ON LIMIT DISTRIBUTIONS OF NORMALIZED TRUNCATED VARIATION, UPWARD TRUNCATED VARIATION AND DOWNWARD TRUNCATED VARIATION PROCESSES

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Abstract. In the paper we introduce the truncated variation, upward truncated variation and downward truncated variation. These are closely related to the total variation but are well-defined even if the latter is infinite. Our aim is to explore their feasibility to studies of stochastic processes. We concentrate on a Brownian motion with drift for which we prove the convergence of the above-mentioned quantities. For example, we study the truncated variation when the truncation parameter c tends to 0. We prove in this case that for "small" c's it is well-approximated by a deterministic process. Moreover we prove that error in this approximation converges weakly (in functional sense) to a Brownian motion. We prove also similar result for truncated variation processes when time parameter is rescaled to infinity. We stress that our methodology is robust. A key to the proofs was a decomposition of the truncated variation (see Lemmas 11 and 12). It can be used for studies of any continuous processes. Some additional results like an analog of the Anscombe-Donsker theorem and the Laplace transform of time to given drawdown by c (and analogously drawup till time) are presented.

1. Introduction

The variation of Brownian paths was the subject of study of many authors (cf. [6], [11], [3], [2] just to name a few, for more detailed account see e.g. [9, Chapter 10]). It is well known that for any $p \leq 2$, $p-$variation of the Brownian motion is a.s. infinite and this arguably gave rise to the development of Itô integral, which alone proves that studies of the variation is of an utmost importance for the stochastic processes theory.

Intuitively, the above motioned infiniteness of the variation stems from "wild behaviour" at small scales. A natural, yet not studied before, way to tackle this problem was introduced in the paper [7]. The idea introduced there was to neglect a moves of a process smaller than a certain (small) $c > 0$. This led to the definition of the *truncated variation*. Let now $f : [a, b] \to \mathbb{R}$ be a continuous function. Its truncated variation on the interval $[a, b]$ is given by

$$\sup_{n} \sup_{a\leq t_1<t_2<\ldots<t_n\leq b} \sum_{i=1}^{n-1} \phi_c \left(|f(t_{i+1}) - f(t_i)|\right),$$

where $\phi_c(x) = \max\{x - c, 0\}$. This defines a functional which can be applied to the paths of any continuous stochastic process. In the paper we study the case of $(W_t)_{t\geq 0}$ being the Brownian motion with drift $\mu \in \mathbb{R}$ and covariance function $\text{cov}(W_s, W_t) = s \wedge t$. The reason for this is twofold. Firstly, W is a widely studied process, which enables us making some explicit computations. Secondly, W is an exemplar case of semimartingales and diffusions. The results for W will shed some light on the

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properties of the truncated variation in these more general classes. We stress that even this paper contains some general results for continuous processes. In separate articles, we plan to apply them for semimartingales and diffusions.

Therefore, the main object of our studies will be the truncated variation of the Brownian motion with drift μ, which is given by

$$TV_c^{\mu}[a,b] := \sup_n \sup_{a\le t_1<t_2<\ldots<t_n\le b} \sum_{i=1}^{n-1} \phi_c\left(\left|W_{t_{i+1}} - W_{t_i}\right|\right), \tag{1.1}$$

for $0 \le a < b < \infty$. We recall a closely related notion of the *upward truncated variation* of W, introduced in [8] and defined by

$$UTV_c^{\mu}[a,b] := \sup_n \sup_{a\le t_1<s_1<t_2<s_2<\ldots<t_n<s_n\le b} \sum_{i=1}^{n} \phi_c\left(W_{s_i} - W_{t_i}\right), \tag{1.2}$$

and, analogously, the *downward truncated variation*:

$$DTV_c^{\mu}[a,b] := \sup_n \sup_{a\le t_1<s_1<t_2<s_2<\ldots<t_n<s_n\le b} \sum_{i=1}^{n} \phi_c\left(W_{t_i} - W_{s_i}\right).$$

The properties of TV_c^{μ}, UTV_c^{μ} and DTV_c^{μ} are known up to some degree. Indeed in the paper [8], the author proved that all three have finite exponential moments. Moreover, there are explicit formulas for the moment generating functions of $UTV_c^{\mu}[0,T]$ and $DTV_c^{\mu}[0,T]$ when T is an exponential random variable, independent of W.

The calculations we mentioned, however, did not give explicit insight into the "nature" of truncated variation. We would like to obtain a more direct description of "infiniteness level" of 1−variation of Brownian paths. A tempting way of answering to this question, followed in this paper, may be to study the divergence of the truncated variation as $c \searrow 0$. We denote $TV_c^{\mu}(t) := TV_c^{\mu}[0,t]$, $UTV_c^{\mu}(t) := UTV_c^{\mu}[0,t]$ and $DTV_c^{\mu}(t) := DTV_c^{\mu}[0,t]$. In the paper we prove that the processes

$$\sqrt{3}\left(TV_c^{\mu}(t) - \frac{t}{c}\right), \tag{1.3}$$

$$\sqrt{3}\left(UTV_c^{\mu}(t) - \left(\frac{1}{c}+\mu\right)\frac{t}{2}\right), \tag{1.4}$$

$$\sqrt{3}\left(DTV_c^{\mu}(t) - \left(\frac{1}{c}-\mu\right)\frac{t}{2}\right), \tag{1.5}$$

converge (in the functional sense) weakly to a standard Brownian motion as $c \to 0$. This is arguably the most surprising result of the paper. It appears that the truncated variation (of the Wiener process) is essentially a deterministic function. We believe that this is closely related to the fact that the quadratic variation $\langle W\rangle_t = t$ is also deterministic.

We prove a very useful decomposition of $TV_c^{\mu}(t), UTV_c^{\mu}(t)$ and $DTV_c^{\mu}(t)$ stated in Lemmas 11 and 12. It is in fact valid for any continuous stochastic process and it will be starting point for our further studies. In this paper setting the decomposition is particularly useful as it is very similar to a renewal process. This is a crux of the proofs of the above-mentioned convergences.

For completeness we also investigate the convergence in distribution of properly normalized processes $TV_c^{\mu}(nt), UTV_c^{\mu}(nt)$ and $DTV_c^{\mu}(nt)$ for fixed c as n tends to infinity. Similarly as before we obtain that the processes

$$\frac{TV_c^{\mu}(nt) - m_1 nt}{\sigma_1\sqrt{n}}, \frac{UTV_c^{\mu}(nt) - m_2 nt}{\sigma_2\sqrt{n}} \text{ and } \frac{DTV_c^{\mu}(nt) - m_3 nt}{\sigma_3\sqrt{n}},$$

converge weakly (in the functional sense) to a standard Brownian motion, for some deterministic constants $m_1, m_2, m_3, \sigma_1, \sigma_2, \sigma_3$. The result is very similar to the case of $c \searrow 0$ however it is more "expected".

For the purposes of our proofs we also prove some results which may be interesting on their own and would be useful tools for some other applications. We calculate the bivariate Laplace transform of the variables

$$T_D(c) := \inf\left\{s : \sup_{0\le u\le s} W_u - W_s = c\right\}$$

and

$$Z_{D-c}(c) = \sup_{0\le t<s\le T_D(c)} \max\{W_s - W_t - c, 0\}.$$

This result is similar to the one of [10], where the bivariate Laplace transform of $T_D(c)$ and $\sup_{0\le t\le T_D(c)} W_t$ was calculated.

We also present a functional convergence theorem (Theorem 13) which may be viewed both as an extension of the Donsker theorem and an analogue of the Anscombe theorem. Although the theorem seems quite standard, up to our knowledge it was not known before.

Let us comment on the organization of the paper. In the next section we present the main results of this paper - theorems about convergence in distribution of the normalized processes $TV_c^{\mu}(t), UTV_c^{\mu}(t)$ and $DTV_c^{\mu}(t)$ as $c \to 0$ and processes $TV_c^{\mu}(nt), UTV_c^{\mu}(nt)$ and $DTV_c^{\mu}(nt)$ as $n \to +\infty$. In the third section we state and prove the lemmas concerning structure of the processes $TV_c^{\mu}(t), UTV_c^{\mu}(t)$ and $DTV_c^{\mu}(t)$. In the fourth section we state and prove the general functional theorem. Finally, the fifth section is devoted to the proofs of the theorems stated in Section 2. The bivariate Laplace transform of $T_D(c)$ and $Z_{D-c}(c)$ is calculated in this section as well.

2. Results

2.1. **Limit in distribution of truncated variation processes.** In this subsection we present the results concerning the limit distribution of the normalized truncated variation, upward truncated variation and downward truncated variation processes. By $B_t, t \ge 0$, we denote a standard Brownian motion.

Now we are ready to present some functional limit theorems. Let us recall the definition of the truncated variation (1.1) and that $TV_c^{\mu}(t) = TV_c^{\mu}([0,t])$. We start with

Theorem 1. *Let $T > 0$. We have*

$$(TV_c^{\mu}(t) - c^{-1}t) \to^d 3^{-1/2}B_t, \quad \textit{as } c \to 0,$$

where $\to^d$ is understood as weak convergence in $\mathcal{C}([0,T],\mathbb{R})$ topology.

Remark 2. The immediate consequence of Theorem 1 is that the variable $\sqrt{3}(TV_c^{\mu}(t) - c^{-1}t)$ converges in law to the variable with normal distribution $\mathcal{N}(0,t)$ as $c \to 0$.

