, f') = 2\omega^E(f, Tf').$$

Therefore, the non degeneracy of U implies that F is ω^E-symplectic. $\square$

This theorem reduces classification of pure complex subspaces to the classification of pairs (F, T) with $F \subset E$ and T a complex structure on F. In particular, in the classification of totally complex subspaces, F is, in addition, $\omega^E|_F$-symplectic and T preserves $\omega^E|_F$.

In the following section we consider the para-complex subspaces. They could be partly treated in a unified way together with the complex subspaces just seen. But the existence of specific characteristics not appearing in the complex case, account for a separate treatment.

3.4. **Para-complex subspaces.**

Definition 3.17. *A subspace $U \subset V$ is called **weakly para-complex** if there exists a para-complex structure $K \in \widetilde{Q}$ such that U is K-invariant i.e $KU \subset U$. We denote such subspaces by (U, K). A **para-complex** subspace (U, K) is a weakly para-complex subspace such that* $\dim U_K^+ = \dim U_K^-$.

Remark 3.18. *The eigenspaces V_K^+, V_K^- of a given para-complex structure $K \in \widetilde{Q}$ are decomposable subspaces (then totally isotropic) of V, i.e. $V_K^+ = h' \otimes E', V_K^- = h'' \otimes E''$ and $E' \oplus E'' = E$. As a first consequence (cfr. footnote in definition (2.4)), any weakly para-complex subspace not para-complex is degenerate.*

The presence of decomposable vectors produces a difference passing from the complex to the weakly para-complex case but, as we will see, a common treatment of both cases is still possible.

Analogously to lemma (3.13) one has

Lemma 3.19. *A weakly para-complex subspace (U, K) of V is pure if and only if there exists a complex structure $I \in \widetilde{Q}$ anti-commuting with K such that $IU \cap U = \{0\}$.*

Remark that *if (U, K) is pure weakly para-complex, then for any compatible complex structure $\tilde{I}$, one has $\tilde{I}U \cap U = \{0\}$.* In fact, let (I, J, K) be the adapted basis of the para-complex subspace (U, K) with $IU \cap U = \{0\}$. Let $\widetilde{Q} \ni \tilde{I} = aI + bJ + cK$, $a^2 - b^2 - c^2 = 1$ be a compatible complex structure. Suppose there exists a non zero $X \in U$ such that $\tilde{I}X \in U$; then $I(aX - bKX) \in U$. This implies $a = \pm b$, hence a contradiction. Then the admissible bases are defined up to a pseudo-rotation in the plane $< I, J >_{\mathbb{R}}$.

Proposition 3.20. *Any weakly para-complex subspace (U, K) is a direct sum $U = U_0 \oplus \tilde{U}$ of the maximal para-quaternionic subspace and of a pure weakly K para-complex subspace $\tilde{U}$. If $\tilde{U}'$ is another K pure weakly para-complex complementary subspace, then $\tilde{U}$ and $\tilde{U}'$ are isomorphic as weakly K-para-complex spaces.*

Assume $\tilde{U} \neq \{0\}$. If $\tilde{U} \not\subseteq U_K^{\pm}$ then the para-complex structure $K \in \widetilde{Q}$ is unique up to sign. Otherwise the family of para-complex structures

$$\tilde{K}_a = aI + aJ \pm K, \quad \textit{if} \quad \tilde{U} \subset {U_K}^+ ; \qquad (\tilde{K}_a = aI - aJ \pm K, \quad \textit{if} \quad \tilde{U} \subset {U_K}^-)$$

preserves U for any adapted basis (I, J, K).

Proof. The proof of the first statement is analogous to that one in the proof of proposition (3.15). Let (U, K) be a weakly para-complex subspace, (I, J, K) an adapted basis and $\tilde{K} = aI + bJ + cK \in \widetilde{Q}$, $\tilde{K}^2 = Id$, $(a^2 - b^2 - c^2 = -1)$ an admissible para-complex structure. For any $X \in U$, the vector $\tilde{K}X$ is in U iff

$$aIX + bJX = I(aX - bKX) \in U. \tag{18}$$

Then, whether $a = b = 0$ i.e. $\tilde{K} = \pm K$, or U is para-quaternionic for $a \neq \pm b$ (in fact in this case the map $X \mapsto aX - bKX$ is an automorphism of U). In case $a = \pm b$, let consider the decomposition $U = U_0 \oplus U'$ into the maximal para-quaternionic subspace U_0 and the weakly pure para-complex component U' respectively. Then condition (18) implies that $aX - bKX = 0$, $\forall X \in U'$ which is verified only if U' is an eigenspace of K. □

Hence also *the description of weakly para-complex subspaces reduces to that one of pure subspaces.* In this case nevertheless there exists a difference regarding the unicity of the para-complex structure. The reason for such a difference is a consequence of the results in the next subsection (see in particular the end of the proof of prop. 3.26).

Definition 3.21. *Let (U, K) be a K-Hermitian para-complex subspace. Then U is called* ***totally para-complex*** *if there exists a complex structure $I \in \widetilde{Q}$ anticommuting with K such that $IU \perp U$ respect to the induced metric $g|_U$.*

Observations analogue to those following the definition (3.14) of totally complex subspaces can be made for totally para-complex subspaces.

Let J be a compatible para-complex structure and (h_1, h_2) a symplectic basis of H such that $J = \mathcal{J} \otimes Id$ i.e. J as in (9).

Theorem 3.22. *With respect to (h_1, h_2), a subspace $U \subseteq V$ is pure weakly J-para-complex iff $U = U^{F,T}$ with T a weakly para-complex structure on $F = p_1(U)$. Then the map*

$$\begin{array}{llll} \phi: & (F,T) & \to & (U^{F,T}, J) \\ & f & \mapsto & h_1 \otimes f + h_2 \otimes Tf \end{array}$$

is an isomorphism of weakly para-complex vector spaces. The subspace U is J-Hermitian if and only if F is g_F-non degenerate hence necessarily para-complex. In this case, the signature of $g|_U$ is always neutral and $\phi : (F, T, g_F) \to (U, J, g|_U)$ is an isomorphism of Hermitian para-complex spaces. In particular T is g_F-skew symmetric.

The para-Kaehler form is given by $\phi^*(g|_U \circ J) = g_F \circ T = -(\omega|_F - \omega|_F(T\cdot, T\cdot))$.

The para-complex subspace (U, J) is totally para-complex if and only if T is $\omega^E|_F$-skew-symmetric $\iff$ the form $\omega^E|_F$ is skew-invariant w.r.t. T i.e.

$$\omega^E|_F(f, f') = -\omega^E|_F(Tf, Tf') \qquad \forall f, f' \in F$$

or equivalently $g_F = -2\omega^E|_F \circ T$, and F is $\omega^E|_F$-symplectic.

Proof. Let (U^k, J) be a pure weakly para-complex subspace in $H \otimes E$, $(h_1, \mathcal{J}h_1 = h_2)$ a symplectic basis and (I, J, K) an adapted basis. Clearly $h_i \otimes E \cap U = \{0\}$, $i = 1, 2$ since U is pure. Then, from proposition (3.2), $U = U^{F,T}$. In particular, w.r.t. a symplectic basis $(h_1, \mathcal{J}h_1 = h_2)$, It is straightforward to verify that $T^2 = Id$. Viceversa the pure subspace $U = U^{F,T} = \{h_1 \otimes f + h_2 \otimes Tf,\ T^2 = Id\}$ is clearly J-invariant.

The eigenspaces V_J^+ and V_J^+ are decomposable subspaces (see remark (3.18) and consequently they are totally isotropic. Then the signature of the induced metric on U (which equals the signature of g_F on F) is $(m, k-2m, m)$ where $m = rk\, g(V_J^+ \times V_J^-)$. From (14), U if J-Hermitian with neutral signature iff F is g_F non degenerate.