Remark 3. This theorem reveals that for small c the truncated variation is almost a deterministic process. Namely, $TV_c^{\mu}(t) \approx c^{-1}t + 3^{-1/2}B_t$ and obviously the first term overwhelms the second one.

For fixed c and rescaled time parameter we have

Theorem 4. *Let $T > 0$ and $c > 0$. We have*

$$\frac{TV_c^\mu(nt) - m_c^\mu nt}{\sigma_c^\mu \sqrt{n}} \to^d B_t, \quad as\, n \to +\infty,$$

where $\to^d$ is understood as weak convergence in $\mathcal{C}([0,T],\mathbb{R})$ topology and

$$m_c^\mu = \begin{cases} \mu \coth(c\mu) & if\, \mu \neq 0, \\ c^{-1} & if\, \mu = 0, \end{cases}$$

$$(\sigma_c^\mu)^2 = \begin{cases} \frac{2-2c\mu\coth(c\mu)}{\sinh^2(c\mu)} + 1 & if\, \mu \neq 0, \\ 1/3 & if\, \mu = 0. \end{cases}$$

Remark 5. The immediate consequence of Theorem 4 is that the variable $(TV_c^\mu(T) - m_c^\mu T) / \left(\sigma_c^\mu \sqrt{T}\right)$ converges in law to the variable with standard normal distribution $\mathcal{N}(0,1)$ as $T \to +\infty$.

Remark 6. We note that when $\mu = 0$ both theorems are equivalent. It is enough to notice that by the scaling property of the Brownian motion we have $TV_c^0(nt) =^d \sqrt{n}\left(TV^0_{c/\sqrt{n}}(t)\right)$.

2.2. **Limit distribution of upward and downward truncated variations processes.** In this subsection we present the results concerning the limit distribution of the normalized upward and downward truncated variations processes. Since $DTV_c^\mu[a,b] =^d UTV_c^{-\mu}[a,b]$ we will only deal with upward truncated variation. We recall the definition (1.2) and that $UTV_c^\mu(t) = UTV_c^\mu([0,t])$. Firstly we analyze the situation when c is small,

Theorem 7. *Let $T > 0$. We have*

$$\left(UTV_c^\mu(t) - \left(\frac{1}{2c} + \frac{\mu}{2}\right)t\right) \to^d 3^{-1/2} B_t, \quad as\, c \to 0,$$

where $\to^d$ is understood as weak convergence in $\mathcal{C}([0,T],\mathbb{R})$ topology.

For fixed c and rescaled time parameter we have

Theorem 8. *Let $T > 0$ and $c > 0$. We have*

$$\frac{UTV_c^\mu(nt) - m_c^\mu nt}{\sigma_c^\mu \sqrt{n}} \to^d B_t, \quad as\, n \to +\infty,$$

where $\to^d$ is understood as weak convergence in $\mathcal{C}([0,T],\mathbb{R})$ topology and

$$m_c^\mu = \begin{cases} \frac{1}{2}\mu(\coth(c\mu) + 1) & if\, \mu \neq 0, \\ (2c)^{-1} & if\, \mu = 0, \end{cases}$$

$$(\sigma_c^\mu)^2 = \begin{cases} \frac{2\exp(4c\mu)(\sinh(2c\mu) - 2c\mu)}{(\exp(2c\mu)-1)^3} & if\, \mu \neq 0, \\ 1/3 & if\, \mu = 0. \end{cases}$$

Remark 9. Analogously as before one checks that both theorems are equivalent if $\mu = 0$. This is a simple consequence of the scaling property of the Brownian motion which yields $UTV_c^0(nt) =^d \sqrt{n}\left(UTV^0_{c/\sqrt{n}}(t)\right)$.

3. A structure of truncated variation, upward truncated variation and downward truncated variation processes

In this section we develop tools to analyze TV, DTV, UTV processes. For the matter of convenience we work with the Winer process with drift W but we stress that all results in this section are valid for any continuous stochastic process.

3.1. A structure of truncated variation process.

Firstly, we will prove that the process $(TV_c^\mu(t))_{t\geq 0}$ has similar structure as a renewal process. To state it more precisely we first define

$$T_D(c) = \inf\left\{s \geq 0 : \sup_{0\leq u\leq s} W_u - W_s = c\right\},$$

$$T_U(c) = \inf\left\{s \geq 0 : W_s - \inf_{0\leq u\leq s} W_u = c\right\},$$

$$T(c) = \min\left\{T_D(c), T_U(c)\right\}.$$

and now let $(T_i(c))_{i=0}^\infty$ be series of stopping times defined in the following way: $T_0 := 0$ and

- if $T(c) = T_D(c)$, then $T_1(c) := T(c)$ and recursively, for $k = 1, 2, ...$,

$$T_{2k}(c) := \inf\left\{s \geq T_{2k-1}(c) : W_s - \inf_{T_{2k-1}(c)\leq u\leq s} W_u = c\right\},$$

$$T_{2k+1}(c) := \inf\left\{s \geq T_{2k}(c) : \sup_{T_{2k}(c)\leq u\leq s} W_u - W_s = c\right\};$$

- if $T(c) = T_U(c)$, then $T_1(c) := 0$, $T_2(c) := T(c)$ and recursively, for $k = 1, 2, ...$,

$$T_{2k+1}(c) := \inf\left\{s \geq T_{2k}(c) : \sup_{T_{2k}(c)\leq u\leq s} W_u - W_s = c\right\},$$

$$T_{2k+2}(c) := \inf\left\{s \geq T_{2k+1}(c) : W_s - \inf_{T_{2k+1}(c)\leq u\leq s} W_u = c\right\}.$$

(Observe that the event $\{T_U(c) = T_D(c)\}$ is impossible, hence the definitions above do not interfere.)

Additionally we define series of times $(S_i(c))_{i=0}^\infty$ (which are not stopping ones): for $k = 0, 1, 2, ...$

- $S_{2k}(c)$ is the first time when the maximum of W_t on the interval $[T_{2k}(c), T_{2k+1}(c)]$ is attained (in particular for $T_1 = 0, S_0 = 0$);
- $S_{2k+1}(c)$ is the first time when the minimum of W_t on the interval $[T_{2k+1}(c), T_{2k+2}(c)]$ is attained.

We have

Lemma 10. *For $k = 1, 2, 3, ...$ the following equalities hold*

$$TV_c^\mu(T_{2k}(c)) = TV_c^\mu(T_{2k-1}(c)) + W_{T_{2k-1}(c)} - \inf_{T_{2k-1}(c)\leq s\leq T_{2k}(c)} W_s,$$

$$TV_c^\mu(T_{2k+1}(c)) = TV_c^\mu(T_{2k}(c)) + \sup_{T_{2k}(c)\leq s\leq T_{2k+1}(c)} W_s - W_{T_{2k}(c)}.$$

Moreover, a partition for which $TV_c^\mu(T_{2k})$ and $TV_c^\mu(T_{2k+1})$ in definition (1.1) *are attained is given by* $S_0(c), S_1(c), \ldots$.

Proof. The proof will follow by induction.

It is easy to see that $TV_c^{\mu}(T(c)) = 0$. Let us consider two cases.

1. First case $T(c) = T_D(c)$.

We start with $k = 1$.

Let $0 = t_0 < t_1 < \dots < t_n \le T_2(c)$ be a partition of the interval $[0, T_2(c)]$. Without the loss of generality we may assume that there is no element t_j such that $\max\left\{\left|W_{t_{j-1}} - W_{t_j}\right| - c, 0\right\} = \max\left\{\left|W_{t_j} - W_{t_{j+1}}\right| - c, 0\right\} = 0$. Indeed, if it would be such element we may skip it and the sum (1.1) will not decrease.

(1) It is easy to see that if $\max\left\{\left|W_{t_{i+1}} - W_{t_i}\right| - c, 0\right\}$ and $\max\left\{\left|W_{t_{i+2}} - W_{t_{i+1}}\right| - c, 0\right\}$ are two consecutive non-zero summands, then

$$\max\left\{\left|W_{t_{i+2}} - W_{t_i}\right| - c, 0\right\} \ge \max\left\{\left|W_{t_{i+1}} - W_{t_i}\right| - c, 0\right\} + \max\left\{\left|W_{t_{i+2}} - W_{t_{i+1}}\right| - c, 0\right\}.$$

In fact, because we are before the first upward move by c we must have $W_{t_{i+1}} - W_{t_i} \le -c, W_{t_{i+2}} - W_{t_{i+1}} \le -c$ hence

$$W_{t_{i+2}} - W_{t_i} = \left(W_{t_{i+2}} - W_{t_{i+1}}\right) + \left(W_{t_{i+1}} - W_{t_i}\right) \le -2c,$$

and

$$\begin{aligned} \max\left\{\left|W_{t_{i+2}} - W_{t_i}\right| - c, 0\right\} &= W_{t_i} - W_{t_{i+2}} - c \\ &\ge \left(W_{t_i} - W_{t_{i+1}} - c\right) + \left(W_{t_{i+1}} - W_{t_{i+2}} - c\right) \\ &= \max\left\{\left|W_{t_{i+1}} - W_{t_i}\right| - c, 0\right\} + \max\left\{\left|W_{t_{i+2}} - W_{t_{i+1}}\right| - c, 0\right\}. \end{aligned}$$