Consider now a J-Hermitian para-complex subspace $U = \{h_1 \otimes f + h_2 \otimes Tf, f \in F\}$. Then $IU = \{h_1 \otimes -Tf + h_2 \otimes f, f \in F\}$. Imposing $U \perp IU$ it follows that the condition for (U, J) to be totally para-complex is given by

$$\omega|_F(f, f') = -\omega|_F(Tf, Tf') \tag{19}$$

($\Leftrightarrow T$ is $\omega^E|_F$-skew-symmetric). The Hermitianicy hypothesis on F implies that F if ω^E-symplectic. Then the decomposition $F = E_1 \oplus E_2$ into into ± 1-eigenspaces of T is a Lagrangian decomposition (i.e. $\omega^E|_{E_1} \equiv 0, \quad \omega^E|_{E_2} \equiv 0$) of the symplectic space F. □

The last theorem reduces classification of weakly pure complex subspaces to that one of pair (F, T) with $F \subseteq E$ and T a weakly para-complex structure on F.

Differently from the pure complex case, where (U, I) admits the form $U^{F,T}$ w.r.t. all symplectic bases of H, in the pure weakly para-complex case, the presence of decomposable vectors in (U, J) and lemma (3.5) allow for some special presentations of (U, J) different from the $U^{F,T}$ form. In particular, using the decomposition of (U, J) into the ± 1 eigenspaces of J on U, we have the following

Proposition 3.23. *Let (U, J) be a pure weakly para-complex subspace with $(h_1, h_2 = \mathcal{J}h_1)$ a symplectic basis. Let moreover (I, J, K) be an adapted basis. The pure weakly para-complex subspace decomposes as*

$$(U, J) = (h'_1 \otimes E_1) \oplus (h'_2 \otimes E_2)$$

where $E_1 \oplus E_2 = F$ is the T ± 1-eigenspaces decomposition of F, $h'_1 = -\frac{1}{\sqrt{2}}(h_1 + h_2)$, $h'_2 = \frac{1}{\sqrt{2}}(h_1 - h_2)$ the symplectic basis of eigenvectors of $\mathcal{J}$ and $h'_1 \otimes E_1 = U_J^+$ and $h'_2 \otimes E_2 = U_J^-$ are the eigenspaces of $J|_U$.

3.5. **Nilpotent subspaces.**

Definition 3.24. *A subspace $U \neq \{0\} \subset H \otimes E$ is called* ***nilpotent*** *if there exists a non zero nilpotent endomorphism $A \in \widetilde{Q}$ which preserves U.*

The nilpotent subspace U will be called also A-nilpotent even if, as we will see later, such a nilpotent endomorphism is never unique.

If U is nilpotent we call **degree of nilpotency of** U the minimum integer n such that $A^n U = \{0\},\ A \in \widetilde{Q}$. Clearly, since $A^2 = 0$, the degree of nilpotency of U is at most 2, and equal to 1 if $U \subset \ker A$.

Proposition 3.25. *A subspace U is nilpotent of degree 1 iff it is a decomposable subspace $h \otimes F,\ F \subset E$. More generally, let $A \in \widetilde{Q}$ be a nilpotent endomorphism and $\ker A = h \otimes E$. A subspace U is A-nilpotent iff, with respect to a symplectic basis $(h_1 \equiv h, h_2)$, one has*

$$h_1 \otimes p_2(U) \subset U.$$

Proof. We first observe that the subspace $p_2(U)$ is invariant for any change of symplectic basis $(h_1, h_2) \mapsto (h_1, h'_2)$, . Fixed a basis (h_1, h_2) in H, let (I, J, K) be the associated standard para-hypercomplex structure. Let U be a A-nilpotent subspace of degree 1 with $A = \alpha I + \beta J + \gamma K$ and $\|A\|^2 \equiv \alpha^2 - \beta^2 - \gamma^2 = 0$. Then, for any $X = h_1 \otimes e_1 + h_2 \otimes e_2 \in U$, condition $AX = 0$ implies $e_2 = \frac{\gamma}{(-\alpha+\beta)} e_1$, i.e. U is the decomposable subspace $(h_1 + \frac{\gamma}{(-\alpha+\beta)} h_2) \otimes p_1(U)$. Viceversa it is clear that any decomposable subspace $U = h \otimes F$ is nilpotent of degree 1; moreover, all $A \in \widetilde{Q}$ with $\ker A = h \otimes E$ annihilate U.

More generally, let U be a A-nilpotent subspace where $A \in \widetilde{Q}$ with $\ker A = h \otimes E$. Let $(h_1 \equiv h, h_2)$ be a symplectic basis of H (then $A(h_2 \otimes E) = h_1 \otimes E$). For any $X = h_1 \otimes e_1 + h_2 \otimes e_2$ with $e_2 \neq 0$ in U the vector $AX \in U$ implies that $h_1 \otimes e_2 \in U$. So, being $E_1 = p_1(U),\ E_2 = p_2(U)$, the A-invariance of U implies that $h_1 \otimes E_2 \subset U$ ($\Rightarrow E_2 \subseteq E_1$). Viceversa, let U be a subspace. If w.r.t. a symplectic basis (h_1, h_2) the subspace $h_1 \otimes p_2(U) \subseteq U$, then, for any $A \in \widetilde{Q},\ \|A\|^2 = 0$ with $\ker A = h_1 \otimes E$, the subspace U is clearly A-nilpotent.

□

Clearly all para-quaternionic subspaces are nilpotent of degree 2 w.r.t. any nilpotent structure in $\widetilde{Q}$.

From previous proposition, we have the following characterization of nilpotent subspaces with respect to proposition (3.4).

Proposition 3.26. *Let $A \in \widetilde{Q}$ be a nilpotent endomorphism such that* $\ker A = h_1 \otimes E$ *where (h_1, h_2) is a symplectic basis. The subspace*

$$U = (H \otimes E_0) \oplus (h_1 \otimes E_1'') \oplus \{h_1 \otimes \bar{e_1} + h_2 \otimes T'\bar{e_1},\ \bar{e_1} \in \bar{E_1}\} \tag{20}$$

with $\bar{E_1} \cap E_1'' = \{0\}$, $T'\bar{E_1} \subset E''$ and T' injective is A-nilpotent. The subspace

$$U' = (h_1 \otimes E_1'') \oplus \{h_1 \otimes \bar{e_1} + h_2 \otimes T'\bar{e_1},\ \bar{e_1} \in \bar{E_1}\}$$

is pure nilpotent of the form $U' = U^{F,T}$ with $F = E_1'' \oplus \bar{E}_1$ and $T = 0 \oplus T' : E_1'' \oplus \bar{E}_1 \to E_1''$.

Viceversa, any A nilpotent subspace can be written in the form (20) i.e. it is direct sum of a para-quaternionic subspace, a decomposable subspace $(h_1 \otimes E_1'')$ and a subspace $\{h_1 \otimes \bar{e_1} + h_2 \otimes T'\bar{e}_1,\ \bar{e_1} \in \bar{E_1}\}$ with T' injective, $\bar{E_1} \cap T\bar{E_1} = \{0\}$ (in next section such a subspace will be called real) and $T'\bar{E_1} \subset E_1''$.

Moreover a sufficient condition for U to be not degenerate is that $p_2(U)$ is ω^E-symplectic.

Proof. The subspace U in (20) is clearly A-nilpotent w.r.t. all $A \in \widetilde{Q}$ such that $\ker A = h_1 \otimes E$. Viceversa if U is a A-nilpotent subspace with $\ker A = h_1 \otimes E$, from proposition (3.25) we have $h_1 \otimes p_2(U) \subset U$ w.r.t. all symplectic basis (h_1, h). Let then fix a basis (h_1, h_2). Let $(h_1 \otimes E) \cap U = h_1 \otimes E_1'$ and $p_1(U) = E_1' \oplus \bar{E}_1$. Then

$$U = (h_1 \otimes E_1') \oplus \{h_2 \otimes e_2 + h_1 \otimes \tilde{T} e_2,\ e_2 \in E_2\},\ E_2 \subseteq E_1',\ E_2 \cap \tilde{T} E_2 = \{0\}, \tag{21}$$

where $\tilde{T} E_2 = \bar{E}_1$ and the complement

$$\tilde{U} = \{h_2 \otimes e_2 + h_1 \otimes \tilde{T} e_2\} \tag{22}$$

is of type $U^{F,T}$ with $\tilde{T} : E_2 \to \bar{E}_1$.

We know that a necessary condition for a subspace U to be nilpotent is the presence of a decomposable subspace in U. More precisely, condition $E_2 \subseteq E_1'$ in (21), implies that whether U contains a para-quaternionic subspace (in case $\tilde{T}$ is not injective) and then $h \otimes E \cap U \neq 0,\ \forall h \in H$, or $U = U^{F,T}$ (if $\tilde{T}$ is injective) and in this case one and only one decomposable subspace is in the pure nilpotent subspace U. (In the next section we will see that, in this second case, the addend $\tilde{U}$ in (22) is a real subspace).