(2) Similarly, if $\max\left\{\left|W_{t_{i+1}} - W_{t_i}\right| - c, 0\right\}$ and $\max\left\{\left|W_{t_{i+3}} - W_{t_{i+2}}\right| - c, 0\right\}$ are two non-zero summands, while $\max\left\{\left|W_{t_{i+2}} - W_{t_{i+1}}\right| - c, 0\right\} = 0$, we have $W_{t_{i+1}} - W_{t_i} \le -c, W_{t_{i+2}} - W_{t_{i+1}} \le -c$ and $W_{t_{i+2}} - W_{t_{i+1}} \le c$, hence

$$\begin{aligned} W_{t_{i+3}} - W_{t_i} &= \left(W_{t_{i+3}} - W_{t_{i+2}}\right) + \left(W_{t_{i+2}} - W_{t_{i+1}}\right) + \left(W_{t_{i+1}} - W_{t_i}\right) \\ &\le -c + c - c \le -c, \\ \max\left\{\left|W_{t_{i+3}} - W_{t_i}\right| - c, 0\right\} &= W_{t_i} - W_{t_{i+3}} - c \\ &\ge \left(W_{t_{i+3}} - W_{t_{i+2}} - c\right) + \left(W_{t_{i+1}} - W_{t_i} - c\right) \\ &= \max\left\{\left|W_{t_{i+1}} - W_{t_i}\right| - c, 0\right\} + \max\left\{\left|W_{t_{i+3}} - W_{t_{i+2}}\right| - c, 0\right\}. \end{aligned}$$

As a result we obtain that the sum $\sum_{i=1}^{n-1} \max\left\{\left|W_{t_{i+1}} - W_{t_i}\right| - c, 0\right\}$ attains its largest value for a **two-element** partition $0 \le t_1 < t_2 \le T_2(c)$.

Since $W_{S_0(c)} = \sup_{0 \le s \le T_2(c)} W_s$ and $W_{S_1(c)} = \inf_{0 \le s \le T_2(c)} W_s$ we get

$$\begin{aligned} TV_c^{\mu}(T_2(c)) &= \sup_n \sup_{0 \le t_1 < t_2 < \dots < t_n \le T_2(c)} \sum_{i=1}^{n-1} \max\left\{\left|W_{t_{i+1}} - W_{t_i}\right| - c, 0\right\} \\ &= \sup_{0 \le t_1 < t_2 \le T_2(c)} \sum_{i=1}^{n-1} \max\left\{\left|W_{t_{i+1}} - W_{t_i}\right| - c, 0\right\} \\ &= \max\left\{\left|W_{S_0(c)} - W_{S_1(c)}\right| - c, 0\right\} \\ &= W_{T_1(c)} - \inf_{T_1(c) \le s \le T_2(c)} W_s \\ &= TV_c^{\mu}(T_1(c)) + W_{T_1(c)} - \inf_{T_1(c) \le s \le T_2(c)} W_s. \end{aligned}$$

(Note that $TV_c^{\mu}(T_1(c)) = TV_c^{\mu}(T(c)) = 0$)

Let us assume that the Lemma 11 holds for some $k \geq 1$. We proceed with the induction step from interval $[0, T_{2k}(c)]$ to the interval $[0, T_{2k+1}(c)]$.

We know that $0 \leq S_0(c) < S_1(c) < ... < S_{2k-1}(c) < T_{2k}(c)$ is the best partition of the interval $[0, T_{2k}(c)]$. We will prove that the best partition of the interval $[0, T_{2k+1}(c)]$ is $0 \leq S_0(c) < S_1(c) < ... < S_{2k-1}(c) < S_{2k}(c)$ i.e.

$$\sum_{i=1}^{n-1} \max\left\{\left|W_{t_{i+1}} - W_{t_i}\right| - c, 0\right\} \leq \sum_{i=0}^{2k-1} \max\left\{\left|W_{S_{i+1}(c)} - W_{S_i(c)}\right| - c, 0\right\},$$

for any partition $0 \leq t_1 < ... < t_n \leq T_{2k+1}(c)$.

Again we will consider several cases.

(1) Firstly let us observe that if there exists such $v \in \{1, 2, ..., n\}$ that $S_{2k-1}(c) \leq t_v < T_{2k}(c)$, then, due to optimality of the partition $0 \leq S_0(c) < S_1(c) < ... < S_{2k-1}(c) < T_{2k}(c)$,

$$\begin{aligned}\sum_{i=1}^{v-1} \max\left\{\left|W_{t_{i+1}} - W_{t_i}\right| - c, 0\right\} &\leq \sum_{i=0}^{2k-2} \max\left\{\left|W_{S_{i+1}(c)} - W_{S_i(c)}\right| - c, 0\right\} \\ &= TV_c^{\mu}(T_{2k}(c)).\end{aligned}$$

Moreover, reasoning similarly as in the proof of the case (1) for $k = 1$, from definitions of $S_{2k-1}(c), T_{2k}(c)$ and $T_{2k+1}(c)$ we obtain that for any $S_{2k-1}(c) \leq s < u \leq T_{2k+1}(c)$, $W_u - W_s \geq -c$ and the sum $\sum_{i=v}^{n} \max\left\{\left|W_{t_{i+1}} - W_{t_i}\right| - c, 0\right\}$ attains its largest value for **two element** partition and can not be larger than $\max\left\{\left|W_{S_{2k}(c)} - W_{S_{2k-1}(c)}\right| - c, 0\right\}$. Collecting these two inequalities, we get

$$\begin{aligned}\sum_{i=1}^{n-1} \max\left\{\left|W_{t_{i+1}} - W_{t_i}\right| - c, 0\right\} &\leq \sum_{i=0}^{2k-1} \max\left\{\left|W_{S_{i+1}(c)} - W_{S_i(c)}\right| - c, 0\right\} \\ &= TV_c^{\mu}(T_{2k}(c)) + \sup_{T_{2k}(c) \leq s \leq T_{2k+1}(c)} W_s - W_{T_{2k}(c)}.\end{aligned}$$

(2) Now we may assume that there is no such indice v that $S_{2k-1}(c) < t_v < T_{2k}(c)$. In this case let v be the largest index such that $t_v \leq S_{2k}$.

We have two subcases.

(a) $W_{t_v} < W_{S_{2k-1}(c)}$.

In this case we have $t_v < S_{2k-2}(c)$ (since $W_{t_v} < W_{S_{2k-1}(c)}$, by definition of $W_{S_{2k-1}(c)}$ as a minimal value of W_t on the interval $[T_{2k-1}(c), T_{2k}(c)]$ we have that $t_v < T_{2k-1}(c)$, but since $W_{S_{2k-2}(c)}$ is the maximal value of W_t on the interval $[T_{2k-2}(c), T_{2k-1}(c)]$ and by definition of $T_{2k-1}(c)$ we must have $t_v < S_{2k-2}(c)$) and we easily find that partition $0 \leq t_1 < ... < t_v < S_{2k-2}(c) < S_{2k-1}(c) < t_{v+1} < ... t_n \leq T_{2k+1}(c)$ gives a larger sum, than the partition $0 \leq t_1 < ... < t_n \leq T_{2k+1}(c)$. So we have a new, better partition which satisfies the conditions of the case (1) above.

(b)$W_{t_v} \geq W_{S_{2k-1}(c)}$.

In this case, if $W_{t_v} \in \left[W_{S_{2k-1}(c)}, W_{t_{v+1}}\right]$ then $\left|W_{t_{v+1}} - W_{t_v}\right| \leq \left|W_{t_{v+1}} - W_{S_{2k-1}(c)}\right|$. If the opposite holds, we get that $\left|W_{t_{v+1}} - W_{t_v}\right| \leq \left|W_{t_v} - W_{S_{2k-1}(c)}\right|$. In both cases we calculate

$$\max\left\{\left|W_{t_{v+1}} - W_{t_v}\right| - c, 0\right\} \leq \max\left\{\left|W_{S_{2k-1}} - W_{t_v}\right| - c, 0\right\} + \max\left\{\left|W_{t_{v+1}} - W_{S_{2k-1}}\right| - c, 0\right\}.$$

So again we have a new, better partition which satisfies the conditions of the case (1) above.

Now we may proceed to the proof of the step: form interval $[0, T_{2k+1}(c)]$ to the interval $[0, T_{2k+2}(c)]$. This case is analogous to the previous one when we consider the process $\tilde{W}_t = -W_t$ instead of W_t.

2. Second case $T(c) = T_U(c)$.

Again this case is analogous to the previous case in such a way, that considering the process $\tilde{W}_t = -W_t$ and the corresponding times $\left(\tilde{T}_i\right)_{i=0}^{\infty}$, $\left(\tilde{S}_i\right)_{i=0}^{\infty}$ we get $T_{i+1} = \tilde{T}_i$, $S_{i+1} = \tilde{S}_i$ for $i = 0, 1, 2, \ldots$ So in this case we obtain the thesis in a similar way as above. □

Let us now define

- for $k = 0, 1, 2, \ldots,$

$$Z_{D,k}(c) := \sup_{T_{2k}(c) \le s \le T_{2k+1}(c)} W_s - W_{T_{2k}(c)}, \tag{3.1}$$

- and similarly

$$Z_{U,k}(c) := W_{T_{2k+2}(c)} - \inf_{T_{2k+1}(c) \le s \le T_{2k+2}(c)} W_s.$$

Now, for $k = 0, 1, 2, \ldots$, we define sequence of random variables

$$D_k(c) := T_{2k+2}(c) - T_{2k}(c),$$

$$Z_k(c) := Z_{D,k}(c) + Z_{U,k}(c).$$

Let us note here that in the case of the Wiener process with drift W obviously $\{D_k(c)\}_k$ and $\{Z_k(c)\}_k$ are i.i.d sequences.