In case U is not pure, let $E_2 = E_0 \oplus E_2'$ and $E_1' = E_0 \oplus E_1''$ with $E_0 = \ker \tilde{T}$ and E_1'', E_2' some complementaries. Then

$$U = (H \otimes E_0) \oplus (h_1 \otimes E_1'') \oplus \{h_2 \otimes e_2' + h_1 \otimes (T')^{-1} e_2', \quad e_2' \in E_2'\} \tag{23}$$

with $(T')^{-1} : E_2' \to \bar{E}_1$ an isomorphism and $E_2' \subset E_1''$.

Let us look for the sufficient condition U to be non degenerate. Let $X_0 \in U$ and suppose $g(X_0, Y) = 0,\ \forall Y \in U$ with $X_0 = h_1 \otimes e_0' + h_2 \otimes e_0 + h_1 \otimes \bar{e}_0$ and $Y = h_1 \otimes f' + h_2 \otimes f + h_1 \otimes \bar{f},\ e_0', f' \in E_1', e_0, f \in E_2, \bar{e_0}, \bar{f} \in \bar{E}_1$ according to the decomposition given in (21). Then

$$-\omega^E(e_0, f' + Tf) + \omega^E(e_0' - Te_0, f) = 0,\ \forall f \in E_2, \forall f' \in E_1'.$$

This implies that $\omega^E(E_2, E_1')$ is degenerate. Then conclusion follows.

□

Note that, from (23), every non trivial pure nilpotent subspace contains a non trivial pure weakly para-complex subspace. Moreover any pure weakly para-complex subspace is direct sum of a pure para-complex subspace and a degree 1 nilpotent subspace.

3.6. **Real subspaces.**

Definition 3.27. *A subspace $U \subset V$ is called **real** if $AU \cap U = \{0\}, \quad \forall A \in \widetilde{Q}$. Equivalently, U does not contain either a non trivial complex or weakly para-complex subspace.*

Let prove the above equivalence. If $AU \cap U = \{0\}$, $\forall A \in \widetilde{Q}$, clearly no non trivial complex or weakly para-complex subspaces are in U. Viceversa let U contain no non trivial complex or weakly para-complex subspaces. Then, as remarked in the previous section, it contains no non trivial nilpotent subspaces as well.

A real subspace U is pure. Also, $\dim U \leq \frac{1}{2} \dim V$.

Definition 3.28. *A non degenerate real subspace $U \subset V$ is called **totally real** if for one and hence for any para-hypercomplex basis (I, J, K) of $\widetilde{Q}$,*

$$IU \perp U, \qquad JU \perp U, \quad KU \perp U$$

or equivalently if the Hermitian product (8) has real values for any admissible basis (I, J, K) of $\widetilde{Q}$ [d].

The implication in the first statement is straightforward to verify. In this case $\dim U \leq \frac{1}{4} \dim V$.

Theorem 3.29. *A subspace $U \subseteq V$ is real iff w.r.t. a symplectic basis (h_1, h_2) it is $U = U^{F,T}$ where the linear map $T : F = E_1 = p_1(U) \to p_2(U)$ is an isomorphism such that, for any non trivial subspace $W \subset F \cap TF$, it is $TW \nsubseteq W$.*

The subspace U is non degenerate if and only F is g_F-non degenerate.

Let $E_2 = TE_1$. The real subspace U is totally real if and only if

$$\omega^E|_{E_1} = \omega^E|_{E_2} \equiv 0 \qquad \text{and} \quad T|_{E_1} \quad \text{is } \omega^E|_F - \text{skew-symmetric.} \tag{24}$$

which implies $E_1 \cap E_2 = \{0\}$.