The immediate consequence of Lemma 10 is

Lemma 11. *For the process $(TV_c^{\mu}(t))_{t \ge 0}$ stopped at (Markov times) $T_{2k+2}(c), k = 0, 1, 2, ...,$ the following equality holds*

$$TV_c^{\mu}(T_{2k+2}(c)) = \sum_{l=0}^{k} Z_l(c).$$

3.2. **A structure of upward and downward truncated variation processes.** Now we will state an analog of Lemma 11 for the upward and downward truncated variation processes.

Let us first define two sequences of stopping times. Let $T_{U,0}(c) = T_{D,0}(c) = 0$ and

- recursively, for $k = 1, 2, \ldots,$

$$T_{D,k}(c) := \inf\left\{ s \ge T_{D,k-1}(c) : \sup_{T_{D,k-1}(c) \le u \le s} W_u - W_s = c \right\},$$

- and analogously

$$T_{U,k}(c) := \inf\left\{ s \ge T_{U,k-1}(c) : W_s - \inf_{T_{U,k-1}(c) \le u \le s} W_u = c \right\},$$

(notice that $T_{D,1}(c) = T_D(c)$). Further, we introduce

- recursively, for $k = 1, 2, \ldots,$

$$Z_{D-c,k}(c) := \sup_{T_{D,k-1}(c) \le t < s \le T_{D,k}(c)} \max\{W_s - W_t - c, 0\},$$

- and analogously

$$Z_{U-c,k}(c) := \sup_{T_{U,k-1}(c) \le t < s \le T_{U,k}(c)} \max\{W_t - W_s - c, 0\}.$$

As the immediate consequence of [8, Lemma 3] we get

Lemma 12. *For $k = 1, 2, 3, ...$ the following equalities hold*

$$UTV_c^{\mu}\left(T_{D,k}\left(c\right)\right) = \sum_{l=1}^{k} Z_{D-c,l}\left(c\right),$$

$$DTV_c^{\mu}\left(T_{U,k}\left(c\right)\right) = \sum_{l=1}^{k} Z_{U-c,l}\left(c\right).$$

4. Anscombe like theorem

Our aim in this section is to present an Anscombe-like functional central limit theorem for renewal processes. For the classical Anscombe theorem and for its functional extensions we refer to [5, Chapter 1, Chapter 2, Chapter 5]. From now one we will use "$\lesssim$" to denote the situation when an equality or inequality holds with a constant K > 0, which is irrelevant for calculations. Our setting is as follows. Let

$$(D_i(c), Z_i(c)), \quad i \in \mathbb{N},$$

be sequences of i.i.d. random vectors indexed by certain parameter $c \in (0, 1]$. We define

$$M_c(t) := \min\left\{n > 0 : \sum_{i=1}^{n} D_i(c) > t\right\},$$

$$P_c(t) := \left(\sum_{i=1}^{M_c(t)} Z_i(c)\right) - \frac{\mathbb{E}Z_1(c)}{\mathbb{E}D_1(c)}t, \quad t \in [0, 1].$$

We will need the following assumptions

(A1) For any $c > 0$ we have $D_1(c) > 0$ a.s. and $\mathbb{E}D_1(c) \to 0$ as $c \to 0$.

(A2) We denote $X_i(c) := Z_i(c) - (\mathbb{E}Z_1(c)/\mathbb{E}D_1(c))D_i(c)$. We have $\mathbb{E}X_i(c) = 0$. Now we assume that there exists $\sigma > 0$ such that

$$\frac{\mathbb{E}X_1(c)^2}{\mathbb{E}D_1(c)} \to \sigma^2, \quad \text{as } c \to 0.$$

(A3) There exists $\delta \in (0, 2]$ such that

$$\frac{\mathbb{E}|X_1(c)|^{2+\delta}}{\mathbb{E}D_1(c)} \to 0, \quad \text{as } c \to 0.$$

(A4) There exists $\delta > 0, C > 0$ such that

$$\mathbb{E}|D_1(c)|^{1+\delta} \le C(\mathbb{E}D_1(c))^{1+\delta}.$$

Theorem 13. *Let $T > 0$ and we assume that (A1)-(A4) hold. Then*

$$P_c \to^d \sigma B, \quad as\, c \to 0,$$

where σ^2 is the same as in (A2), and convergence is understood as the weak convergence in the Skorohod $\mathcal{D}([0, T], \mathbb{R})$ topology.

Remark 14. The theorem should be compared to a few results. However, we will not attempt to present the whole related bibliography referring instead to [5]. Firstly, the convergence of $P_c(1)$ could be proved by the Anscombe theorem [5, Section 1.3 p.16]. Secondly, the result is closely related to the theory of renewal process [5, Chapters 3, 4] - our theorem should be compared with [5, Chapter 4, Theorem 2.3]. Thirdly, one should also bear in mind a functional version of [5, Chapter 4, Theorem 2.3], namely [5, Chapter 5, Theorem 4.1]. Let (D_i, Z_i) be an i.i.d. sequence. We define $D_i(n) := D_i/n$ and $Z_i(n) := Z_i/\sqrt{n}$. Using our theorem we recover the result of [5, Chapter 5, Theorem 4.1]. One should be warned however that our assumptions are a bit stronger, namely in this case (A3) and (A4) are not required. This is not surprising, conditions of the same nature are required in the CLT for triangular arrays. Finally, in the same spirit, if we consider Fact 21, with $X_i(n) = X_i/\sqrt{n}$ and $g(1/n) = n$, for a certain i.i.d. sequence $\{X_i\}_i$, we would get the classic Donsker theorem. Analogously as before (A3) and (A4) could be dropped.

Remark 15. We will now discuss the assumptions. (A1) is pretty obvious it enforces that we indeed we consider a sum of random variables of increasing length. (A2) is also quite straightforward as it require proper normalization, it might be also considered as a requirement of convergence of second moments. Finally, we demand (A3) and (A4) in order to have some control over the distribution as c changes. They are by now means surprising e.g. (A3) is nothing else but the Lyapunov condition. As we noticed in the previous remark these are natural in our setting (though may be dropped in some very special situations).

Remark 16. We stress out that although we work with the Skorohod topology the convergence to a continuous process implies also the convergence in the sup norm.

4.1. **Proof of Anscombe like theorem.** We define

$$S_c(n) := \sum_{i=1}^{n} Z_i(c), \quad V_c(n) := \sum_{i=1}^{n} D_i(c), \quad n \in \mathbb{N}. \tag{4.1}$$

Moreover let us denote $f(c) := \frac{\mathbb{E}Z_1(c)}{\mathbb{E}D_1(c)}$ and we recall that $X_i(c) := Z_i(c) - f(c)D_i(c)$. The proof will be less technical if we consider a "continuous version" of M_c (abusing the notation we keep the name). We define it by declaring M_c to be linear on each segment $(V_c(n), V_c(n+1))$ and putting

$$M_c(V_c(n)) := n + 1, \quad n \in \mathbb{N}.$$

Let us point out that this definition is valid as $D_k(c) > 0$ a.s. Moreover, by construction, M_c is a continuous process therefore after a suitable truncation it can be considered as a random element of $\mathcal{C}([0,T],\mathbb{R})$, $T > 0$. Now we define a family of auxiliary processes which will be crucial for our proof

$$P_c^1(t) := H_c(\lfloor g(c)t \rfloor) + (g(c)t - \lfloor g(c)t \rfloor)(H_c(\lceil g(c)t \rceil) - H_c(\lfloor g(c)t \rfloor)), \quad c > 0, t \in [0, \infty). \tag{4.2}$$

where $H_c(n) := S_c(n) - f(c)V_c(n)$ and $g(c) := (\mathbb{E}D_1(c))^{-1}$.

From now up to further notice we will work only with continuous process. All convergences in distribution are denoted by $\to^d$ and understood as the weak convergence in $\mathcal{C}([0,T],\mathbb{R})$, $\mathcal{C}([0,T],\mathbb{R}^2)$ or $\mathbb{R}^d$, where $T > 0$. For $x \in \mathbb{R}^2$ we will use norm $|x| := |x_1| + |x_2|$. We recall the definition of a module of continuity in $\mathcal{C}([0,T],X)$

$$w(\delta, f) = \sup\left\{|f(x) - f(y)| : x, y \in [0,T],\, |x - y| < \delta\right\}, \tag{4.3}$$

where the choice of $|\cdot|$ will be clear from a context. Before the actual proof we reinforce ourselves with some preliminary facts.