[d] In fact, in [6] such subspace is called a *subspace with real Hermitian product.*

Proof. Let U be a real subspace. Since no non trivial weakly para-complex subspace is in U that it contains no decomposable vectors and from lemma (3.5), fixed any symplectic basis (h_1, h_2) of H, we can write

$$U = U^{F,T} = \{h_1 \otimes e + h_2 \otimes Te,\ e \in F = E_1 = p_1(U)\}.$$

Suppose $T\tilde{F} \subset \tilde{F}$ for some subspace $\tilde{F} \subset W = E_1 \cap E_2$, Then $\tilde{F}$ must be an even dimensional subspace direct sum of 2-dimensional T-invariant real subspaces $\tilde{F}_i$. We show that necessarily $\tilde{F} = \{0\}$.

Let then $\tilde{F} \supseteq \tilde{F}_i =< e, Te >_{\mathbb{R}}$ be a T-invariant plane, with $T(Te) = \lambda e + \mu Te$. Observe that both μ and λ can not be zero. In fact, if $\lambda = 0$ then $T(Te) = \mu Te$ which is excluded since $T|_W$ has no invariant lines. If $\mu = 0$ then $T(Te) = \lambda e$ with $\lambda \leq 0$ since the vectors e, Te are linearly independent. Then the map $\tilde{T} = \frac{T}{\sqrt{|\lambda|}}$ is a complex structure on $< e, Te >$ and the subspace $\tilde{U} = \{h_1 \otimes f + h_2 \otimes \tilde{T} f,\ f \in < e, Te >\}$ is a complex subspace in U. So necessarily $\mu \neq 0$.

Consider the non null vector $X = h_1 \otimes e + h_2 \otimes Te \in U$. For any $A \in \widetilde{Q},\ A = \alpha I + \beta J + \gamma K$ with I, J, K the para-hypercomplex basis associated to the chosen basis (h_1, h_2), by hypothesis, whether $AX = 0$ or $AX \notin U,\ \forall \alpha, \beta, \gamma$. Computing The vector $AX = 0$ only if A is the null map. But, for any γ and by choosing

$$\alpha = \frac{\gamma}{\mu}(\lambda - 1), \qquad \beta = \frac{\gamma}{\mu}(1 + \lambda)$$

the vector $AX \in U$ since, in this case, $Te' = \tilde{e}$, contradiction.

Viceversa, let $U = U^{F,T}$ w.r.t. the symplectic basis (h_1, h_2); denote $E_1 = F,\ E_2 = TF$, and assume that $T : E_1 \to T(E_1) = E_2$ is an isomorphism such that for any non trivial subspace $W \subset E_1 \cap E_2$, it is $TW \subsetneq W$. Let $A = \alpha I + \beta J + \gamma K, \in \widetilde{Q}$. Suppose there exists a non null vector $X = h_1 \otimes e + h_2 \otimes Te \in U$, such that $AX = h_1 \otimes (-\gamma e + (\beta - \alpha)Te) + h_2 \otimes ((\alpha + \beta)e + \gamma Te) \neq 0$ is in U. This implies that $T^2 e \in < e, Te > \subset (E_1 \cap E_2)$, which gives a contradiction.

From (14), the subspace U is non degenerate if and only F is g_F-non degenerate.

Let $U = \{X = h_1 \otimes e + h_2 \otimes Te, \quad e \in E_1\}$ be a totally real subspace in V. Then

$$\begin{aligned} &1) \quad IU = \{Y = -h_1 \otimes Te_1 + h_2 \otimes e_1, \quad e_1 \in E_1\}, \\ &2) \quad JU = \{Y = h_1 \otimes Te_2 + h_2 \otimes e_2, \quad e_2 \in E_1\}, \\ &3) \quad KU = \{Y = -h_1 \otimes e_3 + h_2 \otimes Te_3, \quad e_3 \in E_1\}. \end{aligned} \tag{25}$$

Imposing orthogonality conditions $IU \perp U, \quad JU \perp U, \quad KU \perp U$, we obtain $\omega^E|_{E_1} = \omega^E|_{E_2} \equiv 0$, from 1) and 2), and $\omega^E(e, Te') + \omega^E(Te, e') = 0, \quad \forall e, e' \in E_1$ from 3).

Viceversa, given a pure real subspace $U = U^{F,T}$, from (24) we obtain $IU \perp U, \quad JU \perp U, \quad KU \perp U$. For any $X = h_1 \otimes e + h_2 \otimes Te$ and $Y = h_1 \otimes e' + h_2 \otimes Te'$, the get

$$g(X, Y) = \omega^E(e, Te') - \omega^E(Te, e') = 2\omega^E(e, Te'). \tag{26}$$

Since U is non degenerate, then $\omega^E(E_1 \times E_2)$ is non degenerate hence $E_1 \cap E_2 = \{0\}$. □

3.7. **Decomposition of a generic subspace.** Let $U \subset V$ be a subspace of the para-quaternionic Hermitian vector space $(V = H \otimes E, \tilde{Q} = \mathfrak{sl}(H) \otimes Id, g = \omega^H \otimes \omega^E)$. For any $A \in \widetilde{Q}$ we denote by U_A the maximal A-invariant subspace in U.

The following proposition, whose proof is straightforward by a procedure of successive decompositions, expresses that, by using para-quaternionic, pure complex, weakly pure para-complex, and real subspaces as building blocks, we can construct any subspace $U \subset V$.

Proposition 3.30. *Let U be a subspace in V and U_0 be its maximal para-quaternionic subspace. Then U admits a direct sum decomposition of the form*

$$U = U_0 \oplus U'$$

with

$$U' = U_{I_1} \oplus \ldots \oplus U_{I_p} \oplus U_{J_1} \oplus \ldots \oplus U_{J_q} \oplus U^R,$$

where the U_{I_i}, $i = 1, \ldots, p$, are pure I_i-complex subspaces, the U_{J_j}, $i = 1, \ldots, q$, are J_j-pure weakly para-complex subspaces and U^R is real.

By using as building blocks pure para-complex subspaces instead of pure weakly para-complex, we necessarily need to use also pure nilpotent subspaces.

As an example of the last statement let think of a decomposable subspace $U = h \otimes F$, $h \in H$, $F \subset E$.

The decomposition of the proposition (3.30) is clearly not unique even up to reordering of addends. The first reason depends obviously on the non uniqueness of the complement at each steps of the decomposition. Moreover the decomposition depends on the chosen order of types of subspaces i.e. if we first consider pure complex subspaces and then pure weakly para-complex or the other way round. Not taking into account the metric, we intend to further investigate if the different possible decompositions, choosing the addends by decreasing dimension and fixing the order of the decomposition, are unique up to isomorphisms i.e. have addends of same type and dimension.

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References

[1] V. Cortés , Ch. Mayer, Th. Mohaupt and F.Saueresssig, Special Geometry of Euclidean supersymmetry I: Vector multiplets, *J.Hight Energy Phys.* **028**(3) (2004) 73 pp.

[2] V. Cortés , Ch. Mayer, Th. Mohaupt and F.Saueresssig, Special Geometry of Euclidean supersymmetry II: Hypermultiplets and the c-map, *J.Hight Energy Phys.* **025** (6) (2005) 37 pp.

[3] T.Mohaupt, Special Geometry, Black Holes and Euclidean Supersymmetey, in *Handbook of Pseudo-Riemannian Geometry and Supersymmetry*, ed. V. Cortés. (EMS, Germany, 2010) pp. 149-181.

[4] S. Salamon, Quaternionic Kaehler manifolds, *Invent. Math.* **67** (1982) 143-171.

[5] S. Salamon, Differential geometry of quaternionic manifolds, *Ann. scient. Ec. Norm. Sup.*, 4^e série, **19** (1986) 31-55.

[6] M.Bruni, Su alcune proprietá di geometria euclidea ed hermitiana in uno spazio vettoriale quaternionale, *Ann. Mat. Pura Appl.*, **72** (1965) 59-77.

[7] D.V. Alekseevsky and S. Marchiafava, Quaternionic structures on a manifold and subordinated structures, *Ann. Mat. Pura Appl.* **171** (1996) 205-273.

[8] D.V. Alekseevsky and V. Cortés, The twistor spaces of a para-quaternionic Kaehler manifold, *Osaka J. Math.* **45** (1) (2008) 215-251.

[9] V. Cortés, A new construction of homogeneous quaternionic manifolds and related geometric structures, *Mem. Amer. Math. Soc.* **147** n. 700 (2000) 1-63.

[10] S. Vukmirovic, Para-Quaternionic reduction, (2003) math.DG/0304424.