Lemma 17. *Let $T>0$. For any $f=(f_1,f_2)\in\mathcal{C}([0,T],\mathbb{R}^2)$ we have*

$$w(\delta,f)\le w(\delta,f_1)+w(\delta,f_2).$$

Proof. Let $x,y\in[0,T]$ such that $w(\delta,f)=|f(x)-f(y)|$ we have

$$w(\delta,f)=|f(x)-f(y)|=|f_1(x)-f_1(y)|+|f_2(x)-f_2(y)|\le w(\delta,f_1)+w(\delta,f_2).$$

□

Fact 18. *Let $\{P_c\}_{c\in(0,1]}$ be a sequence of random elements in $\mathcal{C}([0,1],\mathbb{R}^2)$. It is tight if and only if these two conditions hold:*

(1) For any $t\in[0,1]$ set of variables $\{P_c(t)\}_{c\in(0,1]}$ is tight.
(2) For each $\varepsilon,\eta>0$, there exists a $\delta\in(0,1)$ and $c_0>0$ such that

$$\mathbb{P}\left(w(\delta,P_c)\ge\varepsilon\right)\le\eta,\quad c\in(0,c_0).$$

Proof. The proof hinges on an upgraded version of the Arzelà-Ascoli theorem valid for $\mathcal{C}([0,1],\mathbb{R}^2)$. It can be found in [4, Theorem 2.4.3][1]. Now one should proceed along the lines of the proof of [1, Theorem 7.2] with (4.3) instead the one-dimensional modulus of continuity. □

Lemma 19. *Let $c\in(0,1]$, $T>0$ and $\{X_c(t)\}_{t\in[0,T]},\{Y_c(t)\}_{t\in[0,T]}$ be continuous stochastic processes (not necessarily independent). We assume that $Y_c\to^d f$, and $X_i\to^d X$ as $c\to 0$, where f is a deterministic function and X is a certain (continuous) process X. Then*

$$(X_c,Y_c)\to^d(X,f),\qquad \text{as } c\to 0,$$

where $\to^d$ denotes the weak convergence in $\mathcal{C}([0,1],\mathbb{R}^2)$.

Proof. First we recall an elementary fact that if X_c and Y_c were real random variables then convergences $X_c\to^d X$ and $Y_c\to a$ for a certain constant $a\in\mathbb{R}$ would yield $X_i+Y_i\to^d X+a$. By using the Cramér-Wold device we see that this fact holds also if X_c,Y_c,a are $\mathbb{R}^n$-valued. Let now $(t_1,t_2,\dots,t_n)\in[0,T]^n$, the above fact yields

$$\begin{aligned}(4.4)\quad &((X_c(t_1),Y_c(t_1)),(X_c(t_2),Y_c(t_2)),\dots,(X_c(t_n),Y_c(t_n)))\\ &\to^d\left((X(t_1),f(t_1)),(X(t_2),f(t_2)),\dots,(X(t_n),f(t_n))\right),\quad \text{as } c\to 0.\end{aligned}$$

Let $\{c_n\}_{n\ge1}$ be a sequence $c_n\in(0,1]$ and $c_n\to 0$. Using Lemma 17 and Fact 18 we will prove that $\{(X_{c_n},Y_{c_n})\}_n$ is tight in $\mathcal{C}([0,1],\mathbb{R}^2)$. The first condition follows easily from (4.4). Let $\varepsilon,\eta>0$, by Fact 18 and tightness of $\{X_{c_n}\}_n,\{Y_{c_n}\}_n$ there exists $\delta>0$ such that $\mathbb{P}\left(w(\delta,X_{c_n})\ge\varepsilon/2\right)\le\eta/2$ and $\mathbb{P}\left(w(\delta,Y_{c_n})\ge\varepsilon/2\right)\le\eta/2$ for any $n\ge0$. Hence by Lemma 17 we obtain

$$\mathbb{P}\left(w(\delta,(X_{c_n},Y_{c_n})\ge\varepsilon\right)\le\mathbb{P}\left(w(\delta,X_{c_n})+w(\delta,Y_{c_n})\ge\varepsilon\right)\le\mathbb{P}\left(w(\delta,X_{c_n})\ge\varepsilon/2\right)+\mathbb{P}\left(w(\delta,Y_{c_n})\ge\varepsilon/2\right)\le\eta.$$

Now, when the conditions have been verified we know that any sub-sequence of $\{(X_c,Y_c)\}$ contains a further sub-sequence which is weakly convergent in $\mathcal{C}([0,T],\mathbb{R}^2)$. By (4.4) we check that whatever the sequence is the limit has the same distribution, viz. the one of (X,f). This concludes. □

Let $T>0$. We define a functional $F:\mathcal{C}([0,T],\mathbb{R}^2)\to F:\mathcal{C}([0,T],\mathbb{R})$ by

$$F((f,g))(t):=f((g(t)\wedge T)\vee 0).$$

It is obvious that F is well-defined. Moreover,

[1] Let us note that the formulation of the theorem contains a typo. Namely in b) should be $\sup\left\{w_f(\delta;X):f\in A\right\}\to 0$ as $\delta\to 0+$.

Lemma 20. *Functional F is continuous.*

Proof. Let $(f_i, g_i) \in \mathcal{C}([0,T],\mathbb{R}^2)$ such that $(f_i, g_i) \to (f, g)$. This implies (in fact is equivalent to) $f_i \to f$ and $g_i \to g$ in $\mathcal{C}([0,T],\mathbb{R})$. We estimate

$$\begin{aligned}&|f_i((g_i(t)\wedge T)\vee 0) - f((g(t)\wedge T)\vee 0)|\\ &\qquad \le |f_i((g_i(t)\wedge T)\vee 0) - f((g_i(t)\wedge T)\vee 0)| + |f((g_i(t)\wedge T)\vee 0) - f((g(t)\wedge T)\vee 0)|\\ &\qquad\qquad \le \|f_i - f\|_{\mathcal{C}} + w(\|g_i - g\|_{\mathcal{C}}, f),\end{aligned}$$

where $\|f\|_{\mathcal{C}} = \sup_{0\le s\le T}|f(s)|$. Function f is continuous on a compact set, hence is uniformly continuous and $w(\delta, f)\to 0$, as $\delta \to 0$. This concludes. □

We prove now a functional limit theorem.

Fact 21. *Let $T > 0$ and let P^1_c be given by (4.2). We have*

$$P^1_c \to^d \sigma^2 B, \quad \text{as } c \to 0.$$

where $\to^d$ denotes the weak convergence in $\mathcal{C}([0,T],\mathbb{R})$ and σ is given by assumption (A2).

Proof. Firstly, we prove the convergence of finite dimensional distributions. To this end we take $0 \le t_1 < t_2 < \ldots < t_n \le T$ and write

$$\left(P^1_c(t_1), P^1_c(t_2), \ldots, P^1_c(t_n)\right) = (H_c(\lfloor g(c)t_1\rfloor), H_c(\lfloor g(c)t_2\rfloor, \ldots H_c(\lfloor g(c)t_n\rfloor))) + R_c,$$

where R_c is a residual term originating from the second summand of (4.2), consisting of terms of the form $(g(c)t_i - \lfloor g(c)t_i\rfloor)(H_c(\lceil g(c)t_i\rceil) - H_c(\lfloor g(c)t_i\rfloor))$. Let $\varepsilon > 0$ we have $g(c)t_i - \lfloor g(c)t_i\rfloor \le 1$ and we observe that $\mathbb{E}X_1(c) = 0$ hence by assumptions (A1) and (A2) we get

$$\begin{aligned}&\mathbb{P}\left(|(g(c)t_i - \lfloor g(c)t_i\rfloor)(H_c(\lceil g(c)t_i\rceil) - H_c(\lfloor g(c)t_i\rfloor))| \ge \varepsilon\right) \le \mathbb{P}\left(|X_1(c)| \ge \varepsilon\right)\\ &\qquad\qquad \le \frac{Var(X_1(c))}{\varepsilon^2} \to 0, \quad \text{as } c \to 0.\end{aligned}$$

Therefore $R_c \to 0$ in probability and a fortiori in law. We define $K_c(i) := H_c(\lfloor g(c)t_{i+1}\rfloor) - H_c(\lfloor g(c)t_i\rfloor)$ for $i \in \{1, 2, \ldots, n-1\}$. Our aim now is to prove their weak convergence. Assumptions (A1)-(A3) are the assumptions of the classical CLT. One easily checks that

$$K_c(i) \to^d \mathcal{N}(0, \sigma^2(t_{i+1} - t_i)), \quad \text{as } c \to 0.$$

Moreover for c's small enough $K_c(i)$ are independent hence we have proved $P^1_c \to \sigma B$ in the sense of finite-dimensional distributions.

We are now left with the proof of tightness. Let $t \in (0,1)$, $\varepsilon > 0$ and $\gamma \in (0,1)$ (this parameter is to be adjusted), using fact that P^1_c is piecewise linear we have

$$A := \mathbb{P}\left(\sup_{t<s<t+\gamma}|P^1_c(t) - P^1_c(s)| > 4\varepsilon\right) \le \mathbb{P}\left(\max_{i\le g(c)\gamma}\left|\sum_{l=1}^{i} X_l(c) + R(c)\right| > 4\varepsilon\right)$$

where $R(c)$ is a certain remainder term stemming from the second summand of (4.2). By assumption (A3) we know that

$$\mathbb{P}\left(|X_i(c)| > 1\right) \le \mathbb{E}|X_i(c)|^{2+\delta} = g(c)^{-1}h(c),$$

for a certain function $h(c)$ such that $h(c) \to 0$ as $c \to 0$. Moreover,

$$\mathbb{E}|X_i(c)|1_{\{|X_i(c)|>1\}} \le \mathbb{E}|X_i(c)|^{2+\delta}1_{\{|X_i(c)|>1\}} \le g(c)^{-1}h(c). \tag{4.5}$$

We define $Y_i(c) := X_i(c)1_{\{|X_i(c)|\leq 1\}}$. We have $0 = \mathbb{E}X_i(c) = \mathbb{E}Y_i(c) + \mathbb{E}X_i(c)1_{\{|X_i(c)|>1\}}$. By (4.5) we conclude that $|\mathbb{E}Y_i(c)| \leq g(c)^{-1}h(c)$. We now can estimate

$$A \leq \mathbb{P}\left(R(c) > 2\varepsilon\right) + \mathbb{P}\left(\max_{i\leq g(c)\gamma}\left|\sum_{l=1}^{i} Y_l(c)\right| > 2\varepsilon\right) =: I + II.$$

We will deal firstly with the second summand. Obviously

$$II \leq \mathbb{P}\left(\max_{i\leq g(c)\gamma}\left|\sum_{l=1}^{i} Y_l(c)\right| > 2\varepsilon\right) + \mathbb{P}\left(\exists_{i\leq g(c)\gamma}|X_i(c)| \geq 1\right)$$
$$\leq \mathbb{P}\left(\max_{i\leq g(c)\gamma}\left|\sum_{l=1}^{i}\left(Y_l(c) - \mathbb{E}Y_l(c)\right) + i\mathbb{E}Y_l(c)\right| > 2\varepsilon\right) + \gamma h(c).$$

Now, we choose c_0 such that for any $c < c_0$ we have $|g(c)\gamma\mathbb{E}Y_i(c)| \leq \varepsilon$. Thus for $c < c_0$ we may write

$$II \leq \mathbb{P}\left(\max_{i\leq g(c)\gamma}\left|\sum_{l=1}^{i}\left(Y_l(c) - \mathbb{E}Y_l(c)\right)\right| > \varepsilon\right) + \gamma h(c).$$

We put this estimation aside for a moment. For any i.i.d. sequence $\{Q_i\}_i$ such that $\mathbb{E}Q_i = 0$ by simple algebra we get

$$\mathbb{E}\left(\sum_{i=1}^{n} Q_i\right)^4 \lesssim n\mathbb{E}Q_1^4 + n^2\left(\mathbb{E}Q_1^2\right)^2.$$

By (A3) and obvious inequality we have $\mathbb{E}(Y_i(c))^4 \leq \mathbb{E}|X_i(c)|^{2+\delta} \leq g(c)^{-1}h(c)$. Using the Minkowski inequality we get

$$\begin{aligned}\mathbb{E}(Y_i(c) - \mathbb{E}Y_i(c))^4 &\leq \left(\left(\mathbb{E}(Y_i(c))^4\right)^{1/4} + |\mathbb{E}Y_i(c)|\right)^4 \\ &\leq \left(\left(g(c)^{-1}h(c)\right)^{1/4} + g(c)^{-1}h(c)\right)^4 \lesssim g(c)^{-1}h(c).\end{aligned}$$

The last inequality follows from the fact that $g(c)^{-1}h(c) \to 0$. By (A2) we have $\mathbb{E}Y_i(c)^2 \leq \mathbb{E}X_i(c)^2 \lesssim \sigma g(c)^{-1}$. Let $\lambda \geq 0$, using the above facts and the Doob inequality we estimate

$$\mathbb{P}\left(\max_{i\leq g(c)\gamma}\left|\sum_{l=1}^{i}\left(Y_l(c) - \mathbb{E}Y_l(c)\right)\right| > \gamma^{1/2}\lambda\right) \leq K\lambda^{-4}\left(\gamma^{-1}h(c) + 1\right),$$

where K is a certain constant depending only on σ. We pick now $0 < \varepsilon,\, \eta < 1$. We can choose λ such that $K\lambda^{-2} \leq \frac{1}{2}\eta\varepsilon^2$ and $\lambda^{-2} < \varepsilon^{-2}$. Now we fix $\gamma = \varepsilon^2\lambda^{-2}$. Putting these to the above expression we obtain

$$\mathbb{P}\left(\max_{i\leq g(c)\gamma}\left|\sum_{l=1}^{i}\left(Y_l(c) - \mathbb{E}Y_l(c)\right)\right| > \varepsilon\right) \leq \frac{1}{2}\eta\left(\lambda^2\varepsilon^{-2}h(c) + 1\right).$$

When c is small enough we have $\frac{1}{2}\eta\left(\lambda^2\varepsilon^{-2}h(c) + 1\right) \leq \eta$. We may now come back to estimation of A. We know that for any $\varepsilon, \eta > 0$ there exist $\gamma \in (0,1)$ and $c_0 > 0$ that for any $c < c_0$ we have

$$A \leq \mathbb{P}\left(R(c) > 2\varepsilon\right) + \eta + \gamma h(c).$$

The tightness is thus established by appealing to [1, Theorem 7.3]. □

Fact 22. *We have*

$$\frac{M_c}{g(c)} \to^d id, \quad \text{as } c \to 0,$$

where $\to^d$ *denotes the weak convergence in* $\mathcal{C}([0,T],\mathbb{R})$ *and* $id(t) = t$.

Proof. We first define a family of processes

$$U_c(t) := \sum_{i=1}^{\lfloor g(c)t \rfloor} \tilde{D}_i(c) + (g(c)t - \lfloor g(c)t \rfloor)\tilde{D}_{\lfloor g(c)t \rfloor + 1}(c), \quad t \geq 0, \tag{4.6}$$

where $\tilde{D}_i(c) := D_i(c) - \mathbb{E}D_i(c) = D_i(c) - g(c)^{-1}$. Our aim now is to prove that $U_c \to^d 0$.

Firstly let us notice that in assumption (A4) we may always assume $\delta \leq 1$. Indeed, if it is not the case, by the Minkowski inequality we have

$$\mathbb{E}D_i(c)^2 \leq \left(\mathbb{E}D_i(c)^{1+\delta}\right)^{2/(1+\delta)} \leq \left(C(\mathbb{E}D_i(c))^{1+\delta}\right)^{2/(1+\delta)} \lesssim (\mathbb{E}D_i(c))^2.$$

By assumption (A4) and by the Minkowski inequality one can show

$$\mathbb{E}|\tilde{D}_i(c)|^{1+\delta} \lesssim g(c)^{-1-\delta}.$$

Moreover $\mathbb{E}|\tilde{D}_i(c)|1_{\{|\tilde{D}_i(c)| \geq 1\}} \leq \mathbb{E}|\tilde{D}_i(c)|^{1+\delta}$. We define truncated random variables

$$E_i(c) := \tilde{D}_i(c)1_{\{|\tilde{D}_i(c)|<1\}}.$$

Without loss of generality we assume that (A4) holds with $\delta \leq 1$ and we easily get

$$\mathbb{E}(E_i(c))^2 \leq \mathbb{E}|\tilde{D}_i(c)|^{1+\delta} \lesssim g(c)^{-1-\delta}. \tag{4.7}$$

Since $\mathbb{E}E_i(c) = -\mathbb{E}\tilde{D}_i(c)1_{\{|\tilde{D}_i(c)| \geq 1\}}$ hence we get

$$|\mathbb{E}E_i(c)| \leq \mathbb{E}|\tilde{D}_i(c)|^{1+\delta} \lesssim g(c)^{-1-\delta}. \tag{4.8}$$

Let us fix $T > 0, \varepsilon > 0$. We obtain

$$\begin{aligned} I(c) := \mathbb{P}\left(\sup_{0 \leq s \leq T} |U_c(s)| > 2\varepsilon\right) &\leq \mathbb{P}\left(\max_{0 \leq j \leq \lceil g(c)T \rceil} \left|\sum_{i=1}^{j} \tilde{D}_i(c)\right| > 2\varepsilon\right) \\ &\leq \mathbb{P}\left(\max_{0 \leq j \leq \lceil g(c)T \rceil} \left|\sum_{i=1}^{j} E_i(c)\right| > 2\varepsilon\right) + \mathbb{P}\left(\exists_{j \leq \lceil g(c)T \rceil} |\tilde{D}_j(c)| \geq 1\right). \end{aligned}$$

By (4.8) we can choose c small enough to have $\lceil Tg(c) \rceil |\mathbb{E}E_i(c)| \leq \varepsilon$, moreover we apply the Chebyshev inequality and obtain

$$I(c) \leq \mathbb{P}\left(\max_{0 \leq j \leq \lceil g(c)T \rceil} \left|\sum_{i=1}^{j} (E_i(c) - \mathbb{E}E_i(c))\right| > \varepsilon\right) + (g(c)T + 1)\mathbb{E}|\tilde{D}_j(c)|^{1+\delta}.$$

Estimate (4.7) and the Doob inequality for c small enough implies that

$$I(c) \leq \varepsilon^{-2}(g(c)T + 1)\mathrm{Var}\,(E_i(c)) + (g(c)T + 1)\mathbb{E}|\tilde{D}_j(c)|^{1+\delta} \lesssim g(c)^{-\delta} \to 0, \quad \text{as}\, c \to 0.$$

This proves that $\sup_{0 \leq s \leq T} |U_c(s)| \to^d 0$. Now we notice that $(M_c - 1)/g(c) = (t + U_c(t))^{-1}$, where $^{-1}$ denotes the inverse function. This is always well-defined as $t + U_c(t)$ is almost surely strictly increasing. Let us fix $0 < \varepsilon < \frac{1}{2}T$ and $A := \left\{\sup_{0 \leq s \leq 2T} |U_c(s)| \leq \frac{1}{2}\varepsilon\right\}$. On the set A we have $\sup_{0 \leq s \leq T} \left|\frac{M_c(t)}{g(c)} - t\right| \leq \varepsilon$, and $\mathbb{P}(A) \to 1$. □

We are now ready to prove our main result.

Proof. (of Theorem 13) By Fact 21, Fact 22 and Lemma 19 we have

$$(P_c^1, g(c)^{-1}M_c) \to^d (B, id), \quad \text{as } c \to 0,$$

with respect to the topology of $\mathcal{C}([0,2T],\mathbb{R}^2)$. Applying Lemma 20 yields that the family of processes $\left\{P_c^2(t), t\in[0,2T]\right\}_c$, defined by the formula $P_c^2(t) := P_c^1((g(c)^{-1}M_c(t))\wedge 2T)$ converges in $\mathcal{C}([0,2T],\mathbb{R})$ to $\sigma^2 B$. Our final step is to compare this process with P_c. We notice that they agree whenever M_c is an integer, by the construction (4.2) and (A4) we conclude that

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le s\le T}|P_c^2(s)-P_c(s)|>\varepsilon\right) &\le \mathbb{P}\left(\max_{i\le\lceil M_c(T)\rceil}|X_i(c)|>\varepsilon\right)\\ &\le \mathbb{P}\left(\max_{i\le\lceil 2Tg(c)\rceil}|X_i(c)|>\varepsilon\right)+\mathbb{P}\left(M_c(T)>2g(c)T\right)\\ &\le \lceil 2Tg(c)\rceil\varepsilon^{-(2+\delta)}\mathbb{E}|X_i(c)|^{2+\delta}+\mathbb{P}\left(M_c(T)>2g(c)T\right)\to 0,\end{aligned}$$

as $c\to 0$. An application of [1, Theorem 3.1] is enough to prove the convergence in the Skorohod topology. □

5. Proofs of the results of functional convergence

5.1. Truncated variation.

Proof of Theorem 1. The strategy of the proof is to approximate the process TV_c^μ using results of Lemma 11 by a renewal-type process and then use Theorem 13. Let us recall the notation of Section 3 (e.g. (3.1)). By the strong Markov property of Brownian motion we have that $Z_{D,k}(c), k=1,2,...,$ is an i.i.d. sequence and

$$(T_{2k+1}(c)-T_{2k}(c), Z_{D,k}(c)) =^d (T_D(c), Z_D(c)),$$

where $Z_D(c) := W_{T_D(c)} + c$. The formula [10, (1.1)] reads as

$$\mathbb{E}\exp(\alpha Z_D(c)-\beta T_D(c)) = \frac{\delta\exp(-(\alpha+\mu)c)\exp(\alpha c)}{\delta\cosh(\delta c)-(\alpha+\mu)\sinh(\delta c)}, \tag{5.1}$$

where $\delta=\sqrt{\mu^2+2\beta}$. This formula is valid if $\alpha<\coth(\delta c)-\mu$ and $\beta>0$. If $\mu\neq 0$ we may also put $\beta=0$. From (5.1) we easily calculate

$$\mathbb{E}T_D(c) = \frac{e^{2c\mu}-2c\mu-1}{2\mu^2} = c+o(c). \tag{5.2}$$

Further we notice that the distribution of

$$\left(T_{2k+2}(c)-T_{2k+1}(c), Z_{U,k}(c)\right),$$

is the same as the distribution of $(T_U(c), Z_U(c))$, where $Z_U(c) := W_{T_U(c)} - c$, and is the same as $(T_D(c), Z_D(c))$ if we considered a Brownian motion with drift $-\mu$.

For $k=1,2,...$ we also have $(D_k(c), Z_k(c)) =^d (T_D(c)+T_U(c), Z_D(c)+Z_U(c))$ and $(T_D(c), Z_D(c))$, $(T_U(c), Z_U(c))$ are independent.

We will use the renewal theory we developed in Section 4. To this end we define

$$M_c(t) := \min\left\{k : \sum_{l=1}^k D_l(c) > t\right\}, \tag{5.3}$$

and

$$S_c(k) = \sum_{l=1}^{k} Z_l(c).$$

We will apply Theorem 13 which will require verifying conditions (A1) - (A4). Hence we present some auxiliary lemmas.

Lemma 23. *For any $p > 0$ and non-negative random variables X and Y we have*

$$\mathbb{E}(X+Y)^p \leq \mathbb{E}(2\max\{X,Y\})^p \leq 2^p \mathbb{E}(X^p + Y^p).$$

Lemma 24. *We have*

$$\mathbb{E}D_1(c)^4 \lesssim c^8,\ \mathbb{E}Z_1(c)^4 \lesssim c^4.$$

Proof. The proof goes by the simple computation using the Laplace transform (5.1). When $\mu \neq 0$ we have

$$\mathbb{E}T_D(c)^4 = \frac{3e^{8c\mu} + e^{6c\mu}(15-42c\mu) + 6e^{4c\mu}(2-5c\mu)^2 - 18ce^{2c\mu}\mu(4+3c\mu(-3+2c\mu))}{2\mu^8} + \frac{2c\mu(12+c\mu(-3+c\mu-42)(-1+c\mu))}{2\mu^8}.$$

For $\mu = 0$ one has $\mathbb{E}T_D(c)^4 = \frac{277}{21}c^8$. In either case one checks that $\mathbb{E}T_D(c)^4/c^8 \to \frac{277}{21}$, as $c \to 0$, hence $\mathbb{E}T_D(c)^4 \lesssim c^8$. Similarly one checks $\mathbb{E}T_U(c)^4 \lesssim c^8$. Now, by Lemma 23 and definition of $D_1(c)$, $\mathbb{E}D_1(c)^4 \lesssim c^8$.

Analogously, one may check that $Z_D(c)$ has exponential distribution and

$$\mathbb{E}Z_D(c)^4 = \begin{cases} \frac{3(\exp(2c\mu)-1)^4}{2\mu^4} & \text{if } \mu \neq 0, \\ 24c^4 & \text{if } \mu = 0. \end{cases}$$

In either case one checks that $\mathbb{E}Z_D(c)^4/c^4 \to 24$, as $c \to 0$, hence $\mathbb{E}Z_D(c)^4 \lesssim c^4$. Similarly $\mathbb{E}Z_U(c)^4 \lesssim c^4$ and by Lemma 23, $\mathbb{E}Z_1(c)^4 \lesssim c^4$.

We used Mathematica to facilitate above computations. The appropriate Mathematica notebook is available at `http://www.mimuw.edu.pl/~pmilos/calculations.nb.` □

Now we will check the assumptions of Theorem 13. Assumption (A1) is obvious. We have

$$f(c) := \frac{\mathbb{E}Z_1(c)}{\mathbb{E}D_1(c)} = \mu\coth(c\mu) = c^{-1} + O(c). \tag{5.4}$$

We denote the fraction in assumption (A2) as σ_c^μ and calculate

$$(\sigma_c^\mu)^2 = \begin{cases} \frac{2-2c\mu\coth(c\mu)}{\sinh^2(c\mu)} + 1 & \text{if } \mu \neq 0, \\ 1/3 & \text{if } \mu = 0 \end{cases} \to \frac{1}{3} \quad \text{as } c \to 0.$$

Now we proceed to verification of assumption (A3). Using Lemma 23 and 24 we get

$$\mathbb{E}X_1(c)^4 \lesssim \mathbb{E}Z_1(c)^4 + f(c)^4\mathbb{E}D_1(c)^4 \lesssim c^4.$$

We easily check that $\mathbb{E}D_1(c) \approx c^2$ and see that assumption (A3) holds for $\delta = 2$. We are left with assumption (A4). By (5.2) and Lemma 24 it could be easily verified for $\delta = 3$.

Thus, since $f(c) = c^{-1} + O(c)$, by Theorem 13 we obtain:

Corollary 25. *Let S_c, M_c be defined according to (4.1) for the $Z_c(i), D_c(i)$ above. For any $T > 0$ we have*

$$\left(S_c\left(M_c(t)\right) - c^{-1}t\right) \to^d 3^{-1/2}B_t,$$

where $\to^d$ is understood as weak convergence in $\mathcal{C}([0,T],\mathbb{R})$ topology.

The final stage is to compare process TV_c^{μ} and process $S_c\left(M_c(\cdot)\right)$ with the use of Lemma 11. Since $D_0(c)$ has different distribution than $D_k(c)$ for $k = 1, 2, ...$, we introduce two auxiliary objects

$$\tilde{M}_c(t) := \min\left\{k : \sum_{l=0}^{k} D_l(c) > t\right\}$$

and

$$\tilde{S}_c(k) = \sum_{l=0}^{k-1} Z_l(c).$$

These differ from $S_c(k)$ and M_c by starting the summation from $l = 0$. After small changes of the definitions of the appropriate processes we see that the thesis of Theorem 13 holds also in our case and we obtain

$$\left(\tilde{S}_c\left(\tilde{M}_c(t)\right) - c^{-1}t\right) \to^d 3^{-1/2}B_t, \tag{5.5}$$

where $\to^d$ is understood as weak convergence in $\mathcal{C}([0,T],\mathbb{R})$ topology.

From this definition and Lemma 11 we see that the processes TV_{μ}^{c} and $\tilde{S}_c\left(\tilde{M}_c(t)\right)$ coincide at random times $T_{2k}, k = 0, 1, 2, ...$, moreover, both are increasing, hence, for any $t \geq 0$ and $\varepsilon > 0$

$$\mathbb{P}\left(\sup_{t\in[0,T]}\left|TV_{\mu}^{c}(t) - \tilde{S}_c(\tilde{M}_c(t))\right| > \varepsilon\right) \leq \mathbb{P}\left(\sup_{t\in[0,T]} Z_{\tilde{M}_c(t)}(c) > \varepsilon\right). \tag{5.6}$$

Now, by (5.6) we estimate

$$\mathbb{P}\left(\sup_{t\in[0,T]}\left|TV_c^{\mu}(t) - \tilde{S}_c\left(\tilde{M}_c(t)\right)\right| > \varepsilon\right)$$
$$\leq \mathbb{P}\left(\max_{k\leq 2T/\mathbb{E}D_1(c)+1} Z_k(c) \geq \varepsilon\right) + \mathbb{P}\left(\tilde{M}_c(t) \geq \frac{2T}{\mathbb{E}D_1(c)} + 1\right).$$

The first term could be estimated by the Chebyshev inequality and the estimates of $\mathbb{E}Z_1(c)^4$ and $\mathbb{E}D_1(c)$

$$\mathbb{P}\left(\max_{k\leq 2T/\mathbb{E}D_1(c)+1}|Z_i(c)| > \varepsilon\right) \leq \left(\frac{2T}{\mathbb{E}D_1(c)} + 1\right)\frac{\mathbb{E}Z_1(c)^4}{\varepsilon^4} \to 0, \quad \text{as } c \to 0.$$

The convergence of the last term to 0 could be established by simple calculation using assumption (A4). One could also use Fact 22. From this and (5.5) the thesis follows.

Proof of Theorem 4. The strategy of the proof is to find a renewal-type processes G_n which approximates the process in the theorem. In order to prove the convergence of G_n we will use [5, Theorem V.4.1]. In the final step we will show that the approximation error converges to 0.

Let us define a family of processes

$$G_n(t) := \frac{S_c(M_c(nt)) - m_c^{\mu}nt}{\sigma_c^{\mu}\sqrt{n}}, \quad t \geq 0, n \in \mathbb{N},$$

where $m_c^\mu = f(c) = \frac{\mathbb{E}Z_1(c)}{\mathbb{E}D_1(c)}$ and $(\sigma_c^\mu)^2 = \mathrm{Var}((\mathbb{E}Z_1(c))D_1(c) - (\mathbb{E}D_1(c))Z_k(c))(\mathbb{E}D_1(c))^{-3}$ were calculated in the previous subsection. By [5, Theorem V.4.1] we know that $G_n \to^d B$ in the Skorohod topology. Similarly

$$\tilde{G}_n(t) := \frac{\tilde{S}_c(\tilde{M}_c(nt)) - m_c^\mu nt}{\sigma_c^\mu \sqrt{n}} \to B_t, \quad \text{as } n \to \infty.$$

Our final step is to estimate

$$\frac{TV_c^\mu(nt) - \tilde{S}_c(\tilde{M}_c(nt))}{\sigma_c^\mu \sqrt{n}}.$$

Similarly as in the proof of Theorem 1 we estimate

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,T]} \frac{|TV_c^\mu(nt) - \tilde{S}_c(\tilde{M}_c(nt))|}{\sqrt{n}} > \varepsilon\right) &\leq \mathbb{P}\left(\max_{k\leq 2nT/\mathbb{E}D_1(c)+1} |Z_k(c)| > \varepsilon\sqrt{n}\right) \\ &+ \mathbb{P}\left(\tilde{M}_c(nT) > \frac{2nT}{\mathbb{E}D_1(c)} + 1\right).\end{aligned}$$

The first term can be estimated by the Chebyshev inequality

$$\mathbb{P}\left(\max_{k\leq 2nT/\mathbb{E}D_1(c)+1} |Z_k(c)| > \varepsilon\sqrt{n}\right) \leq \left(\frac{2nT}{\mathbb{E}D_1(c)} + 1\right)\frac{\mathbb{E}Z_1(c)^4}{n^2\varepsilon^4} \to 0, \quad \text{as } n \to +\infty.$$

The second one converges to 0 by the law of large numbers. In this way we proved that the limit of the processes in theorem is the same as the one of $\tilde{G}_n$'s.

5.2. **Upward and downward truncated variation.** While the proofs in the previous section rely on Lemma 11, the ones in this section hinge on Lemma 12. The flow of the proofs of this section is much alike the ones in the Section 5.1. The main difficulty is to calculate the of moments $(T_{D,1}(c), Z_{D,1}(c))$.

This will be done with the use of bivariate Laplace transform of $(T_D(c), Z_D(c))$ calculated in the next subsection.

5.2.1. *Bivariate Laplace transform of* $T_D(c)$ *and* $Z_D(c)$. In [8] two-dimensional density of the variables $T_D(c)$ and $\sup_{0\leq s<t\leq T_D(c)}\{W_s - W_t\}$ is calculated. This density is given by [8, formula (11)]. Using it, we unconsciously calculated bivariate Laplace transform $\mathbb{E}\exp(\lambda Z_D(c) + \nu T_D(c))$ which is given in [8] by the formula (20). This formula reads (using notation from [8]) as

$$\mathbb{E}\exp(\lambda Z_D(c) + \nu T_D(c)) = \left(1 - \lambda\frac{L_0^{-W}(-\nu, c)}{T_{-\mu,1}(-\nu, c) + \lambda}\right)\mathbf{E}e^{\nu T_D(c)}, \tag{5.7}$$

where

$$L_0^{-W}(-\nu, c) = \frac{U_\mu(\nu)}{-2\nu}\left\{\frac{e^{\mu c}(U_\mu(\nu)\coth(cU_\mu(\nu)) - \mu)}{\sinh(cU_\mu(\nu))} - \frac{U_\mu(\nu)}{\sinh^2(cU_\mu(\nu))}\right\},$$

$$\mathbf{E}e^{\nu T_D(c)} = \frac{U_\mu(\nu)e^{-\mu c}}{U_\mu(\nu)\cosh(cU_\mu(\nu)) - \mu\sinh(cU_\mu(\nu))},$$

$$T_{-\mu,1}(-\nu, c) = \mu - U_\mu(\nu)\coth(cU_\mu(\nu))$$

and

$$U_\mu(\nu) = \sqrt{\mu^2 - 2\nu}.$$

Substituting the above formulas in (5.7) we obtain [2]:

[2] see also `http://www.mimuw.edu.pl/~pmilos/calculations.nb`.

Corollary 26. *The bivariate Laplace transform of the variable* $(T_D(c), Z_D(c))$ *reads as*

$$\mathbb{E}\exp\left(\lambda Z_D(c) - \nu T_D(c)\right)$$
$$= 1 - \left(1 - \frac{\lambda\nu^{-1}}{\sinh\left(2c\delta\right)/\delta - 2\left(\lambda+\mu\right)\sinh^2\left(c\delta\right)/\delta^2}\right)\left(1 - \frac{e^{-\mu c}}{\cosh\left(c\delta\right) - \mu\sinh\left(c\delta\right)/\delta}\right),$$

where $\delta = \sqrt{\mu^2 + 2\nu}$.

Proof of Theorem 7. This time we adhere to the proof of Theorem 1. We will concentrate on differences leaving the reader the task of filling the rest of details. We calculate

$$f(c) := \frac{\mathbb{E}Z_{D,1}(c)}{\mathbb{E}T_{D,1}(c)} = \frac{1}{2}\mu\left(\coth(c\mu)+1\right) = \frac{1}{2}c^{-1} + \frac{1}{2}\mu + O(c).$$

Further we have

$$\left(\sigma_c^\mu\right)^2 = \begin{cases} \frac{2\exp(4c\mu)(\sinh(2c\mu)-2c\mu)}{(\exp(2c\mu)-1)^3} & \text{if } \mu \neq 0, \\ 1/3 & \text{if } \mu = 0 \end{cases} \to \frac{1}{3} \quad \text{as } c \to 0.$$

To verify the assumption (A3) we firstly notice that $T_{D,1}(c)$ and $Z_{D,1}(c)$ are majorised by $D_1(c)$ and $Z_1(c)$ respectively. Using it we get

$$\mathbb{E}X_1(c)^4 \lesssim \mathbb{E}Z_{D,1}(c)^4 + f(c)^4\mathbb{E}T_{D,1}(c)^4 \lesssim c^4.$$

We easily check that $\mathbb{E}T_{D,1}(c) \approx c^2$ and see that assumption (A3) holds for $\delta = 2$. Assumption (A4) could be easily verified for $\delta = 3$.

Next step - comparison of the process $UTV_c^\mu(t)$ with the appropriate reneval process is even simpler, since the distribution of $Z_{D,1}(c)$ - first term in the sum appearing in the first equation in Lemma 12 - is the same as the distribution of the further terms.

Proof of Theorem 8. This proof goes along the lines of the proof of Theorem 4.

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